Competition under Moment Conditions

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The Game: Competition under Moment Conditions

- $\mathrm{I}(\geq 2)$ players (contestants), $i = 1, ..., I$.
- Player i chooses a random variable X_i with $F_i \in \Delta(\mathcal{R}_+)$ s.t.

$$\int_0^\infty x dF_i(x) \leq m_1, ..., \int_0^\infty x^n dF_i(x) \leq m_n.$$

- The player that produces the highest realization wins.
- Each maximizes the probability of winning: (ignoring atoms)

$$u_i(X^1, ..., X^{\mathrm{I}}) = Pr\{X_i > X_j, \forall j \neq i\} = \int_0^\infty \left[\prod_{j \neq i} F_j(x)\right] dF_i(x).$$

- Look for symmetric Nash equilibria: F^* that solves

$$\max_{F \in \Delta(\mathcal{R}_+)} \int_0^\infty (F^*(x))^{I-1} dF(x) \text{ s.t. } \int_0^\infty x^k dF(x) \leq m_k, \forall k.$$

Applications

1. Product design
   - Vertical differentiation: overall quality
   - Horizontal differentiation: aesthetic design (color, size...)
   - Each design yields a distribution of consumer values
   - Vertical $\Rightarrow$ mean, Horizontal $\Rightarrow$ variance

Applications

2. Research design (contest)
    - How much success (improvement) to aim for?
    - Gradual but likely success vs. Big but risky success
3. Sports
    - Federer/Nadal/Djokovic vs. Isner/Kyrgios/del Potro
    - Constructing a (national soccer) team

Related Literature

- Carrasco, Farinha Luz, Kos, Messner, Monterior, Moreira (2018)
  - Monopoly pricing under adversarial nature
  - Nature chooses the worst distribution under moment conditions
  - Moment conditions are interpreted as sellers' partial knowledge about consumer values.
- Information design
  - The first-moment condition with bounds: Boleslavsky, Cotton (2015), Au (2018), Whitmeyer (2018)...
  - This paper: *polynomialization*, instead of *concavification*.
- Contests
  - The first-moment condition (expected budget) with bounds: Hwang, Koh, and Lu (2020)
  - This paper: general moment conditions

The Model

- $\mathrm{I}(\geq 2)$ players (contestants)
- Player i's strategy: $F_i \in \mathcal{F}(m)$
  - For each $m = (m_1, ..., m_n) \in \mathcal{R}^n_{++}$, define

$$\mathcal{F}(m) \equiv \left\{ F \in \Delta(\mathcal{R}_+) : \int_0^\infty x^k dF(x) \leq m_k, \forall k = 1, ..., n \right\}.$$

- Payoff functions: for each $i = 1, ..., n$, (ignoring atoms)

$$u_i(F_i, F_{-i}) = Pr\{X_i > X_j, \forall j \neq i\} = \int_0^\infty \left(\prod_{j \neq i} F_j(x) \right) dF_i(x).$$

- F^* yields a *symmetric equilibrium* if

$$F^* \in argmax_{F \in \mathcal{F}(m)} u(F, F^*) \equiv \int_0^\infty (F^*(x))^{\mathrm{I}-1} dF(x).$$

Equilibrium Existence

Theorem

For any $\mathrm{I} \geq 2$ *and* $m \in \mathcal{R}^n_{++}$, *there exists a symmetric pure-strategy equilibrium.*

- We apply general results by Reny (1999).
- $\mathcal{F}(m)$ is non-empty, compact (in weak topology), convex.
- $u(F, F^*)$ is linear (so quasi-concave) in F.
- The game is *reciprocal upper-semicontinuous*.
  - $\sum_{i=1}^{n} u_i(F_i, F_{-i}) = 1$ with any strategy profile.
- The game is also *payoff secure*.
  - u_i is continuous in F_{-i}, provided F_i is continuous.
  - Discontinuous F_i can be approximated by continuous ones.

(*) Generalizable for the asymmetric environment.

