Catch-Up: A Rule That Makes Service Sports More Competitive

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Abstract

Service sports include two-player zero-sum games such as volleyball, badminton, and squash. We analyze four rules, including the Standard Rule (SR), in which a player continues to serve until he or she loses. The Catch-Up Rule (CR) gives the serve to the player who has lost the previous point—as opposed to the player who won the previous point, as under SR. We also consider two Trailing Rules that make the server the player who trails in total score. Surprisingly, compared with SR, only CR gives the players the same probability of winning a game while increasing its expected length, thereby making it more competitive and exciting to watch. Unlike one of the Trailing Rules, CR is strategy-proof. By contrast, the rules of tennis fix who serves and when; its tiebreaker, however, keeps play competitive by being fair—not favoring either the player who serves first or who serves second.

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1 Introduction

In service sports, competition between two players (or teams) involves one player serving some object—usually a ball, but a "shuttlecock" in badminton—which the opponent tries to return. Service sports include tennis, table tennis (ping pong), racquetball, squash, badminton, and volleyball.

If the server is successful, he or she wins the point; otherwise—in most, but not all, of these sports—the opponent does. If the competitors are equally skilled, the server (S) generally has a higher probability of winning than the receiver (R). We will say more later about what constitutes winning in various service sports.

In some service sports, such as tennis and table tennis, the serving order is fixed—the rules specify when and for how long each player serves. By contrast, the serving order in most service sports, including racquetball, squash, badminton, and volleyball, is variable: It depends on who won the last point.[1] In these sports, if S won the last point, then S serves on the next point also, whereas if R won, it becomes the new server. In short, the winner of the last point is the next server. We call this the Standard Rule (SR).

In this paper, we analyze three alternatives to SR, all of which are variable. The simplest is the

- Catch-Up Rule (CR): S is the loser of the previous point—instead of the winner, as under SR.[2]

The two other serving rules are Trailing Rules (TRs), which we also consider variable: A player who is behind in points becomes S. Thus, these rules take into account the entire history of play, not just who won or lost the previous point. If there is a tie, then who becomes S depends on the situation immediately prior to the tie:

- TRa: S is the player who was ahead in points prior to the tie;
- TRb: S is the player who was behind in points prior to the tie.

We calculate the players' win probabilities under all four rules. Our only data about the players are their probabilities of winning a point on serve, which we take to be equal exactly when the players are equally skilled. We always assume that all points (rounds) are independent. Among other findings, we prove that SR, CR, and TRa are strategy-proof—neither S nor R can ever benefit from deliberately losing a point. But TRb is

[1] In game theory, a game is defined by "the totality of the rules that describe it" (von Neumann and Morgenstern, 1953, p. 49). The main difference between fixed-order and variable-order serving rules is that when the serving order is variable, the course of play (order of service) may depend on the results on earlier points, whereas when the order is fixed, then so is the course of play. Put another way, the serving order is determined exogenously in fixed-order sports but endogenously in variable-order sports.

[2] The idea of catch-up is incorporated in the game of Catch-Up (see http://game.engineering.nyu.edu/projects/catch-up/ for a playable version), which is analyzed in Isaksen, Ismail, Brams, and Nealen (2015).

strategy-vulnerable: Under TRb, it is possible for a player to increase its probability of winning a game by losing a point deliberately, under certain conditions that we will spell out.

We analyze the probability that each player wins a game by being the first to score a certain number of points (Win-by-One); later, we analyze Win-by-Two, in which the winner is the first player to score at least the requisite number of points and also to be ahead by at least two points at that time. We also assess the effects of the different rules on the expected length of a game, measured by the total number of points until some player wins.

Most service sports, whether they use fixed or variable serving rules, use Win-by-Two. We compare games with and without Win-by-Two to assess the effects of this rule. Although Win-by-Two may prolong a game, it has no substantial effect on the probability of a player's winning if the number of points needed to win is sufficiently large and the players are equally skilled. The main effect of Win-by-Two is to increase the drama and tension of a close game.

The three new serving rules give a break to a player who loses a point, or falls behind, in a game. This change can be expected to make games, especially games between equally skilled players, more competitive—they are more likely to stay close to the end and, hence, to be more exciting to watch.

CR, like SR, is Markovian in basing the serving order only on the outcome of the previous point, whereas TRa and TRb take accumulated scores into account.[3] The two TRs may give an extra advantage to weaker players, which is not true of CR. As we will show, under Win-by-One, CR gives the players exactly the same probabilities of winning a game as SR does. At the same time, CR increases the expected length of games and, therefore, also their competitiveness. For this reason, our major recommendation is that CR replace SR to enhance competition in service sports with variable service rules.

2 Win-by-One

2.1 Probability of Winning

Assume A and B have probabilities p and q, respectively, of winning a point on service. For ease of description, we assume that A is male and B is female. We begin by analyzing the simple case in which the first player to score 2 (out of a possible 3) points wins, which we call Best-of-3.

[3] TRa is the same as the "behind first, alternating order" mechanism of Anbarci, Sun, and Ünver (2015), which says that if a player is behind, it serves next; if the score is tied, the order of service alternates (i.e., switches on the next round). Serving will alternate under this mechanism only if the player who was ahead prior to a tie—and, therefore, whose opponent served successfully to create the tie—then becomes the new server, causing an alternation in S. But this is just TRa, as S is the player who was behind in points prior to the tie.

We always assume that A serves first. Then there are three ways in which A may win: (1) by winning the first two points (which we denote $\underline{AA}$), (2) by winning the first and third points ($\underline{ABA}$), and (3) by winning the second and third points ($\underline{BAA}$). These sequences, which we call win sequences and underscore, indicate which player scored which point, but they do not indicate whether A, say, gained the point because he won on his serve or because B lost on her serve.

For the four rules described in the Introduction (SR, CR, TRa, TRb), we define an outcome as a sequence of server wins and server losses for the players. We use A and B to represent server wins and $\overline{A}$ and $\overline{B}$ to represent server losses. Under each rule, a particular win sequence can be realized only by a specific sequence of wins and losses. For example, the win sequence $\underline{ABA}$, in which A wins the first and third points and B wins the second, is realized as follows under each of the rules:

- SR: A serves first and wins the first point. Since he has just won, he serves again but loses the second point. Then B serves and also fails, so A wins the third point, producing the outcome $A\overline{A}\overline{B}$.

- CR: A serves first and wins the first point. Since B lost the first point, she now serves. She wins the second point, so A, who lost this point, serves again and wins the third point, producing the outcome ABA.

- TRa: A serves first and wins the first point. Since B lost the first point, she is now behind so she serves next. She wins the second point, and the game is now tied, so A, who was previously ahead, serves and wins the third point, producing the outcome ABA.

- TRb: A serves first and wins the first point. Since B lost the first point, she is now behind and serves next. She wins the second point, and the game is now tied, so B, who was previously behind, serves and loses the third point, producing the outcome $AB\overline{B}$.

Because we assume that serves are independent events—in particular, not dependent on the score at any point—we can calculate the probability of win sequence $\underline{ABA}$ for the different rules by multiplying the appropriate probabilities serve-by-serve. To illustrate using SR, the outcome given by win sequence $\underline{ABA}$ has probability $p(1-p)(1-q)$. Adding this probability to the probabilities of win sequences $\underline{AA}$ (p^2) and $\underline{BAA}$ ($p(1-p)(1-q)$) gives the total probability that A wins Best-of-3 under SR:

$$Pr_{SR}(A) = 2p - p^2 - 2pq + 2p^2q \qquad (= 2p - 3p^2 + 2p^3 \text{ when } p = q).$$

The complement of this quantity is, of course, B's probability of winning (recall that A always serves first):

$$Pr_{SR}(B) = 1 - (2p - p^2 - 2pq + 2p^2q) \qquad (= 1 - 2p + 3p^2 - 2p^3 \text{ when } p = q).$$

***A* First:**		***A* Wins**			***B* Wins**		***EL***
Win seq.:	$\underline{AA}$	$\underline{ABA}$	$\underline{BAA}$	$\underline{BB}$	$\underline{BAB}$	$\underline{ABB}$	
SR							
Outcome	AA	$A\overline{AB}$	$\overline{AB}A$	$\overline{A}B$	$\overline{ABA}$	$A\overline{A}B$	
Probability	p^2	$p(1-p)^2$	$p(1-p)^2$	$p(1-p)$	$(1-p)^3$	$p^2(1-p)$	
Sum	$2p-3p^2+2p^3$			$1-2p+3p^2-2p^3$			$3-p$
CR							
Outcome	$A\overline{B}$	ABA	$\overline{A}A\overline{B}$	$\overline{AA}$	$\overline{A}AB$	$AB\overline{A}$	
Probability	$p(1-p)$	p^3	$p(1-p)^2$	$(1-p)^2$	$(1-p)p^2$	$p^2(1-p)$	
Sum	$2p-3p^2+2p^3$			$1-2p+3p^2-2p^3$			$2+p$
TRa							
Outcome	$A\overline{B}$	ABA	$\overline{A}A\overline{B}$	$\overline{AA}$	$\overline{A}AB$	$AB\overline{A}$	
Probability	$p(1-p)$	p^3	$p(1-p)^2$	$(1-p)^2$	$(1-p)p^2$	$p^2(1-p)$	
Sum	$2p-3p^2+2p^3$			$1-2p+3p^2-2p^3$			$2+p$
TRb							
Outcome	$A\overline{B}$	$AB\overline{B}$	$\overline{A}AA$	$\overline{AA}$	$\overline{A}A\overline{A}$	ABB	
Probability	$p(1-p)$	$p^2(1-p)$	$p^2(1-p)$	$(1-p)^2$	$p(1-p)^2$	p^3	
Sum	$p+p^2-2p^3$			$1-p+p^2-2p^3$			$2+p$

Table 1: Probability that each player wins, and Expected Length (EL), for a Best-of-3 game when $p = q$

We focus on the case when $p = q$—the players are equally skilled at serving. (Realistically, $p > 1/2$ in most service sports, though volleyball is usually an exception, as we discuss later.) In Table 1 we show the outcomes, sequences, and probabilities of the different rules for Best-of-3.

If we assume $p = q$, it is easy to see that $Pr_{SR}(A) \geq Pr_{SR}(B)$ if and only if $p \geq \frac{1}{2}$. In particular, in a Best-of-3 game using SR, the initial server (A) and the initial receiver (B) have equal probabilities of winning if and only if $p = \frac{1}{2}$.

