Aggressive Boards and CEO Turnover

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Abstract

This study investigates a communication game between a CEO and a board of directors where the CEO's career concerns can potentially impede value-increasing informative communication. By adopting a policy of aggressive boards (excessive replacement), shareholders can facilitate communication between the CEO and the board. The results are in contrast to the multitude of models which find passive or management-friendly boards to be optimal, and helps to explain empirical results concerning CEO turnover. Additionally, we find that shareholders prefer the board to be more aggressive when the board's advisory capacity is more salient, or when the CEO's ability is difficult to assess.

1 Introduction

A natural tension arises between the CEO of a firm and its board of directors, insofar as the board must sometimes take disciplinary measures on the top executives while simultaneously helping through guidance. One of the board's primary responsibilities is to decide whether to replace or retain the CEO (Lorsch and MacIver (1989)). The board also serves to provide the CEO with guidance and advice concerning the firm's direction, thus benefiting the CEO and shareholders. As Mace et al. (1971) notes, "directors serve as a source of advice and counsel, serve as some sort of discipline, and act in crisis situations" (p. 178). Survey evidence also documents that board members overwhelmingly believe that they help shape the firm's strategic direction (Demb and Neubauer (1992)). However, the CEO and top executives control the non-public information that the board receives.[1] Consequently, the

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[1] This has also been noted by Song and Thakor (2006) and Adams and Ferreira (2007), and has been referenced in the news: "[directors] depend largely on the chief executive and the company's management for

CEO wishes, and is often able, to conceal negative information from the board. The allure to manipulate information places the CEO in an unfortunate predicament: she stands to benefit from the board's guidance and expertise, but in doing so she must communicate potentially unfavorable information about the firm's current operations, which consequently lowers the board's assessment of her ability. Indeed, it has been a significant concern among U.S. public firms that CEOs often fail to effectively communicate with boards by concealing negative information from board members, as exemplified by the infamous cases of Enron and Worldcom.[2]

To further illustrate the dilemma an incumbent manager faces, consider a CEO who observes preliminary information regarding a project that she has been tackling (such as the development of a new product). The preliminary information is negative and hence the CEO is confronted with a problem regarding the best path forward for the project. The CEO can honestly reveal the problem to the board, and in turn she receives the board's expert advice concerning the most viable solution. This allows the CEO to take the best action going forward. However, by communicating honestly the board infers that she is of a low ability, considering that her project was not successful, and this may affect the board's decision to replace the CEO. Alternatively, she can overstate the performance of her project thus far (such as conveying a milder problem), but then the solution offered by the board will not be helpful for her. Conversely, the manager would have no such inhibitions in truthfully communicating good news, and receiving advice on the best action to take following a more successful project (e.g., increasing investment). Overall, the board's guidance is effective as long as the CEO honestly communicates with the board regarding the current status of the firm, however the CEO's reputational concerns may compel her to conceal or misrepresent negative information.

We investigate this interdependency between the advisory and disciplinary roles of the board in a communication game where the CEO has career concerns. The setting is one where the CEO observes private information regarding her ability. She decides whether to truthfully communicate or to misreport her private information to the board, after which the board provides her with information concerning the appropriate direction or solution. Although informative communication between the CEO and the board is value-increasing for the firm, the CEO often has an incentive to overstate her ability. The board then observes

information" (*The Economist*, March 31, 2001). Moreover, as Jensen (1993) notes: "The CEO most always determines the agenda and the information given to the board" (p. 864).

[2]Another example includes the CEO of Kmart misleading the board of directors regarding supplier payments in 2001, see "Former CEO misled board, Kmart says," *Bloomberg News*, February 25, 2003. Larcker and Tayan (2016) discuss cases of CEOs lying to board members about personal information. For empirical evidence on CEO turnover and accounting misreporting, see Hennes et al. (2007).

the firm's output (i.e., a performance measure such as earnings or returns) and decides whether to replace or retain the CEO. This simple setting captures the notion that advice is beneficial for all parties, but the information conveyed may be personally detrimental to the CEO, thus potentially impeding informative communication. By examining this interplay between guidance and replacement, we determine the shareholders' optimal board policy.

The main result of this paper is that *shareholders often prefer the board to be aggressive.* By an aggressive board, we mean that the board prefers to replace the CEO even when the type of the incumbent CEO is above the expected value of the replacement (i.e., excessive replacement). This result is in contrast to the multitude of models which find that management-friendly, or passive, boards are optimal, such as Almazan and Suarez (2003), Adams and Ferreira (2007), Laux (2008), Casamatta and Guembel (2010), Inderst and Mueller (2010), and Dow (2013).[3] In short, an aggressive board often reduces miscommunication between the CEO and the board by adopting an excessively strict performance criteria for the retention of the CEO. The high retention standard discourages a relatively unproductive CEO from manipulating her private information. Shareholders thus benefit from excessive replacement of the CEO as this leads to more informative communication. This is the *disciplining effect* of aggressiveness.

Aggressive boards, however, may also hurt firm value by unnecessarily expelling talented managers. To retain her position, the CEO must satisfy two standards set by the board: through her report to the board and through her output (observed performance). An aggressive board requires excessively high standards for both the report and output, which leads to two kinds of inefficient distortions. First, the standard for the performance measure—a random variable correlated with the ability of the CEO—is so high that the board may erroneously replace a CEO of even the highest ability. Second, by setting a high standard in the report, the board often inefficiently disqualifies a moderately talented CEO that shareholders would rather prefer to retain. [4] These two consequences combined are correspondingly referred to as the *distortion effect* from an aggressive board.

Two additional notable features emerge in the analysis of the unique equilibrium. As mentioned above, the board employs a two-prong replacement strategy, first in the communication and then in the observed output. If the manager's initial report is sufficiently high, then in the second step there exists a unique cutoff level for replacement such that observed output below this threshold results in removal of the CEO. Interestingly, we find that the replacement threshold is independent of the CEO's initial report. As the second feature, the

[3] Hermalin and Weisbach (1998) and Warther (1998) find passive boards as the equilibrium outcome arising from CEO influence over the board.

[4] In our model, the CEO's communication to the board can be thought of as "soft" information, whereas the output measure, such as net income, can be interpreted as "hard" information.

manager's optimal reporting strategy is *non-monotone* in her ability. CEOs of high ability have no incentive to misreport their types and thus truthfully communicate with the board, as expected. In contrast, a CEO of intermediate ability inflates her report (sometimes quite heavily) to conceal information that would otherwise result in her dismissal. Finally, low-ability CEOs report truthfully *even though this leads to certain replacement.* Due to their diminished likelihood of meeting the cutoff—a primary consequence of aggressive boards—these types are truthful so as to maximize value during their current tenure with the firm.

We also provide conditions under which shareholders prefer the board to be aggressive in their replacement of the CEO. We find that shareholders prefer an aggressive board when there is a high mass of low-ability CEOs, or when the manager's private benefit of control is low enough. When there is a sufficiently high concentration of low manager types, shareholders benefit through an aggressive board as it significantly improves truthful communication, and thus overall improves firm value. Likewise, the board does not have to set a high standard when the CEO's thirst for continuation is moderate. In this case, shareholders tend to prefer aggressive boards that frequently induce truthful reporting without severely distorting the retention policy.

The model helps to explain the following empirical results. Taylor (2010) finds empirical support for aggressive boards among large firms. Specifically, he finds that "boards in large firms actually fire *more* CEOs than is optimal for shareholders ex post" (p. 2053, emphasis in original) and that "boards derive a personal net benefit from firing CEOs" (p. 2077). Hence, for large firms Taylor (2010) finds a *negative* entrenchment level, or excessive replacement, and attributes this to boards of large firms being aggressive in their removal of the CEO. Although Taylor (2010) offers a number of possible explanations for the board's net benefit from CEO replacement, our results formally capture this notion and we show how excessive replacement emerges endogenously as the shareholders' optimal policy in equilibrium. Our results are also consistent with the findings of Cornelli et al. (2013), who show that "soft" (non-verifiable) information regarding the CEO's ability is frequently used in replacement decisions by the board. The results show that the board replaces the CEO when the soft information (i.e., the CEO's report to the board) is sufficiently unfavorable. Likewise, the board's optimal strategy of employing a cutoff level for replacement based on the observed performance measure (when the soft information is favorable) is consistent with numerous empirical studies which find that CEOs are more likely to be removed following poor performance.[5] Empirical studies have also documented that poor industry performance also increases the likelihood of forced CEO turnover (e.g., Kaplan and Minton (2012), Jenter and

[5]For the relation between CEO turnover and firm performance, see Warner et al. (1988), Kaplan (1994), Denis and Denis (1995), Huson et al. (2001), Huson et al. (2004), and Faleye et al. (2011), among others.

Kanaan (2015)). In the context of the model, poor industry performance may be interpreted as a high density of poorly performing managers. In this case, a high concentration of weak managers leads to aggressive boards, and thus increased CEO turnover in equilibrium.

In terms of empirical implications, the results of the model provide the following novel predictions: (*i*) the board is more aggressive in their replacement of the CEO in industries or firms where the board's advisory role is more salient; (*ii*) there is greater CEO turnover in firms or industries where CEO performance is relatively more difficult to assess; (*iii*) there is comparatively less CEO turnover in firms or industries where the variance of CEO talent is high. Additional predictions emerge concerning the relation between CEO turnover and the CEO's degree of control over the firm in homogeneous and heterogeneous industries. These predictions are more thoroughly discussed in section 5.

The model resembles a signaling structure whereby the CEO, as the sender, submits a message regarding her ability to the board, as the receiver. We therefore need to apply appropriate equilibrium selection. Notably, we obtain a unique equilibrium after the selection even though this setting often violates standard monotonicity assumptions. In particular, since truthful communication has a special meaning in our model, interesting but ill-behaved *productivity reversal* often occurs in the following sense: a high type with a misreport may become less productive than a low type with a truthful report.[6] Due to the absence of natural monotonicity assumptions, we unavoidably need to handle pathological, non-monotonic actions from the receiver (i.e., the board)—an action of the receiver is a function of the output in our setting. As a methodological contribution, we succeed in constructing a method that allows us to evaluate arbitrary functions as if they are regular and well-behaved (see Lemma 12 in the Appendix). The methods developed here may be more broadly applicable in solving signaling or information-transmission models which similarly involve non-monotonic strategies.

