Robust Pricing with Refunds

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Abstract

We analyze a bilateral trade model where the seller has to make a take-it-or-leave-it offer to the buyer in an environment where the seller does not know what the buyer has learned or will learn about the product fit. We show that a generous return policy reduces the significance of this type of uncertainty and helps the seller to regain market power. We characterize the best-guaranteed profit the seller can obtain by using a generous return policy. We then show that there are no other selling mechanisms that guarantees the seller higher profits. Our result provides a novel rationale behind generous return policies.

1 Introduction

Consumers learn about product characteristics from various sources. Firms are therefore not only uncertain about how much value buyers get from consuming their products, but also how well buyers know whether these products fit their needs. This creates additional uncertainty about buyers' willingness to pay and therefore limits the seller's ability to raise prices. For example, a buyer who considers buying shoes online may be uncertain about the fit or may have already tried them on at another store. Similarly, a consumer booking an airline ticket may already have specific travel itinerary or may still wait for additional information.

In the absence of such uncertainty, e.g., if the seller knows the buyer's information source or has a prior over possible information sources, then the seller's problem is the standard monopoly pricing problem. What this paper analyzes is the environment where the seller neither knows nor has a prior over the buyer's information sources; and consequently, is unsure how the buyer will respond to the offer.

To understand the seller's problem better, suppose that the seller made a non-refundable offer. The worst possible is that buyer's information source turns out to be the one minimizes the probability of generating signals that persuade the

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buyer to purchase at the price. Such imperfect buyer's learning induces the demand that is elastic at the price seller has offered.[1] Therefore, the seller's uncertainty about the buyer's learning process forces the seller to cut the price.

However, if the seller can offer a refund, then it can potentially foster the buyer learning, and thereby diminish the significance of buyer's information source on her purchasing decision.[2] With the possibility of product return, the buyer does not lose much from buying and trying, and the seller can capture the newly-created option value by raising the price. At the same time, for each returned product, the seller incurs the restocking cost. Moreover, the more generous the refund is, the more likely the buyer returns the product. The exact gain and loss crucially depend on the buyer's learning process. If the seller knows the buyer's learning process, then it can design a price-refund pair to bring a balance to this trade-off to maximize it's profit.[3] But if the seller does not know the buyer's information source, how should it design a price-refund pair to maximize the guaranteed profit it can obtain?

To answer this question, we analyze a stylized bilateral trade model: a (female) buyer's valuation to the product is either high or low, the exact value of which unknown to everyone including the buyer. The buyer has access to a (costless) experiment that generates a signal about the buyer's valuation. The seller can offer a refund if the buyer chooses to return the product after the purchase. The product return is costly to the seller. We consider three possible scenarios; the buyer learns the outcome of the experiment (1) before the seller makes an offer; (2) after the seller made a (possibly randomized) offer, but before observing the contract she faces; (3) after observing the offer.[4] As the seller's uncertainty regarding the timing at which the buyer learns increases, the seller's guaranteed profit (weakly) decreases. That is, the guaranteed profit under scenario (1) is the highest, followed by (2) and (3).[5]

For each scenario, our main results (Theorems 1, 2, and 4) characterize the seller's best-guaranteed profit as a function of return costs. If the restocking cost is sufficiently low, which may be realistic for software trial versions, for example, then the seller can obtain the best guaranteed profit by offering a full refund. Such a generous refund makes the demand inelastic, and hence enables the seller to capture a large portion of gains from trade through a high price.

The argument becomes slightly more complex when restocking costs are high. In this case, our optimal policy prescribes high price and almost full refund combination, where restocking cost for the buyer is chosen in a way that buyer who is almost sure to return the product will not like this option. Therefore only buyers whose belief about product fit is high enough will take the refundable option and

[1] See Roesler and Szentes (2017) and Du (2018).

[2] See Ely et al. (2017) and Inderst and Tirosh (2015a) for a similar idea.

[3] See e.g., Inderst and Tirosh (2015a) and Krahmer and Strausz (2015) .

[4] Each scenario respectively captures the environment where the seller knows that (1) the buyer cannot acquire any information, but the seller is uncertain what kind of information she has; (2) the buyer cannot acquire any information after observing the contract she faces, but may acquire additional information after learning the possible distribution over the offers she faces; and (3) nothing about the timing at which the buyer may acquire additional information.

[5] Libgober and Mu (2017) analyzes the scenario (3) without refund but the product is durable.

this makes returns relatively cheap in expectation. And even though the seller has only flattened the highest part of the demand curve, this still reduces the uncertainty about buyer's information and thus raises guaranteed profit.

We also show that these bounds are sharp and robust in two ways. First, we show (Theorem 5) that the seller cannot improve the guaranteed profits by any other selling mechanism, including an arbitrarily sophisticated stochastic menu of options that could screen buyers according to their information. Second, we show (Theorem 6) that if the buyer could choose any information structure,[6] then she would choose the worst-case information structure for the seller. In particular, the buyer can ensure that the seller's profit is kept at the guaranteed profit bound and the trade is efficient.

Related literature: The paper mainly contributes to four branches of economics literature. First, the robust mechanism design literature which follows the Wilson (1987) critique and extends the mechanism design and pricing results to situations where the designer is uncertain about the model details.[7] The closest papers to our work are Roesler and Szentes (2017); Du (2018), who study analogous model, but without the option to offer a return policy. Therefore, their results provide us with a benchmark, which the seller can ensure by never offering an option to return products. We show that having an option to offer return policy will typically strictly increase seller's best-guaranteed profits.

Second, we contribute to the information design literature.[8] There are two information design problems that we solve: we derive worst-case information structure for the seller as well as buyer-optimal information structures. Methodologically we are building on results from Aumann et al. (1995); Kamenica and Gentzkow (2011) and Au and Kawai (2017).

Third, we are adding a novel rationale for return policies. Previous literature on refunds has previously documented many reasons why firms may offer refunds. Grossman (1981); Moorthy and Srinivasan (1995); Inderst and Ottaviani (2013) have argued that warranties and return policies can be used as costly signals for product quality and product fit for the consumer. Che (1996) showed that when consumers are uncertain about product fit and risk-averse, return policy may be used as insurance. Return policies have also studied as price discrimination devices (Zhang, 2013; Escobari and Jindapon, 2014).[9] In Matthews and Persico (2005) trials and refunds reduce the buyer's information acquisition costs and thus allow the seller to control information acquisition. Inderst and Tirosh (2015b) suggested a theory of return policies that is perhaps closest to ours. They argue that return policies work as "metering devices" or two-part tar-

[6]For example, by delegating the information gathering to an assistant, a significant other, or a real-estate agent.

[7] See for example Bergemann and Schlag (2008, 2011); Roesler and Szentes (2017); Auster (2018); Terstiege and Wasser (2017).

[8]See Bergemann and Morris (2017) for a review.

[9]Escobari and Jindapon (2014) also provide some empirical evidence on the use of refundable tickets by airlines. They show that fully refundable ticket is typically about 50% more expensive than a non-refundable ticket, but the difference disappears in the last week before the departure. These facts also fit well to our model predictions.

iffs (see Schmalense (1981)), where generous refunds make different consumers more similar and thus allow the firm to capture more of the surplus by raising prices[10] In our work we focus on another dimension of uncertainty: uncertainty about consumer's information about her valuation rather than uncertainty about her valuation. Moreover, we focus on robust pricing, whereas the previous literature studying refunds has focused on the seller who knows the distribution of the valuations.

Finally, our work is also related to the literature on sequential screening and dynamic mechanism design.[11] This literature has shown that advance sales to still uninformed consumers may help the seller to increase profits (Gale and Holmes, 1992, 1993; Courty and Li, 2000; Eso and Szentes, 2007; Nocke et al., 2011; Gallego and Sahin, 2010; Ely et al., 2016). In contrast to this literature, in our model the main benefit of offering a generous refund policy is not sequential screening, but rather an improvement of guaranteed profits and seller's market power due to reduced uncertainty. In fact, we show that offering menus to screen consumers may be optimal in some situations, but does not guarantee higher profits than a simple stochastic pricing and return policy.

