Optimal patent licensing: from three to two-part tariff

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Abstract

This paper considers patent licensing of a cost reducing innovation in a Cournot oligopoly industry with a quite general demand. A licensing policy allows for combinations of three well-used instruments in practice: upfront fee, unit royalty and ad valorem revenue royalty (a fraction of revenue a licensee pays to the innovator). The set of optimal combinations is characterized for an outside innovator. It is shown that the highest innovator's revenue can be achieved with just two instruments: an upfront fee and a per unit royalty (and in some cases with only an upfront fee). We provide a necessary and sufficient condition for the existence of an optimal two-part tariff policy consisting of upfront fee and ad valorem royalty. Our results are demonstrated for the linear demand case and it is shown that there always exist optimal policies with maximum diffusion (all firms except perhaps one become licensees).

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1 Introduction

The literature on patent licensing typically focuses on the optimal licensing contracts for an innovator (either outside or incumbent) and on their impact on welfare, market structure, and the incentive to innovate.

In practice, the payment of a licensee to the innovator either uses one of the following three instruments: upfront fee, per unit royalty and ad valorem revenue royalty, or it is based on a two-part tariff consisting of upfront fee and one of the royalties. A natural question is why we typically[1] do not observe contracts that include all three instruments– a three part tariff – especially if it benefits the innovator. A second question is why in practice we see multiple licensing contracts. Some are based on a pure upfront fee and some are based on a two part tariff with either per unit royalty or with ad valorem royalty. This paper attempts to provide answers to these questions for cost reducing innovations with an outside innovator.

Sen and Tauman (2007,2018) (hereafter ST) following the model of Kamien and Tauman (1986) and Kamien, Oren and Tauman (1992) study general licensing schemes for a cost reducing innovation in a Cournot oligopoly with homogeneous goods and arbitrary number of firms. ST characterize all optimal contracts for the innovator based on combinations of upfront fee and per unit royalty. However, like many others, they ignore the popular instrument of ad valorem royalty. This paper extends ST and characterizes all optimal licensing policies consisting of the above three instruments. To the best of our knowledge, only Bousquet et al. (1998) study the same three-dimension policies but for a new product innovation with only one (risk averse) potential licensee. The use of royalties in their paper is justified via risk sharing considerations when either demand or cost is uncertain. To pinpoint the significance of uncertainty, Bousquet et al. (1998), unlike ours, avoids any strategic interaction in the downstream market and concentrates only on a monopoly licensee.

For a process (cost-reducing) innovation, a pure upfront fee is the best policy of the innovator in case the industry is a monopoly or in case the innovation is drastic. But for an oligopoly industry, the strategic interaction of firms can lead to an optimal licensing policy for the innovator with positive royalty component even with no uncertainty. Dealing with a Cournot industry of an arbitrary size and a quite general demand function, we first characterize for every number of licensees, the set of all feasible three-dimensional policies which are optimal for the innovator and acceptable for firms. We use this characterization to show that the maximum revenue the innovator can extract with any combination of these three instruments, can be achieved with just two instruments: upfront fee and per unit royalty. Combinations of upfront fee and ad valorem royalty may not be sufficient. We provide a necessary and sufficient condition for the existence of such an optimal two-part tariff scheme. Finally, we illustrate our

[1] An exception is franchising contracts where a "franchisor" allows "franchisee" to use its trademark and duplicate its business in certain locations. Typically the francisor limits the degree of control over the operation of its franchisees. In many restaurants franchises the franchisees pay the franchisor an upfront fee at the time of granting a license, together with a percentage of future revenues (ad valorem royalty, typically around 5-6%). In addition sometimes franchisees are also obliged to use certain inputs (like special sauces) that produced solely by the franchisor who sets their prices. Such a franchising contract is in the spirit of three part tariff.

results in case demand is linear and show that there always exist optimal contracts with maximum diffusion of the innovation.

The multiplicity of optimal licensing contracts may explain the variety of licensing contracts we see in practice. In the real world, depending on the nature of the product, a contract with one royalty type may be more suitable than the other. For instance, per unit royalty typically fits better low value commodities sold in bulk. On the other hand, licensing contracts for goods that are either difficult to be measured by quantities or are of high value and sensitive to a price change are likely to include an ad valorem royalty component. The multiplicity of optimal contracts allows the innovator to choose one fitting best to market specifics.

2 Literature review

There is a vast literature on patent licensing of either cost-reducing, quality improving or new product innovations. The first paper is by Arrow (1962) who answers the question whether a perfectly competitive or a monopolistic industry has a greater incentive to innovate. To that end, he used the pure per unit royalty contract and showed a perfectly competitive industry provides a higher incentive to innovate. Katz and Shapiro (1985,1986) and independently Kamien and Tauman (1984, 1986) and Oren, Kamien and Tauman (1992) consider licensing contracts of either pure up-front fee (f) or pure unit royalty (r). The first two papers consider a general oligopoly while the last three deal with Cournot oligopoly with either linear or general demand structures and with a constant marginal cost. These enable the explicit analysis of the post innovation market structure and the comparison of the pre innovation and post innovation welfare. For instance, in pure up-front fee contracts, all firms are worse off and consumers are better off as the result of the innovation.

Sen and Tauman (2007, 2018) extended the above to two-part tariff contracts (f, r). Surprisingly, very little attention was paid in the literature to ad valorem royalty contracts, (f, v) or v, even though in practice most licensing contracts include ad valorem royalties. Data of French firms find 78% of licensing contracts include royalties, but only 4% of them are per unit royalty and 96% of them are ad valorem royalty (with or without upfront fee) (See Bousquet et al. (1998)).

In most literature that does consider ad valorem royalties, the innovator is one of two incumbent firms (a duopoly). A duopoly with an outside innovator involves three players. There is an important strategic distinction between the two situations. When the innovator is an incumbent firm in a duopoly, the reservation payoff of a potential licensee is a constant independent of licensing policies. If it refuses to purchase a license, there is no licensee at all. This is not the case in a duopoly or more generally with oligopoly with an outside innovator; there the profit a firm obtains without a license depends on the number of other firms having licenses in addition to the licensing policy. This strategic aspect, which affects the willingness to pay for a license, is absent when there is only one potential licensee.

As mentioned in Section 1, Bousquet et al. (1998) is the first paper which considers ad valorem royalty as one component of a licensing contract and the first to consider three-part tariff. The paper deals with a single risk averse producer and an outside innovator. The source of uncertainty has a crucial impact on the type of royalties used in an optimal contract. Without uncertainty, the optimal policy for the innovator is to charge upfront fee only. Hernandez-Murillo and Llobet (2006) consider an outside innovator of a process innovation in a Dixit and Stiglitz (1977) model of monopolistic competition with continuum of firms producing differentiated goods. The new technology affects firms differently and their marginal costs are private information. The special utility structure of a representative consumer allows the analysis of the incentive compatible optimal two-part tariff with ad valorem royalty. Similar to Bousquet et al. (1998), the reservation price of a licensee does not depend on the proportion of other licensees and their model too abstracts from firms' strategic interaction.

San Martín and Saracho (2010, 2015) and Colombo and Filippini (2015) analyze process innovation by a incumbent firm that reduces the cost to zero. While San Martín and Saracho (2010, 2015) deal with product differentiation in a Cournot duopoly with linear demand, Colombo and Filippini (2015) deal with Bertrand duopoly. The zero cost assumption on the new technology reduces the ad valorem revenue royalty to a profit sharing payment, thus avoiding complicated computations. San Martín and Saracho (2010, 2015) show that an incumbent innovator favors ad valorem royalties to per unit royalties and the two contracts are equivalent in case products are perfect substitutes. Colombo and Filippini (2015) prove the opposite with price competition. All these results are in line with our findings applied to homogeneous product.

Heywood, Li and Ye (2014) has a similar model to San Martín and Saracho (2010, 2015) with Cournot duopoly, linear demand, internal innovator and zero post innovation marginal cost. The marginal cost of the licensee is its private information and it is either some commonly known positive constant or it is zero. It is shown that if the innovator restricts itself to contracts which are acceptable only to low cost type, then the upfront fee part of the optimal two-part tariff (whether with per unit royalty or with ad valorem royalty) is zero and the optimal ad valorem royalty contract yields the innovator more than the optimal per unit royalty contract. This is in line with San Martín and Saracho (2010, 2015).

Colombo and Filippini (2016) show numerically in a Cournot duopoly with an internal innovator who reduces the marginal cost to zero, that the best two part tariff with profit sharing contract has zero upfront fee and it is better than any two-part tariff with per unit royalty. This is not ture if the innovator is an outsider. Colombo, Ma, Sen and Tauman (2020) prove that profit sharing contracts with an outside innovator are equivalent to upfront fee contracts and this is the reason we deal only with the three aforementioned instruments and not with profit sharing payments.

Llobet and Padilla (2016) study patent licensing by an outside innovator to a monopoly downstream market and compare the welfare of ad valorem and per unit royalties. Ad valorem royalties tend to lead to lower prices, benefit upstream innovators and do not necessarily hurt downstream producers. This benefit increases when there are multiple innovators contributing complementary technologies.

Ma and Tauman (2020) study a new product innovation whose demand is uncertain. The innovator licenses his technology to a number (of his choice) of risk averse potential Cournot producers by either one of the above three instruments or by two-part tariff with one royalty. They show that the innovator is best off with a pure upfront fee if licensees are risk neutral. Otherwise, a combination of upfront fee and ad valorem royalties is optimal.The innovator and consumers are better off while licensees are worse off with an ad valorem royalty component compared with a per unit royalty component.

Fan et al. (2018) compare in a Cournot duopoly a two-part tariff with a per unit royalty to that with an ad valorem royalty. It is shown that licensing with per unit royalty is more profitable for the innovator if the innovator is more efficient in using the innovation, whereas licensing with ad valorem royalty is more profitable if the licensee uses it more efficiently. Banerjee, Mukherjee and Poddar (2021) look into transportation cost reducing innovation licensed by an insider innovator in a spatial duopoly model. While under two-part tariff contracts, the innovator is better off with the per-unit royalty contract, if a licensing contract allows an up-front fixed-fee, per-unit royalty and ad-valorem royalty, then the combination of all three instruments can do better compared to any two-part tariff.

3 The model

Consider a Cournot oligopoly with n firms where the set of firms is $N = \{1, \ldots, n\}$. For $i \in N$, let q_i be the quantity produced by firm i and $Q = \sum_{i \in N} q_i$. The inverse demand is $p(Q)$. Initially all firms have constant marginal cost c, where $0 < c < p(0)$. An outside innovator $\mathcal{I}$ has a patent for a new technological innovation that reduces the marginal cost from c to $c - \varepsilon$, where $0 < \varepsilon < c$. ε is the magnitude of the innovation.

Following Kamien, Oren, Tauman (1992) (hereafter KOT), we deal with inverse demand functions which satisfy the following two assumptions:

(A1) The total revenue function $Q\,p(Q)$ is strictly concave in Q.

(A2) $p(Q)$ is decreasing and differentiable for $p > 0$ and the price elasticity $\eta(p) = -p\,\frac{Q'(p)}{Q(p)}$ is a non-decreasing function of p.

Licensing policies: $\mathcal{I}$ chooses a licensing policy to license the new technology to some or all firms in N. We consider general policies of the form (k, f, r, v) where $f \geq 0, 0 \leq r \leq \varepsilon$, $k = 1, \ldots, n$, and $0 \leq v \leq 1$. Namely, $\mathcal{I}$ offers to sell at most k licenses at an upfront fee f, unit royalty r and ad valorem royalty v. Any licensee pays $\mathcal{I}$ the fee f upfront, r for every unit it produces and fraction v of its revenue.