The extant theoretical literature investigating equilibrium models of board independence generally conclude that passive boards are better for shareholders. These studies typically consider replacement and advising separately. We first discuss the model in relation to the literature that examines the advising capacity of boards. Adams and Ferreira (2007) consider a model where a privately informed CEO can benefit from communication with the

[6]The problem of productivity reversal always occurs with continuously many types and sometimes even with two types (before we uniquely identify an equilibrium where irregularities are already eliminated). With the productivity reversal, higher outputs do not necessarily increase the likelihood of higher-ability types. Consequently, it becomes intractable to characterize the optimal retention policy for the board. (As we will see later, optimal retention policies are usually a cutoff rule.) See the proofs of Theorem 1 and Lemma 5 for more details. Besides the productivity reversal, it is another anomaly that our model does not satisfy the single-crossing condition between type θ and report $\hat{\theta}$ because, once again, of the special importance of the truthful report.

board. The board, however, can undermine the CEO by committing to a likelihood that the project selection is taken over from the CEO. This leads to less communication between the CEO and the board, inducing shareholders to prefer a less independent, or "friendly," board. The current setting adopts the information transmission assumption in Adams and Ferreira (2007), whereby communication from the CEO is only beneficial when it is truthful. However, we also assume that the information is relevant to the CEO's ability, whereas Adams and Ferreira (2007) assume that the CEO's private information is not payoff-relevant to agents beyond its use in communication. By endogenizing the information communicated into a replacement decision, we obtain results significantly different from those of Adams and Ferreira (2007).

Song and Thakor (2006) examine a model where both the CEO and the board care about their perceived talent. This leads to distorted investment recommendations by the board, as well as less precise information provided to the board by the CEO. While our setting shares some features with Song and Thakor (2006), our primary objective is to examine the interdependency between advising and replacement, and the optimal board policy by shareholders, whereas replacement and turnover are not part of the model in Song and Thakor (2006).

Harris and Raviv (2005, 2008, 2010), Baldenius et al. (2014), and Chakraborty and Yılmaz (2017) consider information transmission between the board and the CEO. Our model adds to this literature by investigating the role of CEO replacement on the CEO's communication with the board.[7]

Several papers considering CEO replacement have found management-friendly boards to be optimal. Almazan and Suarez (2003) investigate a model where shareholders can design a weak or strong board, where the CEO can veto the board's decision of her replacement under a weak board. Almazan and Suarez (2003) find severance pay to be efficient under both board regimes and that weak boards are more desirable for shareholders than strong boards. The present model varies with Almazan and Suarez (2003) in that we assume that the replacement decision stems from the manager's private information, and that this can be learned by the board through both a performance measure and communication.

[7]Baldenius et al. (2014) also consider board monitoring and advising jointly, where shareholders determine the composition of board members. They find that shareholders prefer centralized power by the board when the CEO is known to have a large bias, and prefer to delegate when the CEO's bias is small. Chakraborty and Yılmaz (2017) consider a communication game where both the board and the CEO have private information and where the board's preferences may be aligned with the CEO's. They find that shareholders prefer a manager-aligned board when the manager has decision rights over the project, whereas shareholder-aligned boards are optimal when the board has decision rights. In contrast to these settings, in our model the CEO's decision rights (in the future) depend on the actual information communicated by the CEO to the board.

Laux (2008) and Inderst and Mueller (2010) similarly consider CEO replacement under private information, where the board can induce truth-telling through severance pay. Laux (2008) shows that low type managers report truthfully due to the severance payoff from dismissal, and that less independent boards (analogous to friendly boards in the present paper) are optimal for shareholders. Our setting differs conceptually from severance pay, as the requisite severance level in Laux (2008) does not depend on the CEO's private information, thus allowing the board to offer less severance when the board is friendly. In contrast, the likelihood of replacement, and thus the continuation utility, is affected by the manager's private information in the current setting. This influences our main result that shareholders can maximize truth-telling by adopting a more aggressive board, contrary to the results in Laux (2008). Inderst and Mueller (2010) show that it is more efficient for the board to substitute severance pay with incentive pay; however, as in Laux (2008), the optimal retention policy is friendly (excessive retention) in Inderst and Mueller (2010) since incentive pay is nevertheless costly for the firm. Furthermore, our main focus is on the advising nature of communication and its relation with replacement, whereas advising is absent in Laux (2008) and Inderst and Mueller (2010). Lastly, misreporting does not happen along the equilibrium path in Laux (2008) and Inderst and Mueller (2010), although misreporting demonstrably happens in practice (see fn. 2). Conversely, misreporting and the misreporting interval are key equilibrium outcomes in the current setting that critically affect the main result.

Crémer (1995) considers a principal-agent model with replacement and finds that the principal may prefer an inferior monitoring technology to maintain the agent's incentives to exert costly effort, which implies that replacement decisions are inefficient. While we do not consider an agency setting, the result of inefficient replacement arises in our equilibrium as well, however, the economic forces driving the result are quite different. We find that shareholders prefer inefficient replacement in order to *uncover* the CEO's private information and foster value-increasing communication. In contrast, in Crémer (1995), the principal's acquisition of the agent's private information would rather weaken the agent's incentives. This paper is also related to governance models which incorporate learning of the CEO's ability (e.g., Dominguez-Martinez et al. (2008), Hermalin and Weisbach (2012)). We complement this literature by investigating the interdependency between CEO replacement and the advisory role of boards.

The paper is structured as follows. The next section introduces the model, while section 3 investigates a model with two types to more simply depict some features of the equilibrium. Section 4 solves the model under continuously many types and provides comparative statics analysis. Section 5 discusses empirical predictions and section 6 presents numerical examples. The final section concludes. All proofs are relegated to the Appendix unless otherwise specified.

2 The model

We study a two-period model where the board of a firm decides whether to continue employing the current CEO after observing information concerning her ability in the first period. The board hires a CEO at the beginning of each period if the position is vacant. The CEO in period t learns her productivity $\theta^t \in \Theta$ in this firm (or her fitness to this firm). The type is perfectly persistent so that the period-2 type θ^2 is identical to the period-1 type θ^1 if the board decides to retain the manager at the end of period 1. Otherwise, θ^2 is drawn independently of θ^1. In Section 3, we study a two-type model with $\Theta = \{\theta_H, \theta_L\}$ where $\theta^t = \theta_H$ occurs with probability $\pi \in (0, 1)$. We also study continuous types in Section 4, where the type space is $\Theta = (\underline{\theta}, \overline{\theta})$ with $-\infty \leq \underline{\theta} < \overline{\theta} \leq +\infty$. The distribution of the type θ^t has a probability density function $g(\theta)$, which is positive and continuous on the support $(\underline{\theta}, \overline{\theta})$. The expected value $\mathbb{E}[\theta]$ is finite.

After learning her type θ^t, the manager then sends a report $\hat{\theta}^t \in \Theta$ to the board. We assume that the CEO simply reports her type to the board; however, the results would be qualitatively impervious to assuming that the CEO rather reports a performance signal about a current project that is correlated with her type. We seek to model the essence that the CEO reports a specific concern, or problem, to the board, which conveys implications about her type; the simplest modeling assumption that continues to preserve the economic insights is where the CEO directly reports her type to the board.

The board then advises the manager based on the report $\hat{\theta}^t$. The board is composed of individuals who are adept at solving particular problems, and can offer valuable guidance to the manager. Once the board receives the CEO's report, $\hat{\theta}^t$, it invests time to determine the appropriate course of action, or the state $\omega^t(\hat{\theta}^t)$, and transmits this to the CEO. The state $\omega^t(\hat{\theta}^t)$ can be thought of as the ideal solution or direction for the firm given that the CEO's report $\hat{\theta}^t$ correctly reflects her and her firm's status. Following Adams and Ferreira (2007), we assume that the board's advice is effective only when the CEO truthfully reports her type (i.e., $\hat{\theta}^t = \theta^t$). Otherwise, the board learns an irrelevant state $\omega^t(\hat{\theta}^t)$ that is statistically independent of the relevant state $\omega^t(\theta^t)$. The state $\omega^t(\theta^t)$ is uniformly distributed between 0 and 1.[8] The board's advisory behavior is always truthful and non-strategic. Misreporting thus does not lead to helpful guidance as the solution or direction offered by the board is then not relevant for the current state of affairs. This assumption regarding the reporting

[8]The results hold for any non-degenerate continuous distribution of $\omega^t(\theta^t)$.

stage mirrors that of Adams and Ferreira (2007); however, the key distinction is that we assume that the CEO's private information also conveys the CEO's type.[9]

Given the board's advice, the manager chooses an action $a^t \in [0, 1]$ that affects the cash flow y^t. After misreporting, the CEO is unable to set $a^t = \omega^t(\theta^t)$, and as a result, we assume that this miscommunication impacts firm value in the form of a loss d in cash flows.[10] In contrast, after truthful communication, the CEO is always able to match her action a^t with the relevant state $\omega^t(\theta)$. We can therefore focus on the scenario where the CEO loses d from the cash flow in period t only after a misreport $\hat{\theta}^t \neq \theta^t$ in the same period. Hence, period t cash flow is given by:

$$y^t = \theta^t - d \cdot \mathbf{1}_{\{a^t \neq \omega^t(\theta^t)\}} + \varepsilon^t.$$

The cash flows can be thought of as the outcome of an action a^t, plus the effect of the CEO's ability on firm value, which in this case is the CEO's ability, plus zero-mean noise ε. We assume that the action a^t is not publicly observed; the results would not be qualitatively affected if a^t was observable. We capture the benefits of truthful communication through this parsimonious reduced-form representation so that we may focus the analysis on the manager's reporting strategy, the board's replacement strategy, and the shareholders' board policy.