2 Model

There are a (male) seller who can produce a product at no cost, and a (female) buyer whose valuation for the product is $v \in \{0,1\}$. The buyer's valuation v follows a commonly known distribution such that $\pi = \Pr(v=1)$. No-player, including the buyer, does not know the realization of v. However, the buyer receives a signal about her true valuation v prior to purchase. (The detail will be explained momentarily.)

The seller's (pure) strategy is a contract (p,r) that specifies a sales price p together with a refund r. The seller may use a mixed strategy $\Delta\{(p,r)\}$. Based on the information she has, the buyer decides whether to buy after observing the contract $(p,r) \sim \Delta\{(p,r)\}$. We call the buyer's mixed strategy, i.e., a distribution over contracts, *a policy*; and the realized contract as *an offer*.

If the buyer purchases the product, then she learns the realized value of v. If $r=0$, then the game ends. If $r>0$, then the buyer decides whether or not to return the product. When the product gets returned, the seller needs to incur the commonly known restocking cost $c>0$. We sometimes use $\gamma \equiv \frac{c}{1+c} \in (0,1)$ to denote the normalized restocking cost.

If the buyer keeps the product, or, $r=0$, then her payoff is $v-p$, and the seller's profit is p. If $r>0$ and she returns, then her payoff is $r-p$, and the seller's profit is $p-r-c$, respectively. If she does not buy the product, then her payoff and the seller's profit are both zero.

We can represent a buyer's signal as a posterior $q = \Pr[v=1]$ over $v \in \{0,1\}$ that is a random variable drawn from a cumulative distribution function $F \in$

[10]Similar ideas have studied in other contexts, such as overbooking by airlines (Ely et al., 2016).

[11]See citebergemann2010dynamic,vohra-2012-review for literature reviews.

$\mathcal{F} \equiv \{F : \mathbb{E}_F [q] = \pi\}$.[12] For this reason, we use a distribution over posteriors F to represent the buyer's information structure, and call it *a signal distribution*. For the same token, by a *signal* q, we refer to the realization of the posterior drawn from signal distribution F.

We are interested in the seller's *best profit guarantee* when he is uncertain of the buyer's signal distribution. More precisely, the buyer chooses a signal distribution F from a subset $\mathcal{F}_B$ of $\mathcal{F}$ unknown to the seller; but the seller neither has a prior distribution over $\mathcal{F}_B$ nor observes the buyer's choice of F. The timing at which the buyer chooses a signal distribution affects the seller's best profit guarantee.

Policy-independent signal: The buyer receives signal before the policy is announced, i.e. at $t = 1$ in Figure 1. The signal distribution $F \in \mathcal{F}$ is therefore independent of policy. This captures the environment where the seller knows that buyer cannot acquire any additional information, but she may have information that the seller is unaware of.

Policy-dependent signal: The buyer receives the signal after the policy is announced, i.e. at $t = 2$ in Figure 1. The signal distribution $F_{\Delta(p,r)} \in \mathcal{F}$ may depend on the policy $\Delta(p,r)$, but not the realization of (p,r). For example, the buyer may gather less information when she expects either very low or very high price (on average), or if the refund policy is very generous (with high probability).

Offer-dependent signal: The buyer receives the signal after the offer is realized, i.e. at $t = 3$ in Figure 1. The signal distribution $F_{(p,r)} \in \mathcal{F}$ may depend on the realized offer (p,r). In this case the buyer has full flexibility to adjust information gathering to particular price and refund offer she receives.

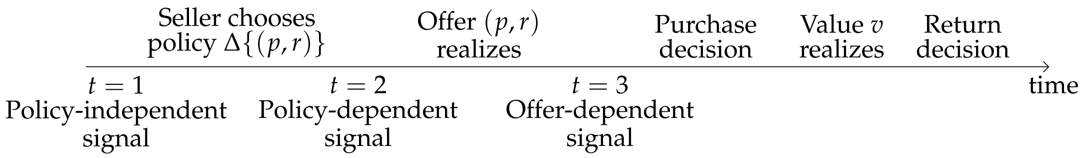


Figure 1: Timing

More formally, let $V((p,r)|F)$ be the seller's expected profit from offer (p,r) and the buyer's signal distribution after $t = 3$ is F. Similarly, $\mathbb{E}_{\Delta\{(p,r)\}} V((p,r)|F)$ represents the seller's expected profit from a policy $\Delta\{(p,r)\}$ when the buyer's signal distribution after $t = 3$ is F. The best profit guarantee the seller can obtain

[12]Notice that $F \in \mathcal{F}$ if and only if $\int_0^1 F(q)\, dq = 1 - \pi$.

when the buyer chooses F at $t = 1, 2, 3$ are, respectively,

$$V_1^* \equiv \sup_{(p,r)} \min_{F \in \mathcal{F}} V((p,r)|F),$$

$$V_2^* \equiv \sup_{\Delta\{(p,r)\}} \min_{F_{\Delta\{(p,r)\}} \in \mathcal{F}} \mathbb{E}_{\Delta\{(p,r)\}} V\left((p,r)|F_{\Delta\{(p,r)\}}\right),$$

$$V_3^* \equiv \sup_{(p,r)} \min_{F_{(p,r)} \in \mathcal{F}} V\left((p,r)|F_{(p,r)}\right).$$

Obviously, $V_1^* \geq V_2^* \geq V_3^*$. We identify V_1^*, V_2^*, and V_3^*. Our results show that $V_1^* = V_2^*$ for all parameter values; and $V_2^* = V_3^*$ if and only if the (normalized) restocking cost γ is small enough. That is, if the restocking cost is sufficiently small for a given prior π, then the timing at which the buyer chooses her signal distribution does not affect the seller's best guaranteed profit. In contrast, when the restocking cost is sufficiently large, whether the buyer chooses a signal distribution after observing an offer do affect the seller's best guaranteed profit. That is, $V_1^* > V_3^*$. Nevertheless, our result that $V_1^* = V_2^*$ informs us that the seller can completely neutralize this negative effect by making a randomized offer and thereby by keeping the buyer in the dark about the actual offer she will face.

3 Analysis

3.1 Offer-Dependent Signal

We start with the case where the information structure may depend on the realized contract (p,r), i.e., the buyer may choose a signal distribution at $t = 3$. Our goal is to identify the best profit guarantee $V_3^* \equiv \sup_{(p,r)} \min_{F_{(p,r)} \in \mathcal{F}} V\left((p,r)|F_{(p,r)}\right)$. With a slight abuse of notation, let $V(q|(p,r))$ be the seller's expected profit when the offer is (p,r) and the buyer's signal is q. Then, $V\left((p,r)|F_{(p,r)}\right) = \mathbb{E}_{F_{(p,r)}}[V(q|(p,r))]$. Also for notational simplicity, we use $\underline{V}_3(p,r)$ to denote $\min_{F_{(p,r)} \in \mathcal{F}} V\left((p,r)|F_{(p,r)}\right)$, so that $V_3^* = \sup_{(p,r)} \underline{V}_3(p,r)$. We say an offer (p,r) is *non-refundable* when $r = 0$; and is *refundable* when $r > 0$.

Suppose that the seller makes a non-refundable offer $(p, 0)$. Then, the buyer with signal q buys if and only if $q \geq p$, i.e.,

$$V(q|(p,0)) = \begin{cases} 0 & q < p \\ p & q > p \end{cases}.$$

To derive $\underline{V}_3(p,0)$, we utilize the concavification approach.[13] Let $\text{con}[-V(\cdot|(p,0))](q)$ be the value of concave closure of $-V(\cdot|(p,0))$ at q.[14] Then $\underline{V}_3(p,0) = -\text{con}[-V(\cdot|(p,0))](\pi)$, which is represented by the red-dotted line in Figure 2. The seller's profit is minimized when the probability of signal being larger than p is minimized, i.e., when $F_{(p,0)}$ minimizes $1 - F_{(p,0)}(p)$ subject to $\mathbb{E}_{F_{(p,0)}}[q] = \pi$.

[13] See Aumann et al. (1995) and Kamenica and Gentzkow (2011).