For any triplet (k, r, v), $\mathcal{I}$ wish to set the upfront fee to extract the maximum possible surplus from licensees. The best way to do this is through an auction, where $\mathcal{I}$ first announces (k, r, v) and then auctions off k licenses (possibly with a minimum bid) so that the upfront fee a licensee pays is its winning bid. Formally,

Definition. A licensing policy $A(k, r, v)$ is defined by:

(1) $\mathcal{I}$ offers the firms to sell at most k licenses in a first price sealed bid auction. $\mathcal{I}$ commits to sell $\min(m, k)$ of licenses, where m is the number of firms that submit bids[2]. Bids must be non-negative.

[2] If $m \leq k$ firms place bids, each of the bidding firms wins a license. If $m > k$ firms place bids, bids

(2) Each licensee pays $\mathcal{I}$ its bid upfront in addition to a per unit royalty r and ad valorem royalty v (a proportion v of its revenue) as soon as productions and sales take place.

The strategic interaction between $\mathcal{I}$ and firms in N is modeled as the *licensing game* Γ that has the following three stages. In Stage 1, $\mathcal{I}$ selects a licensing policy $A(k, r, v)$. In Stage 2, firms simultaneously decide to bid for a license or not and the set of licensees is determined. In Stage 3, firms, licensees and non-licensees, are engaged in a Cournot competition.

Payoffs Consider a licensing policy $A(k, r, v)$. Let L be the set of licensees determined in Stage 2 and $\bar{L} = N \setminus L$ the set of non-licensees. For $i \in L$, denote by f_i the bid at which firm i has won a license. This is the upfront fee it pays. Any licensee i has marginal cost $c - \varepsilon$ and any non-licensee has marginal cost c. Their payoffs are

$$\pi_i = \begin{cases} (1-v)p(Q)q_i - (c-\varepsilon)q_i - rq_i - f_i & \text{if } i \in L \\ [p(Q) - c]q_i & \text{if } i \in \bar{L} \end{cases} \tag{1}$$

The payoff of $\mathcal{I}$ consists of (i) revenue from unit royalty $\sum_{i \in L} rq_i$, (ii) revenue from ad valorem royalty $\sum_{i \in L} vp(Q)q_i$ and (iii) revenue $\sum_{i \in L} f_i$ received from the auction. It is

$$\Pi_I = \sum_{i \in L} [rq_i + vp(Q)q_i + f_i] \tag{2}$$

We confine to Subgame Perfect Nash Equilibrium (SPNE) of Γ.

For $k = 1, 2, ..., n$, let $\Gamma(k, r, v)$ be the subgame of Γ following $\mathcal{I}$'s choice of the licensing policy $A(k, r, v)$.

Remark 1 Observe from (1) that if $v = 1$, a licensee obtains negative or zero payoff. Clearly a negative payoff will not be acceptable to a firm. With $v = 1$, the payoff of a licensee can be zero iff both $q_i = 0$ and $f_i = 0$, but in that case by (2), $\mathcal{I}$ has zero licensing revenue, so such a licensing policy is unacceptable. Henceforth we rule out unacceptable policies, and we only consider policies with $0 \le v < 1$.

Definition. (Arrow(1962)) An innovation is drastic iff the monopoly price under the new technology does not exceed the competitive price, c, under the old technology. Equivalently, an innovation is drastic iff $\varepsilon \ge \frac{c}{\eta(c)}$ ($\varepsilon \ge a - c$, in case the demand is $Q = a - p$).

Note that if the innovation is drastic, the innovator, by selling one license, creates a monopoly (all other firms are driven out of the market) and it extracts the entire monopoly profit under the new technology through just a pure upfront fee. Hence from now on, we assume the innovation is non-drastic.

are arranged in descending order as $f_1 \ge \ldots \ge f_k \ge \ldots \ge f_m$. If $f_k > f_{k+1}$, the firms with k highest bids win licenses. If $f_k = f_{k+1}$, then (a) firms with bids strictly higher than f_k win licenses and (b) a random tie breaking rule is applied for firms who place bid f_k to determine who get the remaining licenses.

3.1 Effective magnitude of the innovation

To determine SPNE of Γ, we start with Stage 3 of this game where firms in N are engaged in a Cournot competition. Since the fee f_i is paid upfront, it does not affect the choice of quantities. So by (1), in Stage 3 the relevant payoff of any firm $i \in L$ is

$$\pi_i = (1-v)p(Q)q_i - (c-\varepsilon)q_i - rq_i = (1-v)[p(Q) - (c-\varepsilon+r)/(1-v)]q_i$$

For $r \geq 0$ and $0 \leq v < 1$, denote

$$\delta(r,v) := [\varepsilon - (r+cv)]/(1-v) \tag{3}$$

Then

$$\pi_i = (1-v)[p(Q) - (c-\delta(r,v))]q_i \tag{4}$$

Since $1-v > 0$, by (4), any licensee in a standard Cournot competition solves in Stage 3 the same problem as a firm with marginal cost $c - \delta(r,v)$. This means that under the policy $A(k,r,v)$, the *effective magnitude* of the innovation is $\delta(r,v)$.[3] Note from (3) that $\delta(r,v)$ is decreasing in both r and v, so $\delta(r,v) \leq \delta(0,0) = \varepsilon$.

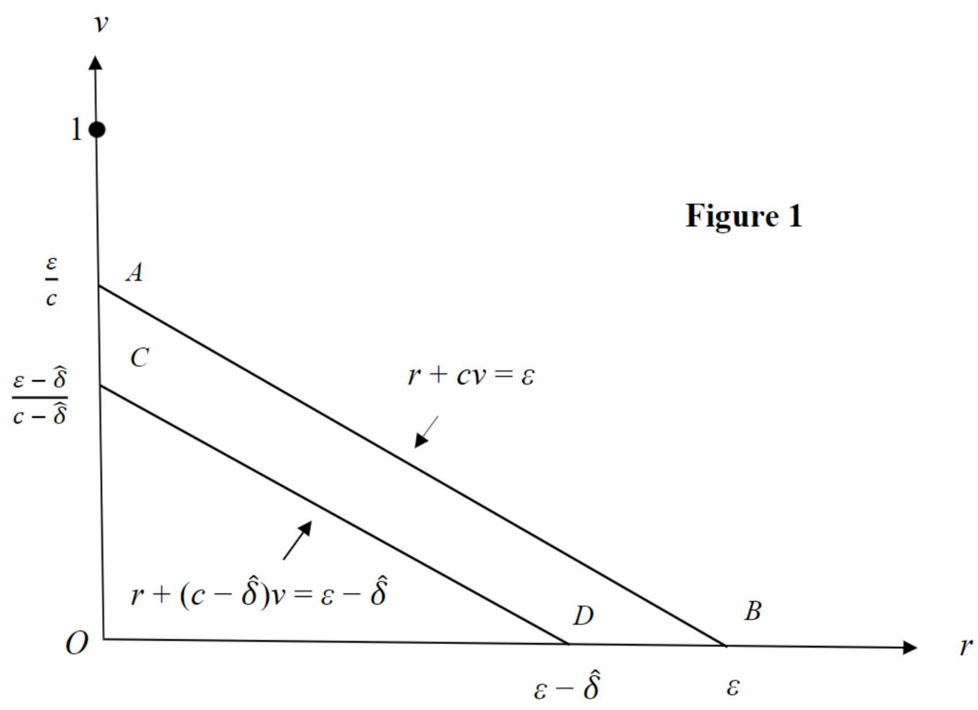


Figure 1 shows that given $k = 1, 2, ..., n$ and δ, $0 < \delta \leq \varepsilon$, there is a continuum of policies (the interval CD) with the same effective magnitude δ of the innovation. By (3), v is linearly depending on r and the set of royalties for all such policies is $RV(\delta) = \{(r,v) | v = \frac{\varepsilon-\delta-r}{c-\delta}, 0 \leq r \leq \varepsilon - \delta\}$.

[3] Since k is fixed and (r,v) is varied, we omit k from $\delta(k,r,v)$ and write $\delta(r,v)$ instead.

Any non-licensee firm has marginal cost c. Assuming no firm will accept a policy that raises its marginal cost,[4] we only consider policies for which the effective magnitude of the innovation is non-negative, so we restrict (r, v) such that $\delta(r, v) \geq 0$.[5] The strategy of $\mathcal{I}$ can also be described as follows: first $\mathcal{I}$ chooses (k, δ), the number of licensees and the effective marginal cost for licensees, $1 \leq k \leq n$, $0 \leq \delta \leq \varepsilon$. Then $\mathcal{I}$ chooses $(r, v) \in RV(\delta)$ and announces the policy $A(k, r, v)$.

Remark 2 For $r \geq 0$ and $0 \leq v < 1$, the unique (r, v) with $\delta(r, v) = \varepsilon$ is $(r = 0, v = 0)$. However, for any $\hat{\delta} \in [0, \varepsilon)$, there is a continuum of (r, v) in $RV(\hat{\delta})$ supporting $\hat{\delta}$. (See intervals AB and CD in Figure 1.)

3.2 Stage 3 Cournot competition

For $k = 0, 1, \ldots, n$ and $\delta \in [0, \varepsilon]$, let $\hat{\mathcal{C}}(k, \delta)$ be the standard Cournot competition with n firms producing one homogeneous good under the inverse demand $p(Q)$, where k firms have marginal cost $c - \delta$ and the remaining $n - k$ firms have marginal cost c. The respective payoffs of a licensee and a non-licensee in $\hat{\mathcal{C}}(k, \delta)$ are $[p(Q) - c + \delta]q_i$ and $[p(Q) - c]q_i$, where $Q = \sum_{i \in N} q_i$. By (4), the third stage of Γ is strategically equivalent to $\hat{\mathcal{C}}(k, \delta)$ where $\delta = \delta(r, v)$ and $(r, v) \in RV(\delta)$. Let $\bar{\phi}(k, \delta)$ and $\underline{\phi}(k, \delta)$ be the Cournot payoffs of a licensee and a non licensee in $\hat{\mathcal{C}}(k, \delta)$, respectively. Note that if $\mathcal{I}$ chooses a policy $A(k, r, v)$, then $\bar{\phi}(k, \delta(r, v))$ and $\underline{\phi}(k, \delta(r, v))$ are the operating profits of a licensee and a non-licensee, respectively. The profit of a licensee is calculated before it pays any royalty or upfront fee to $\mathcal{I}$.

Let $\mathcal{C}(k, \delta)$ be the same game as $\hat{\mathcal{C}}(k, \delta)$ except that the payoff of every licensee in $\mathcal{C}(k, \delta)$ is $(1 - v)$ times its payoff in $\hat{\mathcal{C}}(k, \delta)$ (see (4)). The payoff of a non-licensee remains unchanged. Given the policy $A(k, r, v)$ of $\mathcal{I}$, the third stage of Γ played by the n firms in N is actually the game $\mathcal{C}(k, \delta(r, v))$.

Ignoring licensees' upfront payments to $\mathcal{I}$ in Stage 2, the equilibrium payoffs of firms in $\mathcal{C}(k, \delta(r, v))$ (the third stage of Γ) are$(1 - v)\bar{\phi}(k, \delta(r, v))$ for a licensee and $\underline{\phi}(k, \delta(r, v))$ for a non licensee, respectively.

By (A1) and (A2), the Cournot equilibrium payoffs in $\hat{\mathcal{C}}(k, \delta)$ and in $\mathcal{C}(k, \delta)$ are uniquely determined (see Section 3 of KOT).