The noise ε has mean 0 and follows a distribution $F(\varepsilon)$ with density $f(\varepsilon)$. The density is symmetric (i.e., $f(-\varepsilon) = f(\varepsilon)$), positive, and continuous on the real line. The density also satisfies the following version of the *monotone likelihood ratio property*: whenever $\theta > \theta'$,

$$L(y|\theta, \theta') = \frac{f(y - \theta)}{f(y - \theta')}$$

is continuous and increasing in y, ranging from 0 (at $y = -\infty$) to ∞ (at $y = \infty$). For example, normal distributions with mean 0 satisfy this requirement.

At the end of the first period, the board strategically decides whether to retain or remove the current manager after observing the CEO's message $\hat{\theta}^t$ and output y^t. Let x^t represent the retention decision: $x^t = 0$ indicates that the board has decided to replace the CEO, while $x^t = 1$ corresponds to retention. The board incurs the payoff of c if the CEO is replaced, in addition to the output y^t. That is, the payoff to the board in period t is $y^t + (1 - x^t)c$, and

[9]We note that an alternative modeling specification that qualitatively preserves the main results (but is far more analytically cumbersome) is where the board learns the entire mapping of $\omega(\theta)$ and provides the manager with the entire menu (i.e., for any possible problem the CEO may have, the board provides the solution). However, this is only possible if the action a^t is publicly observable. This does not qualitatively affect the results but rather introduces an additional cost of mimicking, which further incentivizes truth-telling by low-type CEOs. The derivations under this alternative modeling specification are far more analytically onerous.

[10]Alternatively, an equivalent assumption is that cash flows are improved through truthful communication.

the board's total payoff is

$$U_b = \sum_{t=1}^{2} \left[y^t + (1 - x^t)c \right].$$

The parameter c is the cost, or subsidy, that the board receives upon replacing the CEO. To answer our central question, we allow the shareholders of the firm to choose the parameter c before the beginning of the first period to see which level of c endogenously emerges. By controlling c, the shareholders incentivize the board to be more aggressive or passive in their removal of the CEO. Shareholders simply seek to maximize the sum of the firm's return in two periods:[11]

$$U_s = \sum_{t=1}^{2} y^t.$$

Removal can be costly for the board, in which case $c < 0$. A negative c can be thought of as the degree to which the CEO is entrenched in the firm, and so board members must spend more time and energy orchestrating her removal, such as arranging a lengthy takeover bid primarily for the purpose of removing the CEO. Conversely, shareholders can set $c > 0$. This corresponds to the notion that shareholders can approve board members who are predisposed to removing the CEO, or can hold board members more accountable for weak performance in the absence of disciplinary action, thus inducing excessive turnover (Taylor (2010)).[12] This concept of a "negative entrenchment" level (positive c in this case) has also been empirically documented (see Taylor (2010)).[13] CEO removal can also be personally beneficial for board members; this captures the notion that board members have the opportunity to serve as CEO in the event of removal, and hence prefer removing the CEO more often (as empirically documented in Mobbs (2013)). We define $c < 0$ to correspond to the notion of board *friendliness*, or passivity. Similarly, when $c > 0$, we refer to this as board *aggressiveness*, or excessive replacement.

[11]In other words, shareholders have $c = 0$ in their payoff U_s. We can instead allow shareholders to have intrinsic cost (or benefit) of replacing the CEO. This alternative specification does not change the analysis for exogenous c, although a few modifications are needed in calculating the optimal c. To simplify the exposition, we focus on the case that shareholders have $c = 0$. This approach is consistent with our primary objective to examine the possibility and benefit of having aggressive boards.

[12]Taylor (2010) discusses that board members may prefer to remove the CEO excessively in order to protect their reputation or position on the board. His empirical results show that $c > 0$ for large firms with stronger governance. The model here thus develops theoretical underpinnings to help explain this empirical finding.

[13]Taylor (2010) finds that "the degree of entrenchment is significantly lower in recent years and is slightly negative in larger firms" (p. 2053).

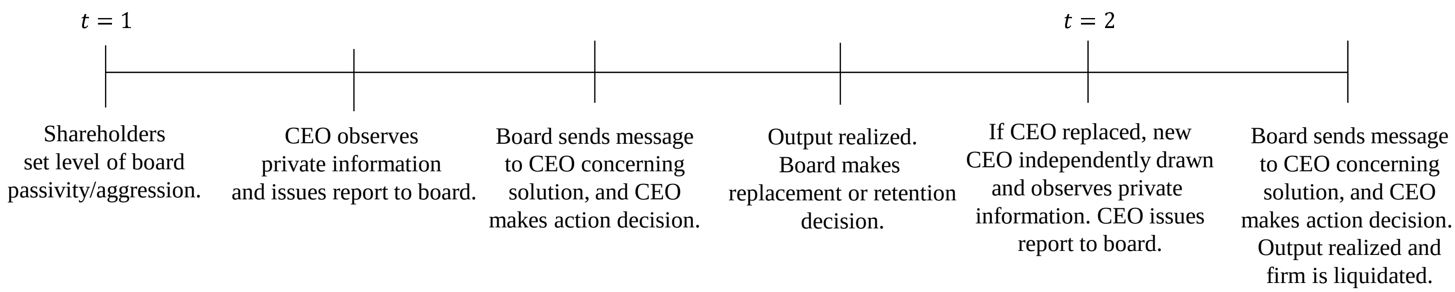

Figure 1: Timeline

The period-1 manager's objective is given by

$$U_m = \left[\chi - d \cdot \mathbf{1}_{\{a^1 \neq \omega^1(\theta^1)\}}\right] + x^1 \cdot \left[\chi - d \cdot \mathbf{1}_{\{a^2 \neq \omega^2(\theta^2)\}}\right],$$

where $\chi > 0$ is a rent from staying in position as CEO for each period. We generally refer to the parameter χ as an "ego rent" but this can also be interpreted as benefits the agent receives from being employed as CEO (sometimes referred to in the literature as private benefits of control). A manager hired in period 2 aims to maximize $\chi - d \cdot \mathbf{1}_{\{a^2 \neq \omega^2(\theta^2)\}}$.

The sequence in each period is summarized as follows:

Stage 1: In the first period only, shareholders determine the cost or benefit, c, that the board receives in the event that the CEO is removed at the end of $t = 1$.

Stage 2: The manager privately observes her type, θ, and submits a report of her type to the board, $\hat{\theta}^t$.

Stage 3: The board learns $\omega^t(\hat{\theta}^t)$, and sends this information to the CEO. The CEO then takes action a^t.

Stage 4: The board observes the firm's output, $y^t = \theta^t - d \cdot \mathbf{1}_{\{a^t \neq \omega^t(\theta^t)\}} + \varepsilon^t$. When $t = 1$, the board then decides whether or not to replace the CEO. If the CEO is replaced, a new manager arrives with type independently drawn. In the case of $t = 2$, the firm is liquidated and payoffs are realized.

We employ perfect Bayesian equilibrium as our solution concept. In particular, the board needs to update its belief over the manager's type θ^t once after observing report $\hat{\theta}$, and once again after observing the output y^t. In this way, we can minimize measure-theoretic anomalies associated with conditional probability. (See Appendix A for more technical details.)

3 The model with two types

We first solve a model with two manager types to illustrate main ideas of the paper. The CEO is either the high type θ_H or the low type θ_L, where $\theta_H > \theta_L$. Let π denote the probability of $\theta = \theta_H$. The manager may report either $\hat{\theta} = \theta_H$ or $\hat{\theta} = \theta_L$.

Under this framework, we can clearly illustrate the key economic trade-offs of an aggressive board. In short, an aggressive board makes excessively strict replacement decisions that discourage the CEO from value-decreasing miscommunication. The reduction of uninformative communication is one of the main benefits of an overly aggressive board. At the same time, it is costly for shareholders to let a talented manager face an inflated retention policy. In this simple two-type setting, we obtain a unique equilibrium in which one of these two effects clearly dominates (for each level of exogenously given c). Although the two-type model captures several important insights, we extend the model to a continuous-type setting following the results in this section. This allows for additional analysis regarding the replacement decision and the optimality of aggressive boards which are not captured in the two-type setting.

Observe that the CEO in the second period has no reason to misreport her type, as there is no replacement decision in the second period. Thus, the CEO's interest is totally aligned with the shareholders' so that she maximizes the firm's value by reporting truthfully and learning the true state $\omega^2(\theta)$. In what follows, we focus on the first period. For ease of exposition, we omit the superscript that indicates period 1.

We primarily focus on equilibria where the CEO with high productivity θ_H reports truthfully to the board. Due to the signaling nature of the model, we must employ an equilibrium selection criterion to preclude an equilibrium where both types are pooled to the low message θ_L. Conversely, it is relatively easy to eliminate the possibility that the manager with θ_H randomizes her report. We postpone the discussion on this issue (until Theorem 1). We allow the manager with low productivity θ_L to report θ_H with probability $\sigma \in [0, 1]$.

Optimal retention policy

We first consider the optimal retention policy of the board where the replacement cost c is exogenously given. Upon replacement, the board incurs the payoff of c, the cost or subsidy of replacement, and then hires a new manager whose expected productivity is $\mathbb{E}[\theta]$. Hence, the board replaces the CEO when the expected value of θ conditional on the current information (report $\hat{\theta}$ and cash flow y) is below $\mathbb{E}[\theta] + c$. This condition is written as:

$$\mu\theta_H + (1 - \mu)\theta_L < \mathbb{E}[\theta] + c,$$

where μ is the posterior probability of θ_H after observing $\hat{\theta}$ and y. The board keeps the manager when the inequality is reversed.

We classify the values of $\mathbb{E}[\theta] + c$ into several cases. When $\mathbb{E}[\theta] + c > \theta_H$, the board is so hostile that the CEO needs to leave her position regardless of the realized cash flow y. Similarly, if $\mathbb{E}[\theta]+c < \theta_L$, the board is extremely friendly and always retains the CEO. When $\mathbb{E}[\theta] + c$ is either θ_H or θ_L, the problem is subtle: continuously many retention policies are sustainable in this stage because the board is indifferent in keeping the type corresponding to the value $\mathbb{E}[\theta]+c$ (either θ_H or θ_L). We begin with the more straightforward and interesting case of $\mathbb{E}[\theta] + c \in (\theta_L, \theta_H)$.