[14] The concave closure of function G is defined by $\text{con}(G)(q) = \sup\{g|(q,g) \in \text{co}(G)\}$, where

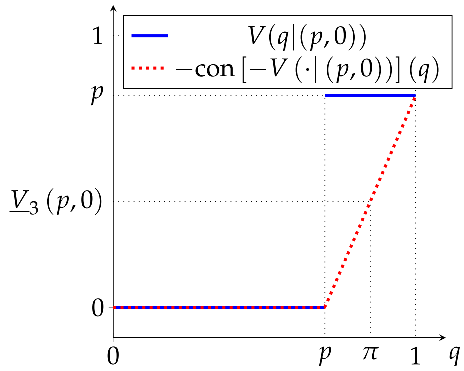


Figure 2: Profit of a non-refundable offer $(p,0)$

If $\pi > p$, then this occurs when the buyer's signal distribution induces two signals p (which results in no trade) and 1 (which results in trade) with probabilities $\frac{1-\pi}{1-p}$, and $\frac{\pi-p}{1-p}$, respectively. In contrast, if $\pi \leq p$, then this occurs when the buyer's signal does not disclose any additional information, i.e., induces signal π (which results in no trade) with probability one. While a higher p results in higher a profit margin should trade occur, it leads to a lower probability of trade. The seller brings a balance to this trade-off by offering $p = 1 - \sqrt{1-\pi}$. More formally,

$$\begin{aligned}\underline{V}_3(p,0) &= -\text{con}\left[-V\left(\cdot|\left(p,0\right)\right)\right](\pi) = \begin{cases} 0 & p \geq \pi \\ p \times \frac{\pi-p}{1-p} & p < \pi \end{cases} \\ &\leq \sup_p \underline{V}_3(p,0) = \sup_{p\in[0,\pi]} p\frac{\pi-p}{1-p} = \left(1-\sqrt{1-\pi}\right)^2. \end{aligned} \tag{1}$$

Next, suppose that the seller allows the buyer to return the product, i.e., makes an offer (p,r) such that $r > 0$. Without loss of generality, we only consider the case where $p \geq r$.[15] Consider the buyer with signal q. Since her payoff from buying is $q \times 1 + (1-q) \times r - p$, she buys if only if her signal is above the marginal signal $\tilde{q}(p,r) \equiv \frac{p-r}{1-r}$; and returns with probability $1-q$ if she buys. The seller's profit from the buyer with signal q is thus

$$V(q|(p,r)) = \begin{cases} 0 & q < \tilde{q}(p,r) \\ v(q;p,r) & q > \tilde{q}(p,r) \end{cases}, \tag{2}$$

where $v(q;p,r) \equiv p - (1-q)(c+r)$.

Observe that if the marginal signal $\tilde{q}(p,r)$ is sufficiently low, i.e., if the refund is sufficiently generous, then the buyer buys even when her signal q is sufficiently low. The buyer with a low signal is likely to return the product, and thereby is

co(G) is the convex hull of the graph of G.

[15]Notice that if $p < r$, then buyer buys irrespective of the value of signal. Therefore, $\underline{V}_3(p,r) < \underline{V}_3(p,r-\varepsilon)$ for a sufficiently small $\varepsilon > 0$.

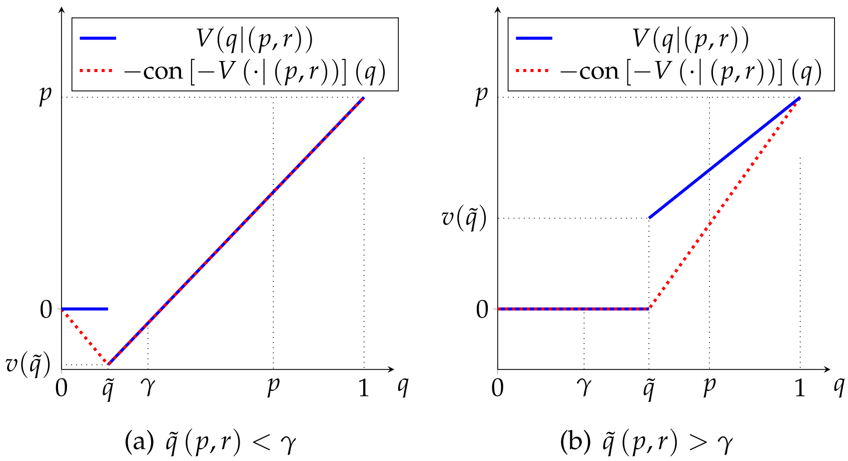


Figure 3: Profit from refundable offer (p, r)

likely to bring the seller an (ex-post) negative profit. More specifically, as captured by the increasing blue-lines in Figures 3(a) and 3(b),

$$v\left(\tilde{q}\left(p,r\right);p,r\right) \leq 0 \text{ if and only if } \tilde{q}\left(p,r\right) \leq \gamma\text{; and}$$
$$v\left(q;p,r\right) \text{ is increasing in } q.$$

Therefore, if $\tilde{q}(p,r) < \gamma$, then as captured by the red dotted-line in Figure 3(a), the seller's profit is minimized when the probability of the seller receiving a signal $\tilde{q}(p,r)$ is maximized. The seller then can improve his guaranteed profit by offering a less generous refund, and thereby increasing the marginal signal.

In contrast, if the marginal signal is sufficiently high so that $\tilde{q}(p,r) > \gamma$, then the buyer who buys the product always brings a positive (ex-post) profit to the seller. The seller's profit is thus minimized when the probability of the seller receiving a signal above $\tilde{q}(p,r)$ is minimized, as captured by the red dotted line in Figure 3(b). The seller then can improve his guaranteed profit by offering a more generous refund, and thereby by lowering the marginal signal.

Combining these observations we conclude that if the seller were to offer a positive refund, then he should set the price p as close to one as possible, and r to $\frac{p-\gamma}{1-\gamma}$ so that the marginal signal is exactly at γ. Hence the profit the seller can guarantee himself by offering a positive refund is $\frac{\pi-\gamma}{1-\gamma}$. Recall that the seller's best guaranteed profit without refund is $\left(1-\sqrt{1-\pi}\right)^2$, as derived in (1). Thus, the seller's best guaranteed profit when the buyer may choose a signal distribution at $t = 3$ is

$$V_3^* = \sup_{p,r} \underline{V}_3\left(p,r\right) = \begin{cases} \frac{\pi-\gamma}{1-\gamma} & \gamma \leq \widehat{\gamma}\left(\pi\right) \\ \left(1-\sqrt{1-\pi}\right)^2 & \gamma \geq \widehat{\gamma}\left(\pi\right) \end{cases}, \tag{3}$$

where $\widehat{\gamma}(\pi) \equiv \frac{2\left(1-\sqrt{1-\pi}\right)}{2-\sqrt{1-\pi}}$. Furthermore, the best guaranteed profit monotonically converges to full surplus from trade π as the restocking cost converges to 0. The resulting bound is depicted on Figure 4.

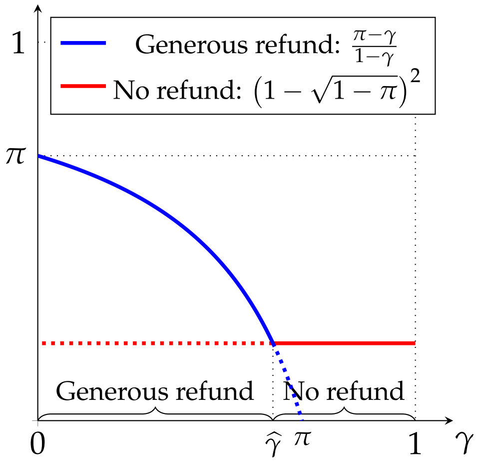


Figure 4: Best guaranteed profit V_3^*

Theorem 1. *Suppose that the buyer may choose a signal distribution at $t = 3$. Then the seller's best guaranteed profit is V_3^* defined in (3). For any $\varepsilon > 0$, the seller's can achieve $(1-\varepsilon)\, V_3^*$ either by offering a generous refund $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$ (when $\gamma \leq \widehat{\gamma}(\pi)$); or no refund $(p, r) = \left(1-\sqrt{1-\pi}, 0\right)$ (when $\gamma \geq \widehat{\gamma}(\pi)$).*