The next lemma, which follows from Lemma 2 of KOT, states useful properties of the Cournot payoffs.

Lemma 1 Suppose (A1) and (A2) hold. For any $0 \leq \delta \leq \varepsilon$

(i) $\bar{\phi}(k, \delta) > \underline{\phi}(k, \delta)$

(ii) $\bar{\phi}(k, \delta)$ is decreasing in k, $k = 1, 2, ..., n - 1$.

[4]It can be shown that if a licensing policy raises the marginal cost of a licensee, *in equilibrium*, a licensee would obtain lower payoff than a non-licensee, so such a policy will not be acceptable. Instead of carrying out equilibrium analysis of subgames that result from such policies, for ease of presentation we assume such policies will not be accepted.

[5]Note that the innovation will effectively increase the production cost if $\delta < 0$.

(iii) $\underline{\phi}(k,\delta)$ is decreasing in k for $k < \frac{c}{\varepsilon\,\eta(c)}$ and $\underline{\phi}(k,\delta) = 0$ for $\frac{c}{\varepsilon\,\eta(c)} \leq k \leq n$.[6]

(iv) Let $P' = \frac{dP}{dQ}$. Then $\bar{\phi}(k,\delta) = -\frac{(p-c+\varepsilon)^2}{P'}$, $\underline{\phi}(k,\delta) = -\frac{(p-c)^2}{P'}$, where p is the Cournot price.

Proof. Follows by (3a), (3b) and Lemma 2 of KOT. ■

3.3 Optimal licensing policies

In this section we characterize for all $k = 1, 2, ..., n$ the optimal licensing policies $A(k, r, v)$ for $\mathcal{I}$ which are allowed use three instruments: upfront fee f (determined by auction), per unit royalty r and ad valorem royalty v. We provide a necessary and sufficient condition for this optimum to be achieved by two-part tariffs (f, r) or (f, v) for $\mathcal{I}$. While there always exists an optimal two-part tariff (f, r) for $\mathcal{I}$, this may not be true for a two-part tariff (f, v). From now on, we refer to an optimal policy as one which maximizes $\mathcal{I}$'s revenue over all equilibrium three-part tariffs $A(k, r, v)$.

In case $n = 1$ (the monopoly case), the optimal licensing policy of $\mathcal{I}$ is clearly a pure upfront fee policy where the fee is the difference between the monopoly profit with the new technology and that with the pre-innovation technology. We therefore deal from now on with $n \geq 2$.

Let $0 \leq \delta \leq \varepsilon$, $0 \leq r \leq \delta$ and $0 \leq v < 1$ s.t. $\delta = \delta(r, v)$ (defined in (3)). If $\mathcal{I}$ intends to sell at most $n-1$ licenses, $k = 1, 2, ..., n-1$, with a per unit royalty r and ad valorem royalty v, it can extract the highest upfront fee by auctioning k licenses with the policy $A(k, r, v)$. Denote

$$WTP(k, r, v) = (1-v)\bar{\phi}(k, \delta(r, v)) - \underline{\phi}(k, \delta(r, v)),\ \ k = 1, 2, ..., n-1$$

We next introduce a third assumption (A3) which enables us to prove uniqueness of the equilibrium outcome in $\Gamma(k, r, v)$ for all $k = 1, 2, ..., n-1$. In particular, if $\mathcal{I}$ auctions off k licenses, there will be more than k bidders bidding for a license and $\mathcal{I}$ will sell exactly k licenses. If $k = n$ and if it is optimal for $\mathcal{I}$ to sell licenses to all firms then (A3) will guarantee a unique subgame perfect equilibrium in Γ (not necessarily in $\Gamma(n, r, v)$) with n licensees. We deal with this case separately.

(A3) For every $0 < \delta \leq \varepsilon$, (i) $\underline{\phi}(0,\delta) < \overline{\phi}(1,\delta)$ and (ii) The ratio $\frac{\underline{\phi}(k-1,\delta)}{\bar{\phi}(k,\delta)}$ is strictly decreasing in $k = 1, 2, ..., n$, as long as $\underline{\phi}(k-1,\delta) > 0$.

The inequality $\underline{\phi}(0,\delta) < \overline{\phi}(1,\delta)$ asserts that if $\delta > 0$, the innovation has some value. Namely, the pre-innovation profit of a firm is less than the post innovation profit of an exclusive licensee. For the other part of (A3), let i be a firm in N. Suppose i faces a competition with $k-1$ licensees and $n-k$ non-licensees. Firm i can be either a non-licensee or a licensee. (A3) requires that the ratio of i's Cournot profit as a non-licensee to its Cournot profit as a licensee is decreasing in k. Namely, the relative disadvantage of i as a non-licensee increases with the number of licensees. If n is sufficiently large,

[6] If $\frac{c}{\varepsilon\,\eta(c)} > n$, then $\underline{\phi}(k,\delta) > 0$ for all $1 \leq k \leq n$.

i.e. $n > \frac{c}{\varepsilon\eta(c)}$, then for $\frac{c}{\varepsilon\eta(c)} \leq k \leq n$, $\underline{\phi}(k,\delta) = 0$ and every non-licensee is driven out of the market. Note that (A3) holds if demand is linear.

Let $k = 1, 2, ..., n$. In the subgame $\Gamma(k, r, v)$, which follows the choice of policy $A(k, r, v)$, firms first make their decisions whether to submit bids and how much to bid. If m firms submit bids, the number of licensees will be $\ell = \min(m, k)$ and the game proceeds with $\mathcal{C}(\ell, \delta(r, v))$.

For $1 \leq k \leq n$, let $S(k)$ be the set of policies $A(k, r, v)$, $r \in [0, \varepsilon]$, $v \in [0, 1)$ with the same number k of licenses for sale. Let $\Gamma(k)$ be defined similar to Γ but where $\mathcal{I}$ is confined to policies in $S(k)$.

Lemma 2 Suppose (A1)-(A3) hold. Let $n \geq 2$ and $k = 1, ..., n-1$. If $A(k, r, v)$ is an equilibrium policy of $\Gamma(k)$, then $0 \leq v \leq 1 - \frac{\underline{\phi}(k,\delta)}{\overline{\phi}(k,\delta)}$, $\delta = \delta(r, v)$. If $\delta > 0$, then at least $k+1$ firms submit the highest bid, which is $WTP(k, r, v)$. If $\delta = 0$, then $v = 0$ and $r = \varepsilon$ and at least k firms submit the highest bid, which is 0.

Proof. Let m be the number of bidders in an equilibrium of $\Gamma(k)$. Suppose first that $m = 0$, then $\mathcal{I}$ obtains 0. But by (A3(i)), $\mathcal{I}$ is better off setting $r = v = 0$ to induce firms to place positive bids.

Suppose $1 \leq m \leq k$ and let $\delta = \delta(r, v)$. For $m \leq k$, $\mathcal{I}$ commits to sell licenses to all bidders, irrespective of their bids. Hence all m bidders place zero bid. There are two

Case (i): $(1-v)\overline{\phi}(m,\delta) < \underline{\phi}(m-1,\delta)$

A licensee is better off deviating to not placing a bid, a contradiction.

Case (ii): $(1-v)\overline{\phi}(m,\delta) \geq \underline{\phi}(m-1,\delta)$

$(ii.1)$ Suppose $\delta > 0$. By (A3(ii)), $1 - v \geq \frac{\underline{\phi}(m-1,\delta)}{\overline{\phi}(m,\delta)} > \frac{\underline{\phi}(m,\delta)}{\overline{\phi}(m+1,\delta)}$, hence $(1-v)\overline{\phi}(m+1,\delta) > \underline{\phi}(m,\delta)$ and if $m < k$, a non-licensee is better off getting a license by placing a zero bid, a contradiction. If $m = k$, a non-licensee is better off getting a license by placing a small positive bid τ, $(0 < \tau < (1-v)\overline{\phi}(m+1,\delta) - \underline{\phi}(m,\delta)$), again a contradiction. We conclude that if $\delta > 0$, $m \geq k+1$. In this case, any deviant firm does not change the number k of licensees.

$(ii.2)$ Suppose $\delta = 0$. Then a licensee and a non-licensee have the same marginal cost and $\overline{\phi}(m,0) = \underline{\phi}(m-1,0)$. Hence Case (ii) holds iff $v = 0$ and $r = \varepsilon$. In this case, every bidder places zero bid. $\mathcal{I}$ is best off with $m \geq k$ to increase licensees' total production. If $m < k$, $\mathcal{I}$ can gain by slightly reducing r to induce more firms to bid.

If a licensee wins a license for a bid less than $WTP(k, r, v)$, it will be outbid by a non-licensee. We conclude that for $\delta > 0$, $WTP(k, r, v) \geq 0$ if $0 \leq v \leq 1 - \frac{\underline{\phi}(k,\delta)}{\overline{\phi}(k,\delta)}$. For the case $\delta = 0$ the last inequality holds as equality since $(1-v)\overline{\phi}(k,0) = \underline{\phi}(k,0)$ and $WTP(k, r, v) = 0$.

■

<u>Remark:</u> By Lemma 2, if $\mathcal{I}$ chooses a contract $A(k, r, v)$, $1 \leq k \leq n-1$, then more than k firms, say m, will place in equilibrium the highest bid, f. Their expected payoff, E_m, is

$$E_m = \frac{k}{m}(\overline{\phi}(k,\delta) - f) + \frac{m-k}{m}\underline{\phi}(k,\delta)$$

where $\delta = \delta(r, v)$. To avoid a deviation of any of these m firms $E_m \geq \underline{\phi}(k, \delta)$ must hold. Equivalently, $\overline{\phi}(k, \delta) - f \geq \underline{\phi}(k, \delta)$. To avoid the deviation of every other firm to any bid above f, $\underline{\phi}(k, \delta) \geq \overline{\phi}(k, \delta) - (f + \xi)$ should hold for all $\xi > 0$. Hence $\underline{\phi}(k, \delta) \geq \overline{\phi}(k, \delta) - f$ must hold and we conclude that $\underline{\phi}(k, \delta) = \overline{\phi}(k, \delta) - f$. This implies that in equilibrium $E_m = \overline{\phi}(k, \delta) - f = \underline{\phi}(k, \delta)$, which does not depend on m.

By Lemma 2, in every equilibrium of $\Gamma(k)$

$$0 \leq v \leq 1 - \underline{\phi}(k, \delta)/\overline{\phi}(k, \delta)$$

Let

$$\gamma(k, \delta) := 1 - \underline{\phi}(k, \delta)/\overline{\phi}(k, \delta), \;\; k = 1, 2, ..., n-1 \tag{5}$$

Then the following constraint

$$0 \leq v \leq \gamma(k, \delta) \tag{6}$$

is the *acceptability constraint* for $k = 1, \ldots, n-1$.