From Table 1, we can compare the influence of the different serving rules on the probabilities that the initial server, A, wins or loses. Notice that SR, CR, and TRa all give the same win probabilities for A and B (shown as "Sum"), but their common value differs from the win probabilities for TRb. It is not hard to show that the former three rules give A a strictly greater probability of winning, except when $p = 0$ or $p = \frac{1}{2}$. In particular, for $p > \frac{1}{2}$, we have

$$Pr_{SR}(A) = Pr_{CR}(A) = Pr_{TRa}(A) > Pr_{TRb}(A).$$

The intuition behind this result is that TRb most helps the player who falls behind. He or she is likely to be B when A serves first and $p > \frac{1}{2}$, making B's probability of winning greater, and A's less, than under the other rules.

But what if a game goes beyond Best-of-3? Most service sports require that the winner be the first player to score 11, 15, or 21 points, not 2. Even for Best-of-5, wherein the winner is the first player to score 3 points, the calculations are considerably more complex and tedious. Instead of three possible ways in which each player can win, there are ten. We carried out these computations and found that for Best-of-5, if $p > \frac{1}{2}$, then

$$Pr_{SR}(A) = Pr_{CR}(A) > Pr_{TRa}(A) > Pr_{TRb}(A).$$

Note that both TRs have lower probabilities of A's winning than SR and CR. But does the equality of A's winning under SR and CR hold generally?[4] Theorem 1 below shows that this is indeed the case.

Theorem 1. *Let $k \geq 1$. In a Best-of-$(2k+1)$ game, $Pr_{SR}(A) = Pr_{CR}(A)$.*

For a proof, see Appendix. The basis of the proof is the idea of serving schedule, which is a record of server wins and server losses organized according to the server. To illustrate, the description of a Best-of-3 game requires a server schedule of length 3, consisting of a record of whether the server won or lost on A's 1st and 2nd serves, and on B's 1st serve. The serving schedule (W, L, L), for instance, records that $A_1 = W$ (i.e., A won on his 1st serve), $A_2 = L$ (A lost on his 2nd serve, and $B_1 = L$ (B lost on her 1st serve.) The idea is that, if the serving schedule is fixed, then both serving rules, SR and CR, give the same outcome as an Auxiliary Rule (AR), in which A serves twice and then B serves once. Specifically,

[4] Kingston (1976) proved that, in a Best-of-$(2k+1)$ game with $k \geq 1$, the probability of A winning under SR is equal to the probability of A winning under any fixed rule that assigns A $k+1$ serves, and B k serves; this proof was simplified, and made more intuitive, by Anderson (1977).

- AR: $< A_1 = W, A_2 = L, B_1 = L >$, outcome $A\overline{A}\overline{B}$. A wins 2-1.
- SR: $< A_1 = W, A_2 = L, B_1 = L >$, outcome $A\overline{A}\overline{B}$. A wins 2-1.
- CR: $< A_1 = W, B_1 = L >$, outcome $A\overline{B}$. A wins 2-0.

Observe that the winner is the same under each rule, despite the differences in outcomes and scores. The basis of our proof is a demonstration that, if the serving schedule is fixed, then the winners under AR, SR, and CR are identical.

The AR service rule is particularly simple, and permits us to establish a formula to determine win probabilities. The fact that the AR, SR, and CR service rules have equal win probabilities makes this representation more useful.

Corollary 1. *The probability that A wins a Best-of-$(2k+1)$ game under service rules AR, SR, and CR is*

$$Pr_{AR}(A) = Pr_{SR}(A) = Pr_{CR}(A) = \sum_{i=1}^{k+1} \sum_{j=0}^{i-1} p^i (1-p)^{k+1-i} q^j (1-q)^{k-j} \binom{k+1}{i} \binom{k}{j}.$$

Proof. Under AR, the winner depends on the results on A's first $k+1$ serves and on B's subsequent k serves. Suppose that A has exactly i server wins among its $k+1$ serves. If $i = 0$, A must lose under AR. If $i = 1, 2, \ldots, k+1$, then A wins under AR if and only if B has exactly j server wins among its first k serves, where $j \leq i-1$. The probability above follows directly. □

2.2 Expected Length of a Game

The different service rules may affect not only the probability of A's winning but also the expected length (EL) of a game. To illustrate the latter calculation, consider SR for Best-of-3 in the case $p = q$. Clearly, the game lasts either two or three serves. Table 1 shows that A will win with probability p^2 after 2 serves and B will win with probability $p(1-p)$ after 2 serves. Therefore the probability that the game ends after two serves is $p^2 + p(1-p) = p$, so the probability that it ends after three serves is $(1-p)$. Hence, the expected length is

$$EL_{SR} = 2p + 3(1-p) = 3 - p.$$

To illustrate, if $p = 0$, servers always lose, so the game will take 3 serves—and 2 switches of server—before the player who starts (A) loses 2 points to 1. On the other hand, if $p = 1$, $EL_{SR} = 2$, because A will win on his first two serves.

By a similar calculation, $EL_{CR} = 2 + p$. We also give results for the two TRs in Table 1, producing the following ranking of expected lengths for Best-of-3 games if $p > \frac{1}{2}$,

$$EL_{TRb} = EL_{TRa} = EL_{CR} > EL_{SR}.$$

Observe that the expected length of a game for Best-of-3 is a minimum under SR. To give an intuition for this conclusion, the player who starts, if he or she is successful, can end play with a second successful serve, which is fairly likely if p is large. On the other hand, CR and the TRs shift the service to the other player, who now has a good chance of evening the score if $p > \frac{1}{2}$.

As noted earlier, we have carried out complete calculations for Best-of-5 games and found that

$$EL_{TRb} = EL_{TRa} > EL_{CR} > EL_{SR}.$$

Thus, both TRs have a greater expected length than CR, and the length of CR in turn exceeds the length of SR.[5]

Theorem 2. *In a Best-of-*$(2k+1)$ *game for any* $k \geq 1$*, the expected length of a game is greater under* CR *than under* SR *if and only if* $p + q > 1$.

The proof of Theorem 2 appears in the Appendix.

Henceforth, we focus on the comparison between SR and CR for two reasons:

1. Under CR, the probability of a player winning is the same as under SR (Theorem 1).

2. Under CR, the length of the game is greater (in expectation) than under SR, provided $p + q > 1$ (Theorem 2).

For these reasons, we believe that service sports currently using the SR rule would benefit from the CR rule. Changes should not introduce radical shifts, such as changing the probability of winning. At the same time, CR would make play more competitive and, therefore, more likely to stimulate fan (and player) interest. CR satisfies both of our criteria.

CR keeps games close by giving a player who loses a point the opportunity to serve and, therefore, to catch up, given $p > \frac{1}{2}$. Consequently, the expected length of games will, on average, be greater under CR than SR. For Best-of-3 games, if $p = \frac{2}{3}$, then $EL_{CR} = \frac{8}{3}$ whereas $EL_{SR} = \frac{7}{3}$. Still, the probability that A wins is the same under both rules ($\frac{16}{27} = 0.592$, from Table 1). By Theorem 1, each player can rest assured that CR, compared with SR, does not affect his or her chances of winning.

It is true under SR and CR that if $p = \frac{2}{3}$, A has almost a 3:2 advantage in probability of winning ($\frac{16}{27} = 0.592$) in Best-of-3. But this is less than the 2:1 advantage A would enjoy if a game were decided by just one serve. Furthermore, A's advantage drops as k increases for Best-of-$(2k+1)$, so games that require more points to win tend to level the playing field.

Because the two TRs do even better than CR in lengthening games, would not one be preferable in making games closer and more competitive? The answer is "yes," but they

[5] In an experiment, Ruffle and Volij (2015) found that the average length of a Best-of-9 game was greater under CR than under SR.

would reduce $Pr(A)$, relative to SR (and CR), and so be a more significant departure from the present rule. For Best-of-3, $Pr_{TRb}(A) = \frac{14}{27} = 0.519$, which is 12.5% lower than $Pr_{SR}(A) = Pr_{CR}(A) = \frac{16}{27} = 0.592$.[6] But TRb has a major strike against it that the other rules do not, which we explore in the next section.

2.3 Incentive Compatibility

As discussed in the Introduction, a rule is strategy-proof or incentive compatible if no player can ever benefit from deliberately losing a point; otherwise, it is strategy-vulnerable.[7]

Theorem 3. *TRb is strategy-vulnerable, whereas SR and CR are strategy-proof. TRa is strategy-proof whenever $p + q > 1$.*

Proof. To show that TRb is strategy-vulnerable, assume a Best-of-3 game played under TRb. We will show that there are values of p and q such that A can increase his probability of winning the game by deliberately losing the first point.

Recall that A serves first. If A loses on the initial serve, the score is 0-1, so under TRb, A serves again. If he wins the second point (with probability p), he ties the score at 1-1 and serves once more, again winning with probability p. Thus, by losing deliberately, A wins the game with probability p^2.

If A does not deliberately lose his first serve, then there are three outcomes in which he will win the game:

- $A\overline{B}$ with probability $p(1-q)$
- $AB\overline{B}$ with probability $pq(1-q)$
- $\overline{A}AA$ with probability $(1-p)p^2$.

Therefore, A's probability of winning the game is greater when A deliberately loses the first serve if and only if

$$p^2 > (p - pq) + (pq - pq^2) + (p^2 - p^3),$$

[6]We recognize that it may be desirable to eliminate entirely the first-server advantage, but this is difficult to accomplish in a service sport, as the ball must be put into play somehow. In fact, the first-server advantage decreases as the number of points required to win increases. For Best-of-5, for example, $Pr_{SR}(A) = Pr_{CR}(A) = \frac{46}{81} = 0.568$ when $p = \frac{2}{3}$, which is less than 0.592 for Best-of-3. Thus, increasing the number of points needed to win diminishes the probabilistic advantage of the player who serves first in a game. Of course, any game is ex-ante fair if the first server is chosen randomly according to a coin toss, but it would be desirable to align ex-ante and ex-post fairness. As we show later, the serving rules of the tiebreaker in tennis achieve this alignment exactly.

[7]For an informative analysis of strategizing in sports competitions, and a discussion of its possible occurrence in the 2012 Olympic badminton competition, see Pauly (2014).

which is equivalent to $p(1-p^2-q^2) < 0$. Because $p > 0$, it follows that A maximizes the probability that he wins the game by deliberately losing his first serve when

$$p^2 + q^2 > 1.$$

In (p, q)-space, this inequality describes the exterior of a circle of radius 1 centered at $(p, q) = (0, 0)$. Because the probabilities p and q can be any numbers within the unit square of this space, it is possible that (p, q) lies outside this circle. If so, A's best strategy is to deliberately lose his initial serve.