We observe that the posterior μ is increasing in y (unless $\sigma = 0$, i.e., no mimicry by θ_L). The posterior probability of θ_H is given by:

$$\mu(y;\sigma) = \frac{\pi f(y-\theta_H)}{\pi f(y-\theta_H) + (1-\pi)\sigma f(y-(\theta_L - d))} = \frac{\pi}{\pi + (1-\pi)\sigma R(y)}, \tag{1}$$

where $R(y) = f(y-(\theta_L - d))/f(y-\theta_H)$. The likelihood ratio $R(y)$ is decreasing due to the monotone likelihood ratio property. Hence, $\mu(y;\sigma)$ is increasing in y whenever $\sigma > 0$.

The monotone belief implies that the board's retention policy after the good report $\hat{\theta} = \theta_H$ takes the form of a *cutoff rule*. As y increases, the posterior probability improves and eventually the expected value $\mu\theta_H + (1-\mu)\theta_L$ exceeds the threshold $\mathbb{E}[\theta]+c \in (\theta_L, \theta_H)$. This occurs when μ is equal to

$$\mu^*(c) = \frac{\{\mathbb{E}[\theta] + c\} - \theta_L}{\theta_H - \theta_L} = \pi + \frac{c}{\theta_H - \theta_L}.$$

Due to the monotonicity of μ^*, we can uniquely find a threshold $k \in [-\infty, +\infty]$ such that $\mu(y;\sigma) > \mu^*$ for $y > k$ and $\mu(y;\sigma) < \mu^*(c)$ for $y < k$. That is, the CEO remains in her position if the observed cash flow y exceeds the *cutoff* k and the board replaces her when $y < k$.

For each value of $\sigma \in (0, 1]$, we have found a unique cutoff $k(\sigma)$ as the best response to the low-type CEO's mixed strategy σ. The best-response function

$$k(\sigma) = R^{-1}\left(\frac{\pi}{1-\pi} \cdot \frac{1-\mu^*(c)}{\mu^*(c)} \cdot \sigma^{-1}\right),$$

is continuous and decreasing on $(0, 1]$. Even at $\sigma = 0$, the function $k(\sigma)$ is continuous because $k(\sigma)$ goes to $-\infty$ as $\sigma \to 0$ and $k = -\infty$ is the optimal cutoff when $\sigma = 0$.

Misreporting

We now analyze the reporting behavior of the CEO with low productivity θ_L given the board employs cutoff k after observing the good report $\hat{\theta} = \theta_H$. If the CEO truthfully reports her low productivity, then the board becomes confident that the manager's type is θ_L and replaces her for sure (i.e., $k = +\infty$).[14] On the other hand, after misreporting her type, the CEO may keep her position and gain the private benefits χ at the cost d of productivity loss due to the inaccurate advice from the board. The manager is successful in retention if the outcome $y = \theta_L - d + \varepsilon$ exceeds the cutoff k. Thus, the probability of retention is $1 - F(k + d - \theta_L)$. When the low-ability CEO mixes her report, she must be indifferent between the two choices:

$$\big\{1 - F(k + d - \theta_L)\big\} \cdot \chi = d. \tag{2}$$

The left-hand side represents the benefit of misreporting, whereas the right-hand side is the cost, or productivity loss.

By solving the indifference condition (2) in k, we uniquely obtain a cutoff k_0 that makes θ_L indifferent between misreporting and truth-telling:

$$k_0 = \theta_L - d + F^{-1}(1 - d/\chi).$$

If the cutoff k for the good report $\hat{\theta} = \theta_H$ is higher than this threshold cutoff k_0, the low-type CEO rather prefers to tell the truth even though she will face certain replacement. If $k < k_0$, then θ_L chooses to mimic θ_H to have an acceptable likelihood of retention. We therefore obtain the following best response correspondence for each value of cutoff k for the good report $\hat{\theta} = \theta_H$:

$$\sigma(k) = \begin{cases} \{0\} & \text{if } k > k_0 \\ [0,1] & \text{if } k = k_0 \\ \{1\} & \text{if } k < k_0. \end{cases} \tag{3}$$

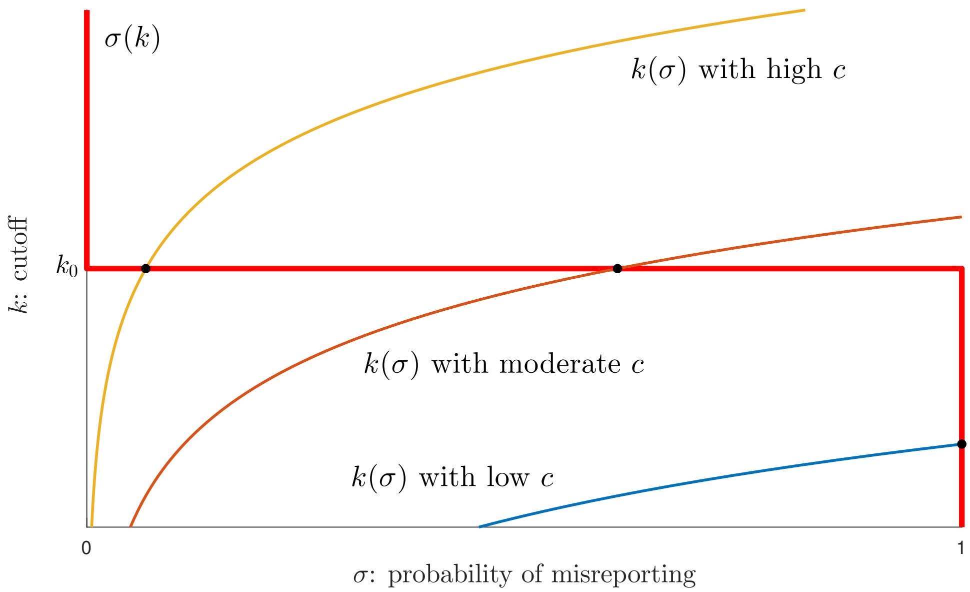

Figure 2: The fixed point problem to find (k^*, σ^*).

Equilibrium with an exogenous board policy c

An equilibrium is represented by a pair of the cutoff k^* and the mixed strategy σ^* that satisfies $\sigma^* \in \sigma(k^*)$ and $k^* = k(\sigma^*)$. As illustrated in Figure 2, the function $k(\sigma)$ is an increasing function of σ; an increase in σ decreases μ in equation (1) so that the board must increase k to keep $\mu(k;\theta_H)$ equal to μ^*. Hence, we can find a unique fixed point.

Before claiming the game has a unique equilibrium, we need to consider the possibility of non-plausible equilibria where both types report $\hat{\theta} = \theta_L$. Such equilibria do not survive the D1 criterion. We consider the model with exogenous c as a signaling game by interpreting the report $\hat{\theta}$ as the message from the sender (the CEO) and the retention policy as the action from the receiver (the board). We apply the D1 criterion to such a signaling game (see Appendix A for more details). In the Appendix, we show that, when the message θ_H is an out-of-equilibrium message, it is type θ_H that benefits the most from employing that message because of d, the disadvantage from misreporting.

Theorem 1. *The two-type model with exogenously given* $c \in (\theta_L - \mathbb{E}[\theta], \theta_H - \mathbb{E}[\theta])$ *has a unique equilibrium that survives the D1 criterion. In the unique equilibrium,*

[14]This argument implicitly assumes that the manager with θ_L truthfully reports with positive probability (i.e., $\sigma < 1$). If not, the board may have an arbitrary belief when $\hat{\theta} = \theta_L$. Nevertheless, the following argument on uniqueness still works in the sense that such improvement in belief for $\hat{\theta} = \theta_L$ further reduces the set of possible equilibria.

- *the high-type CEO sends a truthful report for sure, but the low-type CEO sends report θ_H with probability σ^*;*
- *after report θ_H, the board employs a cutoff k^*; and*
- *after report θ_L, the board assigns probability 1 on type θ_L and replaces the CEO for sure.*

The parameters σ^* and k^* are characterized by

(i) $\mu(k^*;1) = \mu^*(c)$ *and* $\sigma^* = 1$ *when* $\mu^*(c) < \mu(k_0;1)$*; and*

(ii) $k^* = k_0$ *and* $\mu(k_0;\sigma^*) = \mu^*(c)$ *when* $\mu^*(c) > \mu(k_0;1)$.

Furthermore, there exists $\hat{c} \in (\theta_L - \mathbb{E}[\theta], \theta_H - \mathbb{E}[\theta])$ such that $c > \hat{c}$ implies $\mu^*(c) \geq \mu(k_0;1)$ and $c < \hat{c}$ implies $\mu^*(c) \leq \mu(k_0;1)$.

Theorem 1 provides existence and uniqueness under the D1 criterion, and also specifies that there exists a level $\hat{c}$ in the relevant domain $(\theta_L - \mathbb{E}[\theta], \theta_H - \mathbb{E}[\theta])$ which induces partial separation of the high type. As formally shown in Theorem 1 and as illustrated by Figure 2, there are two possible classes of D1 equilibria. The first type of equilibrium appears on the vertical area to the right in Figure 2: the low-type manager always misreports (i.e., $\sigma = 1$) and the cutoff k is implicitly determined by $\mu(k;1) = \mu^*$. The second type is represented by the horizontal area. Contrary to the first, in this type of equilibrium, the cutoff is unchangeably set to k_0 whereas σ is implicitly given by $\mu(k_0;\sigma) = \mu^*$; here, $k = k_0$ is plugged into the condition $\mu(k;\sigma) = \mu^*$.