3.2 Policy-Independent Signal

We now consider the case where the buyer may choose a signal distribution only at $t = 1$. Recall that given an offer (p, r), the marginal signal is $\tilde{q}(p, r) = \frac{p-r}{1-r}$. Therefore,

$$V_1((p, r)\,|F) = p\left(1 - F(\tilde{q}(p, r))\right) - \mathbf{1}_{r>0}(c + r)\int_{\tilde{q}(p,r)}^{1}(1-q)dF(q).$$

Our goal is to identify $V_1^* \equiv \sup_{(p,r)} \min_{F\in\mathcal{F}} V((p, r)\,|F)$. To this end, we introduce a few notations. First, to identify the profit from a non-refundable offer, define

$$G_V(q) \equiv \begin{cases} 0 & q \in [0, V) \\ 1 - \frac{V}{q} & q \in [V, 1) \\ 1 & q = 1 \end{cases}. \tag{4}$$

Then, for a given buyer's signal distribution F, the highest profit the seller can obtain by a non-refundable offer is identified by $\inf\{V : F(q) \geq G_V(q) \text{ for all } q\}$, which we denote by $V_{NR}(F)$.[16] Also define

$$F_V(q) \equiv \begin{cases} G_V(q) & q \in [0, \gamma) \\ 1 - \Phi(V) & q \in [\gamma, 1) \\ 1 & q = 1 \end{cases}. \tag{5}$$

[16]This comes from the same argument as in Roesler and Szentes (2017). The seller's profit from offering $(p, 0)$ is $V((p, 0)\,|F) = p\left(1 - F(p) + \Delta F(p)\right)$. Furthermore, since $F(q) \geq G_{V_{NR}}(q)$ for all q, we have $V((p, 0)\,|F) \leq p\left(1 - G_{V_F}(p)\right) = V_{NR}(F)$. Therefore, $\sup_p V((p, 0)\,|F) = V_{NR}(F)$.

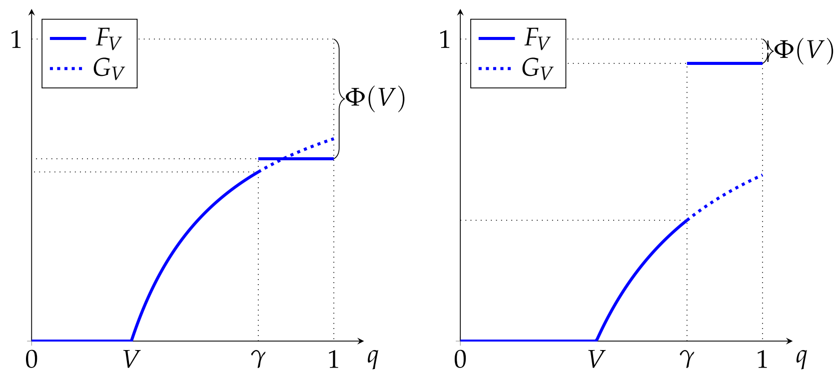
where

$$\Phi(V) = \begin{cases} \frac{V\left(\ln\frac{V}{\gamma}-1\right)+\pi}{1-\gamma} & V \in [0,\gamma) \\ \frac{\pi-\gamma}{1-\gamma} & V \in [\gamma,1) \end{cases}.$$

Notice that $\int_0^1 F_V(q)\,dq = 1-\pi$. Therefore, if the buyer's signal distribution is F_V, then $\Phi(V)$ is (the supremum of) the seller's profit from a generous refund $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$. Observe that since $\Phi(V)$ is strictly decreasing in V on $[0,\gamma)$, and $\lim_{V\to\gamma}\Phi(V) = \frac{\pi-\gamma}{1-\gamma}$, either (i) $\Phi(V) > V$ for all $V \in [0,\gamma)$ (when $\frac{\pi-\gamma}{1-\gamma} > \gamma$, or equivalently $\gamma < 1-\sqrt{1-\pi}$), or (ii) there exists unique $\tilde{V} = \frac{\pi-\tilde{V}}{1-\gamma-\ln\frac{\tilde{V}}{\gamma}} \in [0,\gamma]$ such that $\Phi\left(\tilde{V}\right) = \tilde{V}$ (when $\gamma \geq 1-\sqrt{1-\pi}$). Let V^* be the largest $V \in [0,\gamma]$ such that $\Phi(V) \geq V$, i.e.,

$$V^* \equiv \begin{cases} \gamma & \gamma < 1-\sqrt{1-\pi} \\ \tilde{V} & \gamma \geq 1-\sqrt{1-\pi} \end{cases}.$$

Figiures 5(a) and 5(b) illustrate F_{V^*} when $\gamma < 1-\sqrt{1-\pi}$ and $\gamma \geq 1-\sqrt{1-\pi}$, respectively.



(a) $\gamma < 1-\sqrt{1-\pi}$ (b) $\gamma \geq 1-\sqrt{1-\pi}$

Figure 5: F_{V^*}

It is straightforward to observe that $\Phi(V^*) = \sup_{(p,r)} V((p,r)|F_{V^*})$ by construction of F_{V^*}. Below, we show that

$$\sup_{(p,r)} V((p,r)|F) \geq \sup_{(p,r)} V((p,r)|F_{V^*}) \text{ for all } F \in \mathfrak{F}.$$

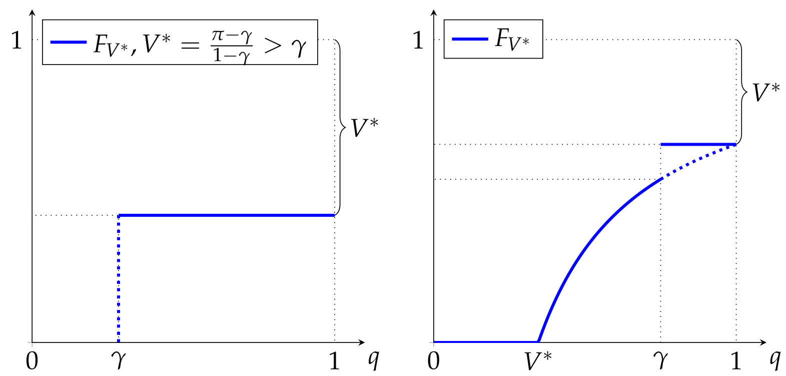
That is, $V_1^* = \Phi(V^*)$. To see this, first suppose there exists a distribution F such that $F(q) < F_{V^*}(q)$ for some $q \in [0,\gamma)$. Then, since $V_{NR}(F) > V^*$, we have $\sup_{(p,r)} V((p,r)|F) > \sup_{(p,r)} V((p,r)|F_{V^*}(q))$. Next, suppose that $F(q) \geq F_{V^*}(q)$ for all $q \in [0,\gamma)$. Recall that when the seller offers a generous refund $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$, the buyer buys if and only if $q \geq \gamma$; and the seller's profit from the buyer with signal q is $(1-\varepsilon)\frac{q-\gamma}{1-\gamma}$. Therefore, the seller's profit from a generous refund when the buyer's signal distribution is F is (weakly) higher than $(1-\varepsilon)\,\Phi(V^*)$. Consequently, $\sup_{(p,r)} V((p,r)\,|F) \geq \sup_{(p,r)} V((p,r)\,|F_{V^*}(q))$.

We note that $\Phi(V^*)$ is the unique solution $V_1^* \in (0,\gamma]$ that solves the following equation:[17]

$$V_1^* = \frac{\pi - V_1^*}{1-\gamma-\ln\frac{V_1^*}{\gamma}}. \tag{6}$$

We therefore have the following theorem.

Theorem 2. *Suppose that the buyer may choose signal distribution at $t=1$. Then, the best guaranteed profit for the seller V_1^* is defined by (6).*



(a) V_1^* (blue) and V_3^* (red)

(b) V_1^* for various priors

Figure 6: Best guaranteed profit V_1^*

A few observations follow. The best guaranteed profit when the seller cannot offer any refund, $\sup_p \min_{F\in\mathcal{F}} V((p,0)\,|F)$ (i.e., the brown dotted-line in Figure 6(a)), is strictly lower than $V_1^* = \sup_{(p,r)} \min_{F\in\mathcal{F}} V((p,r)\,|F)$ for any level of restocking cost γ. That is, being able to offer a refund strictly improves the seller's best guaranteed profit. This is a clear contrast to the case where the buyer may choose a signal distribution in $t=3$, where the refund improves the best guaranteed profit if and only if the (normalized) restocking cost γ is sufficiently small, i.e., $\gamma < \widehat{\gamma}(\pi)$.