We next show that SR is strategy-proof. In a Best-of-$(2k+1)$ game played under SR, let (C, x, y) denote a state in which player C ($C = A$ or B) is the server, A's score is x and B's y. For example, when the game starts, the state is $(A, 0, 0)$. Let $W_{AS}(C, x, y)$ denote A's win probability from state (C, x, y). It is clear that A's win probability $W_{AS}(A, x, y)$ is increasing in x.

To show that a game played under SR is strategy-proof for A, we must show that $W_{AS}(A, x+1, y) \geq W_{AS}(B, x, y+1)$ for any x and y. Now

$$W_{AS}(A, x, y) = pW_{AS}(A, x+1, y) + (1-p)W_{AS}(B, x, y+1),$$

which implies that

$$p\left[W_{AS}(A, x+1, y) - W_{AS}(A, x, y)\right] + (1-p)\left[W_{AS}(B, x, y+1) - W_{AS}(A, x, y)\right] = 0.$$

Because $0 < p < 1$ and $W_{AS}(A, x+1, y) \geq W_{AS}(A, x, y)$, the first term is non-negative, so the second must be non-positive. It follows that

$$W_{AS}(B, x, y+1) \leq W_{AS}(A, x, y) \leq W_{AS}(A, x+1, y),$$

as required. Thus, A cannot gain under SR by deliberately losing a serve, so SR is strategy-proof for A. By an analogous argument, SR is also strategy-proof for B.

We next show that CR is strategy-proof. We use the same notation for states, and denote A's win probability from state (C, x, y) by $W_{AC}(C, x, y)$ and B's by $W_{BC}(C, x, y)$. As in the SR case, it is clear that $W_{AC}(C, x, y)$ is an increasing function of x.

To show that a game played under CR is strategy-proof for A, we must show that $W_{AC}(B, x+1, y) \geq W_{AC}(A, x, y+1)$ for any x and y. Now

$$W_{AC}(A, x, y) = pW_{AC}(B, x+1, y) + (1-p)W_{AC}(A, x, y+1),$$

which implies that

$$p\left[W_{AC}(B, x+1, y) - W_{AC}(A, x, y)\right] + (1-p)\left[W_{AC}(A, x, y+1) - W_{AC}(A, x, y)\right] = 0.$$

Again, it follows that

$$W_{AC}(A, x, y+1) \leq W_{AC}(A, x, y) \leq W_{AC}(B, x+1, y),$$

as required. Thus a game played under CR is strategyproof for A, and by analogy it is strategyproof for B.

The proof that TRa is strategy-proof is left for the Appendix. □

To illustrate the difference between TRa and TRb and indicate its implications for strategy-proofness, suppose that the score is tied and that A has the serve. If A deliberately loses, B will be ahead by one point, so A will serve the second point. If A is successful, the score is again tied, and TRb awards the next serve to A, whereas TRa gives it to B, who was ahead prior to the most recent tie. If $p > 1 - q$, the extra serve is an advantage to A.

Suppose that the players are equally skilled at serving ($p = q$) in the Best-of-3 counterexample establishing that TRb is not strategy-proof. Then the condition for deliberately losing to be advantageous under TRb is $p > 0.707$. Thus, strategy-vulnerability arises in this simple game if both players have a server win probability greater than about 0.71, which is high but not unrealistic in most service sports.

From numerical calculations, we know that for Best-of-5 (and longer) games, neither TRa (nor the strategy-vulnerable TRb) has the same win probability for A as SR , whereas by Theorem 1 CR maintains it in Win-by-One games of any length. This seems a good reason for focusing on CR as the most viable alternative to SR, especially because our calculations show that CR increases the expected length of Best-of-5 and longer games, thereby making them more competitive.

Most service sports are not Win-by-One but Win-by-Two (racquetball is an exception, discussed in section 3). If $k + 1$ points are required to win, and if the two players tie at k points, then one player must outscore his or her opponent by two points in a tiebreaker in order to win.

We next analyze Win-by-Two's effect on A's probability of winning and the expected length of a game.[8] We note that the service rule, which we take to be SR or CR, affects the tiebreak; starting in a tied position, SR gives one player the chance of winning on two consecutive serves, whereas under CR a player who loses a tiebreak must lose at least once on serve. We later consider sports with fixed rules for serving and ask how Win-by-Two affects them.

3 Win-by-Two

To illustrate Win-by-Two, consider Best-of-3, in which a player wins by being the first to receive 2 points *and* by achieving at least two more points than the opponent. In other words, 2-0 is winning but 2-1 is not; if a 1-1 tie occurs, the winner will be the first player to lead by 2 points.

[8]Before tiebreakers for sets were introduced into tennis in the 1970s, Kemeny and Snell (1960; reprinted 1976, pp. 161-164) showed how the effect of being more skilled in winning a point in tennis ramifies to a game, a set, and a match. For example, a player with a probability of 0.51 (0.60) of winning a point—whether it served or not—had a probability of 0.525 (0.736) of winning a game, a probability of 0.573 (0.966) of winning a set, and a probability of 0.635 (0.9996) of winning a match (in men's competition, or Best-of-5 for sets). In tennis, to win a game or a set now requires, respectively, winning by at least two points or at least two games (if there is no tiebreaker). In effect, tiebreakers change the margin by which a player must win a set from at least two games to at least two points in the tiebreaker.

3.1 Standard Rule Tiebreaker

Consider the ways in which a 1-1 tie can occur under SR. There are two sequences that produce such a tie:

- $A\overline{A}$ with probability $p(1-p)$
- $\overline{AB}$ with probability $(1-p)(1-q)$.

Thus, the probability of a 1-1 tie is

$$(p-p^2)+(1-p-q+pq)=1+pq-p^2-q,$$

or $1-p$ if $p=q$. As p approaches 1 (and $p=q$), the probability of a tie approaches 0, whereas if $p=\frac{2}{3}$, the probability is $\frac{1}{3}$.

We now assume that $p=q$. In a Best-of-$(2k+1)$ tiebreaker played under SR, recall that $Pr_{SR}(A)$ is the probability that A wins when he serves first. Then $1-Pr_{SR}(A)$ is the probability that B wins when A serves first. Since $p=q$, $1-Pr_{SR}(A)$ is also the probability that A wins the tiebreaker when B serves first. Then it follows that $Pr_{SR}(A)$ must satisfy the following recursion:

$$Pr_{SR}(A)=p^2+[p(1-p)(1-Pr_{SR}(A))]+[(1-p)^2Pr_{SR}(A)].$$

The first term on the right-hand side gives the probability that A wins the first two points. The second term gives the probability of sequence $A\overline{A}$—so A wins initially and then loses, re-creating a tie—times the probability that A wins when B serves first in the next tiebreaker. The third term gives the probability of sequence $\overline{AB}$—in which A loses initially and then B loses—re-creating a tie—times the probability that A wins the next tiebreaker. Because $p>0$, this equation can be solved for $Pr_{SR}(A)$ to yield

$$Pr_{SR}(A)=\frac{1}{3-2p}.$$

Table 2 presents the probabilities and expected lengths under SR and CR (see subsection 3.2) corresponding to various values of p. Under SR, observe that A's probability of winning is greater than p when $p=\frac{1}{3}$ and less when $p=\frac{2}{3}$. More generally, it is easy to show that $Pr_{SR}(A)>p$ if $0<p<\frac{1}{2}$, and $Pr_{SR}(A)<p$ if $\frac{1}{2}<p<1$. Thus, A does worse in a tiebreaker than if a single serve decides a tied game when $\frac{1}{2}<p<1$.

When $p=1$, A will win the tiebreaker with certainty, because he serves first and will win every point. At the other extreme, when p approaches 0, $Pr_{SR}(A)$ approaches $\frac{1}{3}$. But we do not claim that $Pr_{SR}(A)=\frac{1}{3}$ when $p=0$, because the above equation for $Pr_{SR}(A)$ cannot be solved in this case. This corresponds to the observation that when $p=0$, the server will always lose, so neither player will ever win two points in a row. Because the game never ends, we cannot assign a probability that A (or B) wins.

p	$Pr_{SR}(A)$	$Pr_{CR}(A)$	EL_{SR}	EL_{CR}
0	undefined	0.00	∞	2.00
1/4	0.40	0.33	8.00	2.67
1/3	0.43	0.40	6.00	3.00
1/2	0.50	0.50	4.00	4.00
2/3	0.60	0.57	3.00	6.00
3/4	0.67	0.60	2.67	8.00
1	1.00	undefined	2.00	∞

Table 2: Probability (Pr) that A Wins, and Expected Length of a Game (EL), under SR and CR for various values of p

When there is a tie at the end of a game, Win-by-Two not only prolongs the game over Win-by-One but also changes the players' win probabilities. If $p = \frac{2}{3}$, under Win-by-One, sequence $A\overline{A}$ has twice the probability of producing a tie as $\overline{AB}$. Consequently, B has twice the probability of serving the tie-breaking point (and, therefore, winning) than A does under Win-by-One.

But under Win-by-Two, if $p = \frac{2}{3}$ and B serves first in the tiebreaker, B's probability of winning is not $\frac{2}{3}$, as it is under Win-by-One, but $\frac{3}{5}$. Because we assume $p > \frac{1}{2}$, the player who serves first in the tiebreaker is hurt—compared with Win-by-One—except when $p = 1$.[9]

By how much, on average, does Win-by-Two prolong a game? Recall our assumption that $p = q$. It follows that the expected length (EL) of a tiebreaker does not depend on which player serves first (for notational convenience below, we assume that A serves first). The following recursion gives expected length (EL) under SR when $p > 0$:

$$EL_{SR} = [p^2 + (1-p)p](2) + [p(1-p) + (1-p)^2](EL_{SR} + 2).$$

The first term on the right-hand side of the equation contains the probability that A wins an immediate victory (sequence AA) or that B wins an immediate victory (sequence $\overline{A}B$), both of which contribute length 2. The second term contains the probability that, because of ties, the tiebreaker increases the expected length by two serves (in sequences $A\overline{A}$ and $\overline{AB}$), which implies that the conditional expected length is $EL_{SR} + 2$.

[9] In gambling, the player with a higher probability of winning individual games does better the more games are played. Here, however, when the players have the same probability p of winning points when they serve, the player who goes first, and therefore would seem to be advantaged under SR when $p > \frac{1}{2}$, does not do as well under Win-by-Two as under Win-by-One. This apparent paradox is explained by the fact that Win-by-Two gives the second player a chance to come back after losing a point, an opportunity not available under Win-by-One.