Theorem 1 and Figure 2 also show that these two equilibria continuously arise. When c is sufficiently small (i.e., smaller than $\hat{c}$), the equilibrium of the first kind emerges. As c increases, while the probability σ of misreporting is kept equal to 1, the equilibrium cutoff increases and eventually reaches k_0—the corner in Figure 2—when c touches the threshold value $\hat{c}$. After that, the equilibrium moves to the horizontal area and reduces the probability σ of misreporting while keeping the value of the equilibrium cutoff k equal to k_0. As $\mathbb{E}[\theta] + c$ approaches θ_H, the misreporting probability σ eventually converges to 0. This argument is formally shown in Theorem 1.

We briefly discuss what happens when $\mathbb{E}[\theta] + c > \theta_H$ and when $\mathbb{E}[\theta] + c < \theta_H$.

Proposition 1. *Consider the two-type model with exogenously given c.*

(i) *When $\mathbb{E}[\theta] + c > \theta_H$, the game has a unique equilibrium where the board always replaces the CEO regardless of her report and output.*

(ii) *When* $\mathbb{E}[\theta] + c < \theta_L$*, the game has a unique equilibrium where the board always keeps the CEO regardless of her report and output.*

When the board is extremely hostile (case (i)), the manager has no chance to remain in the firm and thus attempts to maximize the value of the firm during her tenure by reporting truthfully. Similarly, if the board is extremely friendly (case (ii)), the manager faces no threat of replacement and thus her objective is again totally aligned with that of the shareholders.

Multiple equilibria emerge in the knife-edge cases of $\mathbb{E}[\theta] + c = \theta_H$ and $\mathbb{E}[\theta] + c = \theta_L$. Some of the equilibria are pathological and make the shareholders' payoff discontinuous with respect to c on $[\theta_L - \mathbb{E}[\theta], \theta_H - \mathbb{E}[\theta]]$, although they do not survive after we introduce small perturbations to the model.[15] To avoid complications associated with multiple equilibria, we allow equilibria that are continuously connected to the equilibria described in Theorem 1.[16] More specifically, we allow only the following equilibria:

(i) When $\mathbb{E}[\theta] + c = \theta_H$, the board uses the cutoff k_0 when the CEO reports θ_H and replaces her when θ_L. The CEO always reports her true type.

(ii) When $\mathbb{E}[\theta] + c = \theta_L$, the board keeps the CEO when she reports θ_H and replaces her when she reports θ_L. The CEO always reports θ_H.

Endogenous board policy

We now investigate the degree of board hostility or friendliness, c, that the shareholders endogenously determine. We ultimately observe that only two values of c survive as candidates of the optimal choice: $c = 0$ (*neutral* board) and $c = \theta_H - \mathbb{E}[\theta]$ (*maximally aggressive* board). As Figure 3 illustrates, the maximally aggressive board $c = \theta_H - \mathbb{E}[\theta]$ arises as the primary candidate of the solution. Consider $\hat{c}$ as in Theorem 1. The maximum aggressiveness $c = \theta_H - \mathbb{E}[\theta]$ dominates any other $c \in (\hat{c}, \theta_H - \mathbb{E}[\theta])$, as they all induce the same cutoff k_0, but the maximally aggressive board minimizes the misreporting probability σ to 0 unlike any other value of c.

When the ego rent χ is large (see the case of $\chi = 10$), the objective function of the shareholders is no longer monotonic and the neutral board $c = 0$ emerges as another peak. Shareholders are able to manipulate the aggressiveness or friendliness of the board through changing c. Consequently, a marginal change in c affects values of σ or k (either one, but not both). Therefore, through changes in the value of c, shareholders are able to indirectly

[15] For example, we can replace the atomic type θ_i with a uniform distribution on $(\theta_i - \Delta, \theta_i + \Delta)$ with small Δ. The equilibrium multiplicity disappears as we see in Section 4.

[16] That is, we eliminate equilibria that are discontinuous from neighboring equilibria. In other words, we impose some sort of lower hemi-continuity on equilibria in the two knife-edge cases.

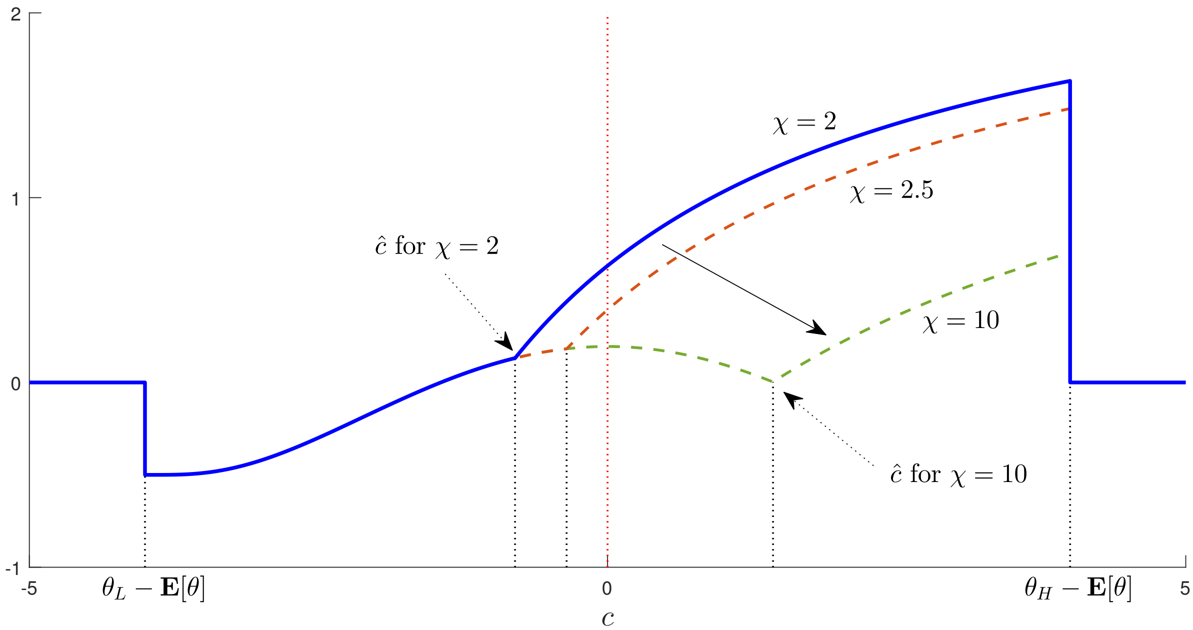

Figure 3: The normalized objective function of the shareholders when $\theta_H = 9$, $\theta_L = 1$, $\pi = 1/2$, $d = 1$, and $\varepsilon \sim N(0, 10)$.

manipulate k or σ. As illustrated in Figure 4, the neutral board, $c = 0$, tends to be optimal when χ is large. Hence, we ultimately see that two potential peaks emerge in the shareholders' payoff. The first is at $c = \hat{c}$ and the second at $c = \theta_H - \mathbb{E}(\theta)$.

We formalize the above arguments. We first formulate the objective function of the shareholders. Recall that the shareholders' objective is to maximize the sum of outputs in the two periods. Given σ, the expected value of the output in period 1 is

$$\tilde{V}_1(\sigma) = \mathbb{E}[\theta] - (1 - \pi)\sigma d.$$

The expected value of period-2 output depends on how likely the manager of each type remains in the firm. The probability of retention is $1 - F(k - \theta_H)$ $(= F(\theta_H - k))$ for type θ_H, $F(\theta_L - k - d)$ for type θ_L who misreports, and 0 for type θ_L who reports truthfully. Since the expected productivity is $\mathbb{E}[\theta]$ after replacement, the expected value of the output in period 2 is:

$$\begin{aligned} \tilde{V}_2(\sigma, k) &= \mathbb{E}[\theta] + \pi(\theta_H - \mathbb{E}[\theta])F(\theta_H - k) + \sigma \cdot (1 - \pi)(\theta_L - \mathbb{E}[\theta])F(\theta_L - k - d) \\ &= \mathbb{E}[\theta] + \Delta\theta \cdot \pi(1 - \pi)\left[F(\theta_H - k) - \sigma F(\theta_L - k - d)\right], \end{aligned}$$

where $\Delta\theta = \theta_H - \theta_L$. It is noteworthy that we can combine the terms for the two types

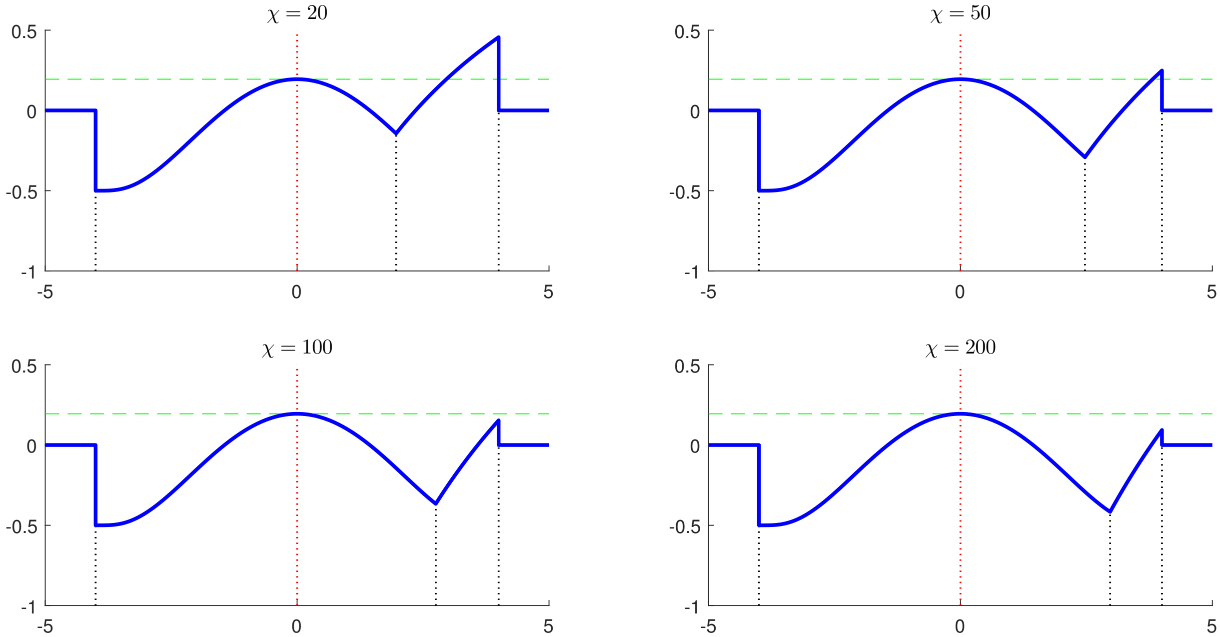

Figure 4: The normalized objective function of the shareholders when $\theta_H = 9$, $\theta_L = 1$, $\pi = 1/2$, $d = 1$, and $\varepsilon \sim N(0, 10)$.

despite the different weights (π and $1-\pi$) because $\theta_H - \mathbb{E}[\theta] = (1-\pi)\cdot\Delta\theta$ is proportional to $1-\pi$ and $\theta_L - \mathbb{E}[\theta] = \pi \cdot \Delta\theta$ is proportional to π. By combining these two values, we obtain the following objective function (after subtracting the constant $\mathbb{E}[\theta]$ twice for normalization):

$$V(\sigma, k) = (1-\pi)\left\{\Delta\theta \cdot \pi \left[F(\theta_H - k) - \sigma F(\theta_L - k - d)\right] - d\sigma\right\}.$$

The shareholders' objective is to maximize this function V.