Competition under the First-Moment Constraint

Proposition

If $m = m_1 \in \mathcal{R}$, then there exists a unique equilibrium in which $F^*(x)^{\mathrm{I}-1} = \lambda_1 x$ (linear) over its support $[0, \overline{x}]$.

- Two conditions for two unknowns, λ_1 and $\overline{x}$:

$$F^*(\overline{x})^{\mathrm{I}-1} = \lambda_1 \overline{x} = 1 \text{ and } \int_0^{\overline{x}} x dF^*(x) = \frac{\overline{x}}{\mathrm{I}} = m_1.$$

- Solving the two equations,

$$\overline{x} = \mathrm{I} m_1,\ \lambda_1 = \frac{1}{\mathrm{I} m_1} \Rightarrow F^*(x) = \left(\frac{x}{\mathrm{I} m_1}\right)^{1/(\mathrm{I}-1)}, \forall x \in [0, \mathrm{I} m_1].$$

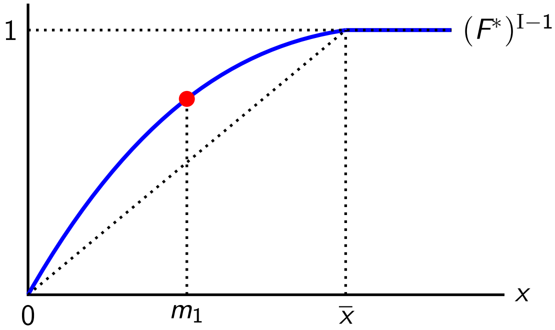
- Linearity is due to concavification.

Only the First Moment: Why Linear $(F^*)^{I-1}$?

- Recall that player i solves

$$\max_{F_i \in \mathcal{F}(m)} \int_0^\infty (F^*(x))^{I-1} dF_i(x).$$

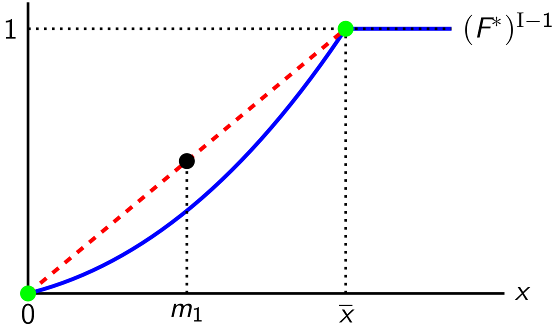
- Suppose that $(F^*)^{I-1}$ is concave.



- δ_{m_1} (degenerate) is optimal, but $F^* \neq \delta_{m_1}$.

Only the First Moment: Why Linear $(F^*)^{\mathrm{I}-1}$?

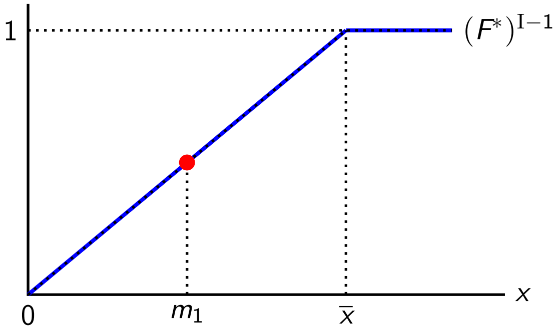
- Suppose that $(F^*)^{\mathrm{I}-1}$ is convex.



- Binary distribution (on 0 and $\overline{x}$) is optimal, but $F^* \neq$ binary.

Only the First Moment: Why Linear $(F^*)^{\mathrm{I}-1}$?

- Suppose that $(F^*)^{\mathrm{I}-1}$ is linear.



- Player i is *indifferent* among all distributions such that

$$supp(F) \subset [0, \bar{x}] \text{ and } \int_0^\infty x dF(x) = m_1.$$

Competition under Two Moment Constraints

- Now suppose that $m = (m_1, m_2) \in \mathcal{R}^2_{++}$.
- m_2 is sufficiently large $\Rightarrow$ m_2 irrelevant.