Next, when the restocking cost γ is small, i.e., $\gamma < 1-\sqrt{1-\pi}$, whether the buyer may choose a signal distribution at $t=1$ or $t=3$ does not affect the seller's best guaranteed profit. That is, the blue-line and the red dotted-line in Figure 6(a) coincide. This is because when γ is small, or equivalently the prior π is sufficiently large, the cost of offering a generous refund $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$ is not significant. Furthermore, as we have seen in the previous section, the worst signal distribution for a generous refund $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$ does not depend on ε,

[17] $V_1^* = \frac{-\pi}{W_{-1}\left(-\frac{\pi}{\gamma}e^{\gamma-2}\right)}$, where $W_{-1}(\cdot)$ denotes the lower branch of the Lambert's W.

and has the support that consists of γ and 1.[18] This is, however, also is the worst distribution when the buyer chooses signal distribution only before she observes the policy, as depicted in Figure 5(a). Therefore, knowing that the buyer cannot choose a signal distribution at $t = 3$ does not help the seller improving his best guaranteed profit.

As one can see in Figure 6(b), the seller's best guaranteed profit strictly increases as the prior π goes up (for a given restocking cost γ); and as the restocking cost γ goes down (for a given prior π).

We now construct the seller's strategy that achieves V_1^*. When the restocking cost is small, i.e., $\gamma < 1 - \sqrt{1-\pi}$, we already know that a generous refund offer $\left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$ achieves $(1-\varepsilon)\, V_1^*$. Therefore, we limit our attention to the case where $\gamma \geq 1 - \sqrt{1-\pi}$.

We note that for any seller's pure strategy (p, r) such that $V\left((p,r)\,|F_V^*\right) \geq V_1^* - \varepsilon$, there exists a signal distributiuon F such that $V_1^* > V\left((p,r)\,|F\right)$. However, we show that, for any $\tilde{\varepsilon} > 0$, the seller can guarantee himself a profit $\tilde{\varepsilon}$ away from V_1^* by the following policy $S^\varepsilon(p,r)$, which we call *the exponential pricing policy with a generous refund*: the seller makes (i) offers without refunds $(p, 0)$ on $p \in [V_1^*, \gamma)$ with density $s_0^\varepsilon(p) = \frac{1}{p\left(1-\gamma-\ln\frac{V_1^*}{\gamma}\right)}$; and (ii) a generous refund $(p,r) = \left(1-\varepsilon, 1-\frac{\varepsilon}{1-\gamma}\right)$ with probability $s_r = 1 - \int_{V_1^*}^{\gamma} s_0^\varepsilon(p)\,dp = \frac{1-\gamma}{1-\gamma-\ln\frac{V_1^*}{\gamma}}$.[19]

Theorem 3. *The seller achieves the profit that is $\tilde{\varepsilon}$ away from the best guaranteed profit V_1^* either by (i) offering a generous refund (when $\gamma \leq 1-\sqrt{1-\pi}$); or (ii) using the exponential pricing policy with with a generous refund $S^\varepsilon(p,r)$ for a sufficiently small ε (when $\gamma > 1-\sqrt{1-\pi}$).*

Proof. In the appendix. □

3.3 Policy-Dependent Signal

We now consider the case where the buyer may choose a signal distribution only at $t = 2$. Our goal is to identify

$$V_2^* \equiv \sup_{\Delta\{(p,r)\}} \min_{F_{\Delta\{(p,r)\}}\in\mathcal{F}} \mathbb{E}_{\Delta\{(p,r)\}} V\left((p,r)\,|F_{\Delta\{(p,r)\}}\right).$$

As an immediate corollary of Theorem 3, we have the following theorem.

Theorem 4. *Suppose that the buyer may choose a signal distribution only at $t = 2$. Then, for any restocking cost γ, $V_1^* = V_2^*$. The seller achieves the profit that is $\tilde{\varepsilon}$ away from the best guaranteed profit either by (i) offering a generous refund (when $\gamma \leq 1-\sqrt{1-\pi}$); or (ii) using the exponential pricing policy with with a generous refund $S^\varepsilon(p,r)$ for a sufficiently small ε (when $\gamma > 1-\sqrt{1-\pi}$).*

[18]See Figure 3.

[19]This is a variant of mechanism found in Du (2018).

3.4 Direct Mechanism

We have identified the best guaranteed profit that the seller can achieve with a uniform pricing with refunds. One may wonder if the seller can improve his best guaranteed profit by using a more intricate mechanism that potentially screens the buyer based on her signal q. Below we show that there exists no such a mechanism.

To this end, consider the environment where the buyer may choose a signal distribution only at $t = 1$. We first note that, for any mechanism, there exists an outcome-equivalent simple static direct mechanism with refunds.

Definition 1. *We say a direct mechanism $M \equiv \{p(q), \{\alpha_0(q), \alpha_r(q)\}\}_{q\in[0,1]}$ is a direct mechanism with refunds if, for each buyer's signal $q \in [0,1]$, the mechanism specifies (i) $p(q)$: the transfer from the buyer to the seller; (ii) $\alpha_0(q) \in [0,1]$: the probability that the buyer receives the product without an option to return; and (iii) $\alpha_r(q) \in [1-\alpha_0(q), 1]$: the probability that the buyer receives the product with an option to return with refund $r = 1$.*

Lemma 1. *For any outcome the seller can induce by an indirect mechanism, there exists an outcome-equivalent direct mechanism with refunds that is individually rational and incentive compatible.*

Proof. In the appendix. □

Under the direct mechanism with refunds M, the buyer's payoff of reporting q' when her signal is q is

$$\begin{aligned} U(q'; q|M) &\equiv \left(\alpha_0(q') + \alpha_r(q')\right) q + \alpha_r(q')(1-q) - p(q) \\ &= q \times \alpha_0(q') - \left(p(q) - \alpha_r(q')\right). \end{aligned}$$

The seller's profit from the buyer with signal q is

$$\begin{aligned} R(q|M) &\equiv p(q) - \alpha_r(q')(1-q)(c+1) \\ &= p(q) - \alpha_r(q')\frac{1-q}{1-\gamma}, \end{aligned}$$

and hence his profit when the buyer's signal distribution is F is

$$V(M; F) \equiv \int_0^1 R(q|M)\, dF(q).$$

Thus seller's objective is to find a mechanism $M = \{p(q), \{\alpha_0(q), \alpha_r(q)\}\}_{q\in[0,1]}$ that solves

$$\max_{M\in\mathcal{M}} \min_{F\in\mathcal{F}} V(M; F) \tag{7}$$

where $\mathcal{M}$ is the set of all direct mechanisms with refunds M that satisfy the following two conditions:

$$\begin{aligned} &\text{IC: } U(q; q|M) \geq U(q'; q|M) \text{ for all } q' \text{ and } q \\ &\text{IR: } U(q; q|M) \geq 0 \text{ for all } q. \end{aligned}$$

By adopting the standard argument, we can simplify the seller's problem to the one in which he only chooses an increasing function $\alpha_0(\cdot)$ (instead of M, which is a triplet of functions, that is individually rational and incentive compatible). To formally state this result, for a function $\alpha_0 : [0,1] \to [0,1]$, define $\underline{R}(\pi;\alpha_0) \equiv -\mathrm{con}\left[-R(q;\alpha_0)\right](\pi)$, where

$$R(q;\alpha_0) \equiv \begin{cases} q \times \alpha_0(q) - \int_0^q \alpha_0(\tilde{q})\, d\tilde{q} & \text{if } q < \gamma \\ q \times \alpha_0(q) + \frac{q-\gamma}{1-\gamma}(1-\alpha_0(q)) - \int_0^q \alpha_0(\tilde{q})\, d\tilde{q} & \text{if } q \geq \gamma \end{cases}. \tag{8}$$

We then have:

Lemma 2. *Take $\alpha_0^*(q) \in \arg\max_{\alpha_0 \in \mathcal{A}} \underline{R}(\pi;\alpha_0)$, where $\mathcal{A}$ is the set of all increasing functions from $[0,1]$ to $[0,1]$. Then $M^* = \{p^*(q), \{\alpha_0^*(q), \alpha_r^*(q)\}\}_{q\in[0,1]}$, where $\alpha_r^*(q) = \mathbf{1}_{q\in[\gamma,1]} \times (1-\alpha_0(q))$, and $p^*(q) = q\alpha_0^*(q) + \alpha_r^*(q) - \int_0^q \alpha_0^*(\tilde{q})\, d\tilde{q}$, is a solution to the problem (7). Furthermore, there exists $V \in [0,\gamma]$ such that*

$$R(q;\alpha_0^*) = \begin{cases} 0 & \text{if } q < V \\ \frac{q-V}{1-\gamma-\ln\frac{V}{\gamma}} & \text{if } q \geq V \end{cases}.$$

and the seller's best guaranteed profit is

$$\max_{M\in\mathcal{M}} \min_{F\in\mathcal{F}} V(M;F) = \max_{V\in[0,\gamma]} \frac{\pi - V}{1-\gamma-\ln\frac{V}{\gamma}}.$$

Proof. In Appendix. □

Theorem 5. $\max_{M\in\mathcal{M}} \min_{F\in\mathcal{F}} V(M;F) = V_1^* = V_2^*$. *That is, the best guaranteed profit under any mechanism is bounded from above by V_1^*.*

Proof. It is straightforward to verify that $\frac{\pi-V}{1-\gamma-\ln\frac{V}{\gamma}}$ is maximized at $V = V_1^*$. Furthermore $V_1^* = \frac{\pi-V_1^*}{1-\gamma-\ln\frac{V_1^*}{\gamma}}$ by (6). Therefore, we have the required result. □

4 Feasible Outcomes

In this section, we identify the information structure that maximizes the buyer's welfare, as well as outcomes that can be supported by some information structure.

We first characterize the buyer-optimal outcome. Without loss of generality, we assume that the buyer chooses an information structure at $t = 1$ in Figure 1, and observe a signal at $t = 3$. The buyer's choice becomes public information, and it is assumed that the buyer cannot change the information structure she has chosen after $t = 1$.

Given our focus on the buyer-optimal outcome, we assume that when the seller is indifferent between two or more offers, the seller makes an offer that induces a higher buyer's payoff.

Then there are two possible scenarios that results in different buyer's payoffs.

Policy-independent signal The buyer chooses a single signal distribution $F \in \mathcal{F}$ at $t = 1$. The seller chooses an offer (p, r) to maximize his profit.

Offer-dependent signal The buyer chooses a set of signal distributions $\left\{F_{(p,r)}\right\}_{(p,r)}$ at $t = 1$ that maps each possible seller's offer (p, r) into a possibly different signal distribution $F_{(p,r)} \in \mathcal{F}$. The seller chooses an offer (p, r) to maximize his profit.

More formally, let $U((p, r)|F)$ be the buyer's payoff when her signal distribution at $t = 3$ is $F \in \mathcal{F}$, and the seller's offer is (p, r). Also, let $V((p, r)|F)$ be the seller's profit from offer (p, r) and the buyer's signal distribution after $t = 3$ is F. Then, the buyer-optimal outcomes when the signals are, policy-independent, and offer-dependent, are respectively characterized as

$$U_1^* \equiv \max_{F\in\mathcal{F}} U((p_1, r_1)|F) \text{ s.t. } (p_1, r_1) \in \arg\max_{(p,r)} V((p, r)|F) \tag{9}$$

$$U_3^* \equiv \max_{F_{(p,r)}\in\mathcal{F}} U\left((p_3, r_3)|F_{(p_3,r_3)}\right) \text{ s.t. } (p_3, r_3) \in \arg\max_{(p,r)} V\left((p, r)|F_{(p,r)}\right)$$

Our goal is to identify U_1^* and U_3^*. Before proceeding to with the analysis, we make two observations. First, $U_1^* \leq U_3^*$. Furthermore, the total surplus is bounded from above by π. However, the preceding analysis identifies that the seller's best guaranteed profit is V_1^* when the signal is policy-independent (Theorem 2); and V_3^* $(\leq V_1^*)$ when the signal is offer-dependent (Theorem 1). Therefore,

$$U_1^* \leq \pi - V_1^* \text{ and } U_3^* \leq \pi - V_3^*.$$

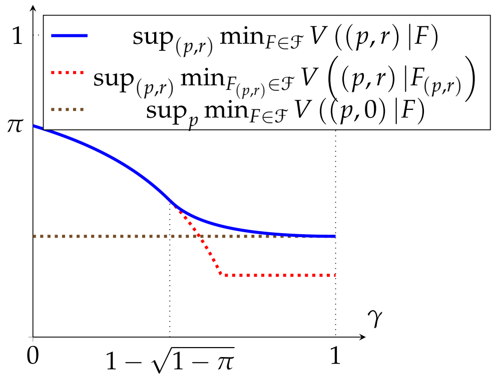
The theorem below shows that the buyer can obtain the respective upper bounds.

Theorem 6. *If the buyer can choose and commit to a policy-independent signal distribution at $t = 1$, then her payoff under the buyer-optimal outcome is $U_1^* = \pi - V_1^*$. If she can choose and commit to a set of offer-depend signal distributions at $t = 1$, then, her payoff under the buyer-optimal outcome is $U_3^* = \pi - V_3^*$.*

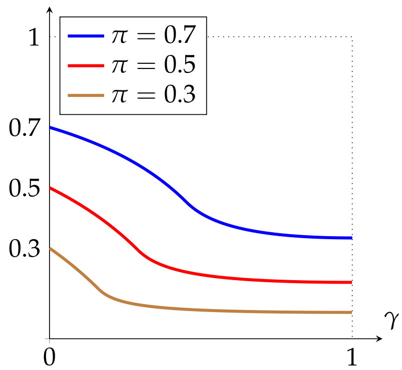
Proof. In the Appendix. □

We note that there exist multiple policy-independent signal distributions (or sets of offer-dependent signal distributions) that support result in a same buyer-optimal outcomes. Also, as an immediate corollary, we can characterize pairs of buyer's payoff and seller's profit that can be supported by some information structure. More precisely, we say $\left(\hat{U}, \hat{F}\right)$ is a *feasible outcome* supported by a policy-independent signal F if $(\hat{p}, \hat{r}) \in \arg\max_{(p,r)} V((p, r)|F)$; $\hat{V} = V((\hat{p}, \hat{r})|F)$; and $U = \hat{U}((\hat{p}, \hat{r})|F)$. Similarly, we can define a feasible outcome supported by a set of offer-dependent signal distributions. Then, the set of feasible outcomes O_1 supported by some policy-independent signals; and the set of outcomes O_3 supported by some set of offer-dependent signal distributions are, respectively,

$$O_1 \equiv \{(U, V) : V \in [V_1^*, \pi - U], U \in [0, \pi - V_1^*]\}$$
$$\text{and } O_3 \equiv \{(U, V) : V \in [V_3^*, \pi - U], U \in [0, \pi - V_3^*]\}.$$



(a) O_1: Policy-dependent signal



(b) O_3: Offer-dependent signal

Figure 7: Feasible outcomes

5 Discussion

In this paper, we analyzed a bilateral trade problem, where the seller does not know what kind of information the buyer has about her own valuation. We showed that offering a well-designed return policy ensures the seller larger profits. We characterized the best-guaranteed profit bounds under various assumptions and showed how to achieve these bounds by simple stochastic pricing policy. The bounds are also sharp, there are no other mechanisms that ensure the seller higher profits. Moreover, the worst-case information structure that we identify is also natural in the sense that if the buyer would be able to pre-commit to an information structure, then it would be optimal for the buyer.

Although we only focused on a single buyer-seller interaction, it is natural to expect that the same ideas extend in other settings. An alternative interpretation of our buyer-optimality result is that if a buyer could ex-ante delegate the information gathering to a third party[20] knowing that the seller will respond to any information structure by optimal pricing and return policy, then our best-guaranteed profit bound gives a constraint for buyer optimality. We show that in fact, the buyer can always choose an information structure and an equilibrium that leads to efficient outcome and guarantees the seller his best guaranteed profit and nothing more.