Let $k = 1, 2, ..., n-1$, $0 \leq \delta \leq \varepsilon$ and $(r, v) \in RV(\delta)$. By (2) and Lemma 2, in every equilibrium of $\Gamma(k, r, v)$, $\mathcal{I}$ obtains from each licensee:

$$\begin{aligned}\frac{1}{k}\Pi_I(k, r, v) =& r\overline{q}(k, \delta) + vp(k, \delta)\overline{q}(k, \delta) \\ &+ (1 - v)[p(k, \delta) - c + \delta]\overline{q}(k, \delta) - [p(k, \delta) - c]\underline{q}(k, \delta)\end{aligned}$$

By (3), $(1 - v)\delta = \varepsilon - (r + cv)$. Hence

$$r - (1 - v)c + (1 - v)\delta(r, v) = r - (1 - v)c + \varepsilon - (r + cv) = \varepsilon - c.$$

This implies

$$\frac{1}{k}\Pi_I(k, r, v) = [p(k, \delta) - c + \varepsilon]\overline{q}(k, \delta) - [p(k, \delta) - c]\underline{q}(k, \delta).$$

So the payoff of $\mathcal{I}$, $\Pi_I(k, r, v)$, depends on k and δ only and with some abuse of notations, we can replace it by $\Pi_I(k, \delta)$.

$$\begin{aligned}\Pi_I(k, \delta) =& k[p(k, \delta) - c + \varepsilon]\overline{q}(k, \delta) - k[p(k, \delta) - c]\underline{q}(k, \delta) \\ =& [p(k, \delta) - c + \varepsilon]Q(k, \delta) - n\underline{\phi}(k, \delta) - (n - k)\underline{q}(k, \delta)\varepsilon\end{aligned} \tag{7}$$

We conclude that fixing k, $k = 1, \ldots, n-1$, $\mathcal{I}$ is best off maximizing $\Pi_I(k, \delta)$ over $\delta \in [0, \varepsilon]$ and then choosing $(r, v) \in RV(\delta)$ subject to the acceptability constraint (6). Let $F : \mathbb{R}_+ \to \mathbb{R}$ be the profit of a monopolist as a function of price p under marginal cost $c - \varepsilon$, namely,

$$F(p) := [p - (c - \varepsilon)]Q(p) \tag{8}$$

By (7) and (8), the payoff of $\mathcal{I}$ under $0 \le \delta \le \varepsilon$, $(r,v) \in RV(\delta)$ and $\delta = \delta(r,v)$ is

$$\Pi_I(k,\delta) = F(p(k,\delta)) - n\underline{\phi}(k,\delta) - (n-k)\underline{q}(k,\delta)\varepsilon \tag{9}$$

Let

$$\delta^*(k) \in \arg\max_{0\le\delta\le\varepsilon} \Pi_I(k,\delta),\ k = 1,2,...,n-1$$

Since $(r,v) \in RV(\delta^*(k))$ iff $v = \frac{\varepsilon-\delta^*(k)-r}{c-\delta^*(k)}$ for $0 \le r \le \varepsilon$, if $\delta^*(k) < \varepsilon$, there is a continuum of pairs (r,v) in $RV(\delta^*(k))$.

For $k = 1,2,...,n-1$, the policy $A(k,r,v)$ is optimal for $\mathcal{I}$ iff (i) $(r,v) \in RV(\delta^*(k))$ and (ii) (r,v) satisfies the acceptability constraint (6).

Optimizing for each k separately is useful in case the innovator has other considerations not specified in our model. For example, in practice the implementation of a new technology may require a training cost. This may affect $\mathcal{I}$'s optimal number of licensees. If there are no such considerations $\mathcal{I}$ simply chooses k s.t. $k \in \arg\max_{1\le k\le n-1} \Pi_I(k,\delta^*(k))$.
2mm

For the characterization of the optimal acceptable contracts, we next define an interval, a subset of $[0,\varepsilon]$, of all per unit royalties r that together with v s.t. $\delta(r,v) = \delta^*(k)$ (given in (3)) define an optimal policy $A(k,r,v)$ for any $k = 1,2,...,n-1$. Let

$$R(k) \equiv \left[\max\left(0, \varepsilon - \delta^*(k) - (c-\delta^*(k))\cdot\gamma(k,\delta^*(k))\right), \varepsilon - \delta^*(k)\right],\ k = 1,2,...,n-1$$

Typically, the three instruments f, r and v of an optimal policy $A(k,r,v)$ can be all positive. As an example (see Section 4), if the demand function is $Q = a - p$ and $n > \max(\frac{a-c}{\varepsilon}, 2 + \frac{a-c}{2\varepsilon})$, then it can be shown that $\delta^*(k) < \varepsilon$ and for all k with $\frac{a-c}{\varepsilon} \le k \le n$, $0 < r < \varepsilon - \delta^*(k)$ and $v = \frac{\varepsilon-\delta^*(k)-r}{c-\delta^*(k)}$, the contract $A(k,r,v)$ is optimal and all three instruments f, r and v can be positive.

Proposition 1 Suppose (A1)-(A3) hold. If $\mathcal{I}$ auctions off k licenses $k = 1,2,...,n-1$, then $A(k,r,v)$ is an optimal policy for $\mathcal{I}$ iff $r \in R(k)$ and $v = \frac{\varepsilon-\delta^*(k)-r}{c-\delta^*(k)}$. In particular,

(i) $A(k, r = \varepsilon - \delta^*(k), v = 0)$ is optimal for $\mathcal{I}$.

(ii) $A(k, r = 0, v = \frac{\varepsilon-\delta^*(k)}{c-\delta^*(k)})$ is optimal for $\mathcal{I}$ iff $\frac{\underline{\phi}(k,\delta^*(k))}{\overline{\phi}(k,\delta^*(k))} \le \frac{c-\varepsilon}{c-\delta^*(k)}$.

Proof. See the Appendix. ■

Fixing k, $k = 1,2,...,n-1$, to be the number of licensees, by Proposition 1, the highest revenue the innovator can achieve with the three instruments (f,v,r) can always be achieved with a two-part tariff (f,r)[7] and without any ad valorem royalty. On the

[7] f is the winning bid for a license in the auction for k licenses.

other hand, a two-part tariff (f, v) can achieve this maximum if and only if

$$\frac{\underline{\underline{\phi}}(k, \delta^*(k))}{\bar{\phi}(k, \delta^*(k))} \leq \frac{c - \varepsilon}{c - \delta^*(k)} \tag{10}$$

Alternatively, by part (iv) of Lemma 2, (10) is equivalent to

$$\left(\frac{p - c}{p - c + \delta^*(k)}\right)^2 \leq \frac{c - \varepsilon}{c - \delta^*(k)} \tag{11}$$

where p is the Cournot price in a market with k licensees and $n-k$ non-licensees, if the effective magnitude of the innovation is $\delta^*(k)$. It can be easily shown (see KOT, page 24) that the pre-innovation Cournot price is increasing in c. Hence the post innovation Cournot price is decreasing in δ, the effective magnitude of the innovation. Replacing $\delta^*(k)$ by δ, If $p \leq c$, then every non-licensee is driven out of the market and (10) certainly holds. By Lemma 1, $p \leq c$ iff $\delta \leq \frac{c}{k\,\eta(c)}$. If $\delta > \frac{c}{k\,\eta(c)}$, the left hand side of (11) is decreasing in δ from 1 to a level below 1; the right hand side of (11) is increasing in δ from $\frac{c-\varepsilon}{c}$ to 1. Hence the intersection of the two sides of (11) is at a unique point, say $\bar{\delta}_k \in (0, \varepsilon]$, and (11) holds iff $\delta^*(k)$ is larger than or equal to $\bar{\delta}_k$. In other words, a two-part tariff (f, v) is a revenue maximizing policy (with $r = 0$) iff $v = \frac{\varepsilon - \delta^*(k)}{c - \delta^*(k)}$ and $\delta^*(k) \in [\bar{\delta}_k, \varepsilon]$. Note however that if $\delta^*(k) = \varepsilon$, then $r = v = 0$ and the optimal policy is a pure upfront fee. We summarize the above in the next corollary.

Corollary 1 Suppose (A1)-(A3) hold and let $k = 1, ..., n-1$ be the number of licensees. If $\delta^*(k) < \frac{c}{k\,\eta(c)}$, then $p(k, \delta^*(k)) > c$, and there exists $\bar{\delta}_k \in (0, \varepsilon]$ such that an optimal two-part tariff policy (f, v) with $v > 0$ exists iff $\delta^*(k) \geq \bar{\delta}_k$. If $\frac{c}{k\,\eta(c)} \leq \delta^*(k) \leq \varepsilon$, then all non-licensees are driven out of the market and $\mathcal{I}$ can obtain its highest revenue by a two-part tariff (f, v) where $v = \frac{\varepsilon - \delta^*(k)}{c - \delta^*(k)}$.

That is, a two-part tariff (f, v) with $v > 0$ (and $r = 0$) cannot be optimal for $\mathcal{I}$ if $\delta^*(k)$, the effective cost reduction rate, is relatively low. Otherwise, if $\delta^*(k)$ is relatively high, $\mathcal{I}$ can obtain its highest revenue by a two-part tariff (f, v) with $v > 0$ iff $\delta^*(k) < \varepsilon$. In contrast, irrespective of how large $\delta^*(k)$ is, $\mathcal{I}$ can extract its highest revenue with a two-part tariff (f, r) with $r = \varepsilon - \delta^*(k)$ and $v = 0$.

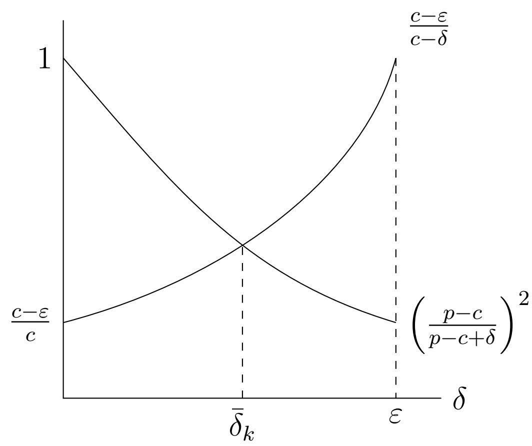
The case $k = n$ is different. The policy $A(n, r, v)$ is a commitment of $\mathcal{I}$ to sell licenses to <u>*all*</u> bidding firms regardless of their bids. This implies zero bid by all firms. For every $k = 1, 2, ..., n$, let

$$\overline{WTP}(k, r, v) = (1 - v)\bar{\phi}(k, \delta(r, v)) - \underline{\phi}(k - 1, \delta(r, v))$$

Then in case $k = n$, each firm is willing to pay

$$\overline{WTP}(n, r, v) = (1 - v)\bar{\phi}(n, \delta(r, v)) - \underline{\phi}(n - 1, \delta(r, v)) \tag{12}$$

Suppose $\overline{WTP}(n, r, v) > 0$. If the innovator chooses $A(n, r, v)$, it is best off setting $\overline{WTP}(n, r, v)$ as a minimum reservation price. Equivalently, it can sell license not by auction but rather by a fixed fee equal to $\overline{WTP}(n, r, v)$. Every firm obtains a license



The case where $p > c$

if it pays this fee in addition to a commitment to pay $\mathcal{I}$ the royalties (r, v). Note that $\overline{WTP}(n, r, v)$ is the willingness to pay of a firm for a license if it knows that all other firms have an access to the new technology. A deviant firm reduces the number of licensees to $n-1$ and it will obtain $\underline{\phi}(n-1, \delta(r, v))$.

If $\mathcal{I}$ intends to sell n licenses using the policy $A(n, r, v)$ with the upfront fee $\overline{WTP}(n, r, v)$, the corresponding subgame $\Gamma(n, r, v)$ of Γ may have multiple equilibrium points with different number of licensees, in which case $\mathcal{I}$ cannot ensure that all n firms purchase licenses. Following the policy $A(n, r, v)$ with the upfront fee $\overline{WTP}(n, r, v)$, if the inequality

$$\overline{WTP}(k, r, v) \geq \overline{WTP}(n, r, v,) \geq \overline{WTP}(k+1, r, v)$$

holds for some $k \leq n-2$, then in addition to the equilibrium of $\Gamma(n, r, v)$ where all n firms purchase licenses, there is another equilibrium where exactly k firms purchase licenses. Clearly, $\mathcal{I}$ is better off with the "n-licensee-equilibrium" where it obtains $\frac{n}{n-k}$ times its payoff with the "k-licensee-equilibrium". Nevertheless this problem is avoided in any subgame perfect equilibrium of Γ. Lemma 3 guarantees that an equilibrium of $\Gamma(n, r, v)$ which is induced by any subgame perfect equilibrium Γ, must have exactly n licensees.