Proposition

F^* continues to be the unique equilibrium under $\mathcal{F}(m_1, m_2)$ if and only if $m_2 \geq \overline{m}_2 \equiv (\mathrm{I}m_1)^2/(2\mathrm{I}-1)$.

- Recall

$$F^*(x) = \left(\frac{x}{Im_1}\right)^{1/(\mathrm{I}-1)} \quad \text{for } x \in [0, Im_1].$$

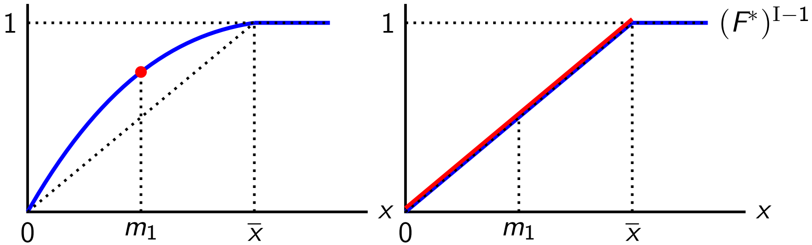
- Let

$$\overline{m}_2 = \int_0^\infty x^2 dF^*(x) = \frac{(\mathrm{I}m_1)^2}{2\mathrm{I}-1}.$$

- If $m_2 \geq \overline{m}_2$, then F^* satisfies the second-moment constraint.

Binding Second Moment

- What happens if $m_2 < \overline{m}_2$?
- $(F^*)^{I-1}$ now should be convex (over its support)!



- $(F^*)^{I-1}$ concave: not possible as before
- $(F^*)^{I-1}$ linear: also not possible.
- Intuitively, the second moment matters when a player wishes to disperse her values, which is when $(F^*)^{I-1}$ is convex.

Binding Second Moment

- How convex should $(F^*)^{\mathrm{I}-1}$ be?

Proposition

If $m_2 < \overline{m}_2$, then there exists a unique equilibrium in which $supp(F^*) = [0, \bar{x}]$ and for some $\lambda_1 (\geq 0)$ and $\lambda_2 (> 0)$,

$$F^*(x)^{\mathrm{I}-1} = \lambda_1 x + \lambda_2 x^2, \forall x \in [0, \bar{x}].$$

More concisely (combining with the previous proposition)

Proposition

For any $m \in \mathcal{R}^2_{++}$, there exists a unique equilibrium in which $supp(F^*) = [0, \bar{x}]$ and for some $\lambda_1 (\geq 0)$ and $\lambda_2 (\geq 0)$,

$$F^*(x)^{\mathrm{I}-1} = \lambda_1 x + \lambda_2 x^2, \forall x \in [0, \bar{x}].$$

Binding Second Moment: Why Quadratic?

- Consider the following programming problem:

$$\max_{F\in\Delta(\mathcal{R}_+)} \int_0^\infty u(x)dF(x) \text{ s.t. } \int_0^\infty x dF(x) \le m_1, \ \int_0^\infty x^2 dF(x) \le m_2.$$

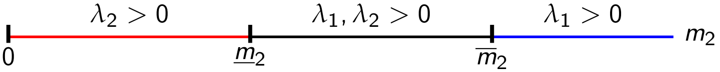
- Heuristically,

$$\begin{aligned}
\mathcal{L} &= \int_0^\infty u(x)dF(x) \\
&+ \lambda_1\left(m_1 - \int_0^\infty x dF(x)\right) + \lambda_2\left(m_2 - \int_0^\infty x^2 dF(x)\right) \\
&= \int_0^\infty (u(x) - \lambda_1 x - \lambda_2 x^2)dF(x) + \lambda_1 m_1 + \lambda_2 m_2.
\end{aligned}$$

- Each player is "indifferent" over a set of distributions if

$$F^*(x)^{\mathrm{I}-1} = u(x) = \lambda_1 x + \lambda_2 x^2.$$

Competition under Two Moment Constraints



$$\text{where } \underline{m}_2 \equiv \frac{((\mathrm{I}+1)m_1)^2}{4\mathrm{I}} \text{ and } \overline{m}_2 \equiv \frac{(\mathrm{I}m_1)^2}{2\mathrm{I}-1}.$$