Assuming that $p > 0$, this relation can be solved to yield

$$EL_{SR}(A) = \frac{2}{p}.$$

Clearly, if $p = 1$, A immediately wins the tiebreaker with 2 successful serves, whereas it takes 3 and then 4, on average, as p approaches $\frac{1}{2}$ from above. If p dips below $\frac{1}{2}$, which is unrealistic in most service sports, the tiebreaker grows increasingly long. Again, the length increases without bound as p approaches 0; in the limit, the tiebreaker under SR never ends.

3.2 Catch-Up Rule Tiebreaker

We suggested earlier that CR is a viable alternative to SR, so we next compute the probability that A wins under CR using the following recursion, which mirrors the one for SR:

$$Pr_{CR}(A) = p(1-p) + p^2 Pr_{CR}(A) + [(1-p)(p)[1 - Pr_{CR}(A)].$$

The first term on the right-hand side is the probability of sequence $A\overline{B}$, in which A wins the first point and B loses the second point, so A wins at the outset. The second term gives the probability of sequence AB—A wins the first point and B the second, creating a tie—times the probability that A wins eventually. The third term gives the probability of sequence $\overline{A}A$—so A loses initially and then wins, creating a tie—times the probability that A wins when B serves first in the tiebreaker, the complement of the probability that A wins when he serves first.

This equation can be solved for $Pr_{CR}(A)$ provided $p < 1$, in which case

$$Pr_{CR}(A) = \frac{2p}{1+2p}.$$

Note that when $p = 1$, $Pr_{CR}(A)$ is undefined—not $\frac{2}{3}$—because the game will never end since each win by one player leads to a win by the other player, precluding either player from ever winning by two points. For various values of $p < 1$, Table 2 gives the corresponding probabilities.

What is the expected length of a tiebreaker under CR? When $p = q < 1$, EL_{CR} satisfies the recursion

$$EL_{CR} = [p(1-p) + (1-p)^2](2) + [p^2 + (1-p)p](EL_{CR} + 2),$$

which is justified by reasoning similar to that given earlier for EL_{SR}. This equation can be solved for EL_{CR} only if $p < 1$, in which case

$$EL_{CR} = \frac{2}{1-p}.$$

Unlike EL_{SR}, EL_{CR} increases with p, but it approaches infinity at $p = 1$, because the tiebreaker never ends when both players alternate successful serves.

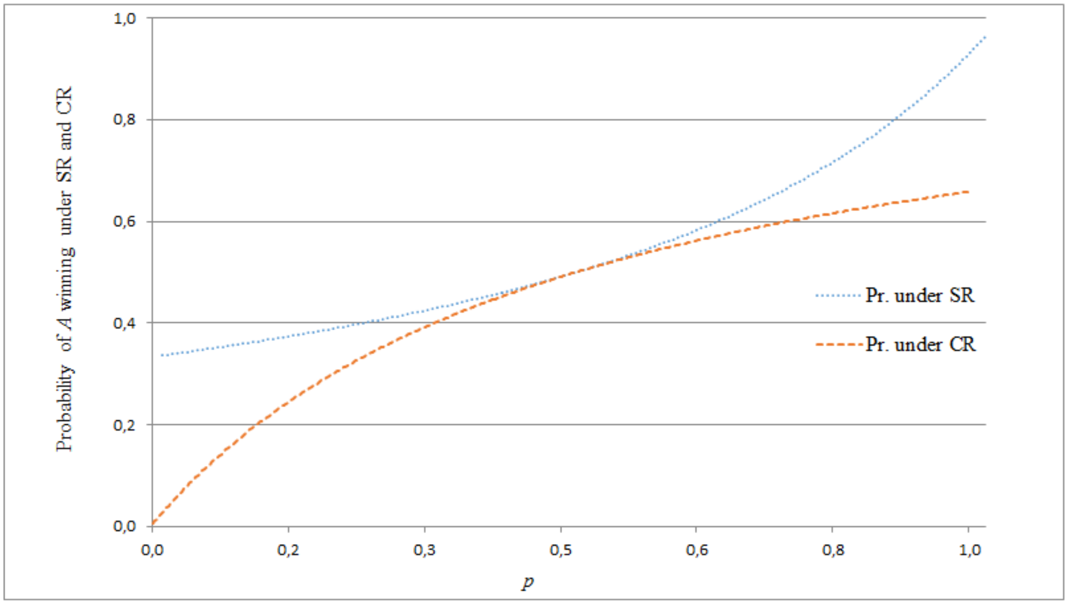

Figure 1: Graph of $Pr_{SR}(A)$ and $Pr_{CR}(A)$ as a Function of p

3.3 Comparison of Tiebreakers

We compare $Pr_{SR}(A)$ and $Pr_{CR}(A)$ in Figure 1. Observe that both are increasing in p, though at different rates. The two lines touch at $p = \frac{1}{2}$, where both probabilities equal $\frac{1}{2}$. But as p approaches 1, $Pr_{SR}(A)$ increases at an increasing rate toward 1, whereas $Pr_{CR}(A)$ increases at a decreasing rate toward $\frac{2}{3}$. When $p > \frac{1}{2}$, A has a greater advantage under SR, because he is fairly likely to win at the outset with two successful serves, compared to under CR where, if his first serve is successful, B (who has equal probability of success) serves next.

The graph in Figure 2 shows the inverse relationship between EL_{SR} and EL_{CR} as a function of p. If $p = \frac{2}{3}$, which is a realistic value in several service sports, the expected length of the tiebreaker is greater under CR than under SR (6 vs. 3), which is consistent with our earlier finding for Win-by-One: $EL_{CR} > EL_{SR}$ for Best-of-$(2k+1)$ (Theorem 2). Compared with SR, CR adds 3 serves, on average, to the tiebreaker, and also increases the expected length of the game prior to any tiebreaker, making a tiebreaker that much more likely.

But we emphasize that in the Win-by-One (Best-of-$2k+1$) game, $Pr_{SR}(A) = Pr_{CR}(A)$ (Theorem 1). Thus, compared with SR, CR does not change the probability that A or B wins in the regular game. But if there is a tie in this game, the players' win probabilities may be different in an SR tiebreaker versus a CR tiebreaker. For example, if $p = \frac{2}{3}$ and A is the first player to serve in the tiebreaker, he has a probability of $\frac{3}{5} = 0.600$ of winning under SR and a probability of $\frac{4}{7} = 0.571$ of winning under CR.

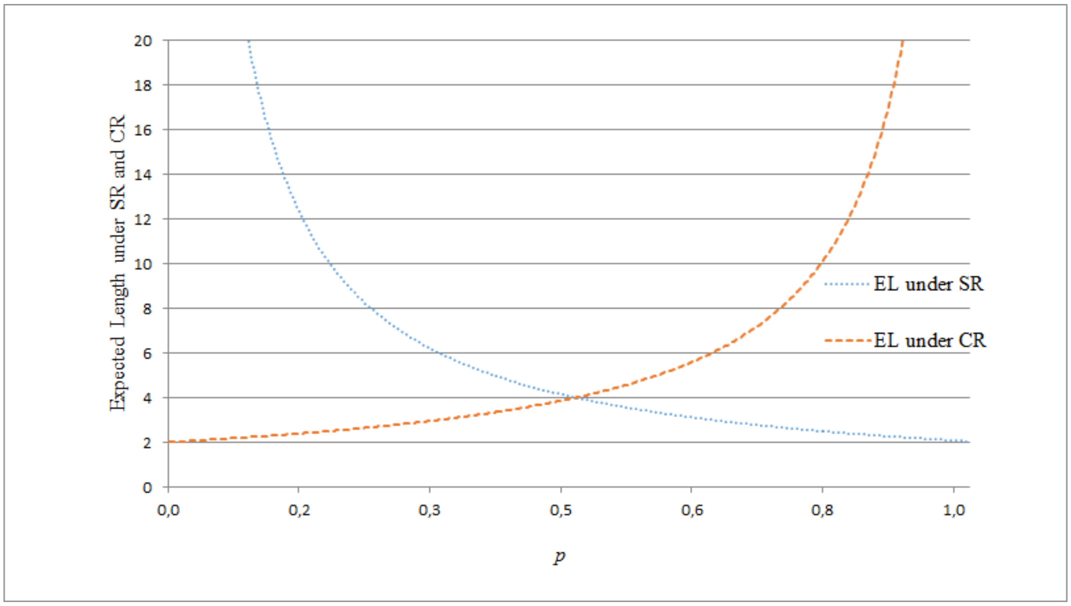

Figure 2: Graph of EL_{SR} and EL_{CR} as a Function of p

3.4 Numerical Comparisons

Clearly, serving first in the tiebreaker is a benefit under both rules. For Best-of-3, we showed in section 2 that if $p = \frac{2}{3}$, the probability of a tie is $\frac{1}{3}$, and B is twice as likely as A to serve first in the tiebreaker under SR.

In Table 3, we extend Best-of-3 to the more realistic cases of Best-of-11 and Best-of-21 for both Win-by-One (WB1) and Win-by-Two (WB2) for the cases $p = \frac{2}{3}$ and $p = \frac{3}{4}$. Under WB1, the first player to score $m = 6$ or 11 points wins; under WB2, if there is a 5-5 or 10-10 tie, there is a tiebreaker, which continues until one player is ahead by 2 points.

We compare $Pr_{SR}(A)$ and $Pr_{CR}(A)$, which assume WB1, with the probabilities $Qr_{CR}(A)$ and $Qr_{CR}(A)$, which assume WB2, for $p = \frac{2}{3}$ and $\frac{3}{4}$. The latter probabilities take into account the probability of a tie, $Pr(T)$, which adjusts the probabilities of A and B winning.

For the values of $m = 3$, 11, and 21 and $p = \frac{2}{3}$ and $\frac{3}{4}$, we summarize below how the probabilities of winning and the expected lengths of a game are affected by SR and CR and Win-by-One and Win-by-Two:

1. As m increases from $m = 3$ to 11 to 21, the probability of a tie, $Pr(T)$, decreases by more than a factor of two for both $p = \frac{2}{3}$ and $p = \frac{3}{4}$. Even at $m = 21$, however, $Pr(T)$ is always at least 12% for SR and at least 25% for CR, indicating that, especially for CR, Win-by-Two will often end in a tiebreaker.