Lemma 3 Suppose (A1)-(A3) hold. Consider a subgame perfect equilibrium of Γ where $\mathcal{I}$ selects the policy $A(n, r, v)$, $0 \leq r \leq \varepsilon$, $0 \leq v < 1$, together with the upfront fee $\overline{WTP}(n, r, v)$. Then <u>all</u> firms purchasing licenses is the only equilibrium outcome of the subgame $\Gamma(n, r, v)$.

We do not claim that $\Gamma(n, r, v)$ has a unique equilibrium. As argued before, $\Gamma(n, r, v)$ may have besides the equilibrium, e_1, where every firm pays $\mathcal{I}$ the fee $\overline{WTP}(n, r, v)$ in addition to the royalties r and v, another equilibrium, e_2, with only k licensees, $k < n$, where firms pay the same royalties (r, v) and the same upfront fee $\overline{WTP}(n, r, v)$ as in e_1. Lemma 3 states that following the choice of policy $A(n, r, v)$, in any subgame perfect equilibrium of Γ, e_1 must be played in $\Gamma(n, r, v)$.

Proof of Lemma 3. Suppose e is a subgame perfect equilibrium of Γ and suppose $\mathcal{I}$ chooses in the first stage the policy $A(n,r,v)$ with the upfront fee $\overline{WTP}(n,r,v)$. Suppose to the contrary that in the second stage of e, only $k < n$ firms purchase licenses in $\Gamma(n,r,v)$. $\mathcal{I}$ then obtains $k\,\overline{WTP}(n,r,v)$ in e. If $\mathcal{I}$ deviates from $A(n,r,v)$ to $A(n-1,r,v)$, then by Lemma 2, all firms bid $\overline{WTP}(n-1,r,v) = (1-v)\overline{\phi}(n-1,\delta(r,v)) - \underline{\phi}(n-1,\delta(r,v))$, and $n-1$ of them are randomly selected as licensees. The innovator will obtain

$$\Pi_I = (n-1)[(1-v)\overline{\phi}(n-1,\delta(r,v)) - \underline{\phi}(n-1,\delta(r,v))]$$

Since $n-1 \geq k$ and by Lemma 1, $\overline{\phi}(n-1,\delta(r,v)) > \overline{\phi}(n,\delta(r,v))$, it follows that $\Pi_I > k\,\overline{WTP}(n,\delta(r,v))$, a contradiction. ■

Since

$$(1-v)\overline{\phi}(n,\delta) \gtreqless \underline{\phi}(n-1,\delta) \Leftrightarrow v \lesseqgtr 1 - \underline{\phi}(n-1,\delta)/\overline{\phi}(n,\delta).$$

By (13), the willingness to pay of a licensee if $k = n$ is non-negative iff $0 \leq v \leq 1 - \underline{\phi}(n-1,\delta)/\overline{\phi}(n,\delta)$. A policy $A(n,r,v)$ is *acceptable* iff for $0 \leq \delta \leq \varepsilon$ and $\delta = \delta(r,v)$

$$0 \leq v \leq 1 - \underline{\phi}(n-1,\delta)/\overline{\phi}(n,\delta),$$

Let $\gamma(n,\delta) = 1 - \underline{\phi}(n-1,\delta)/\overline{\phi}(n,\delta)$. Then

$$0 \leq v \leq \gamma(n,\delta) \tag{13}$$

is the acceptability constraint for $k = n$.

In every equilibrium of $\Gamma(n,r,v)$, $\mathcal{I}$ set an upfront fee $f = \overline{WTP}(n,r,v)$. By (12), $\mathcal{I}$ obtains from each licensee:

$$\begin{aligned}\frac{1}{n}\Pi_I(n,r,v) =& r\overline{q}(n,\delta) + vp(n,\delta)\overline{q}(n,\delta)\\ &+ (1-v)[p(n,\delta) - c + \delta]\overline{q}(n,\delta) - [p(n-1,\delta) - c]\underline{q}(n-1,\delta)\end{aligned}$$

Rearranging terms, by(8), the payoff of $\mathcal{I}$ under $0 \leq \delta \leq \varepsilon$ and $(r,v) \in RV(\delta)$ is

$$\Pi_I(n,\delta) = F(p(n,\delta)) - n\underline{\phi}(n-1,\delta) \tag{14}$$

which is again a function of n and δ.

Proposition 2 Suppose (A1)-(A3) hold. Suppose that in a subgame perfect equilibrium of Γ, the innovator offers to sell licenses to all n firms for an upfront fee $\overline{WTP}(n,r,v) = (1-v)\overline{\phi}(n,\delta(r,v)) - \underline{\phi}(n-1,\delta(r,v))$ (in addition to the two royalties r and v). Then $A(n,r,v)$ is optimal for $\mathcal{I}$ iff $r \in R(n)$ and $v = \frac{\varepsilon-\delta^*(n)-r}{c-\delta^*(n)}$. In particular, $A(n,\varepsilon-\delta^*(n),0)$ is always optimal; and $A(n,0,\frac{\varepsilon-\delta^*(n)}{c-\delta^*(n)})$ is optimal iff $\frac{\underline{\phi}(n-1,\delta^*(n))}{\overline{\phi}(n,\delta^*(n))} \leq \frac{c-\varepsilon}{c-\delta^*(n)}$.

Proof. Follows from Lemma 3 along similar lines as the proof of Proposition 1. ■

Similar to Corollary 1, we have

Corollary 2 Suppose (A1)-(A3) hold. Suppose $p(k,\delta) > c$. Then there exists $\bar{\delta}_n \in (0,\varepsilon]$ such that an optimal two-part tariff policy (f,v) with $v > 0$ exists iff $\delta^*(n) \geq \bar{\delta}_n$.

The next corollary asserts that the optimal two-part tariff policy with ad valorem royalty (if such optimal policy exists) and the optimal two-part tariff policy with per unit royalty have the same number of licensees.

Corollary 3 Suppose the innovator restricts itself to two-part tariff policies only (whether with per unit royalyt or with ad valorem royalty). Let k^* be an optimal number of licensees for $\mathcal{I}$ subject to all policies of the form $A(k,r,v)$. Then

(i) The best two-part tariff policy for $\mathcal{I}$ with per unit royalty generates at least as high revenue for $\mathcal{I}$ as the best two-part tariff with ad valorem royalty.

(ii) The two-part tariff $A(k^*, r = \varepsilon - \delta^*(k^*), v = 0)$ maximizes $\mathcal{I}$'s revenue.

(iii) If some two-part tariff with ad valorem royalty maximizes $\mathcal{I}$'s revenue, then so is the policy $A(k^*, r = 0, v = \frac{\varepsilon-\delta^*(k^*)}{c-\delta^*(k^*)})$. Namely the two optimal two-part tariff policies use the same number of licensees.

In the next section we show for the linear demand case that for some values of a, c and ε, no acceptable two-part tariff (f,v) is optimal for $\mathcal{I}$.

4 Example: The linear demand case

In this section, we analyze the special case where the demand function is linear,

$$p(Q) = a - Q \text{ if } Q < a \quad \text{and} \quad p(Q) = 0 \text{ if } Q \geq a.$$

Lemma 4 *For $k = 1,\ldots,n$ and $\delta \in [0,\varepsilon]$, the unique equilibrium of $\mathcal{C}(k,\delta)$ satisfies the following*

(i) For $k = 1,\ldots,n-1$:

$$\bar{q}(k,\delta) = \begin{cases} \frac{a-c+\delta}{k+1} & \text{if } \delta \geq \frac{a-c}{k} \\ \frac{a-c+(n-k+1)\delta}{n+1} & \text{if } 0 \leq \delta \leq \frac{a-c}{k} \end{cases}$$

$$\underline{q}(k,\delta) = \begin{cases} 0, & \text{if } \delta \geq \frac{a-c}{k} \\ \frac{a-c-k\delta}{n+1} & \text{if } 0 \leq \delta \leq \frac{a-c}{k} \end{cases}$$

$$p(k,\delta) = \begin{cases} \frac{a+k(c-\delta)}{k+1} & \text{if } \delta \geq \frac{a-c}{k} \\ \frac{a+nc-k\delta}{n+1} & \text{if } 0 \leq \delta \leq \frac{a-c}{k} \end{cases}$$

$p(k,\delta)$ is linear and decreasing in δ. In particular, $p(k,\delta) \lesseqqgtr c \Leftrightarrow \delta \gtreqqless \frac{a-c}{k}$.

(ii) For $k = n$:

$$\bar{q}(n,\delta) = \frac{a-c+\delta}{n+1}, \;\; p(n,\delta) = \frac{a+n(c-\delta)}{n+1}$$

(iii) In all cases $\bar{\phi}(k,\delta) = [\bar{q}(k,\delta)]^2$ and $\underline{\phi}(k,\delta) = [\underline{q}(k,\delta)]^2$.

Proof: easy to verify. ■

Lemma 5 Let $r \in [0,\varepsilon]$ and $v \in [0,1)$.

(i) $A(k,r,v)$ is acceptable for $k \leq n-1$ iff

$$\text{either } \;\frac{a-c}{k} < \delta(r,v) \leq \varepsilon \;\text{ or }\; \begin{cases} 0 \leq \delta(r,v) \leq \min\{\varepsilon, \frac{a-c}{k}\} \\ \text{and } v \leq 1 - \left[\frac{a-c-k\delta}{a-c+(n-k+1)\delta}\right]^2 \end{cases}$$

(ii) $A(n,r,v)$ is acceptable iff

$$\text{either } \;\frac{a-c}{n-1} < \delta(r,v) \leq \varepsilon \;\text{ or }\; \begin{cases} 0 \leq \delta(r,v) \leq \min\{\varepsilon, \frac{a-c}{n-1}\} \\ \text{and } v \leq 1 - \left[\frac{a-c-(n-1)\delta}{a-c+\delta}\right]^2 \end{cases}$$

Proof: Follows by Lemma 4, (6), (13) and standard computations. ■
The next proposition shows that for the linear demand case there are optimal policies for $\mathcal{I}$ with maximum diffusion of the innovation.

Proposition 3 There exists an optimal policy for $\mathcal{I}$ with either $n-1$ or n licensees.

Proof: See the Appendix.
The next lemma enables us to characterize these optimal policies.

Lemma 6 Let $n \geq 3$.

