Liberal parentalism
online appendix

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1 Proof of the Theorem

For simplicity, we first present the proof for the case where the choice set of each self is one-dimensional, $X_i \subset \mathbb{R}^{k_i}$ with $k_i = 1$, $i = 1, ..., n$, and then elaborate on how to read the same proof for the case of any finite $k_i \geq 1$.

If $\hat{x} = (\hat{x}_1, ..., \hat{x}_n)$ is an interior subgame-perfect equilibrium path, then at $\hat{x}$ the following n first-order conditions obtain:

$$\begin{aligned} \frac{\partial u_n}{\partial x_n} &= 0 \\ \frac{\partial u_{n-1}}{\partial x_{n-1}} + \frac{\partial u_{n-1}}{\partial x_n}\frac{\partial b_n}{\partial x_{n-1}} &= 0 \\ \frac{\partial u_{n-2}}{\partial x_{n-2}} + \frac{\partial u_{n-2}}{\partial x_{n-1}}\frac{\partial b_{n-1}}{\partial x_{n-2}} + \frac{\partial u_{n-2}}{\partial x_n}\left(\frac{\partial b_n}{\partial x_{n-2}} + \frac{\partial b_n}{\partial x_{n-1}}\frac{\partial b_{n-1}}{\partial x_{n-2}}\right) &= 0 \\ &\vdots \end{aligned}$$

To simplify notation in the sequel, define the matrix of direct and indirect effects

$$h = \begin{pmatrix} 1 & & \cdots & & & \\ 0 & 1 & & & & \vdots \\ \vdots & & \ddots & & & \\ 0 & & 0 & 1 & \frac{\partial b_{n-1}}{\partial x_{n-2}} & \left(\frac{\partial b_n}{\partial x_{n-2}} + \frac{\partial b_n}{\partial x_{n-1}}\frac{\partial b_{n-1}}{\partial x_{n-2}}\right) \\ 0 & \cdots & & 0 & 1 & \frac{\partial b_n}{\partial x_{n-1}} \\ 0 & & \cdots & & 0 & 1 \end{pmatrix}$$

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so that the above system becomes at $\hat{x}$

$$F_n(x;u) \equiv \frac{\partial u_n}{\partial x_n} = 0 \tag{BI_{n}}$$

$$F_{n-1}(x;u) \equiv \frac{\partial u_{n-1}}{\partial x_{n-1}} + \frac{\partial u_{n-1}}{\partial x_n} h_{n-1,n} = 0 \tag{$BI_{\text{n-1}}$}$$

$$F_{n-2}(x;u) \equiv \frac{\partial u_{n-2}}{\partial x_{n-2}} + \frac{\partial u_{n-2}}{\partial x_{n-1}} h_{n-2,n-1} + \frac{\partial u_{n-2}}{\partial x_n} h_{n-2,n} = 0 \tag{$BI_{\text{n-2}}$}$$

$$\vdots$$

If $\hat{x}$ is also Pareto optimal, then there exists $\hat{\theta} = \left(\hat{\theta}_1, ..., \hat{\theta}_n\right)$ such that at $\left(\hat{x}, \hat{\theta}\right)$

$$G_1(x, \theta; u) \equiv \sum_{i=1}^{n} \theta_i \frac{\partial u_i}{\partial x_1} = 0 \tag{PO_1}$$

$$\vdots$$

$$G_n(x, \theta; u) \equiv \sum_{i=1}^{n} \theta_i \frac{\partial u_i}{\partial x_n} = 0 \tag{PO_n}$$

and

$$G_{n+1}(\theta) = \sum_{i=1}^{n} \theta_i^2 - 1 = 0 \tag{$PO_{\text{n+1}}$}$$

Indeed, if there doesn't exist $\left(\hat{\theta}_1, ..., \hat{\theta}_n\right)$ satisfying (PO_{n+1}) such that $(PO_1), ..., (PO_n)$ hold, the Jacobian Du has full row rank at $\hat{x}$, and one can find a direction $\begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix}$ such that $Du \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_n \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, so that perturbing $\hat{x}_i$ in the direction δ_i for $i = 1, ..., n$ improves all of $u_1, ...u_n$, thus constituting a Pareto improvement.