2. For $p = \frac{2}{3}$, the probability of a tie, $Pr(T)$, is twice as great under CR as SR; it

$p = \frac{2}{3}$	Best-of-m	$Pr(A)$	$Pr(B)$	$Qr(A)$	$Qr(B)$	$Pr(T)$	$EL(WB1)$	$EL(WB2)$
	$m = 3$	0.593	0.407	0.600	0.400	0.333	2.333	3.000
SR	$m = 11$	0.544	0.456	0.544	0.456	0.173	8.650	8.995
	$m = 21$	0.531	0.469	0.531	0.469	0.124	17.251	17.500
	$m = 3$	0.593	0.407	0.571	0.429	0.667	2.667	6.000
CR	$m = 11$	0.544	0.456	0.542	0.458	0.346	9.825	11.553
	$m = 21$	0.531	0.469	0.531	0.469	0.248	19.126	20.368
$p = \frac{3}{4}$	Best-of-m	$Pr(A)$	$Pr(B)$	$Qr(A)$	$Qr(B)$	$Pr(T)$	$EL(WB1)$	$EL(WB2)$
	$m = 3$	0.656	0.344	0.667	0.333	0.250	2.250	2.667
SR	$m = 11$	0.573	0.427	0.574	0.426	0.139	8.240	8.472
	$m = 21$	0.551	0.449	0.552	0.448	0.101	16.540	16.708
	$m = 3$	0.656	0.344	0.600	0.400	0.750	2.750	8.000
CR	$m = 11$	0.573	0.427	0.567	0.433	0.418	10.080	13.006
	$m = 21$	0.551	0.449	0.550	0.450	0.303	19.513	21.631

Table 3: Probability (Pr) that A Wins and B Wins in Best-of-m, and (Qr) that A Wins and B Wins in Best-of-m with a tiebreaker, and that the tiebreaker is implemented (T), and Expected Length of a game (EL), for $p = \frac{2}{3}$ and $\frac{3}{4}$

is three times greater for $p = \frac{3}{4}$. Thus, games are much closer under CR than SR because of the much greater frequency of ties under CR.

3. For Best-of-11 and Best-of-21, the probability of A's winning under Win-by-One ($Pr(A)$) and Win-by-Two ($Qr(A)$) falls within a narrow range (53-57%), whether CR or SR is used, echoing Theorem 1 that $Pr(A)$ for CR and SR are equal for Win-by-One. They stay quite close for Win-by-Two.

4. The expected length of a game, EL, is always greater under Win-by-Two than under Win-by-One. Whereas EL for Win-by-Two never exceeds EL for Win-by-One by more than one game under SR, under CR this difference may be two or more games, whether $p = \frac{2}{3}$ or $\frac{3}{4}$. Clearly, the tiebreaker under Win-by-Two may significantly extend the average length of a game under CR, rending it more competitive.

All the variable-rule service sports mentioned in section 1—except for racquetball, which uses Win-by-One—currently use SR coupled with Win-by-Two. The normal winning score of a game in squash is 11 (Best-of-21) and of badminton is 21 (Best-of-41), but a tiebreaker comes into play when there is a tie at 10-10 (squash) or 20-20 (badminton).[10] In volleyball, the winning score is 25; in a tiebreaker it is the receiving team, not the serving team, that is advantaged, because it can set up spike as a return, which is usually successful (Schilling, 2009).

[10]In badminton, if there is still a tie at 29-29, a single "golden point" determines the winner (see www.bwfbadminton.org).

The winning score in racquetball is 15, but unlike the other variable-rule service sports, a player scores points only when he or she serves, which prolongs games if the server is unsuccessful. This scoring rule was formerly used in badminton, squash, and volleyball, lengthening games in each sport by as much as a factor of two. The rule was abandoned because this prolongation led to problems in tournaments and discouraged television coverage (Barrow, 2012, p. 103). We are not sure why it persists in racquetball, and why this is apparently the only service sport to use Win-by-One.

We believe that all the aforementioned variable-rule service sports would benefit from CR. Games would be extended and, hence, be more competitive, without significantly altering win probabilities. Specifically, CR gives identical win probabilities to SR under Win-by-One, and very similar win probabilities under Win-by-Two.

4 The Fixed Rules of Table Tennis and Tennis

Both table tennis and tennis use Win-by-Two, but neither uses SR. Instead, each uses a fixed rule. In table tennis, the players alternate, each serving on two consecutive points, independent of the score and of who wins any point. The winning score is 11 unless there is a 10-10 tie, in which case there is a Win-by-Two tiebreaker, in which the players alternate, but serve on one point instead of two.

There would be little or no advantage to serving first (on two points) in table tennis if there were little or no advantage to serving, as seems to be the case. If $p = \frac{1}{2}$, the playing field is level, so neither player gains a probabilistic advantage from being the first double server.

Tennis is a different story.[11] It is generally acknowledged that servers have an advantage, perhaps ranging from about $p = \frac{3}{5}$ to $p = \frac{3}{4}$ in a professional match. In tennis, points are organized into games, games into sets, and sets into matches. Win-by-Two applies to games and sets. In the tiebreaker for sets, which occurs after a 6-6 tie in games, one player begins by serving once, after which the players alternate serving twice in a row.[12]

A tennis tiebreaker begins with one player—for us, A—serving. Either A wins the point or B does. Regardless of who wins the first point, there is now a fixed alternating sequence of double serves, $BBAABB\ldots$ for as long as is necessary until one player goes two points ahead of the other. When A starts, the entire sequence can be viewed as one of two alternating single serves, broken by the slashes shown below,

$$AB/BA/AB/BA\ldots$$

Between each pair of adjacent slashes, the order of A and B switches as one moves from left to right. We call the serves between adjacent slashes a block.

[11] Part of this section is adapted from Brams and Ismail (2016).

[12] Arguably, the tiebreaker creates a balance of forces: A is advantaged by serving at the outset, but B is then given a chance to catch up, and even move ahead, by next having two serves in a row.

The first player to score 7 points, and win by a margin of at least two points, wins the tiebreaker. Thus, if the players tie at 6-6, a score of 7-6 is not winning. In this case, the tiebreaker would continue until one player goes ahead by two points (e.g., at 8-6, 9-7, etc.).

Notice that, after reaching a 6-6 tie, a win by one player can occur only after the players have played an even number of points, which is at the end of an AB or BA block. This ensures that a player can win only by winning twice in a block—once on the player's own serve, and once on the opponent's. The tiebreaker continues as long as the players continue to split blocks after a 6-6 tie, because a player can lead by two points only by winning both serves in a block. When one player is finally ahead by two points, thereby winning the tiebreaker, the players must have had exactly the same number of serves.

The fixed order of serving in the tennis tiebreaker, which is what precludes a player from winning simply because he or she had more serves, is not the only fixed rule that satisfies this property. The strict alternation of single serves,

$$AB/AB/AB/AB\ldots,$$

or what Brams and Taylor (1999) call "balanced alternation,"

$$AB/BA/BA/AB\ldots,$$

are two of many alternating sequences that create adjacent AB or BA blocks.[13] All are fair—they ensure that the losing player does not lose only because he or she had fewer serves—for the same reason that the tennis sequence is fair.[14]

Variable-rule service sports, including badminton, squash, racquetball, and volleyball, do not necessarily equalize the number of times the two players or teams serve. Under SR, if A holds its serve throughout a game or a tiebreaker, he can win by serving all the time.

This cannot happen under CR, because A loses his serve when he wins; he can hold his serve only when he loses. But this is not to say that A cannot win by serving more often. For example, in Best-of-3 CR, A can beat B 2-1 by serving twice. But the tiebreaker rule in tennis, or any other fixed rule in which there are alternating blocks of AB and BA, does ensure that the winner did not benefit by having more serves.

[13]Brams and Taylor (1999, p. 38) refer to balanced alternation as "taking turns taking turns taking turns ..." This sequence was proposed and analyzed by several scholars, and is also known as the Prouhet-Thue-Morse (PTM) sequence (Palacios-Huerta, 2014, pp. 82-85). Notice that the tennis sequence maximizes the number of double repetitions when written as $A/BB/AA/BB/\ldots$, because after the first serve by one player, there are alternating double serves by each player. This minimizes changeover time and thus the "jerkiness" of switching servers.

[14]It is true that if A serves first in a block, he may win after serving one more time than B if the tiebreaker does not go to a 6-6 tie. For example, assume that the score in the tiebreaker is 6-4 in favor of A, and it is A's turn to serve. If A wins the tiebreaker at 7-4, he will have had one more serve than B. However, his win did not depend on having one more serve, because he is now 3 points ahead. By comparison, if the tiebreaker had gone to 6-6, A could not win at 7-6 with one more serve; his win in this case would have to occur at the end of a block, when both players have had the same number of serves.

5 Summary and Conclusions

We have analyzed four rules for service sports—including the Standard Rule (SR) currently used in many service sports—that make serving variable, or dependent on a player's (or team's) previous performance. The three new rules we analyzed all give a player, who loses a point or falls behind in a game, the opportunity to catch up by serving, which is advantageous in most service sports. The Catch-Up Rule (CR) gives a player this opportunity if he or she just lost a point—instead of just winning a point, as under SR. Each of the Trailing Rules (TRs) make the server the player who trails; if there is a tie, the server is the player who previously was ahead (TRa) or behind (TRb).

For Win-by-One, we showed that SR and CR give the players the same probability of a win, independent of the number of points needed to win. We proved, and illustrated with numerical calculations, that the expected length of a game is greater under CR than under SR, rendering it more likely to stay close to the end.

By contrast, the two TRs give the player who was not the first server, B, a greater probability of winning, making it less likely that they would be acceptable to strong players, especially those who are used to SR and have done well under it. In addition, TRb is not strategy-proof: We exhibited an instance in which a player can benefit by deliberately losing, which also makes it less appealing.

We analyzed the effects of Win-by-Two, which most service sports currently combine with SR, showing that it is compatible with CR. We showed that the expected length of a game, especially under CR, is always greater under Win-by-Two than Win-by-One.

On the other hand, Win-by-Two, compared with Win-by-One, has little effect on the probability of winning (compare $Qr(A)$ and $Qr(B)$ with $Pr(A)$ and $Pr(B)$ in Table 3).[15] The latter property should make CR more acceptable to the powers-that-be in the different sports who, generally speaking, eschew radical changes that may have unpredictable consequences. But they want to foster competitiveness in their sports, which CR does.

Table tennis and tennis use fixed rules for serving, which specify when the players serve and how many serves they have. We focused on the tiebreaker in tennis, showing that it was fair in the sense of precluding a player from winning simply as a result of having served more than his or her opponent. CR does not offer this guarantee in variable-rule sports, although it does tend to equalize the number of times that each player serves.

There is little doubt that suspense is created, which renders play more exciting and unpredictable, by making who serves next dependent on the success or failure of the server—rather than fixing in advance who serves and when. But a sport can still generate keen competition, as the tennis tiebreaker does, without leaving uncertain who will be the next server. Thus, the rules of tennis, and in particular the tiebreaker, seem well-chosen; they create both fairness and suspense.