(i) The unique maximizer of $\Pi_I(n-1,\delta)$ with respect to δ is

$$\delta^*(n-1) = \begin{cases} \varepsilon & \text{if } \varepsilon < \frac{a-c}{2(n-2)} \\ \frac{a-c+2\varepsilon}{2(n-1)} & \text{if } \frac{a-c}{2(n-2)} \leq \varepsilon \leq \frac{a-c}{2} \\ \frac{a-c}{n-1} & \text{if } \varepsilon > \frac{a-c}{2} \end{cases}$$

(ii) The unique maximizer of $\Pi_I(n,\delta)$ with respect to δ is

$$\delta^*(n) = \begin{cases} \varepsilon & \text{if } \varepsilon < \frac{a-c}{2n-1} \\ \frac{(n-1)(a-c)+(n+1)\varepsilon}{2(n^2-n+1)} & \text{if } \varepsilon \geq \frac{a-c}{2n-1} \end{cases}$$

(iii) There exists an optimal policy $A(k, r, v)$ with $k \geq n-1$. It satisfies for all $\varepsilon \in (0, c)$

$$\delta^*(k) = \frac{\varepsilon - (r + cv)}{1 - v} \quad \text{and} \quad \begin{cases} v \leq 1 - \left[\frac{a-c-(n-1)\delta^*(n-1)}{a-c+2\delta^*(n-1)}\right]^2 & \text{for } k = n-1 \\ v \leq 1 - \left[\frac{a-c-(n-1)\delta^*(n)}{a-c+\delta^*(n)}\right]^2 & \text{for } k = n \end{cases}$$

Note that by Lemma 6 (iii), for $n \geq 3$, all policies $A(k, r, 0)$ satisfying $r = \varepsilon - \delta^*(k)$, $k \in \{n-1, n\}$ are optimal. That is, a two-part tariff (f, r) is sufficient to achieve the highest revenue of $\mathcal{I}$. However, two-part tariff policies of the form $A(k, 0, v)$ cannot achieve the maximum revenue of $\mathcal{I}$ for all k, $1 \leq k \leq n$, since it requires (i.) $v = \frac{\varepsilon-\delta^*(k)}{c-\delta^*(k)}$ or equivalently $\frac{\varepsilon - cv}{1-v} = \delta^*(k)$ and (ii.) $v \leq 1 - \left[\frac{a-c-\frac{(n-1)(\varepsilon-cv)}{1-v}}{a-c+\frac{2(\varepsilon-cv)}{1-v}}\right]^2$ for $k = n-1$ and $v \leq 1 - \left[\frac{a-c-\frac{(n-1)(\varepsilon-cv)}{1-v}}{a-c+\frac{\varepsilon-cv}{1-v}}\right]^2$ for $k = n$. For large n ($n \geq 7$) and large ε ($\frac{a-c}{u(n)} < \varepsilon < c$), there exists an optimal two-part tariff (f, v) with $v^* = \frac{(n-1)\varepsilon-(a-c)}{nc-a}$ and $r = 0$. In this case, $p = c$ and hence (11) holds. Moreover, as n grows indefinitely, the aggregate profits of all producers decreases to 0. The innovator can collect only negligible upfront fees and almost all its revenue comes from ad valorem royalties in a rate approaching $\frac{\varepsilon}{c}$.

Proposition 4 The acceptable optimal policies for $n \geq 3$ with the maximum diffusion of the innovation is given in the following table[8].

Table 1: Equilibrium policy of $\mathcal{I}$ for industries of size $n \geq 3$

$n = 3$	$\varepsilon \in (0, \frac{a-c}{5}]$	$\varepsilon \in (\frac{a-c}{5}, \frac{(4-\sqrt{7})(a-c)}{6}]$	$\varepsilon \in (\frac{(4-\sqrt{7})(a-c)}{6}, \frac{a-c}{2}]$	$\varepsilon \in (\frac{a-c}{2}, a-c]$
Optimal $(k, \delta^*(k))$	$k = 3, \delta^* = \varepsilon$	$k = 3, \delta^* = \frac{a-c+2\varepsilon}{7}$	$k = 2, \delta^* = \varepsilon$	$k = 2, \delta^* = \frac{a-c}{2}$
$n = 4, 5$	$\varepsilon \in (0, \frac{2(a-c)}{n^2+n-3}]$	$\varepsilon \in (\frac{2(a-c)}{n^2+n-3}, \frac{a-c}{2n-4}]$	$\varepsilon \in (\frac{a-c}{2n-4}, \frac{a-c}{2}]$	$\varepsilon \in (\frac{a-c}{2}, a-c]$
Optimal $(k, \delta^*(k))$	$k = n, \delta^* = \varepsilon$	$k = n-1, \delta^* = \varepsilon$	$k = n-1, \delta^* = \frac{a-c+2\varepsilon}{2(n-1)}$	$k = n-1, \delta^* = \frac{a-c}{n-1}$

$n = 6$	$\varepsilon \in (0, \frac{2(a-c)}{39}]$	$\varepsilon \in (\frac{2(a-c)}{39}, \frac{a-c}{8}]$	$\varepsilon \in (\frac{a-c}{8}, \frac{a-c}{d_1}]$	$\varepsilon \in (\frac{a-c}{d_1}, \frac{a-c}{d_2}]$	$\varepsilon \in (\frac{a-c}{d_2}, \frac{a-c}{2}]$	$\varepsilon \in (\frac{a-c}{2}, a-c]$
Optimal $(k, \delta^*(k))$	$k = 6$, $\delta^* = \varepsilon$	$k = 5$, $\delta^* = \varepsilon$	$k = 5$, $\delta^* = \frac{a-c+2\varepsilon}{10}$	$k = 6$, $\delta^* = \frac{5(a-c)+7\varepsilon}{62}$	$k = 5$, $\delta^* = \frac{a-c+2\varepsilon}{10}$	$k = 6$, $\delta^* = \frac{a-c}{5}$

$n \geq 7$	$\varepsilon \in (0, \frac{2(a-c)}{n^2+n-3}]$	$\varepsilon \in (\frac{2(a-c)}{n^2+n-3}, \frac{a-c}{2n-4}]$	$\varepsilon \in (\frac{a-c}{2n-4}, \frac{a-c}{v(n)}]$	$\varepsilon \in (\frac{a-c}{v(n)}, \frac{a-c}{u(n)}]$	$\varepsilon \in (\frac{a-c}{u(n)}, a-c]$
Optimal $(k, \delta^*(k))$	$k = n$, $\delta^* = \varepsilon$	$k = n-1$, $\delta^* = \varepsilon$	$k = n-1$, $\delta^* = \frac{a-c+2\varepsilon}{2(n-1)}$	$k = n$, $\delta^* = \frac{(n-1)(a-c)+(n+1)\varepsilon}{2(n^2-n+1)}$	$k = n-1$, $\delta^* = \frac{a-c}{n-1}$

where $d_1 \equiv \frac{210+\sqrt{5642}}{67}$, $d_2 \equiv \frac{210-\sqrt{5642}}{67}$,
$v(n) := \frac{n^3-n+\sqrt{(n+1)(n^2-n+1)(n^3-6n^2+5n-4)}}{2n^2-n+1}$, $u(n) := \frac{(n+1)(1+\sqrt{n^2-n+1})^2}{n(n-1)^2}$.

[8] Table 1 is consistent with Table A.5 in Sen and Tauman (2007) for $v = 0$.

Proof: see the Appendix.

Remark 3. (i) Since the innovation is non-drastic, $\varepsilon < a - c$.
(ii) Since $\varepsilon < c$, in case $c < \frac{a}{2}$ ($\leftrightarrow c < a - c$) some of the larger regions of ε in Table 1 may be empty.

We next characterize all acceptable optimal licensing policy for a Cournot Duopoly.

Proposition 5 Suppose $n = 2$.

(i) *If $\varepsilon < \frac{a-c}{3}$, the unique optimal policy for $\mathcal{I}$ is $A(2,0,0)$, that is, $\mathcal{I}$ sells licenses to both firms using pure upfront fee.*

(ii) *If $\varepsilon > \frac{(\sqrt{2}+1)(a-c)}{3}$, the unique optimal policy for $\mathcal{I}$ is $A(1,0,0)$, that is, $\mathcal{I}$ sells a license to only one firm using pure upfront fee.*

(iii) *If $\frac{a-c}{3} < \varepsilon < \frac{(\sqrt{2}+1)(a-c)}{3}$, there is a continuum of optimal policies for $\mathcal{I}$. Under each one of these policies, both firms receive licenses. Any $A(2,r,v)$ s.t. $\frac{\varepsilon-(r+cv)}{1-v} = \frac{a-c+3\varepsilon}{6}$ and $0 \le v \le 1 - \left(\frac{5a-5c-3\varepsilon}{7a-7c+3\varepsilon}\right)^2$ is optimal for $\mathcal{I}$.*

Proof: See the Appendix.

Table 2: Equilibrium policy of $\mathcal{I}$ for a duopoly industry

$n = 2$	$\varepsilon \in (0, \frac{a-c}{3}]$	$\varepsilon \in (\frac{a-c}{3}, \frac{(\sqrt{2}+1)(a-c)}{3}]$	$\varepsilon \in (\frac{(\sqrt{2}+1)(a-c)}{3}, a-c]$
Optimal $(k, \delta^*(k))$	$k = 2, \delta^* = \varepsilon$	$k = 2, \delta^* = \frac{a-c+3\varepsilon}{6}$	$k = 1, \delta^* = \varepsilon$
Optimal $A(k,r,0)$	$k = 2, r = 0$	$k = 2, r = \frac{3\varepsilon-(a-c)}{6}$	$k = 1, r = 0$
Optimal $A(k,0,v)$	$k = 2, v = 0$	$k = 2, v = \frac{3\varepsilon-(a-c)}{-a+7c-3\varepsilon}$ if $\frac{6(c-\varepsilon)}{-a+7c-3\varepsilon} \ge \left(\frac{5a-5c-3\varepsilon}{7a-7c+3\varepsilon}\right)^2$	$k = 1, v = 0$

By Proposition 5, in a Cournot Duopoly with linear demand and for intermediate values of ε, the innovator can generate its highest revenue by selling two licenses charging a two-part tariff with either a per unit royalty (f, r), $r > 0$, or with an ad valorem royalty (f, v), $v > 0$. For example, if $a = 1, c = \frac{1}{2}$ and $\varepsilon = \frac{1}{3}$, then $A(2, 0, \frac{1}{3})$ and $A(2, \frac{1}{12}, 0)$ are optimal for $\mathcal{I}$.

If for instance $a = 2.5, c = 1$ and $\varepsilon = 0.97$, then the two-part tariff policy with a per unit royalty $A(2, 0.235, 0)$ is optimal for $\mathcal{I}$. However, there is no optimal two-part tariff policy with an ad valorem royalty. If there is one, it must satisfy $v^* = \frac{\varepsilon-\delta^*(2)}{c-\delta^*(2)} = 0.887$, but $A(2, 0, 0.887)$ is not acceptable. (Licensees will not agree to pay $\mathcal{I}$ 88.7% of their revenues.)

Note that by Table 1 and Table 2 above, if ε is relatively small, the unique optimal policy of $\mathcal{I}$ is a pure upfront fee (since $\hat{\delta}^* = \varepsilon$ implies by (3) $r = v = 0$).

5 Conclusion

The paper focuses on optimal licensing schemes for an outside innovator of a process innovation. The paper deals with quite general demand and the results are demonstrated in case demand is linear. In practice, the most common licensing schemes consist of either one of the following three instruments: upfront fee, per unit royalty and ad valorem royalty, or a combination of upfront fee and one of the two royalties (two-part tariff).

It is shown that the optimal two-part tariff with per unit royalties is as good as the optimal scheme with all three instruments. This is not always the case for two-part tariffs with ad valorem royalties. The reason is intuitive. Any two-part tariff scheme with a per unit royalty that does not exceed the magnitude of the innovation together with an appropriate upfront fee is acceptable by firms. But schemes with high ad valorem royalty are not acceptable for them. Interestingly, if an optimal two-part tariff scheme with positive ad valorem royalty exists, it typically involves the same number of licensees as an optimal two-part tariff with positive per unit royalty.

The possible multiplicity of optimal two-part tariff schemes may explain the variety fo licensing schemes we observe in practice. In reality, the more suitable scheme depends on the nature of the product. Low value commodities sold in bulk typically are sold by a per unit royalty. However, products that are either difficult to be measured by quantities or are of high value are sensitive to price fluctuations and they are typically sold by ad valorem royalties. Sometimes, even when ad valorem schemes are less efficient for the innovator, they are more convenient to use and the trade-off between efficiency and practicality sometimes is in favor of the latter. The degree of efficiency loss when using ad valorem royalty is another interesting and perhaps challenging topic for future research.