1. $m_2 > \overline{m}_2$:

$$F^*(x) = \lambda_1 x = \frac{x}{\mathrm{I}m_1} \text{ over } [0, \mathrm{I}m_1].$$

2. $m_2 \in (\underline{m}_2, \overline{m}_2)$:

$$F^*(x) = \lambda_1 x + \lambda_2 x^2 \text{ over } [0, \overline{x}].$$

3. $m_2 \in (0, \underline{m}_2)$:

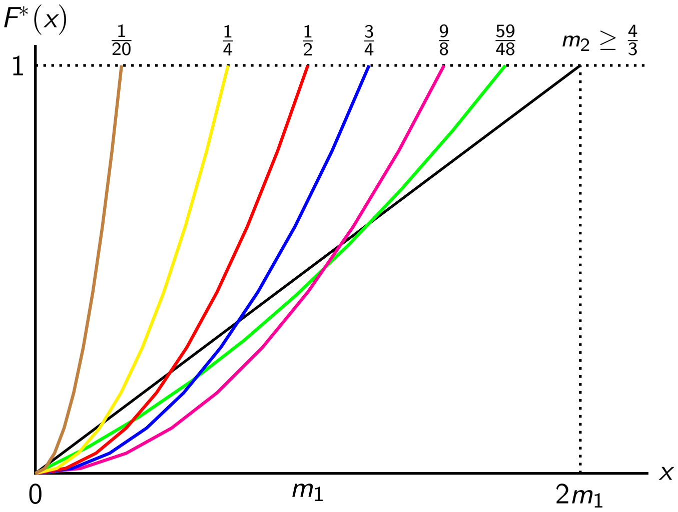
$$F^*(x) = \lambda_2 x^2 = \frac{x^2}{\mathrm{I}m_2} \text{ over } [0, \sqrt{\mathrm{I}m_2}].$$

Closed-form Solution for the Intermediate Case

2. If $\mathrm{I} = 2$, then

$$
\begin{aligned}
\lambda_1 &= \frac{-9m_1^3 - 3m_1^2\sqrt{9m_1^2 - 6m_2} + 9m_1 m_2 + 2m_2\sqrt{9m_1^2 - 6m_2}}{3m_2^2} \\
\lambda_2 &= \frac{18m_1^4 + 6m_1^3\sqrt{9m_1^2 - 6m_2} - 18m_1^2 m_2 - 4m_1\sqrt{9m_1^2 - 6m_2}\,m_2 + 3m_2^2}{6m_2^3} \\
\bar{x} &= 3m_1 - \sqrt{9m_1^2 - 6m_2}.
\end{aligned}
$$

Equilibrium Distribution: $I = 2$, $m_1 = 1$



General Characterization: n Constraints

Theorem

For any $m = (m_1, ..., m_n) \in \mathcal{R}^n_{++}$, $F^* \in \mathcal{F}(m)$ yields a symmetric equilibrium if and only if *there exist $\bar{x}$ and $\lambda \in \mathcal{R}^n_+ / \{\mathbf{0}\}$ s.t.*

(i) *Convex support from 0: $supp(F^*) = [0, \bar{x}]$*

(ii) *Polynomial representation:*

$$F^*(x)^{\mathrm{I}-1} = \lambda_1 x + ... + \lambda_n x^n, \forall x \in [0, \bar{x}].$$

(iii) *Complementary slackness:*

$$\lambda_k \left(m_k - \int_0^\infty x^k dF^*(x) \right) = 0, \forall k = 1, ..., n.$$