[15] CR would make the penalty shootout of soccer fairer by giving the team that loses the coin toss—and which, therefore, usually must kick second in the shootout—an opportunity to kick first on some kicks (Brams and Ismail, 2016).

Appendix

5.1 Theorem 1: SR and CR Give Equal Win Probabilities

We define a serving schedule for a Best-of-$(2k+1)$ game to be

$$(A_1, \ldots, A_{k+1}, B_1, \ldots, B_k) \in \{W, L\}^{2k+1},$$

where for $i = 1, 2, \ldots, k+1$, A_i records the result of A's i^{th} serve, with W representing a win for A and L a loss for A, and for $j = 1, 2, \ldots, k, B_j$ represents the result of B's j^{th} serve, where now W represents a win for B and L represents a loss for B.

The auxiliary rule, AR, is a serving rule for a Best-of-$(2k+1)$ game in which A serves $k+1$ times consecutively, and then B serves k times consecutively. For convenience, we can assume that AR continues for $2k+1$ serves, even after one player accumulates $k+1$ points. The basis of our proof is a demonstration that, if the serving schedule is fixed, then games played under all three serving rules, AR, SR, and CR, are won by the same player.

Theorem 1. *Let $k \geq 1$. In a Best-of-$(2k+1)$ game, $Pr_{SR}(A) = Pr_{CR}(A)$.*

Proof. Suppose we have already demonstrated that, for any serving schedule, the three service rules—AR, SR, and CR—all give the same winner. It follows that the subset of service schedules under which A wins under SR must be identical to the subset of service schedules under which A wins under CR. Moreover, regardless of the serving rule, any serving schedule that contains r wins for A as server and s wins for B as server must be associated with the probability $p^r(1-p)^{k+1-r}q^s(1-q)^{k-s}$. Because the probability that a player wins under a service rule must equal the sum of the probabilities of all the service schedules in which the player wins under that rule, the proof of the theorem will be complete.

Fix a serving schedule $(A_1, \ldots, A_{k+1}, B_1, \ldots, B_k) \in \{W, L\}^{2k+1}$. Let a be the number of A's server losses and b be the number of B's server losses. (For example, for $k = 2$ and serving schedule (W, L, L), $a = 1$ and $b = 1$.)

Under service rule AR, there are $2k+1$ serves in total. Then A must accumulate $k+1+b-a$ points and B must accumulate $k-b+a$ points. Clearly, A has strictly more points than B if and only if $b \geq a$.

Now consider service rule SR, under which service switches whenever the server loses. First we prove that, if $b \geq a$, A will have the opportunity to serve at least $k+1$ times. A serves until he loses, and then B serves until she loses, so immediately after B's first loss, A has also lost once and is to serve next. Repeating, immediately after B's a^{th} loss, A has also lost a times and is to serve next. Either A's prior loss was his $(k+1)^{\text{st}}$ serve, or A wins every serve from this point on, including A's $(k+1)^{\text{st}}$ serve. After this serve, A has $k+1-a+b \geq k+1$ points, so A must have won either on this serve or earlier.

Now suppose that $a > b$ under service rule SR. Then, after B's b^{th} loss, A has also lost b times and is to serve. Moreover, A must lose again (since $b < a$), and then B becomes server and continues to serve (without losing) until B has served k times. By then, B will have gained $k - b + a \geq k + 1$ points, so B will have won. In conclusion, under SR, A wins if and only if $b \geq a$.

Consider now service rule CR, under which A serves until he wins, then B serves until she wins, etc. Clearly, B cannot win a point on her own serve until after A has won a point on his own serve. Repeating, if B has just won a point on serve, then A must have already won the same number of points on serve as B.

Again assume that $b \geq a$. Note that A has $k + 1 - a$ server wins in his first $k + 1$ serves, B has $k - b$ server wins in her first k serves, and $k + 1 - a > k - b$. Consider the situation under CR immediately after B's $(k - b)^{\text{th}}$ server win. (All of B's serves after this point up to and including the k^{th} serve must be losses.) At this point, both A and B have $k - b$ server wins. Suppose that A also has $a' \leq a$ server losses, and that B has $b' \leq b$ server losses. Then the score at this point must be $(k - b + b', k - b + a')$.

Immediately after B's $(k-b)^{\text{th}}$ server win, A is on serve. For A, the schedule up to A's $(k+1)^{\text{st}}$ serve must contain $a-a'$ server losses and $k+1-a-(k-b) = b-a+1 > 0$ server wins. Immediately after A's next server win, which is A's $(k - b + 1)^{\text{st}}$ server win, A's score is $k-b+b'+1$, and B's score is at most $k-b+a'+(a-a') = k-b+a \leq k$. Then B is on serve, and all of B's $b-b'$ remaining serves (up to and including B's k^{th} serve) must be server losses. Therefore, after B's k^{th} serve, A's score is $k-b+b'+1+(b-b') = k+1$, and A wins.

If $b < a$, an analogous argument shows that B wins under CR. This completes the proof that, under any serving schedule, the winner under AR is the same as the winner under SR and under CR. □

5.2 Theorem 2: Expected Lengths under SR and CR

Lemma 1. *Let $1 \leq t \leq s \leq r$. If a subset of s dots is selected uniformly from a sequence of r dots, the expected position of the t^{th} selected dot in the full sequence is*

$$t\left(\frac{r+1}{s+1}\right).$$

Proof. The t^{th} selected dot can be in any position from t to $r - s + t$ in the original sequence. For the dot to be in position x, the selection can be carried out in $\binom{x-1}{t-1}\binom{r-x}{s-t}$ ways. Note that

$$H(r,s,t) = \sum_{x=t}^{r-s+t} \binom{x-1}{t-1}\binom{r-x}{s-t} = \binom{r}{s},$$

which is identity (3.3) of Gould (1972, p. 22). This identity holds whenever $1 \leq t \leq s \leq r$; note that the value of $H(r,s,t)$ does not depend on t.

Clearly, the expected position of the t^{th} selected dot is

$$E = \frac{1}{\binom{r}{s}} \sum_{x=t}^{r-s+t} x \binom{x-1}{t-1} \binom{r-x}{s-t}$$

Denote the above summation by K. To evaluate K, note first that $x\binom{x-1}{t-1} = t\binom{x}{t}$. Next, substitute $R = r+1$, $S = s+1$, and $T = t+1$ to obtain

$$K = \sum_{x=t}^{r-s+t} t\binom{x}{t}\binom{r-x}{s-t} = t \sum_{x=T-1}^{R-S+T-1} \binom{x}{T-1}\binom{R-1-x}{S-T}.$$

Now set $y = x+1$ and note that $x = T-1$ corresponds to $y = T$, and $x = R-S+T-1$ corresponds to $y = R-S+T$. Then

$$K = t \sum_{y=T}^{R-S+T} \binom{y-1}{T-1}\binom{R-y}{S-T} = tH(R,S,T) = t\binom{R}{S}.$$

using Gould's identity. To complete the proof, note that

$$\binom{R}{S} = \binom{r+1}{s+1} = \frac{r+1}{s+1}\binom{r}{s},$$

so that $E = \frac{K}{\binom{r}{s}} = t\left(\frac{r+1}{s+1}\right)$, as required. □

Theorem 2. *If $0 < p, q < 1$ and $k \geq 1$, then the expected length of Best-of-$(2k+1)$ game under CR is greater than under SR if and only if $p + q > 1$.*

Proof. As defined earlier, a service schedule is a sequence of exactly $k+1$ wins and losses on A's serves and exactly k wins and losses on B's serves. A service schedule has parameters (n, m) if it contains n server wins for A and m server wins for B. The probability of any particular service schedule with parameters (n, m) is

$$Pr_0(n,m) = p^n(1-p)^{k+1-n}q^m(1-q)^{k-m}.$$

The probability that *some* service schedule with parameters (n, m) occurs is

$$Pr(n,m) = \binom{k+1}{n}\binom{k}{m}p^n(1-p)^{k+1-n}q^m(1-q)^{k-m}.$$

Consider a service schedule with parameters (n, m). Using SR, if $n > m$, then A wins and exhausts his part of the service schedule. B uses her part of the schedule through her $(k+1-n)^{\text{th}}$ loss. From the Lemma with $r = k$, $s = k-m$, $t = k+1-n$, the expected length of the game is

$$EL^0_{SR}(n,m) = k+1+(k+1-n)\left(\frac{k+1}{k+1-m}\right) = 2(k+1)-(n-m)\left(\frac{m}{k+1-m}+1\right).$$

However, if $n \le m$, then B wins and exhausts her part of the service schedule, while A uses his part of the schedule through his $(k+1-m)^{\text{th}}$ loss. From the Lemma with $r = k+1$, $s = k+1-n$, $t = k+1-m$, the expected length of the game is

$$EL^0_{SR}(n,m) = k + (k+1-m)\left(\frac{k+1+1}{k+1-n+1}\right) = 2(k+1) - (m-n+1)\left(\frac{n}{k+1-n+1} + 1\right).$$

Using CR, if $n > m$, then A wins, B exhausts her part of the service schedule, and A uses his schedule through his $(m+1)^{\text{st}}$ win. From the Lemma with $r = k+1$, $s = n$, $t = m+1$, the expected length of the game is

$$EL^0_{CR}(n,m) = k + (m+1)\left(\frac{k+1+1}{n+1}\right) = 2(k+1) - (n-m)\left(\frac{k+1-n}{n+1} + 1\right).$$

But if $n \le m$, then B wins, A exhausts his part of the service schedule, and B uses her part through her n^{th} win. From Lemma 1 with $r = k$, $s = m$, $t = n$, the expected length of the game is[16]

$$EL^0_{CR}(n,m) = k + 1 + n\left(\frac{k+1}{m+1}\right) = 2(k+1) - (m-n+1)\left(\frac{k-m}{m+1} + 1\right).$$

Combining the first two formulas, we can write the expected length of a game played under SR, [17]

$$\begin{aligned} EL_{SR} = &\sum_{n>m} Pr(n,m)\left[2(k+1) - (n-m)\left(\frac{m}{k+1-m} + 1\right)\right] \\ &+ \sum_{n\le m} Pr(n,m)\left[2(k+1) - (m-n+1)\left(\frac{n}{k+1-n+1} + 1\right)\right], \end{aligned}$$

and the expected length of a game played under CR,

$$\begin{aligned} EL_{CR} = &\sum_{n>m} Pr(n,m)\left[2(k+1) - (n-m)\left(\frac{k+1-n}{n+1} + 1\right)\right] \\ &+ \sum_{n\le m} Pr(n,m)\left[2(k+1) - (m-n+1)\left(\frac{k-m}{m+1} + 1\right)\right]. \end{aligned}$$

[16] By using the substitutions $n \to k+1-n$ and $m \to k-m$, we could make the second and fourth ELs look more like the first and third, which would add some appealing symmetry to the following calculations, but then would not immediately simplify them.