We will distinguish two families of cases: (i) when there is a minimal index $a \geq 1$ and a maximal index $b > a$ such that $\hat{\theta}_a \neq 0$ and $\hat{\theta}_b \neq 0$; (ii) when there is a unique index $1 \leq c \leq n$ for which $\hat{\theta}_c \neq 0$ while $\hat{\theta}_j = 0 \; \forall j \neq c$.

For every case of the first family, we will show that for every perturbation direction $(\pi_1, ..., \pi_n, p_1, ..., p_n, p_{n+1})$ for $(F_1, ..., F_n, G_1, ...G_n, G_{n+1})$ there exists

$$Y = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n}$$

such that with the path of utility tuples

$$u^t = \begin{pmatrix} u_1^t \\ \vdots \\ u_n^t \end{pmatrix} \equiv u + tY \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

defined for $t \in [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$, and with

$$G_{n+1}^{t} = G_{n+1} + t p_{n+1}$$

we get at $t = 0$

$$\frac{\partial F_1\left(\hat{x}; u^0\right)}{\partial t} = \pi_1$$
$$\vdots$$
$$\frac{\partial F_n\left(\hat{x}; u^0\right)}{\partial t} = \pi_n$$
$$\frac{\partial G_1\left(\hat{x}, \hat{\theta}; u^0\right)}{\partial t} = p_1$$
$$\vdots$$
$$\frac{\partial G_n\left(\hat{x}, \hat{\theta}; u^0\right)}{\partial t} = p_n$$
$$\frac{\partial G_{n+1}^{0}\left(\hat{\theta}\right)}{\partial t} = p_{n+1}$$

The last equation is immediate by the definition of G_{n+1}^{t}. For the first $2n$ equations to hold, choose Y to be

$$Y = \begin{pmatrix} \pi_1 & 0 & \cdots & 0 & & & \cdots & 0 \\ 0 & \ddots & & 0 & & & & 0 \\ \vdots & & \ddots & \vdots & & & & \vdots \\ \frac{p_1 - \hat{\theta}_1 \pi_1}{\hat{\theta}_a} & \cdots & \cdots & \pi_a - \sum\limits_{i=a+1}^{n} \frac{p_i - \hat{\theta}_i \pi_i}{\hat{\theta}_a} h_{a,i} & \cdots & \frac{p_b - \hat{\theta}_b \pi_b}{\hat{\theta}_a} & \cdots & \frac{p_n - \hat{\theta}_n \pi_n}{\hat{\theta}_a} \\ \vdots & & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \cdots & \frac{p_a - \hat{\theta}_a\left(\pi_a - \sum\limits_{i=a+1}^{n} \frac{p_i - \hat{\theta}_i \pi_i}{\hat{\theta}_a} h_{a,i}\right)}{\hat{\theta}_b} & 0 & \pi_b & & 0 \\ \vdots & & & \vdots & & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & & 0 & & \pi_n \end{pmatrix}$$

In words,

(1) for $i \neq a$ set $y_{ii} = \pi_i$, thus perturbing F_i at $t = 0$ by π_i for $i \neq a$;

(2) for $i \neq a$, (1) perturbs G_i at $t = 0$ by $\hat{\theta}_i \pi_i$, so set $y_{ai} = \frac{p_i - \hat{\theta}_i \pi_i}{\hat{\theta}_a}$ in order to eventually end up perturbing G_i at $t = 0$ by p_i;

(3) Together over all $i \neq a$, (2) upsets F_a at $t = 0$ by $\sum_{i>a} \frac{p_i - \hat{\theta}_i \pi_i}{\hat{\theta}_a} h_{a,i}$ (only, $h_{a,i}$ itself was not perturbed, because it depends on partial cross-derivatives of u_r w.r.t. x_s for $r > s > a$, and these were not perturbed); so in order to eventually perturb F_i at $t = 0$ by π_a, set $y_{aa} = \pi_a - \sum_{i>a} \frac{p_i - \hat{\theta}_i \pi_i}{\hat{\theta}_a} h_{a,i}$;