Proof of Sufficiency

$$\begin{aligned} u(F^*, F^*) &= \int_0^\infty F^*(x)^{\mathrm{I}-1} dF^*(x) \\ &= \int_0^{\bar{x}} F^*(x)^{\mathrm{I}-1} dF^*(x) \quad (\because \text{(i)}) \\ &= \int_0^{\bar{x}} (\lambda_1 x + ... + \lambda_n x^n) dF^*(x) \quad (\because \text{(ii)}) \\ &= \lambda_1 m_1 + ... + \lambda_n x_m \quad (\because \text{(iii)}). \end{aligned}$$

$$\begin{aligned} u(F, F^*) &= \int_0^\infty F^*(x)^{\mathrm{I}-1} dF(x) \\ &= \int_0^\infty \min\{\lambda_1 x + ... + \lambda_n x^n, 1\} dF(x) \quad (\because \text{(ii)}) \\ &\leq \int_0^\infty (\lambda_1 x + ... + \lambda_n x^n) dF(x) \\ &= \lambda_1 \int_0^\infty x dF(x) + ... + \lambda_n \int_0^\infty x^n dF(x) \\ &\leq \lambda_1 m_1 + ... + \lambda_n m_n \quad (\because \ \lambda_k > 0 \ \& \ F \in \mathcal{F}(m)). \end{aligned}$$

Competition under Three Moment Constraints

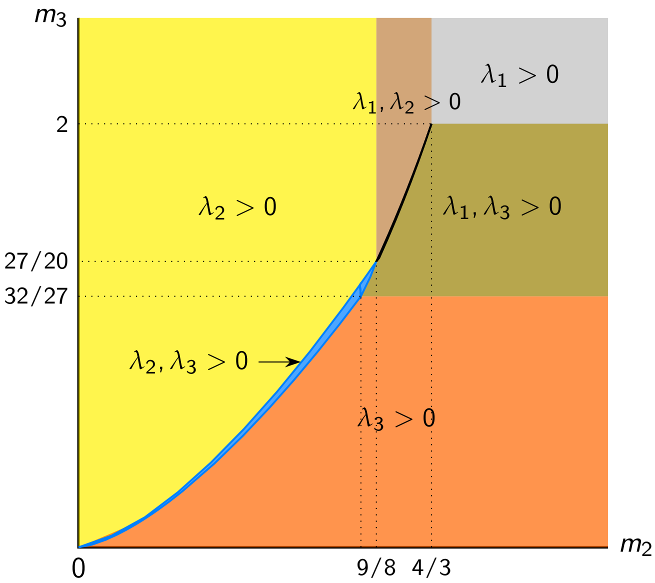
- Consider $m = (m_1, m_2, m_3)$ and $I = 2$ (for simplicity).
- 7 cases to consider:
  1. $\lambda_1 > 0$, $\lambda_2 = \lambda_3 = 0$
  2. $\lambda_2 > 0$, $\lambda_1 = \lambda_3 = 0$
  3. $\lambda_3 > 0$, $\lambda_1 = \lambda_2 = 0$
  4. $\lambda_1, \lambda_2 > 0$, $\lambda_3 = 0$
  5. $\lambda_1, \lambda_3 > 0$, $\lambda_2 = 0$
  6. $\lambda_2, \lambda_3 > 0$, $\lambda_1 = 0$
  7. $\lambda_1, \lambda_2, \lambda_3 > 0$
- For example, if $\lambda_3 > 0$, $\lambda_1 = \lambda_2 = 0$, then

$$F^*(x) = \frac{x^3}{2m_3} \text{ over } x \in [0, \sqrt[3]{2m_3}].$$

This is an equilibrium if and only if

$$m_1 \geq \int_0^{\bar{x}} x dF^*(x) = \frac{3(2m_3)^{1/3}}{4} \text{ and } m_2 \geq \int_0^{\bar{x}} x^2 dF^*(x) = \frac{3(2m_3)^{2/3}}{5}.$$

Competition under Three Moment Constraints ($m_1 = 1$)



Intense Competition

Corollary

Fix $m \in \mathcal{R}^n_{++}$. *If* I *(the number of players) is sufficiently large, then only the last-moment condition is binding (i.e.,* $\lambda_k = 0, \forall k$ *and* $\lambda_n > 0$*), in which case*

$$(F^*(x))^{\mathrm{I}-1} = \frac{x^n}{\mathrm{I}m_n} \text{ for } x \in \left[0, \sqrt[n]{\mathrm{I}m_n}\right].$$