[17] We take the sum $\sum_{n>m}$ to include terms for every integer pair (n,m) with $n > m$. If $n > k+1$ or $m > k$, we take $Pr(n,m) = 0$, so the sum includes only finitely many nonzero terms. Similarly, the sum $\sum_{n\le m}$ includes terms for every integer pair (n,m) with $n \le m$. In general, we interpret the binomial coefficient $\binom{a}{b}$ to be zero whenever $b < 0$ or $b > a$.

When we subtract these expressions, many terms cancel:

$$EL_{CR} - EL_{SR} = \sum_{n>m} Pr(n,m)(n-m)\left[\frac{m}{k+1-m} - \frac{k+1-n}{n+1}\right] + \sum_{n\le m} Pr(n,m)(m-n+1)\left[\frac{n}{k+1-n+1} - \frac{k-m}{m+1}\right].$$

Our objective now is to show that this quantity, the expected number of points by which the length of a CR game exceeds the length of an SR game, has the same sign as $p+q-1$.

To simplify this expected difference in lengths, we extract the binomial coefficients $\binom{k+1}{n}$ and $\binom{k}{m}$ from the probability factors and use the following identities:

$$\binom{k}{m}\cdot\frac{m}{k+1-m} = \binom{k}{m-1} \qquad \binom{k+1}{n}\cdot\frac{k+1-n}{n+1} = \binom{k+1}{n+1}$$

$$\binom{k+1}{n}\cdot\frac{n}{k+1-n+1} = \binom{k+1}{n-1} \qquad \binom{k}{m}\cdot\frac{k-m}{m+1} = \binom{k}{m+1}$$

The result is

$$EL_{CR} - EL_{SR} = \sum_{n>m} Pr_0(n,m)(n-m)\left[\binom{k+1}{n}\binom{k}{m-1} - \binom{k+1}{n+1}\binom{k}{m}\right] + \sum_{n\le m} Pr_0(n,m)(m-n+1)\left[\binom{k+1}{n-1}\binom{k}{m} - \binom{k+1}{n}\binom{k}{m+1}\right].$$

Now split the sums to obtain

$$\begin{aligned} EL_{CR} - EL_{SR} = &\sum_{n>m} Pr_0(n,m)(n-m)\binom{k+1}{n}\binom{k}{m-1} \\ &- \sum_{n>m} Pr_0(n,m)(n-m)\binom{k+1}{n+1}\binom{k}{m} \\ &+ \sum_{n\le m} Pr_0(n,m)(m-n+1)\binom{k+1}{n-1}\binom{k}{m} \\ &- \sum_{n\le m} Pr_0(n,m)(m-n+1)\binom{k+1}{n}\binom{k}{m+1}. \end{aligned}$$

To analyze the expression for $EL_{CR} - EL_{SR}$, we first shift indices in the second summation by replacing n by $n-1$ and m by $m-1$ throughout. Of course, the condition

$n > m$ and the factor $n - m$ are not affected. Then

$$\begin{aligned}
&\sum_{n>m} Pr_0(n,m)(n-m)\binom{k+1}{n+1}\binom{k}{m} \\
&= \sum_{n>m} Pr_0(n-1,m-1)(n-m)\binom{k+1}{n}\binom{k}{m-1} \\
&= \frac{1-p}{p}\,\frac{1-q}{q} \sum_{n>m} Pr_0(n,m)(n-m)\binom{k+1}{n}\binom{k}{m-1},
\end{aligned}$$

where in the last step we have used the relationship

$$Pr_0(n-1,m-1) = \frac{1-p}{p}\,\frac{1-q}{q}\; Pr_0(n,m)$$

to make the second summation match the first. Similarly, we work on the fourth summation, shifting the indices in the same way—replacing n by $n-1$ and m by $m-1$ throughout. Again, the condition $n \leq m$ and the factor $m - n + 1$ are not affected, and in the end the fourth summation matches the third.

$$\begin{aligned}
&\sum_{n\leq m} Pr_0(n,m)(m-n+1)\binom{k+1}{n}\binom{k}{m+1} \\
&= \sum_{n\leq m} Pr_0(n-1,m-1)(m-n+1)\binom{k+1}{n-1}\binom{k}{m} \\
&= \frac{1-p}{p}\,\frac{1-q}{q} \sum_{n\leq m} Pr_0(n,m)(m-n+1)\binom{k+1}{n-1}\binom{k}{m}.
\end{aligned}$$

Incorporating these changes into our expression for $EL_{CR} - EL_{SR}$ gives

$$\begin{aligned}
&EL_{CR} - EL_{SR} \\
&= \left(1 - \frac{1-p}{p}\,\frac{1-q}{q}\right) \sum_{n>m} Pr_0(n,m)(n-m)\binom{k+1}{n}\binom{k}{m-1} \\
&+ \left(1 - \frac{1-p}{p}\,\frac{1-q}{q}\right) \sum_{n\leq m} Pr_0(n,m)(m-n+1)\binom{k+1}{n-1}\binom{k}{m}.
\end{aligned}$$

This completes the proof, because the sums include only positive terms, and the common factor is

$$1 - \frac{1-p}{p}\,\frac{1-q}{q} = \frac{p+q-1}{pq},$$

which has the same sign as $p + q - 1$. □

5.3 Theorem 3: TRa is strategyproof if $p+q>1$

Theorem 3. *TRb is strategy-vulnerable, whereas SR and CR are strategy-proof. TRa is strategy-proof whenever $p+q>1$.*

See the text for the proof that TRb is strategy-vulnerable and that SR and CR are strategy-proof. Here we consider only a Best-of-$(2k+1)$ game played under TRa, and use notation similar to the text: (C,x,y) denotes a state in which player C ($C=A$ or B) is about to serve, A's current score is x, and B's is y. Let $W_A(C,x,y)$ denote A's conditional win probability given that the game reaches state (C,x,y). Assume that $p+q>1$.

Given any state, call the state that was its immediate predecessor its *parent*. Call two states *siblings* if they have the same parent. For example, state $(A,0,0)$ is the parent of siblings $(B,1,0)$ and $(A,0,1)$. Let $0<x\le k+1$ and $0\le y\le k+1$ with $x+y\le 2k+1$. Then any state (C,x,y) must have a sibling $(C',x-1,y+1)$, namely the state that would have arisen had B, and not A, won the last point.

To show that the Best-of-$(2k+1)$ game played under TRa is strategy-proof for A, we must show that $W_A(C,x,y)\ge W_A(C',x-1,y+1)$ whenever (C,x,y) and $(C',x-1,y+1)$ are siblings. For purposes of induction, note that, if x and y satisfy $0\le x\le k+1$, $0\le y\le k+1$, and $x+y\le 2k+1$, then $W_A(C,x,y)=1$ if $x=k+1$ and $W_A(C,x,y)=0$ if $y=k+1$, where $C=A$ or B. In these cases, (C,x,y) is a *terminal state* and can be written (x,y), as there are no more serves.

Lemma 2. *Suppose that $x\ne y$ and the state is (C,x,y). Then either $x<y$ and $C=A$, or $x>y$ and $C=B$.*

Proof. Under TRa, the player about to serve must be the player with the lower score. □

Lemma 3. *The states (A,x,y) and (B,x,y) can both arise if and only if $0<x=y<k+1$. In this case, the state is (A,x,x) if the parent was $(B,x,x-1)$, and the state is (B,x,x) if the parent was $(A,x-1,x)$.*

Proof. The first statement follows from Lemma 2. The second is a paraphrase of TRa. □

Note that Lemma 3 fails for TRb; in fact, it captures the difference between TRa and TRb. For example, under TRb, if A loses at $(B,1,0)$, the next state will be $(B,1,1)$ (it would be $(A,1,1)$ under TRa), and if A wins at $(A,0,1)$, the next state will be $(A,1,1)$ (rather than $(B,1,1)$ under TRa).

Lemma 4. $W_A(A,x,x)>W_A(B,x,x)$ *if and only if* $W_A(B,x+1,x)>W_A(A,x,x+1)$.

Proof. First notice that $W_A(A,x,x)=pW_A(B,x+1,x)+(1-p)W_A(A,x,x+1)$ and $W_A(B,x,x)=(1-q)W_A(B,x+1,x)+qW_A(A,x,x+1)$. Therefore

$$W_A(A,x,x)-W_A(B,x,x)=(p+q-1)\left[W_A(B,x+1,x)-W_A(A,x,x+1)\right]$$

and the claim follows from the assumption that $p+q>1$. □

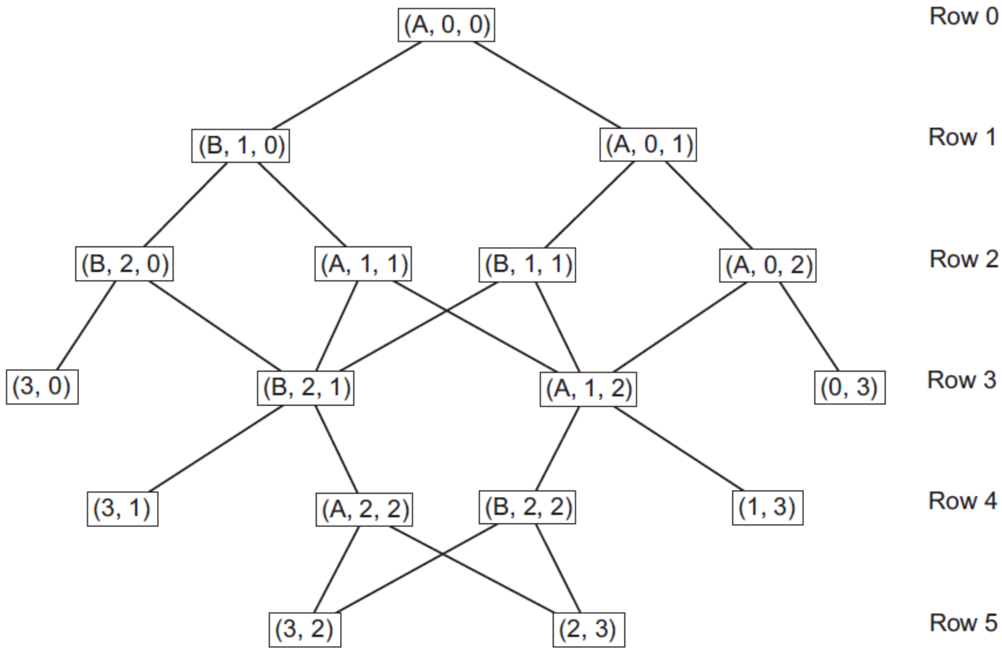

Figure 3: Best-of-5 TRa Probability Tree

We now consider the probability tree of a Best-of-$(2k+1)$ game played under TRa. The Best-of-5 tree ($k = 2$) is shown in Figure 3. The nodes (states) are labeled (C, x, y), where C is the player to serve, x is the score of A, and y is the score of B. Because there are no servers after the game has been won, the terminal node are labeled either $(k+1, y)$ or $(x, k+1)$. The probability tree is rooted at $(A, 0, 0)$ and is binary—every non-terminal node is parent to exactly two nodes. For non-terminal nodes where A serves, the left-hand outgoing arc has probability p and the right-hand outgoing arc has probability $1-p$; for non-terminal nodes where B serves, the two outgoing arcs have probability $1-q$ and q, respectively.

