Supplemental Notes

for the presentation of "Weak Monotone Comparative Statics"

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Some Concepts from Lattice Theory

- Consider a **partially ordered** set X w.r.t. a *partial order* $\geq$
  - Namely $\geq$ is a binary relation on X that is reflesive, transitive and anti-symmetric.
- X is a **lattice** if for any $x, x' \in X$, $x \vee x' \in X$ and $x \wedge x' \in X$.

  For any $x, x' \in X$ and $X' \subset X$, let $x \vee_X x' := \inf\{x'' \in X : x'' \geq x \text{ and } x'' \geq x'\}$ and $\sup_X X' := \inf\{z \in X : z \geq x, \forall x \in X'\}$. $x \wedge_X x'$ and $\inf_X X'$ are analogously defined with $\geq$ and inf being replaced by $\leq$ and sup.
- X is a **complete lattice** if, for any $X' \subset X$, $\sup_X X' \in X$ and $\inf_X X' \in X$.
- A subset $X' \subset X$ is a **sublattice** of X if, for any $x, x' \in X'$, $x \wedge_X x' \in X'$ and $x \vee_X x' \in X'$.
- A subset $X' \subset X$ is a **complete sublattice** of X if $\sup_X Y \in X'$ and $\inf_X Y \in X'$ for all $Y \subseteq X'$.

Conditions for Strong-Set Monotone Comparative Statics

- Consider a pair of functions $u, v : X \to \mathbb{R}$.
- v **MS dominates** u, or $v \succeq_{MS} u$:
  1. v **single-crossing dominates** u: for any $x'' > x'$, $u(x'') - u(x') \geq (>)0 \Rightarrow v(x'') - v(x') \geq (>)0$;
  2. $f = u, v$ is **quasi-supermodular**: for any $x', x'' \in X$, $f(x'') - f(x' \wedge x'') \geq (>)0 \Rightarrow f(x' \vee x'') - f(x') \geq (>)0$.
- A **cardinal strengthening of MS**:
  1. **increasing-differences dominates** u if, for any $x' > x$, $v(x') - v(x) \geq u(x') - u(x)$.
  2. $f = u, v$ is **supermodular** if, for any $x, x' \in X$, $f(x \vee x') - f(x) \geq f(x') - f(x \wedge x')$.

Lemma

If u and v are supermodular, and v increasing-differences dominates u, then v MS dominates u.

sMCS results

Theorem (Migrom-Shannon)

Suppose v MS dominates u. Then,

$$M_{X''}(v) := \arg\max_{x \in X''} v(x) \geq_{ss} \arg\max_{x \in X'} u(x) =: M_{X'}(u),$$

for any $X'' \geq_{ss} X'$.

Some Topological Facts

- Let X be a partially ordered set, equipped with $\geq$.
- We consider **natural topology**—a topology in which the upper and lower contour sets—that is, $\{x \in X : x \geq a\}$ and $\{x \in X : x \leq a\}$ for each $a \in X$—are closed.
- The **order interval topology** is the smallest/coarsest natural topology.

Theorem (Frink, Birkhoff)

A sublattice of X is complete if and only if it is compact in the order interval topology on X.

Theorem (Topkis Theroem 2.3.1)

A sublattice of $\mathbb{R}^n$ is complete if and only if it is compact in the Euclidean topology on $\mathbb{R}^n$.