A similar analysis could be used in other problems, such as social planner's problem or platform design. A designer (for example, a platform such as eBay or Airbnb) could choose information that will be revealed to the buyers about the product. The seller chooses pricing and return policy optimally and the buyer makes an optimal purchase and return decisions. As we derive the Pareto-optimality frontier, it is natural to expect that the optimal outcomes for such problems will be on the frontier and coincide with our results.

[20]Such as an assistant, a significant other, a real estate agent, or an algorithm.

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A Proofs for Section 3 (Analysis)

Proof of Theorem 3:
When F is the buyer's information structure, the seller's payoff of using policy S^{ε} is

$$\mathbb{E}_{S^{\varepsilon}}\left[V\left((p,r)\,|F\right)\right] = \int_{V_1^*}^{\gamma} p\left(1-F\left(p\right)\right) s_0^{\varepsilon}\left(p\right) dp + s_r \int_{\gamma}^{1}\left(\left(1-\varepsilon\right)-\left(1-q\right)\left(\left(1-\frac{\varepsilon}{1-\gamma}\right)+c\right)\right) dF\left(q\right).$$

Then,

$$\int_{V_1^*}^{\gamma} p\left(1-F\left(p\right)\right) s_0^{\varepsilon}\left(p\right) dp = \frac{1}{1-\gamma+\ln\frac{V_1^*}{\gamma}}\int_{V_1^*}^{\gamma}\left(1-F\left(p\right)\right) dp \geq \frac{1}{1-\gamma+\ln\frac{V_1^*}{\gamma}}\int_{0}^{\gamma}\left(1-F\left(p\right)\right) dp,$$

and

$$\begin{aligned}
&s_r \int_{\gamma}^{1}\left(\left(1-\varepsilon\right)-\left(1-q\right)\left(\left(1-\frac{\varepsilon}{1-\gamma}\right)+c\right)\right) dF\left(q\right)\\
&= \frac{1-\gamma}{1-\gamma+\ln\frac{V_1^*}{\gamma}}\left(1-\left(1+c\right)\int_{\gamma}^{1}F\left(q\right)dq-\varepsilon\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right)\right)\\
&= \frac{1-\gamma}{1-\gamma+\ln\frac{V_1^*}{\gamma}}\left(1-\frac{1}{1-\gamma}\int_{\gamma}^{1}F\left(q\right)dq-\varepsilon\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right)\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}_{S^{\varepsilon}}\left[V\left((p,r)\,|F\right)\right] &\geq \frac{\left(-\int_0^1 F\left(p\right)dp - V_1^* + 1\right) - \varepsilon\left(1-\gamma\right)\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right)}{1-\gamma-\ln\frac{V_1^*}{\gamma}}\\
&= \frac{-\left(1-\pi\right)-V_1^*+1-\varepsilon\left(1-\gamma\right)\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right)}{1-\gamma-\ln\frac{V_1^*}{\gamma}}\\
&= \frac{\pi-V_1^*}{1-\gamma-\ln\frac{V_1^*}{\gamma}} - \frac{\varepsilon\left(1-\gamma\right)}{1-\gamma-\ln\frac{V_1^*}{\gamma}}\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right)\\
&= V_1^* - \frac{\varepsilon\left(1-\gamma\right)}{1-\gamma-\ln\frac{V_1^*}{\gamma}}\int_{\gamma}^{1}\left(1+\frac{1-q}{1-\gamma}\right)dF\left(q\right)
\end{aligned}$$

We thus have the required result.

Proof of Lemma 1: Any outcome the seller can induce by an indirect mechanism can be induced by an individually-rational and incentive-compatible direct mechanism $\Phi = \left\{\alpha_q, p_q, \left\{\left(\kappa_q^0, \tau_q^0\right), \left(\kappa_q^1, \tau_q^1\right)\right\}\right\}$ that specifies, for each $q \in [0,1]$, (i) α_q: the probability that the buyer receives the product; (ii) p_q: the transfer from the buyer to the seller; and (iii) $\left\{\kappa_q^v, \tau_q^v\right\}_{v\in\{0,1\}}$: the direct mechanism that specifies, for each buyer's true valuation $v \in \{0,1\}$, (a) κ_q^v: the probability the buyer keeps the product; and (b) τ_q^v: the transfer from the seller to the buyer with the following properties: (IC) $\kappa_q^1 + \tau_q^1 \geq 1$ and $\kappa_q^0 + \tau_q^0 \geq 0$ and (IR) $\kappa_q^1 + \tau_q^1 \geq \kappa_q^0 + \tau_q^0$ and $\tau_q^0 \geq \tau_q^1$. Notice that for any stochastic direct mechanisms (over $\left\{\alpha_q, p_q, \left\{\left(\kappa_q^0, \tau_q^0\right), \left(\kappa_q^1, \tau_q^1\right)\right\}\right\}$), there exists an outcome-equivalent deterministic direct mechanism. So without loss, we limit our attention to the deterministic mechanisms.

Under this direct mechanism $\Phi = \left\{\alpha_q, p_q, \left\{\left(\kappa_q^0, \tau_q^0\right), \left(\kappa_q^1, \tau_q^1\right)\right\}\right\}$, the payoff of the buyer with signal q and the seller's profit from her are, respectively,

$$\begin{aligned}
u(\Phi) &\equiv \alpha_q \left(q\left(\kappa_q^1 + \tau_q^1\right) + (1-q)\,\tau_q^0\right) - p_q, \\
v(q|\Phi) &\equiv p_q - \alpha_q \left[q\left(\left(1-\kappa_q^1\right)c + \tau_q^1\right) - (1-q)\left(\left(1-\kappa_q^0\right)c + \tau_q^0\right)\right].
\end{aligned}$$

Notice that without loss, we can assume that $\kappa_q^1 = 1$, $\tau_q^1 = 0$, $\tau_q^0 = 1$. To see this, consider $\tilde{\Phi} \equiv \left\{\alpha_q, \tilde{p}_q, \left\{\left(\tilde{\kappa}_q^0, \tilde{\tau}_q^0\right), \left(\tilde{\kappa}_q^1, \tilde{\tau}_q^1\right)\right\}\right\}$ such that $\left(\tilde{\kappa}_q^1, \tilde{\tau}_q^1\right) \neq (1,0)$; and $\Phi \equiv \left\{\alpha_q, p_q, \left\{\left(\kappa_q^0, \tau_q^0\right), \left(\kappa_q^1, \tau_q^1\right)\right\}\right\}$ such that (i) $\left(\kappa_q^1, \tau_q^1\right) = (1,0)$; (ii) $\left(\kappa_q^0, \tau_q^0\right) = \left(\tilde{\kappa}_q^0, 1\right)$; (iii) $p_q = \tilde{p}_q - \alpha_q q\left(\tilde{\kappa}_q^1 + \tilde{\tau}_q^1 - 1\right) + \alpha_q(1-q)\left(\tilde{\tau}_q^0 - 1\right)$. Then,

$$\begin{aligned}
u\left(\tilde{\Phi}\right) &= \alpha_q\left(q\left(\tilde{\kappa}_q^1 + \tilde{\tau}_q^1\right) + (1-q)\,\tilde{\tau}_q^0\right) - \tilde{p}_q \\
&= \alpha_q\left(q + (1-q)\right) - p_q = u(\Phi)
\end{aligned}$$

and

$$\begin{aligned}
v\left(q|\tilde{\Phi}\right) &= \tilde{p}_q - \alpha_q\left[q\left(\left(1-\tilde{\kappa}_q^1\right)c + \tilde{\tau}_q^1\right) - (1-q)\left(\left(1-\tilde{\kappa}_q^0\right)c + \tilde{\tau}_q^0\right)\right] \\
&= p_q - \alpha_q\left[q\left(\left(1-\tilde{\kappa}_q^1\right)(c+1)\right) + (1-q)\left(\left(1-\tilde{\kappa}_q^0\right)c + 1\right)\right] \\
&\leq p_q - \alpha_q(1-q)\left(\left(1-\tilde{\kappa}_q^0\right)c + 1\right) = v(q|\Phi).
\end{aligned}$$

If we denote $\alpha_0(q) = \alpha_q(1-q)\,\tilde{\kappa}_q^0$, and $\alpha_r(q) = \alpha_q\left(q + (1-q)\left(1-\tilde{\kappa}_q^0\right)\right)$ so that $\alpha_0(q) + \alpha_r(q) = \alpha_q$, then we have the required result.