The probability tree for the Best-of-$(2k+1)$ game played under TRa has $2k+2$ rows, numbered 0 (at the top) to $2k+1$ (at the bottom). The top row contains only the initial state, $(A, 0, 0)$, and the bottom row contains only terminal nodes. At every node in row ℓ, the players' scores sum to ℓ.

In the upper half of the tree, above row $k+1$, there are no terminal nodes. Let $1 \leq \ell < k+1$. If ℓ is odd, every state in row ℓ is of the form $(C, x, \ell - x)$. By Lemma 2, there are $\ell + 1$ states in row ℓ, from $(B, \ell, 0)$ to $(A, 0, \ell)$. If ℓ is even, Lemmata 2 and 3 show that row ℓ contains $\ell + 2$ states, from $(B, \ell, 0)$ to $(A, 0, \ell)$, including both $(A, \frac{\ell}{2}, \frac{\ell}{2})$ and $(B, \frac{\ell}{2}, \frac{\ell}{2})$. We place $(A, \frac{\ell}{2}, \frac{\ell}{2})$ to the left of $(B, \frac{\ell}{2}, \frac{\ell}{2})$.

In the lower half of the tree—row $k+1$ and below—each row begins and ends with a terminal node. Let $k+1 \leq \ell \leq 2k+1$. Then the first entry in row ℓ is the terminal node

$(k+1, \ell-k-1)$, and the last entry is the terminal node $(\ell-k-1, k+1)$. If ℓ is odd, every non-terminal node in row ℓ is a state of the form $(C, x, \ell-x)$ for $x = k, k-1, \ldots, \ell-k$. (If $\ell = 2k+1$, the only nodes in row ℓ are two terminal nodes.) By Lemma 2, row ℓ contains $2k-\ell+3$ nodes in total. If ℓ is even, row ℓ contains $2k-\ell+4$ nodes, from the terminal node $(k, \ell-k)$ to the terminal node $(\ell-k, k)$, including all possible states of the form $(C, x, \ell-x)$ for $x = k, k-1, \ldots, \ell-k$. Among these states are both $(A, \frac{\ell}{2}, \frac{\ell}{2})$ and $(B, \frac{\ell}{2}, \frac{\ell}{2})$, with the former state on the left.

As the figure illustrates, sibling states have a unique parent unless they are of the form $(B, x+1, x)$ and $(A, x, x+1)$, in which case both (A, x, x) and (B, x, x) are parents. Thus, $(B, x+1, x)$ and $(A, x, x+1)$ could be called "double siblings."

The method of proof is to show by induction that the function $W_A(\cdot)$ is decreasing on each row as one reads from left to right. This will prove that TRa is strategy-proof for A, because it will show that in every state A does better by winning the next point rather than losing it to obtain the sibling state. Since $W_B(\cdot) = 1 - W_A(\cdot)$, the proof also shows that $W_B(\cdot)$ is increasing on each row, and therefore that TRa is strategy-proof for B.

Proof. To begin the induction, observe that $W_A(\cdot)$ is decreasing on row $2k+1$ since $W_A(k+1, k) = 1$ and $W_A(k, k+1) = 0$. Now consider row $2k$, which begins with a terminal node $(k+1, k-1)$, where $W_A(k+1, k-1) = 1$, and ends with a terminal node $(k-1, k+1)$, where $W_A(k-1, k+1) = 0$. Row $2k$ contains 4 nodes; its second entry is (A, k, k) and its third is (B, k, k). To apply Lemma 4 with $x = k$, note that $(B, k+1, k)$ and $(A, k, k+1)$ are the terminal nodes $(k+1, k)$ and $(k, k+1)$, and $W_A(k+1, k) = 1 > W_A(k, k+1) = 0$. Lemma 4 now implies that $W_A(A, k, k) > W_A(B, k, k)$, as required.

Now consider any row ℓ, and assume that $W_A(\cdot)$ has been shown to be strictly decreasing on row $\ell+1$. If $\ell \geq k+1$, row ℓ begins and ends with a terminal node, for which $W_A(k+1, \ell-k-1) = 1$ and $W_A(\ell-k-1, k+1) = 0$; any other node in row ℓ is a non-terminal node. Suppose that $(C, x, \ell-x)$ and $(C', x', \ell-x')$ are adjacent nodes in row ℓ and that $(C, x, \ell-x)$ is on the left. First suppose that $x = x'$. Then by Lemma 3, ℓ must be even, and the two nodes must be $(A, \frac{\ell}{2}, \frac{\ell}{2})$ (on the left) and $(B, \frac{\ell}{2}, \frac{\ell}{2})$ (on the right). The induction assumption and Lemma 4 now implies that $W_A(A, \frac{\ell}{2}, \frac{\ell}{2}) > W_A(B, \frac{\ell}{2}, \frac{\ell}{2})$.

Otherwise, consecutive nodes $(C, x, \ell-x)$ and $(C', x', \ell-x')$ in row ℓ must satisfy $x' = x-1$, by Lemmata 2 and 3. Thus we are comparing $W_A(C, x, \ell-x)$ with $W_A(C', x-1, \ell-x+1)$. Now

$$\begin{aligned} W_A(C, x, \ell-x) &= rW_A(D, x+1, \ell-x) + (1-r)W_A(D', x, \ell-x+1) \\ &> W_A(D', x, \ell-x+1), \\ W_A(C', x-1, \ell-x+1) &= sW_A(E, x, \ell-x+1) + (1-s)W_A(E', x-1, \ell-x+2) \\ &> W_A(E, x, \ell-x+1), \end{aligned}$$

where $0 < r, s < 1$ because each of r and s must equal one of p, q, $1-p$, and $1-q$.

According to Lemma 3, $(D', x, \ell-x+1) = (E, x, \ell-x+1)$ unless $x = \ell-x+1$, i.e. $x = \frac{\ell+1}{2}$ (which of course requires that ℓ be odd). First assume that $x \neq \frac{\ell+1}{2}$. Then we have shown that

$$W_A(C, x, \ell - x) > W_A(D', x, \ell - x + 1)$$
$$= W_A(E, x, \ell - x + 1) > W_A(C', x - 1, \ell - x + 1),$$

as required.

Now assume that $x = \frac{\ell+1}{2}$. Then the original states $(C, x, \ell - x)$ and $(C', x-1, \ell-x+1)$ must have been $(B, \frac{\ell+1}{2}, \frac{\ell-1}{2})$ and $(A, \frac{\ell-1}{2}, \frac{\ell+1}{2})$. Moreover, $(D', x, \ell-x+1) = (A, \frac{\ell+1}{2}, \frac{\ell+1}{2})$ and $(E, x, \ell - x + 1) = (B, \frac{\ell+1}{2}, \frac{\ell+1}{2})$. Recall that (A, x, x) always appears to the left of (B, x, x). Thus the assumption that $W_A(\cdot)$ is decreasing on row $\ell + 1$ implies that $W_A(A, \frac{\ell+1}{2}, \frac{\ell+1}{2}) > W_A(B, \frac{\ell+1}{2}, \frac{\ell+1}{2})$. By Lemma 4, we have shown that

$$W_A(B, \frac{\ell+1}{2}, \frac{\ell-1}{2}) > W_A(A, \frac{\ell+1}{2}, \frac{\ell+1}{2})$$
$$> W_A(B, \frac{\ell+1}{2}, \frac{\ell+1}{2}) > W_A(A, \frac{\ell-1}{2}, \frac{\ell+1}{2}),$$

completing the proof of Theorem 3. □

References

Anbarci, N., C.-J. Sun, and M. U. Ünver (2015, January). Designing Fair Tiebreak Mechanisms: The Case of FIFA Penalty Shootouts. *http://ssrn.com/abstract=2558979*.

Anderson, C. (1977). Note on the advantage of first serve. *Journal of Combinatorial Theory, Series A 23*(3), 363.

Barrow, J. (2012). *Mathletics: 100 Amazing Things You Didn't Know about the World of Sports.* W. W. Norton.

Brams, S. and A. Taylor (1999). *The Win-win Solution: Guaranteeing Fair Shares to Everybody.* W.W. Norton.

Brams, S. J. and M. S. Ismail (2016). Making the Rules of Sports Fairer. *https://ssrn.com/abstract=2737672*.

Gould, H. (1972). *Combinatorial identities: A standardized set of tables listing 500 binomial coefficient summations.* Morgantown, W Va.

Isaksen, A., M. Ismail, S. J. Brams, and A. Nealen (2015). Catch-Up: A Game in Which the Lead Alternates. *Game & Puzzle Design 2*(1), 38–49.

Kemeny, J. and J. Snell (1976). *Finite Markov Chains.* Undergraduate Texts in Mathematics. Springer New York.

Kingston, J. (1976). Comparison of scoring systems in two-sided competitions. *Journal of Combinatorial Theory, Series A 20*(3), 357–362.

Palacios-Huerta, I. (2014). *Beautiful Game Theory: How Soccer Can Help Economics.* Princeton University Press.

Pauly, M. (2013). Can strategizing in round-robin subtournaments be avoided? *Social Choice and Welfare 43*(1), 29–46.

Ruffle, B. J. and O. Volij (2015). First-mover advantage in best-of series: an experimental comparison of role-assignment rules. *International Journal of Game Theory*, 1–38.

Schilling, M. F. (2009). Does momentum exist in competitive volleyball? *CHANCE 22*(4), 29–35.

von Neumann, J. and O. Morgenstern (1944). *Theory of Games and Economic Behavior* (1953, Third ed.). Princeton University Press.