Proof.

$$\int_0^\infty x^k dF^*(x) = \int_0^\infty x^k d\left(\frac{x^n}{\mathrm{I}m_n}\right) = \ldots = \frac{nm_n^{k/n}\mathrm{I}^{k/n}}{n + k(\mathrm{I}-1)}.$$

If $k < n$, then this k-th moment vanishes as $\mathrm{I} \to \infty$. □

Conclusion

- This paper: competition under moment conditions
  - General equilibrium existence
  - A necessary and sufficient condition for symmetric equilibrium
  - Full characterization for $n = 2, 3$ or I sufficiently large
- In progress
  - Asymmetric environment: $2 \times n$ case
  - Social vs. Private incentives regarding distribution design
  - Endogenizing moment conditions
- Future: other economic problems with moment conditions

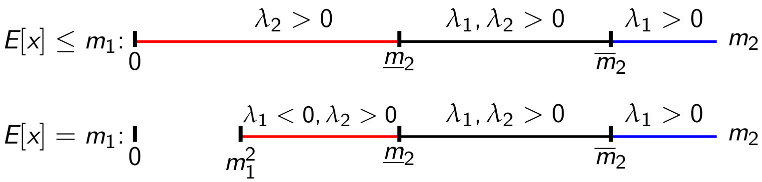
Discussion: Binding nConstraints

- Suppose that F must satisfy

$$\int_0^\infty x dF(x) = m_1 \text{ and } \int_0^\infty x^2 dF(x) \leq m_2.$$

- In this case,

$$m_2 \geq m_1^2 \Leftrightarrow Var(x) = m_2 - m_1^2 \geq 0.$$



Discussion: Binding Constraints

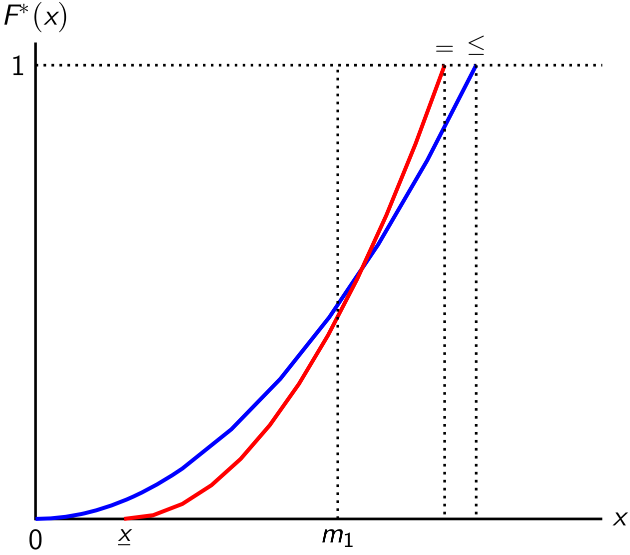
- Suppose that $m_2 \in (m_1^2, \underline{m}_2)$.
- Then, $\lambda_1 < 0$.
  - A player wishes to disperse her values, which is inhibited by the binding first-moment constraint.
- Unique symmetric equilibrium: $\underline{x} > 0$ and

$$F^*(x) = \left(\frac{x - \underline{x}}{\overline{x} - \underline{x}}\right)^{\frac{2}{I-1}} \text{ over } x \in [\underline{x}, \overline{x}].$$

- If $I = 2$, then

$$\underline{x} = m_1 - 2\sqrt{2(m_2 - m_1^2)} \text{ and } \overline{x} = m_1 + \sqrt{2(m_2 - m_1^2)}.$$

Equilibrium Distribution: $\mathrm{I} = 2$, $m_1 = 1$, $m_2 = 17/16$



Equilibrium Distribution: $\mathrm{I} = 2$, $m_1 = 1$, $m_2 = 1.02$