Proof of Lemma 2: We first show that for any direct mechanism with refunds $\tilde{M} \in \mathcal{M}$, there exists another direct mechanism with refunds $M \in \mathcal{M}$ with the following properties: (i) $R\left(q|\tilde{M}\right) \leq R(q|M)$ for all q; (ii) $\alpha_0(q)$ is increasing, and (iii) $R(q|M) = R(q;\alpha_0)$.

By the standard argument, the incentive compatibility condition is equivalent to that

$$\alpha_0(q) \text{ is increasing in } q \text{ and } U(q;q|M) = \int_0^q \alpha_0(\tilde{q})\,d\tilde{q}.$$

Since $\int_0^q \alpha_0(\tilde{q})\, d\tilde{q} = q\alpha_0(q) - (p(q) - \alpha_r(q))$,

$$\begin{aligned} R(q|M) &= p(q) - a_r(q)\frac{1-q}{1-\gamma} \\ &= q\alpha_0(q) + \alpha_r(q) - \int_0^q \alpha_0(\tilde{q})\, d\tilde{q} - a_r(q)\frac{1-q}{1-\gamma} \\ &= q\alpha_0(q) + \frac{q-\gamma}{1-\gamma}\alpha_r(q) - \int_0^q \alpha_0(\tilde{q})\, d\tilde{q}. \end{aligned}$$

Observe that $R(0) \leq 0$. Therefore, if $q = 0$, then the seller chooses $\alpha_0(q) = \alpha_r(q) = 0$. Next, if $q \in (0, \gamma)$, then since $\frac{q-\gamma}{1-\gamma} < 0$, $\alpha_r(q) = 0$. If $q = \gamma$, $R(\gamma|M)$ does not depend on $\alpha_r(q)$. Therefore, $\alpha_r(q) = 1 - \alpha_0(q)$. Similarly, if $q \in (\gamma, 1]$, then since $\frac{q-\gamma}{1-\gamma} > 0$, $\alpha_r(q) = 1 - \alpha_0(q)$. We thus can conclude that

$$\max_{M \in \mathcal{M}} \min_{F \in \mathcal{F}} V(M; F) = \max_{\alpha_0 \in \mathcal{A}} \underline{R}(\pi; \alpha_0).$$

Take $\alpha_0^*(q) \in \arg\max_{\alpha_0 \in \mathcal{A}} \underline{R}(\pi; \alpha_0)$, M^*, and the the lower linear envelope function of $\underline{R}(q; \alpha_0^*)$ at $q = \pi$ that has the steepest slope denoted by $L(q) \equiv \chi(q - V)$. Observe that $\operatorname{supp}\{F^*\} \subset Q \equiv \{q : R(q; \alpha_0^*) = L(q)\}$ for any $F^* \in \arg\max_{F \in \mathcal{F}} \min_{F \in \mathcal{F}} V(M^*; F)$, and hence $\alpha_0(q)$ is strictly increasing at q only if $q \in \operatorname{supp}\{F^*\}$. Therefore implies $R(q; \alpha_0^*) = \max\{0, L(q)\}$ for all $q \in [0, 1]$.

By (8), $R(q; \alpha_0^*) = \chi(q - V)$ on $q \in [V, 1]$ implies that for some k,

$$\alpha_0^*(q) = \begin{cases} \chi \ln \frac{q}{V} & q \in [V, \gamma] \\ 1 - (1-\gamma)\chi + k \times (-(1-q))^{-\frac{1}{\gamma}} & q \in [\gamma, 1] \end{cases}.$$

However, since $(-(1-q))^{-\frac{1}{\gamma}}$ is not real for any $q \in [\gamma, 1]$, we have $k = 0$. Thus $\chi \ln \frac{\gamma}{V} = 1 - (1-\gamma)\chi$, or $\chi = \frac{1}{1-\gamma-\ln\frac{V}{\gamma}}$, equivalently. We thus have $\max_{\alpha_0 \in \mathcal{A}} \underline{R}(\pi; \alpha_0) = \max_{V \in [0,\gamma]} \frac{\pi - V}{1-\gamma-\ln\frac{V}{\gamma}}$.

B Proofs for Section 4 (Feasible Outcomes)

Proof of Theorem 7: We start with the case where $\gamma \geq 1 - \sqrt{1-\pi}$. Consider $F_{V_1^*}(q)$, where F_V is defined in (5) and V_1^* in Theorem 2. $F_{V_1^*}(q)$ is illustrated in Figure 5(b), and $\sup_{(p,r)} V\left((p,r)\,|F_{V_1^*}\right) = V_1^*$. Notice that $V\left((V_1^*, 0)\,|F_{V_1^*}\right) = V_1^*$. Furthermore, since $\operatorname{supp}\left\{F_{V_1^*}(q)\right\} = [V_1^*, \gamma] \cup \{1\}$, the buyer's payoff when $(p, r) = (V_1^*, 0)$ is $\pi - V_1^*$. We thus have $U_1^* = \pi - V_1^*$.

Next, we consider the case where $\gamma < 1 - \sqrt{1-\pi}$. Define

$$F^*(q) \equiv \begin{cases} 0 & q \in [0, V_1^*) \\ 1-\gamma & q = [V_1^*, 1) \\ 1 & q = 1 \end{cases}.$$

The seller's profit from offering $(V_1^*, 0)$ is $V_1^* = \frac{\pi-\gamma}{1-\gamma}$. The seller's profit from $(p, r), r > 0$ is bounded from V_1^*. To see this, notice that if $\tilde{q}(p, r) > V_1^*$, then

$$V((p,r)|F^*) = \gamma p \leq \gamma \leq \frac{\pi - \gamma}{1-\gamma} = V_1^*,$$

and if $\tilde{q}(p, r) \leq V_1^*$, then since $\tilde{q}(p, r) = \frac{p-r}{1-r}$,

$$\begin{aligned} V((p,r)|F^*) &= p - (1-\gamma)(1-\tilde{q}(p,r))\left(\frac{p-\tilde{q}(p,r)}{1-\tilde{q}(p,r)} + \frac{\gamma}{1-\gamma}\right) \text{ (increasing in } \tilde{q}(p,r)\text{)} \\ &\leq p - (1-\gamma)(1-V_1^*)\left(\frac{p-V_1^*}{1-V_1^*} + \frac{\gamma}{1-\gamma}\right) \text{ (increasing in } p\text{)} \\ &\leq 1 - (1-\gamma)(1-V_1^*)\left(\frac{1}{1-\gamma}\right) = V_1^*. \end{aligned}$$

Therefore, $(V_1^*, 0) \in \arg\max_{(p,r)} V((p,r)|F^*)$. Since $\text{supp}\{F^*(q)\} = \{V_1^*, 1\}$, the buyer's payoff when $(V_1^*, 0)$ is $\pi - V_1^*$.

Consider the following set of offer-dependent signal distributions $\left\{F^*_{(p,r)}\right\}$. If $(p, r) = (V_3^*, 0)$, then $F^*_{(p,r)}$ has an atom of size one at $q = \pi$, i.e., an uninformative signal; and if $(p, r) \neq (V_3^*, 0)$, then $F^*_{(p,r)} \in \arg\min_{F_{(p,r)} \in \mathcal{F}} V\left((p,r)|F_{(p,r)}\right)$. Then $V\left((V_3^*, 0)|F^*_{(V_3^*,0)}\right) = V_3^*$ and $V\left((p,r)|F^*_{(p,r)}\right) \leq V_3^*$ for all $(p, r) \neq (V_3^*, 0)$ by Theorem 1. Furthermore, under $F_{(V_3^*,0)}$, the buyer's posterior is $q = \pi > V_3^*$ the buyer's payoff is $\pi - V_3^*$.