This paper does not shed light on post innovation market structure when demand is general. For the linear demand case, as is shown in Sen-Tauman (2007), there always exist optimal two-part tariff schemes with almost full diffusion of the innovation and is shown in this paper they are optimal even if ad valorem royalty can be used as a third instrument.

Analyzing the case of an incumbent innovator is another interesting future research. Suppose the innovator $\mathcal{I}$ is also an incumbent firm, increasing the total number of producers from n to $n+1$. Assuming firms $1, .., k$ have licenses, the payoff function of the incumbent innovator I has 4 components: (i) its own operating profit $[p(Q)-(c-\varepsilon)]q_I$, (ii) unit royalty revenue $r(q_1+..+q_k)$, (iii) ad valorem royalty revenue $vp(Q)(q_1+...+q_k)$ and (iv) fixed fee f.

At the Cournot stage, the payoff of I is therefore:

$\Pi_I = [p(Q) - (c - \varepsilon)]q_I + r(q_1 + .. + q_k) + vp(Q)(q_1 + ... + q_k)$

The choice of q_I affects price $p(Q)$, so the ad valorem royalty is affected by choice of q_I. Therefore, unlike the outside innovator case where r and v affect I's payoff only through $\delta(r, v)$, the incumbent innovator's payoff depends separately on r and v. Hence the "from three to two part tariff" result does not hold for an incumbent innovator. The strategic interaction of an incumbent innovator and competing firms will be another interesting topic.

6 Appendix

Proof of Proposition 1. Let $k = 1, 2, ..., n-1$. Since $\Pi_I(k, \delta)$ is maximized for $\delta = \delta^*(k)$, the optimal policies $A(k, r, v)$ are those where $\delta^*(k) = \delta(r, v)$ and for which the acceptability condition (6) holds.
By (3) and (6),

$$v \leq \gamma(k, \delta^*(k)) \text{ and } v = \frac{\varepsilon - \delta^*(k) - r}{c - \delta^*(k)} \tag{15}$$

Equivalently, (15) holds with $r \geq \varepsilon - \delta^*(k) - (c - \delta^*(k)) \cdot \gamma(k, \delta^*(k))$. Since $\delta^*(k)$ is the effective marginal cost, $r \in [0, \varepsilon - \delta^*(k)]$. We conclude that $r \in R(k)$ and v is given by (15), as claimed.
Taking $v = 0$, we have by (15) $r = \varepsilon - \delta^*(k)$ and certainly (6) holds. So $A(k, \varepsilon - \delta^*(k), 0)$ is optimal. Finally, suppose $r = 0$. By (15), $v = \frac{\varepsilon - \delta^*(k)}{c - \delta^*(k)}$. The policy $A(k, r, v)$ is optimal iff $0 \in R(k)$. Namely, $\gamma(k, \delta^*(k)) \geq \frac{\varepsilon - \delta^*(k)}{c - \delta^*(k)}$. ■

Proof of Proposition 3. We first provide some useful inequalities that enable us to characterize optimal licensing contracts for the linear demand case.

Lemma 7 (i) If for some $k = 1, \ldots, n-1$, $A(k, r, v)$ is optimal for $\mathcal{I}$, then

$$0 \leq \delta(r, v) \leq \min\{\varepsilon, \frac{a-c}{k}\}.$$

(ii) If $A(n, r, v)$ is optimal for $\mathcal{I}$, then $0 \leq \delta(r, v) \leq \min\{\varepsilon, \frac{a-c}{n-1}\}$.

Proof. (i) Suppose $A(k, r, v)$ is optimal for $\mathcal{I}$. Since $0 \leq \delta(r, v) \leq \varepsilon$, the result is immediate if $\varepsilon \leq \frac{a-c}{k}$. Suppose $\varepsilon > \frac{a-c}{k}$ and suppose to the contrary $\delta(r, v) > \frac{a-c}{k}$. By Lemma 5, $A(k, r, v)$ with $\delta(r, v) > \frac{a-c}{k}$ is acceptable. By Lemma 4, for any such policy $\underline{q}(k, \delta) = 0$ and $\underline{\phi}(k, \delta) = 0$, so by (9), the payoff of $\mathcal{I}$ is $\Pi_I(k, \delta) = F(p(k, \delta))$, where F is defined by (8). Note that $F(p)$ (the monopolist's profit under marginal cost $c - \varepsilon$) is increasing for $p < p_M(\varepsilon) \equiv (a + c - \varepsilon)/2$ (the monopoly price). Since the innovation is non drastic, we have $c < p_M(\varepsilon)$. By Lemma 4, $p(k, \delta)$ is decreasing in δ for all $\delta > 0$, hence $F(p(k, \delta))$ is decreasing for $\delta > \frac{a-c}{k}$, implying $F\left(p(k, \frac{a-c}{k})\right) > F(p(k, \delta))$ for $\delta > \frac{a-c}{k}$. But by Lemma 5(i), for all $v < 1$, $A(k, r, v)$ with $\delta(r, v) = \frac{a-c}{k}$ is acceptable, contradicting the fact that $A(k, r, v)$ with $\delta(r, v) > \frac{a-c}{k}$ is optimal for $\mathcal{I}$.

(ii) As in (i), the result is immediate if $\varepsilon \leq \frac{a-c}{n-1}$. Suppose $\varepsilon > \frac{a-c}{n-1}$ and suppose to the contrary $\delta(r, v) > \frac{a-c}{n-1}$. By Lemma 5 (ii), $A(n, r, v)$ with $\delta(r, v) > \frac{a-c}{n-1}$ is acceptable. By Lemma 4, for any such policy $\underline{q}(n-1, \delta) = 0$ and $\underline{\phi}(n-1, \delta) = 0$, so by (14), the payoff of $\mathcal{I}$ is $\Pi_I(n, \delta) = F(p(n, \delta))$. Since $\delta(r, v) > \frac{a-c}{n-1}$, we have $p(n, \delta) < c$ and by similar arguments as in (i.), $F\left(p(n, \frac{a-c}{n-1})\right) > F(p(n, \delta))$ for $\delta > \frac{a-c}{n-1}$. By Lemma 5(ii), for all $v < 1$, $A(n, r, v)$ with $\delta(r, v) = \frac{a-c}{n-1}$ is acceptable, contradicting the fact that $A(n, r, v)$ with $\delta(r, v) > \frac{a-c}{n-1}$ is optimal for $\mathcal{I}$.

This contradicts the fact that $A(n, r, v)$ is optimal for $\mathcal{I}$.

■

Lemma 8 Let $n \geq 3$ and $k = 1, \dots, n-2$. Suppose $0 \leq \delta(r, v) \leq \min\{\varepsilon, \frac{a-c}{k}\}$ and $A(k, r, v)$ is acceptable. Denote

$$\tilde{\delta} = \frac{k}{n-1}\delta, \;\; \tilde{v} = 1 - \left(\frac{(n-1)\sqrt{1-v}}{(n-k-1)\sqrt{1-v}+k} \right)^2$$

and let $\tilde{r}$ be s.t. $\tilde{\delta}(\tilde{r}, \tilde{v}) = \frac{k}{n-1}\delta(r, v)$. Let $\delta = \delta(r, v)$ and $\tilde{\delta} = \tilde{\delta}(r, v)$.

(i) $0 \leq \tilde{\delta} \leq \min\{\varepsilon, \frac{a-c}{n-1}\}$

(ii) $A(n-1, \tilde{r}, \tilde{v})$ is acceptable.

(iii) $\Pi_I(n-1, \tilde{\delta}) \geq \Pi_I(k, \delta)$, and equality holds if and only if $\varepsilon \geq \frac{a-c}{k}$ and $\delta = \frac{a-c}{k}$. In that case $\tilde{\delta} = \frac{a-c}{n-1} < \varepsilon$, $p(n-1, \tilde{\delta}) = p(k, \delta) = c$, $\underline{q}(n-1, \tilde{\delta}) = \underline{q}(k, \delta) = 0$ and $\Pi_I(n-1, \tilde{\delta}) = \Pi_I(k, \delta) = F(c)$.

Proof. i. Note that $(n-1)\tilde{\delta} = k\delta$. Since $n-1 > k$, we have $(n-1)\varepsilon > k\varepsilon \geq k\delta = (n-1)\tilde{\delta}$, implying $\tilde{\delta} < \varepsilon$. Since $\delta \leq \frac{a-c}{k}$, then $\tilde{\delta} = \frac{k}{n-1}\delta \leq \frac{a-c}{n-1}$.

ii. Since $A(k, r, v)$ is acceptable, by (6) and Lemma 4 we have $1-v \geq \left[\frac{a-c-k\delta}{a-c+(n-k+1)\delta}\right]^2$, that is ,

$$\left(\frac{k\sqrt{1-\tilde{v}}}{(n-1)-(n-k-1)\sqrt{1-\tilde{v}}} \right)^2 \geq \left[\frac{a-c-k\delta}{a-c+(n-k+1)\delta} \right]^2$$

.

Equivalently,

$$\tilde{v} \leq 1 - \left(\frac{a-c-(n-1)\tilde{\delta}}{a-c+2\tilde{\delta}} \right)^2$$

By Lemma 5, $A(n-1, \tilde{r}, \tilde{v})$ is acceptable.

iii. By (14) and Lemma 4,

$$\Pi_I(n-1, \tilde{\delta}) - \Pi_I(k, \delta) = (n-1-k)\varepsilon \underline{q}(k, \delta) \geq 0.$$

Note that $\Pi_I(n-1, \tilde{\delta}) = \Pi_I(k, \delta)$ if and only if $\underline{q}(k, \delta) = 0$. Since $\delta \leq \frac{a-c}{k}$, by Lemma 4, $\underline{q}(k, y) = 0$ if and only if $\varepsilon \geq \delta = \frac{a-c}{k}$. Other parts of iii follow trivially. ■

Proposition 3 follows immediately from Lemma 8 (iii). ■

Proof of Lemma 6 and Proposition 4. By Lemma 7, for the policies $A(n-1, r, v)$ or $A(n, r, v)$ to be optimal, it is enough to consider $\delta \in [0, \min\{\varepsilon, \frac{a-c}{n-1}\}]$.
First let $k = n-1$. By (9) and Lemma 4 (i),

$$\Pi_I(n-1, \delta) = \frac{n-1}{n+1}[(a-c+2\varepsilon)\delta - (n-1)\delta^2 + (a-c)\varepsilon].$$

We have

$$\frac{\partial \Pi_I}{\partial \delta}(n-1,\delta) \gtreqless 0 \Leftrightarrow \delta \lesseqgtr \frac{a-c+2\varepsilon}{2(n-1)}$$

Since $\frac{a-c+2\varepsilon}{2(n-1)} \le \frac{a-c}{n-1}$ iff $\varepsilon \le \frac{a-c}{2}$ and $\frac{a-c+2\varepsilon}{2(n-1)} \le \varepsilon$ iff $\varepsilon \ge \frac{a-c}{2(n-2)}$, then $\delta^*(n-1) = \frac{a-c+2\varepsilon}{2(n-1)}$ iff $\frac{a-c}{2(n-2)} \le \varepsilon \le \frac{a-c}{2}$. Otherwise, $\Pi_I(n-1,\delta)$ is increasing in δ for $\delta \in [0, \min\{\varepsilon, \frac{a-c}{n-1}\}]$. So we have

$$\delta^*(n-1) = \begin{cases} \varepsilon & \text{if } \varepsilon < \frac{a-c}{2(n-2)} \\ \frac{a-c+2\varepsilon}{2(n-1)} & \text{if } \frac{a-c}{2(n-2)} \le \varepsilon \le \frac{a-c}{2} \\ \frac{a-c}{n-1} & \text{if } \frac{a-c}{2} < \varepsilon < a-c \end{cases}$$