Now, we investigate the optimal value of c in the interval $[\hat{c}, \theta_H - \mathbb{E}[\theta]]$, where the cutoff k is a constant ($k^* = k_0$ by Theorem 1) but the misreporting rate σ decreases as c increases, as illustrated in Figure 2. We refer to this reduction of misreporting probability as the *disciplinary effect*, and to this interval as the *disciplined region*. The objective function $V(\sigma, k)$ is decreasing in σ, and σ shrinks to 0 as $c \to \theta_H - \mathbb{E}[\theta]$. Therefore, the maximally aggressive c is optimal for shareholders in the disciplined region $[\hat{c}, \theta_H - \mathbb{E}[\theta]]$:

Lemma 1. *Assume that the board employs the cutoff k_0 when $\mathbb{E}[\theta] + c = \theta_H$. On the interval $[\hat{c}, \theta_H - \mathbb{E}[\theta]]$, the optimal value of c for the shareholders is $c = \theta_H - \mathbb{E}[\theta]$.*

In contrast, we refer to the region $[\theta_L - \mathbb{E}[\theta], \hat{c}]$ as the *incorrigible region*, as changes in c do not affect the value of σ, but rather affect the value of the cutoff k. In this region, through controlling the board, shareholders wish to achieve the optimal value of k under the constraint that $\sigma = 1$ is unchangeable. As we see below, the solution is simple: the board should share the same interest with shareholders (i.e., $c = 0$) whenever possible. In this way, the board never makes a biased decision—such a bias is useful in the disciplined region, but not in the incorrigible region—and thus the equilibrium cutoff k is optimal for shareholders. In other words, by choosing a value of c other than $c = 0$, shareholders end up with distorting the retention decision of the board and reducing the payoff for themselves. We refer to this effect as the *distortion effect*.

To see this point more explicitly, consider the problem of maximizing shareholders' value $V(\sigma, k)$ given $\sigma = 1$. The first-order condition

$$\pi(\theta_H - \theta_L)\left\{f(k + d - \theta_L) - f(k - \theta_H)\right\} = 0,$$

is equivalent to $R(k) \equiv f(k - (\theta_L - d))/f(k - \theta_H) = 1$. When $c = 0$, $R(k)$ is indeed equal to 1 in equilibrium because the unique solution of the indifference condition

$$\pi = \mu^* = \mu(y; 1) = \frac{\pi}{\pi + (1 - \pi)R(k)},$$

is $R(k) = 1$. Therefore, $c = 0$ is the optimal choice if the neutral board is included in the incorrigible region (i.e., $0 \leq \hat{c}$).

The next lemma describes the optimal value of c in the incorrigible region not only for the case of $\hat{c} \geq 0$ but also for $\hat{c} \leq 0$.

Lemma 2. *On the interval* $(\theta_L - \mathbb{E}[\theta], \hat{c}]$*, the optimal value of* c *for the shareholders is* $c = 0$ *if* $0 \leq \hat{c}$*; otherwise,* $c = \hat{c}$*.*

Lemma 2 states that, in the incorrigible region, shareholders set the board to be neutral when $\hat{c} \geq 0$, and set the board to be friendly when $\hat{c} \leq 0$ by setting $c = \hat{c}$ in the latter case. However, we ultimately show (in the next theorem) that the friendly board which occurs when $\hat{c} \leq 0$ is dominated by an aggressive board after we consider the whole interval $(\theta_L - \mathbb{E}[\theta], \theta_H - \mathbb{E}[\theta])$. This occurs because, even though $c = \hat{c}$ is optimal within the incorrigible region, setting $c = \hat{c}$ is the *worst* choice in the disciplined region. In contrast, in the case of $\hat{c} > 0$, the payoff function V of shareholders has two peaks, as illustrated in Figure 4. Hence, on the interval $(\theta_L - \mathbb{E}[\theta], \theta_H - \mathbb{E}[\theta])$, two choices for the shareholders' optimal board policy emerge: either neutral ($c = 0$) or aggressive ($c > 0$).

Before concluding that there are only two possibilities, we briefly discuss the extreme cases covered in Proposition 1. These cases are dominated by the maximally aggressive board $c = \theta_H - \mathbb{E}[\theta]$, which has the same imitation probability σ but a more effective retention policy (i.e., cutoff k_0).

We now characterize the necessary and sufficient condition for the optimality of the maximally aggressive board. In the statement below, we can interpret the variable $m = F^{-1}(1 - d/\chi)$ as a measure of the CEO's *intrinsic misreporting incentive*: as $\chi \to \infty$, the value of m goes to infinity, whereas m goes to $-\infty$ as the cost of miscommunication, d, approaches the value of the private benefits, χ. The condition $m = m_*$ provides a threshold in terms of the intrinsic incentive of misreporting for the CEO of low ability.

Theorem 2. *Let* $m = F^{-1}(1 - d/\chi)$ *and* $m_* = (\Delta\theta + d)/2$.

(i) *When* $m \leq m_*$, *the payoff of the shareholders is increasing in* c *on* $(\theta_L - \mathbb{E}[\theta], \theta_H - \mathbb{E}[\theta])$. *Consequently, the maximally aggressive board is optimal.*

(ii) *When* $m > m_*$, *the payoff of the shareholders has two peaks on* $(\theta_L - \mathbb{E}[\theta], \theta_H - \mathbb{E}[\theta])$: $c = 0$ *and* $c = \theta_H - \mathbb{E}[\theta]$. *The aggressive board,* $c = \theta_H - \mathbb{E}[\theta]$, *is optimal if and only if*

$$F(2m_* - m) + \frac{d}{\pi \cdot \Delta\theta} \geq 2F(m_*) - 1. \tag{4}$$

It is the unique optimum when the inequality is strict.

Figure 3 is helpful in understanding part (i) of Theorem 2. Let us start from $c = \theta_L - \mathbb{E}[\theta]$ and gradually increase the value of c. The value of the shareholders' objective function initially increases because c approaches the neutral level $c = 0$ in the incorrigible region. If $\hat{c} \leq 0$, the value of c reaches $\hat{c}$ before the objective function starts to decline. Now the value of c enters the disciplined region $[\hat{c}, \theta_H - \mathbb{E}[\theta]]$ and the objective function continues to increase. Thus, as long as $\hat{c} \leq 0$, the objective function is increasing on $[\theta_L - \mathbb{E}[\theta], \theta_H - \mathbb{E}[\theta]]$. We show that the condition $\hat{c} \leq 0$ is equivalent to $m \leq m_*$ in the Appendix.

In part (ii) of Theorem 2, we obtain condition (4) by simply comparing the two cases. When $\hat{c} > 0$, the equilibrium with $c = 0$ is given by $\sigma = 1$ and $R(k) = 1$. We can explicitly solve the equation $R(k) = 1$. With its unique solution $k^{**} = (\theta_H + \theta_L - d)/2$, we can explicitly calculate the value of $V(\sigma, k)$ when $c = 0$:

$$\begin{aligned} V_N = V(1, k^{**}) &= \Delta\theta \cdot \pi(1-\pi)\left[F(m_*) - F(-m_*) - \frac{d}{\pi \cdot \Delta\theta}\right] \\ &= \Delta\theta \cdot \pi(1-\pi)\left[2F(m_*) - 1 - \frac{d}{\pi \cdot \Delta\theta}\right]. \end{aligned}$$

When the board is maximally aggressive (i.e., $\mathbb{E}[\theta] + c = \theta_H$), $\sigma = 0$ and the shareholders' payoff is given as:

$$V_A = V(0, k_0) = \Delta\theta \cdot \pi(1-\pi)F(\theta_H - k_0) = \Delta\theta \cdot \pi(1-\pi)F(2m_* - m).$$

Condition (4) is determined by comparing V_A and V_N.

Properties of equilibrium

We obtain several notable implications from Theorem 2. First, shareholders choose an aggressive board when the value of communication, d, is sufficiently large (note that $m_* - m$ is increasing in d and goes to ∞ as $d \to \chi$). When d is close enough to χ, the CEO is discouraged from misreporting and thus the cutoff k_0, with which the CEO is indifferent between truth-telling and misreporting, decreases. As the indifference cutoff k_0 decreases, it eventually goes below the cutoff *that is optimal for the shareholders*. Because the board's replacement policy becomes unacceptably lenient, the shareholders rather induce the board to be more aggressive by raising c.

Second, the probability of the high type, π, has a negative effect on board aggressiveness, as the right-hand side of (4) is increasing in π. Intuitively, when the probability that the CEO is of type θ_L is low, so is the ex ante likelihood of misreporting. Shareholders then focus on avoiding distortionary retention policies and eventually choose a neutral board when π is large enough. Put differently, the disciplinary effect from aggression—truthful reporting from low types—dissipates as π increases. As π approaches 0, the left-hand side of condition (4) unboundedly increases and condition (4) is satisfied. Hence, shareholders prefer the board to be more aggressive when there is a high probability mass of low types.