(4) finally, (3) upsets G_a at $t=0$ by $\hat{\theta}_a\left(\pi_a-\sum_{i>a}\frac{p_i-\hat{\theta}_i\pi_i}{\hat{\theta}_a}h_{a,i}\right)$, so in order to eventually perturb G_a at $t=0$ by p_a, set

$$y_{ba}=\frac{p_a-\hat{\theta}_a\left(\pi_a-\sum_{i>a}\frac{p_i-\hat{\theta}_i\pi_i}{\hat{\theta}_a}h_{a,i}\right)}{\hat{\theta}_b}$$

(This does not upset F_b, which only depends on partial derivatives of u_{ib} w.r.t. x_s for $s\geq b$, whereas $a<b$.)

(5) All the remaining entries of Y are zero.

Therefore, by the transversality theorem there exists an open and dense subset $\mathcal{U}_{a,b}\subseteq\mathcal{U}$ of utility profiles u for which at any solution $\left(\hat{x},\hat{\theta}\right)$ of $(BI_1),...,(BI_n),(PO_1),...,(PO_n),(PO_{n+1})$ for which $\arg\min_i\left(\hat{\theta}_i\neq 0\right)=a$ and $\arg\max_i\left(\hat{\theta}_i\neq 0\right)=b$, the $(2n+1)\times 2n$ matrix

$$\begin{pmatrix}
\frac{\partial F_1(\hat{x};u)}{\partial x_1} & \cdots & \frac{\partial F_1(\hat{x};u)}{\partial x_n} & \frac{\partial F_1(\hat{x};u)}{\partial \theta_1} & \cdots & \frac{\partial F_1(\hat{x};u)}{\partial \theta_n}\\
\vdots & & \vdots & \vdots & & \vdots\\
\frac{\partial F_n(\hat{x};u)}{\partial x_1} & \cdots & \frac{\partial F_n(\hat{x};u)}{\partial x_n} & \frac{\partial F_n(\hat{x};u)}{\partial \theta_1} & \cdots & \frac{\partial F_n(\hat{x};u)}{\partial \theta_n}\\
\frac{\partial G_1\left(\hat{x},\hat{\theta};u\right)}{\partial x_1} & \cdots & \frac{\partial G_1\left(\hat{x},\hat{\theta};u\right)}{\partial x_n} & \frac{\partial G_1\left(\hat{x},\hat{\theta};u\right)}{\partial \theta_1} & \cdots & \frac{\partial G_1\left(\hat{x},\hat{\theta};u\right)}{\partial \theta_n}\\
\vdots & & \vdots & \vdots & & \vdots\\
\frac{\partial G_n\left(\hat{x},\hat{\theta};u\right)}{\partial x_1} & \cdots & \frac{\partial G_n\left(\hat{x},\hat{\theta};u\right)}{\partial x_n} & \frac{\partial G_n\left(\hat{x},\hat{\theta};u\right)}{\partial \theta_1} & \cdots & \frac{\partial G_n\left(\hat{x},\hat{\theta};u\right)}{\partial \theta_n}\\
\frac{\partial G_{n+1}\left(\hat{\theta}\right)}{\partial x_1} & \cdots & \frac{\partial G_{n+1}\left(\hat{\theta}\right)}{\partial x_n} & \frac{\partial G_{n+1}\left(\hat{\theta}\right)}{\partial \theta_1} & \cdots & \frac{\partial G_{n+1}\left(\hat{\theta}\right)}{\partial \theta_n}
\end{pmatrix}\qquad(\bigstar)$$

has full row rank.[1]