Therefore,

$$\Pi_I(n-1,\delta^*(n-1)) = \begin{cases} \frac{(-n^2+n+4)n\varepsilon^2+2(a-c)n(n+1)\varepsilon+(a-c)^2}{(n+2)^2} & \text{if } \varepsilon < \frac{a-c}{2(n-2)} \\ \frac{1}{4(n+1)}[4\varepsilon^2+4(a-c)\varepsilon n+(a-c)^2] & \text{if } \frac{a-c}{2(n-2)} \le \varepsilon \le \frac{a-c}{2} \\ (a-c)\varepsilon & \text{if } \frac{a-c}{2} < \varepsilon < a-c \end{cases} \tag{16}$$

Now let $k = n$. By (14) and Lemma 4 (ii),

$$\Pi_I(n,\delta) = \frac{n}{(n+1)^2}\{[(n-1)(a-c)+(n+1)\varepsilon]\delta - (n^2-n+1)\delta^2 + (a-c)\varepsilon(n+1)\}$$

So we have

$$\frac{\partial \Pi_I}{\partial \delta}(n,\delta) \gtreqless 0 \Leftrightarrow \delta \lesseqgtr \frac{(n-1)(a-c)+(n+1)\varepsilon}{2(n^2-n+1)}$$

Since $\frac{(n-1)(a-c)+(n+1)\varepsilon}{2(n^2-n+1)} < \frac{a-c}{n-1}$ for all $\varepsilon < a-c$ (non-drastic innovation) and $\frac{(n-1)(a-c)+(n+1)\varepsilon}{2(n^2-n+1)} \le \varepsilon$ iff $\varepsilon \ge \frac{a-c}{2n-1}$, then $\delta^*(n) = \frac{(n-1)(a-c)+(n+1)\varepsilon}{2(n^2-n+1)}$ iff $\varepsilon \ge \frac{a-c}{2n-1}$. Otherwise, $\Pi_I(n,\delta)$ is increasing in δ for $\delta \in [0,\varepsilon]$. So we have

$$\delta^*(n) = \begin{cases} \varepsilon & \text{if } \varepsilon < \frac{a-c}{2n-1} \\ \frac{(n-1)(a-c)+(n+1)\varepsilon}{2(n^2-n+1)} & \text{if } \frac{a-c}{2n-1} \le \varepsilon < a-c \end{cases}$$

Therefore,

$$\Pi_I(n,\delta^*(n)) = \begin{cases} \frac{(-n^3+n+1)\varepsilon^2+2(n^2+n+1)(a-c)\varepsilon+(a-c)^2}{(n+2)^2} & \text{if } \varepsilon < \frac{a-c}{2n-1} \\ \frac{(n^3+4)(a-c+\varepsilon)^2+4n^2(n+1)^2(a-c)\varepsilon}{4(n+2)^2(n^2-n+1)} & \text{if } \frac{a-c}{2n-1} \le \varepsilon < a-c \end{cases} \tag{17}$$

Since $\frac{a-c}{2n-1} < \frac{a-c}{2(n-2)} < \frac{a-c}{2}$, comparing (16) and (17), the difference is

$$\Pi_I(n,\delta^*(n)) - \Pi_I(n-1,\delta^*(n-1))$$
$$= \begin{cases} \frac{\varepsilon}{(n+2)^2}[-(n^2+n-3)\varepsilon + 2(a-c)] \begin{cases} >0 & \text{if } \varepsilon < \frac{2(a-c)}{n^2+n-3} \\ <0 & \text{if } \varepsilon > \frac{2(a-c)}{n^2+n-3} \end{cases} & \text{if } \varepsilon < \frac{a-c}{2n-1} \\ \frac{(4n^5-16n^4+13n^3+6n^2-15n+12)\varepsilon^2 - 2(a-c)(2n^4-5n^3+3n-4)\varepsilon + (a-c)^2 n(n-1)^2}{4(n+1)^2(n^2-n+1)} & \text{if } \frac{a-c}{2n-1} \le \varepsilon \le \frac{a-c}{2(n-2)} \\ \begin{cases} <0 & \text{if } n=3 \text{ and } \varepsilon > \frac{(4-\sqrt{7})(a-c)}{6} \text{ or if } n \ge 4 \\ >0 & \text{if } n=3 \text{ and } \varepsilon < \frac{(4-\sqrt{7})(a-c)}{6} \end{cases} & \\ \frac{(-3n^3+2n^2+n-4)\varepsilon^2 + 2(a-c)n(n^2-1)\varepsilon - (a-c)^2(2n^2-n+1)}{4(n+1)(n+2)^2(n^2-n+1)} & \text{if } \frac{a-c}{2(n-2)} < \varepsilon \le \frac{a-c}{2} \\ \begin{cases} <0 & \text{if } n \le 5 \text{ or if } n=6 \text{ and } \varepsilon \in (\frac{a-c}{8}, \frac{a-c}{d_1}] \cup (\frac{a-c}{d_2}, \frac{a-c}{2}] \\ & \text{or if } n \ge 7 \text{ and } \varepsilon \in (\frac{a-c}{2n-4}, \frac{a-c}{v(n)}] \cup (\frac{a-c}{u(n)}, \frac{a-c}{2}] \\ >0 & \text{otherwise} \end{cases} & \\ \frac{(n^3+4)(a-c-\varepsilon)^2}{4(n+2)^2(n^2-n+1)} > 0 & \text{if } \frac{a-c}{2} < \varepsilon < a-c \end{cases}$$

where

$$d_1 \equiv \frac{210+\sqrt{5642}}{67},\ d_2 \equiv \frac{210-\sqrt{5642}}{67},$$
$$v(n) := \frac{n^3-n+\sqrt{(n+1)(n^2-n+1)(n^3-6n^2+5n-4)}}{2n^2-n+1},$$
$$u(n) := \frac{(n+1)(1+\sqrt{n^2-n+1})^2}{n(n-1)^2}$$

Table 1 summarizes the above results.

■

Proof of Proposition 5. Suppose $n = 2$. We separately consider policies with $k = 1$ and $k = 2$. Then we compare the best acceptable policies of $\mathcal{I}$ in these two cases.

Suppose $k = 1$: Consider a policy $A(1, r, v)$ in a duopoly. By (9) and (6), the innovator's problem is

$$\max_{\delta} \Pi_I(1,\delta) = F(p(1,\delta)) - 2\underline{\phi}(1,\delta) - \underline{q}(1,\delta)\varepsilon$$
$$s.t.\ \ 0 \le v \le 1 - \underline{\phi}(1,\delta)/\bar{\phi}(1,\delta)$$

By Lemma 4, the innovator's problem becomes

$$\max_{\delta} \Pi_I(1,\delta) = \frac{1}{3}\left[\delta(a-c-\delta) + \varepsilon(a-c+2\delta)\right]$$
$$s.t.\ \ 0 \le v \le \frac{3\delta^2 + 6\delta(a-c)}{(a-c+2\delta)^2} \tag{18}$$

By Lemma 7, δ in (18) can take any value in $[0, \varepsilon]$.
Since $\delta \le \varepsilon$, $\frac{\partial \Pi_I}{\partial \delta}(1,\delta) = \frac{a-c+2(\varepsilon-\delta)}{3} > 0$ and the unique maximizer of $\Pi_I(1,\delta)$ is $\delta = \varepsilon$. This corresponds to the pure upfront fee policy with the unit royalty $r = 0$ and $v = 0$

(the licensee keeps its entire revenue). The innovator obtains

$$\Pi_I(1,\varepsilon) = [2(a-c)+\varepsilon]\varepsilon/3 \tag{19}$$

Suppose $k=2$: Consider a policy $A(2,r,v)$ in a duopoly. By (14) and (6), the innovator's problem is

$$\begin{aligned}\max_{\delta}\ \Pi_I(2,\delta) =& F(p(2,\delta)) - 2\underline{\phi}(1,\delta)\\ & s.t.\ \ v \leq 1-\underline{\phi}(1,\delta)/\bar{\phi}(2,\delta)\end{aligned}$$

By Lemma 4, the innovator's problem becomes

$$\begin{aligned}\max_{\delta}\ \Pi_I(2,\delta) =& \frac{2}{9}\left[-3(c-\delta)^2-(a-7c+3\varepsilon)(c-\delta)+(c+3\varepsilon)a-4c^2\right]\\ & s.t.\ \ 0\leq v\leq \frac{4(a-c)\delta}{(a-c+\delta)^2}\end{aligned} \tag{20}$$

Note that $\frac{\partial \Pi_I}{\partial\delta}(2,\delta)=\frac{2}{9}(3\varepsilon-6\delta+a-c)$, so

$$\frac{\partial \Pi_I}{\partial\delta}(2,\delta) \gtreqless 0 \Leftrightarrow \delta \lesseqgtr \frac{3\varepsilon+a-c}{6}.$$

By Lemma 7, δ in (20) can take any value in $[0,\varepsilon]$. The maximizer of $\Pi_I(2,\delta)$ is $\delta^*=\frac{3\varepsilon+a-c}{6}$ provided that $0\leq\frac{3\varepsilon+a-c}{6}\leq\varepsilon$ (equivalently, $\varepsilon\geq\frac{a-c}{3}$), and it is $\delta^*=\varepsilon$ if $\varepsilon<\frac{a-c}{3}$.

Proof of part (i). Suppose $\varepsilon<\frac{a-c}{3}$ and $k=2$. Then $\delta^*=\varepsilon$ and this implies $r=v=0$ and $\Pi_I(2,\varepsilon)=8\varepsilon(a-c)/9$. By (19),

$$\Pi_I(2,\varepsilon)-\Pi_I(1,\varepsilon)=\varepsilon(2a-2c-3\varepsilon)/9>\varepsilon(a-c-3\varepsilon)/9>0.$$

Hence the acceptable optimal policy of $\mathcal{I}$ for $n=2$ and $\varepsilon<\frac{a-c}{3}$ is $A(2,0,0)$.

Proof of parts (ii) and (iii). Let $\varepsilon\geq(a-c)/3$.
In this case (20) holds and $\Pi_I(2,\delta)$ is maximized at $\delta^*=\frac{3\varepsilon+a-c}{6}$.
Note that

$$\Pi_I(2,\tfrac{3\varepsilon+a-c}{6})=[(a-c)^2+42(a-c)\varepsilon+9\varepsilon^2]/54 \tag{21}$$

Comparing this with the equilibrium payoff of $\mathcal{I}$ following $A(1,0,0)$ (the best policy for $k=1$), by (19)

$$\Pi_I(2,\tfrac{3\varepsilon+a-c}{6})-\Pi_I(1,\varepsilon)=\kappa(\varepsilon):=[(a-c)^2+6(a-c)\varepsilon-9\varepsilon^2]/54$$

which is positive iff $\varepsilon<\frac{(\sqrt{2}+1)\varepsilon}{3}$. For $A(2,r,v)$ with $\delta(r,v)=\frac{3\varepsilon+a-c}{6}$ to be acceptable (i.e. to satisfy (20)), $v\leq 1-\left(\frac{5a-5c-3\varepsilon}{7a-7c+3\varepsilon}\right)^2$ should hold.

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