Third, shareholders optimally choose the aggressive board when the difference $\Delta\theta = \theta_H - \theta_L$ is small enough to satisfy inequality (4). In an extreme situation where θ_H and θ_L are almost identical, the choice of cutoff k is mostly unimportant to the shareholders because they can hire a similarly talented CEO even after wrongly replacing a high-type one. In other words, the distortion effect disappears as $\Delta\theta$ shrinks to 0, while the disciplinary effect is intact.

Lastly, the aggressive board is optimal when the private benefit of control χ is sufficiently small. Intuitively, a decrease in χ reduces the low type CEO's incentive to misreport while keeping the shareholders' preferences over the board's replacement cutoff unchanged. Consequently, as occurs with the value loss d, the cutoff k_0 that makes the low-type CEO indifferent eventually becomes lower than the standard the shareholders would set, resulting in shareholders setting a more aggressive board.

To summarize the main qualitative implications of our discrete setting, we find that *shareholders are more likely to choose an aggressive board, and thus induce higher rates of CEO turnover, when (i) the benefit from informative communication d increases; (ii) there is a greater concentration of low-ability CEOs; (iii) CEO ability becomes more homogeneous; and (iv) the private benefit χ decreases.* While we find several interesting properties of the equilibrium of this setting, the continuous-type model we analyze in Section 4 provides additional economic implications that build from this analysis. We present a thorough discussion of the empirical predictions which emerge from our analysis (and their corresponding relation to the extant empirical literature) in Section 5.

Discussion

One of the strengths of the two-type model is that we can more directly observe the two disparate, countervailing effects of aggressive boards. Moreover, as shown above, we observe these two effects distinctly separated in two regions. This allows us to cleanly observe how changes in the shareholders' choice of c affect the board's cutoff k, and the manager's reporting strategy σ. The first effect from a policy of aggressive boards is on the probability of misreporting. We find that the low type's equilibrium probability of misreporting, σ, decreases in the disciplined region as c increases. Intuitively, a stricter retention policy disincentivizes low-quality CEOs to engage in value-reducing mimicry. Hence, aggressiveness is beneficial in this sense as it induces truthful communication.

The second effect is the costly distortion in the retention policy. By making the board more aggressive, a higher hurdle for retention is adopted. Although the higher cutoff disciplines the manager with low productivity, excessive replacement decisions can be detrimental for firm value as talented CEOs are mistakenly removed. We characterize conditions under which the first effect dominates and the aggressive board is optimal. In the case where the second effect dominates, the results show that a neutral board is optimal. Perhaps surprisingly, Theorem 2 shows that friendly boards do not emerge along the equilibrium path.

Board aggressiveness as an ex ante optimal policy by shareholders can help to explain the relatively high rate of performance-induced turnover among CEOs, as documented by Jenter and Lewellen (2017)). Moreover, the results help to explain the findings of Taylor (2010), who documents excessive replacement among large firms and that boards receive a net personal benefit from excessive removal of the CEO (p. 2077). The results are also consistent with the findings of Cornelli et al. (2013), who show that soft information (the CEO's report in our setting) is a salient factor in the board's CEO replacement decision. The two-type model can also be interpreted as a setting where high-quality executives stand out from the pool of lower-quality, or "average," executives. When the gap between the ability of the two types (i.e., $\Delta\theta$) is significantly distinct, an aggressive board is likely to emerge according to Theorem 2. Moreover, as noted above, shareholders prefer a more aggressive board when the pool of low types, $1-\pi$, is high. This implies that during times of overall poor managerial quality—as may be related to poor industry performance—shareholders would rather prefer the board to be more aggressive in their replacement of the firm's top executives. This is consistent with the findings of Kaplan and Minton (2012) and Jenter and Kanaan (2015).

While the two-type model illustrates the main incentives underlying the shareholders' optimal board policy, the continuous-type setting captures additional insights, properties, and empirical predictions concerning aggressive boards. In the two-type setting, we see that, if the board is made aggressive by shareholders, they adopt the maximally aggressive retention policy. Below, we show that this extreme form of hostility does not arise in the continuous-type model, and that misreporting always occurs on the equilibrium path. Moreover, Theorem 4 below shows that an aggressive board policy endogenously emerges. Our closed-form characterization of c in the following section allows us to conduct additional comparative statics analysis on the shareholders' optimal policy.

4 The model with continuous types

In this section, we study a model with continuously many types $(\underline{\theta}, \overline{\theta})$. Recall that the distribution of these continuous types has a density function $g(\theta)$.

4.1 Equilibrium with exogenous aggressiveness

As in the two-type setting, we begin the analysis of the continuous model where c is exogenously given and then examine the case of endogenously determined c. In the second period, the manager has no incentive to misreport. This occurs for the same reason as in the two-type model; there is no replacement decision at that point and hence the manager cannot benefit from misreporting. The action selected in period 2 by a CEO of type θ is thus $a^2 = \omega^2(\theta)$. In the ensuing analysis, we consider incentives in the first period.

Also, we focus on the case of $c \in [\underline{\theta} - \mathbb{E}[\theta], \overline{\theta} - \mathbb{E}[\theta]]$. As we have seen in the two-type model (Proposition 1), only trivial equilibria emerge with extremely large or small c.[17] We state the result for the alternative case and then switch to the primary case of $c \in [\underline{\theta} - \mathbb{E}[\theta], \overline{\theta} - \mathbb{E}[\theta]]$.

[17]The two exemplary equilibria stated in Proposition 2 survive both D1 and D2 criteria because there is no out-of-equilibrium message.

Proposition 2. *Suppose* $c < \underline{\theta} - \mathbb{E}[\theta]$ *or* $c > \overline{\theta} - \mathbb{E}[\theta]$. *Then, the CEO always reports her true type. The board always retains the CEO if* $c < \underline{\theta} - \mathbb{E}[\theta]$. *When* $c > \overline{\theta} - \mathbb{E}[\theta]$, *the board always replaces the CEO.*

Structure of equilibria

Like the two-type model, the CEO does not necessarily report her true type in the first period. Suppose that the board believes that reports are truthful. Then, by replacing the manager with report θ, the board forgoes the period-2 expected output θ under the current manager but instead gains the subsidy or cost c and the expected period-2 output $\mathbb{E}[\theta]$ after replacement. Therefore, the board replaces the CEO when the report θ is less than $\mathbb{E}[\theta] + c$, whereas the CEO with $\theta > \mathbb{E}[\theta] + c$ remains in her position.

We naturally conjecture that the threshold $\theta_1 = \mathbb{E}[\theta] + c$ determines the behavior of the manager in the following manner. The manager with type $\theta > \theta_1$ truthfully reports her type to the board, presumably because she does not need to hide her type in order to survive in the current firm. On the other hand, the CEO with type $\theta < \theta_1$ sometimes reports a message higher than θ_1 in order to have a chance to remain in the firm.

Given that the reports above θ_1 pool different types, the board sometimes must replace the CEO such that productive managers are retained with a higher likelihood than less productive ones. As in the model with two types, the optimal retention policy is a cutoff rule.

Lemma 3. *Let* $\hat{\theta} \in (\theta_1, \overline{\theta})$. *After observing report* $\hat{\theta}$, *the board's optimal retention policy is a cutoff rule with threshold* $k(\hat{\theta}) \in [-\infty, +\infty]$ *regardless of the posterior belief* $\beta_{\hat{\theta}}$.

Observe that the cutoff $k(\hat{\theta})$ needs to be uniform across messages that the CEO uses when misreporting her type. When reporting $\hat{\theta} \neq \theta$, the manager with type θ only cares about the survival probability

$$\Pr\{\theta - d + \varepsilon > k(\hat{\theta})\} = 1 - F(k(\hat{\theta}) + d - \theta),$$

which is decreasing in $k(\hat{\theta})$. Thus, the CEO always chooses a message $\hat{\theta}$ with the lowest cutoff $k(\theta) = \inf_s k(s)$ and never uses $\hat{\theta}$ with a higher cutoff when misreporting.

We eventually claim that the above argument applies to all reports above θ_1; i.e., all of these reports have the same cutoff level. However, it is still potentially possible at this stage that some or even all reports above θ_1 are out-of-equilibrium and, after such reports, the board has a very pessimistic belief and an extremely strict cutoff. We temporarily allow such implausible beliefs and cutoffs in the next lemma (Lemma 4). However, we soon claim a fuller statement (Lemma 6) after equilibrium selection (Lemma 5).

Lemma 4. *In any equilibrium with exogenous c, it occurs with probability 1 that, whenever the manager with type θ chooses a misreport $\hat{\theta} \in (\theta_1, \overline{\theta}) \setminus \{\theta\}$, the cutoff $k(\hat{\theta})$ associated with the misreport is equal to* $\inf_{s \in (\theta_1, \overline{\theta})} k(s)$.

As in the case of two types, we employ the D1 criterion (see Appendix A) to eliminate the anomaly associated with out-of-equilibrium reports and to ensure truthful reporting from types $\theta > \theta_1$. As we see soon, when the CEO reports an out-of-equilibrium message $\hat{\theta} \in (\theta_1, \infty)$, the board must believe that the CEO's type is above θ_1 after we apply the D1 criterion. If this is the case, the board sets $k(\hat{\theta}) = -\infty$ (i.e., no replacement) and consequently the type $\hat{\theta}$ (as well as many other types) begins to use the message $\hat{\theta}$ to utilize the extremely friendly retention policy; consequently, message $\hat{\theta}$ is no longer out-of-equilibrium. Once out-of-equilibrium messages disappear from the interval $(\theta_1, \overline{\theta})$, all CEO types in this interval report truthfully and face the uniform cutoff $k_* = \inf_s k(s)$. (See the proof of Lemma 5 for details.)