As for cases of the second family, where at a solution $\left(\hat{x},\hat{\theta}\right)$, $\hat{\theta}_c$ is the only non-zero entry in $\hat{\theta}$, $(PO_1),...,(PO_n)$ amount to $\nabla u_c=\left(\frac{\partial u_c}{\partial x_1},...,\frac{\partial u_c}{\partial x_n}\right)=0$, which turns (BI_c) to an identity, $0=0$. In such a case renounce therefore the expression F_c, but also read the remaining $2n$ expressions $(F_j)_{j\neq c},(G_i)_{i=1}^{n+1}$ as functions of the $n+1$ variables $x_1,...,x_n,\theta_c$ only, i.e. with $\theta_j\equiv 0$ for $j\neq c$. Then for every corresponding rates of perturbation $\left((\pi_j)_{j\neq c},(p_i)_{i=1}^{n+1}\right)$, if we choose

$$Y=\begin{pmatrix}
\pi_1 & \cdots & 0 & \cdots & 0 & \cdots & 0\\
\vdots & \ddots & \vdots & & \vdots & & \vdots\\
0 & \cdots & \pi_j & \cdots & 0 & \cdots & 0\\
\vdots & & \vdots & \ddots & \vdots & & \vdots\\
\frac{p_1}{\hat{\theta}_c} & \cdots & \frac{p_j}{\hat{\theta}_c} & \cdots & \frac{p_c}{\hat{\theta}_c} & & \frac{p_n}{\hat{\theta}_c}\\
\vdots & & \vdots & & \vdots & \ddots & \vdots\\
0 & \cdots & 0 & \cdots & 0 & \cdots & \pi_n
\end{pmatrix}$$

[1] Hirsch (1976), p. 74, theorem 2.1 (b), with $M=\mathbb{R}^{2n}$, $N=\mathbb{R}^{2n+1}$, $L=X\times[0,1]^n$ and $A=\{0\}$. For this case the theorem then states that $\pitchfork^2_{X\times[0,1]^n}\left(\mathbb{R}^{2n},\mathbb{R}^{2n+1};\{0\}\right)$ is open and dense within $C^2\left(\mathbb{R}^{2n},\mathbb{R}^{2n+1}\right)$.

then at $t=0$ we have $\frac{\partial F_j\left(\hat{x};u^0\right)}{\partial t}=\pi_j$ for $j\neq c$, $\frac{\partial G_i\left(\hat{x},\hat{\theta}_c;u^0\right)}{\partial t}=p_i$ for $i=1,...,n$, and $\frac{\partial G^0_{n+1}\left(\hat{\theta}_c\right)}{\partial t}=p_{n+1}$.

Therefore, by the transversality theorem there exists an open and dense subset $\mathcal{U}_c\subseteq\mathcal{U}$ of utility profiles u for which at any solution $\left(\hat{x},\hat{\theta}\right)$ of $(BI_1),...,(BI_n),(PO_1),...,(PO_n),(PO_{n+1})$ for which $\hat{\theta}_c\neq 0$ and $\hat{\theta}_j=0$ for $j\neq c$, the $2n\times(n+1)$ matrix

$$\begin{pmatrix} \frac{\partial F_1(\hat{x};u)}{\partial x_1} & \cdots & \frac{\partial F_1(\hat{x};u)}{\partial x_n} & \frac{\partial F_1(\hat{x};u)}{\partial \theta_c} \\ \vdots & & \vdots & \vdots \\ \frac{\partial F_{c-1}(\hat{x};u)}{\partial x_1} & \cdots & \frac{\partial F_{c-1}(\hat{x};u)}{\partial x_n} & \frac{\partial F_{c-1}(\hat{x};u)}{\partial \theta_c} \\ \frac{\partial F_{c+1}(\hat{x};u)}{\partial x_1} & \cdots & \frac{\partial F_{c+}(\hat{x};u)}{\partial x_n} & \frac{\partial F_{c+}(\hat{x};u)}{\partial \theta_c} \\ \vdots & & \vdots & \vdots \\ \frac{\partial F_n(\hat{x};u)}{\partial x_1} & \cdots & \frac{\partial F_n(\hat{x};u)}{\partial x_n} & \frac{\partial F_n(\hat{x};u)}{\partial \theta_c} \\ \frac{\partial G_1\left(\hat{x},\hat{\theta}_c;u\right)}{\partial x_1} & \cdots & \frac{\partial G_1\left(\hat{x},\hat{\theta}_c;u\right)}{\partial x_n} & \frac{\partial G_1\left(\hat{x},\hat{\theta}_c;u\right)}{\partial \theta_c} \\ \vdots & & \vdots & \vdots \\ \frac{\partial G_n\left(\hat{x},\hat{\theta}_c;u\right)}{\partial x_1} & \cdots & \frac{\partial G_n\left(\hat{x},\hat{\theta}_c;u\right)}{\partial x_n} & \frac{\partial G_n\left(\hat{x},\hat{\theta}_c;u\right)}{\partial \theta_c} \\ \frac{\partial G_{n+1}\left(\hat{\theta}_c\right)}{\partial x_1} & \cdots & \frac{\partial G_{n+1}\left(\hat{\theta}_c\right)}{\partial x_n} & \frac{\partial G_{n+1}\left(\hat{\theta}_c\right)}{\partial \theta_c} \end{pmatrix} \qquad (\bigstar\bigstar)$$

has full row rank.

The intersection of the finitely many open and dense sets

$$\mathcal{U}_0=\left(\bigcap_{1\leq a<b\leq n}\mathcal{U}_{a,b}\right)\cap\left(\bigcap_{c=1}^{n}\mathcal{U}_c\right)$$

is open and dense. For every utility tuple $u\in\mathcal{U}_0$, at every solution $\left(\hat{x},\hat{\theta}\right)$ of $(BI_1),...,(BI_n)$, $(PO_1),...,(PO_n),(PO_{n+1})$, either $(\bigstar)$ or $(\bigstar\bigstar)$ has full row rank, which is impossible because both $(\bigstar)$ and $(\bigstar\bigstar)$ have more rows than columns. Hence for generic u (namely for $u\in\mathcal{U}_0$), no interior backward induction SPE path $\hat{x}$ is Pareto optimal.

When choice variables are multi-dimensional, $x_i=(x_{i,1},...,x_{i,k_i})$, $i=1,...,n$, the same proof applies verbatim with the following caveats: $\frac{\partial u_i}{\partial x_j}=\left(\frac{\partial u_i}{\partial x_{j,1}},\cdots\frac{\partial u_i}{\partial x_{j,k_j}}\right)$, $\frac{\partial b_i}{\partial x_j}=\begin{pmatrix}\frac{\partial b_{i,1}}{\partial x_{j,1}} & \cdots & \frac{\partial b_{i,1}}{\partial x_{j,k_j}}\\ \vdots & & \vdots \\ \frac{\partial b_{i,k_i}}{\partial x_{j,1}} & \cdots & \frac{\partial b_{i,k_i}}{\partial x_{j,k_j}}\end{pmatrix}$, $F_i=(F_{i,1},...,F_{i,k_i})$, $\frac{\partial F_i}{\partial x_j}=\begin{pmatrix}\frac{\partial F_{i,1}}{\partial x_{j,1}} & \cdots & \frac{\partial F_{i,1}}{\partial x_{j,k_j}}\\ \vdots & & \vdots \\ \frac{\partial F_{i,k_i}}{\partial x_{j,1}} & \cdots & \frac{\partial F_{i,k_i}}{\partial x_{j,k_j}}\end{pmatrix}$, $\pi_i=(\pi_{i,1},...,\pi_{i,k_i})$, $G_i=(G_{i,1},...,G_{i,k_i})$, $\frac{\partial G_i}{\partial x_j}=\begin{pmatrix}\frac{\partial G_{i,1}}{\partial x_{j,1}} & \cdots & \frac{\partial G_{i,1}}{\partial x_{j,k_j}}\\ \vdots & & \vdots \\ \frac{\partial G_{i,k_i}}{\partial x_{j,1}} & \cdots & \frac{\partial G_{i,k_i}}{\partial x_{j,k_j}}\end{pmatrix}$, $p_i=(p_{i,1},...,p_{i,k_i})$, $\delta_i=(\delta_{i,1},...,\delta_{i,k_i})$; θ_i remains a scalar. ■