To illustrate the D1 criterion in the present setting, consider an (ideal) equilibrium where the manager always encounters a uniform cutoff k_* after misreporting her type on the equilibrium path.[18] We aim to show that if $\hat{\theta} \in (\theta_1, \overline{\theta})$ is an out-of-equilibrium message, then it is type $\hat{\theta}$ that benefits the most from this message among all other types. First, observe that the manager with type θ can get at least the following payoff from misreporting:

$$U_*(\theta) = (1 - F(k_* + d - \theta))\chi + \{2\theta + \chi\},$$

in equilibrium under cutoff k_*. To guarantee this payoff or better, the out-of-equilibrium message $\hat{\theta}$ must result in a cutoff $k(\hat{\theta}) \leq k_*$ if the manager has type $\theta \neq \hat{\theta}$. When $\theta = \hat{\theta}$, the type $\hat{\theta}$ receives a much higher payoff with the truthful report $\hat{\theta}$ than the equilibrium payoff $U_*(\hat{\theta})$, as this type can uniquely boost her output y with the message $\hat{\theta}$. As a result, by the D1 criterion, the type $\hat{\theta}$ is the only type that deserves a probability weight. The actual proof is somewhat more involved than the above discussion.[19] We ultimately obtain the following result:

[18]This simple structure may not arise in a presumably implausible equilibrium where some types above θ_1 choose messages lower than θ_1. We eliminate such pathological cases in the proof of Lemma 5.

[19]In the derivation, we cannot assume that all misreporting types face some cutoff rule; Lemma 3 applies only to messages above θ_1, and the other messages may induce intractable retention policies. Nevertheless, thanks to Lemma 3, at least the retention policy after $\hat{\theta}$ is tractable even though the other side—the retention policy each type faces in equilibrium—may be pathological.

Lemma 5. *In any equilibrium that survives the D1 criterion, the manager with type above θ_1 truthfully reports her type.*

We now know that no report above θ_1 is an out-of-equilibrium message. In other words, we have overcame the problem of implausibly pessimistic beliefs and can strengthen the statement of Lemma 4:[20]

Lemma 6. *In any equilibrium that survives the D1 criterion, the board (almost surely) employs cutoff $k_* = \min_{\hat{\theta} \in (\theta_1, \bar{\theta})} k(\hat{\theta})$ after receiving a report above θ_1.*

Lemmas 5 and 6 significantly simplify the analysis in several ways. First, as mentioned above, all messages above θ_1 are on the equilibrium path. Hence, we no longer need to worry about equilibrium selection on these high messages. Second, we can partition the type space $\Theta = (\underline{\theta}, \overline{\theta})$ into two disparate intervals: types above θ_1 and types below. What remains is to analyze the behavior of CEO types in the interval $(\underline{\theta}, \theta_1]$.

Third, the manager with type $\theta \in (\underline{\theta}, \theta_1]$ will be replaced for sure unless she pretends to have a high type $\hat{\theta} \in (\theta_1, \overline{\theta})$. More precisely, it cannot occur (except on an event of probability 0) that the CEO has type $\theta \in (\underline{\theta}, \theta_1]$, reports $\hat{\theta} \in (\underline{\theta}, \theta_1]$, and is retained. Thus, we can essentially assume that types $\theta \in (\underline{\theta}, \theta_1]$ have only two choices: the truthful report θ or some misreport $\hat{\theta} \in (\theta_1, \overline{\theta})$. However, it is guaranteed only *on the equilibrium path* that the board replaces the manager for sure after message $\hat{\theta} \in (\underline{\theta}, \theta_1]$: the message $\hat{\theta} \in (\underline{\theta}, \theta_1]$ can be out of equilibrium, and after this message, the board may still form a modestly optimistic belief and set a cutoff that is not infinitely strict (i.e., $k(\hat{\theta}) < \infty$) but high enough to discourage every type from using this message. We ultimately show that such optimistic beliefs cannot survive equilibrium selection (Theorem 3). Meanwhile, we simply assume in the exposition that *any report $\hat{\theta} \in (\underline{\theta}, \theta_1]$ results in the removal of the manager for sure* (i.e., $k(\hat{\theta}) = +\infty$).

In sum, we have characterized the structure of equilibria for types and reports higher than θ_1. The board employs an identical cutoff k_* whenever the message $\hat{\theta}$ is higher than θ_1 (Lemma 6). Also, the corresponding types $\theta > \theta_1$ submit truthful reports to the board (Lemma 5). It is still unclear if, as expected, the board removes the CEO after receiving a message below θ_1. We postpone the analysis of this question as it requires another stage of equilibrium selection. We instead investigate the reporting behavior of manager types below θ_1, assuming that any report below θ_1 certainly induces CEO replacement. We then return to the equilibrium selection problem (Theorem 3).

[20] We actually prove Lemma 6 in the proof of Lemma 5.

Equilibrium decisions

As in the model with two types, the manager with $\theta < \theta_1$ faces two choices. If the manager reports her true type, she obtains the informational gain d but the board removes her before the next period with probability one. By reporting something above θ_1, the manager forgoes the gain d but has a positive chance to remain in the current firm. Since the probability of retention is $1 - F(k_* + d - \theta)$ in the latter case, the indifference condition is

$$\left\{1 - F(k_* + d - \theta)\right\} \chi = d.$$

By solving this equation, we obtain the threshold type with which the manager is indifferent between the above two choices:

$$\theta_2(k_*) = k_* + d + F^{-1}(d/\chi). \tag{5}$$

Types below this threshold θ_2 cannot gain a satisfactory retention rate even after misreporting (i.e., $\{1 - F(k_* + d - \theta)\}\chi < d$). These types thus rather prefer to report truthfully. In contrast, types above θ_2 but below θ_1 prefer to inflate their report because $\{1 - F(k_* + d - \theta)\}\chi > d$. The following lemma summarizes this argument.[21]

Lemma 7. *Suppose $\theta < \theta_1$. In any equilibrium that survives the D1 criterion, the manager truthfully reports her type if $\theta < \theta_2$; and reports a message above θ_1 if $\theta > \theta_2$.*

The value of θ_2 is thus the threshold such that types below this level report truthfully and are replaced with certainty. As we see shortly, shareholders can induce informative communication (i.e., truthful reports) from types lower than θ_2 by raising this threshold θ_2. This is achieved by setting a more aggressive board and consequently raising θ_1 (at the expense of misreporting by intermediate types). This feature is analogous to the *disciplinary effect* we observed in the two-type model.

We then investigate how the uniform cutoff k_* is determined by the board given this reporting behavior. We saw above that the cutoff levels $k(\hat{\theta})$ for reports $\hat{\theta} > \theta_1$ must be some uniform level k_* (Lemma 6), but each cutoff level $k(\hat{\theta})$ needs to be a solution of the optimization problem for the board and thus depends on the posterior belief after observing report $\hat{\theta}$. Thus, if the posterior beliefs for such reports are not properly aligned—e.g., when certain messages attract too many (or too few) misreporting types—the board may employ several different cutoffs, which never occurs in equilibrium due to Lemma 6. In what follows,

[21] We allow $\theta_2 > \theta_1$ and $\theta_2 < \underline{\theta}$. The latter case does not cause any problem as long as we set $g(\theta) = 0$ for $\theta < \underline{\theta}$. We can easily see that $\theta_2 > \theta_1$ never occurs in equilibrium; if k_* is so high that θ_2 exceeds θ_1, the manager never misreports her type and thus k_* goes down to $-\infty$.

we instead require the board to choose some uniform cutoff k_* for (almost) every $\hat{\theta} > \theta_1$ and find a necessary condition for each value of $\hat{\theta}$. In this way, we can eventually obtain a single, useful condition that determines the value of k_* as a function of θ_2, and the function $k_*(\theta_2)$ works as if it is the board's best response function.

For ease of exposition, we focus on an equilibrium where all types in the misreporting interval (θ_2, θ_1) employ the same density function $h(\hat{\theta})$ in choosing a misreport $\hat{\theta}$.[22] We first calculate the posterior belief of the board after observing $\hat{\theta} \in (\theta_1, \overline{\theta})$:

$$\text{Prob}\left\{\theta = \hat{\theta} \,\middle|\, \hat{\theta}\right\} = \frac{g(\hat{\theta})}{g(\hat{\theta}) + h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta)\,\mathrm{d}\theta} = \frac{g(\hat{\theta})}{Q(\hat{\theta})}$$

and,

$$\text{Prob}\left\{\theta \neq \hat{\theta} \,\middle|\, \hat{\theta}\right\} = \frac{h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta)\,\mathrm{d}\theta}{g(\hat{\theta}) + h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta)\,\mathrm{d}\theta} = \frac{h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta)\,\mathrm{d}\theta}{Q(\hat{\theta})},$$

where $Q(\hat{\theta}) = g(\hat{\theta}) + h(\hat{\theta}) \int_{\theta_2}^{\theta_1} g(\theta)\,\mathrm{d}\theta$ represents the probability (density) that the manager chooses report $\hat{\theta}$. Also note $\int_{\theta_2}^{\theta_1} g(\theta)\,\mathrm{d}\theta$ is the unconditional probability of misreporting. More specifically, the posterior probability of $\theta \le x$ is

$$\text{Prob}\left\{\theta \le x \,\middle|\, \hat{\theta}\right\} = \frac{h(\hat{\theta})}{Q(\hat{\theta})} \int_{\theta_2}^{x} g(\theta)\,\mathrm{d}\theta,$$

for all $x \in (\theta_2, \theta_1)$. Hence, type $\theta \in (\theta_2, \theta_1)$ has a density $h(\hat{\theta})g(\theta)/Q(\hat{\theta})$ conditional on report $\hat{\theta}$, while the truthful type $\theta = \hat{\theta}$ has probability $g(\hat{\theta})/Q(\hat{\theta})$ as an atom.

Given the above posterior belief, the board must be indifferent between keeping and replacing the manager after observing output $y = k_*$. The corresponding indifference condition is

$$\hat{\theta} \cdot \underbrace{f(k_* - \hat{\theta}) \frac{g(\hat{\theta})}{Q(\hat{\theta})}}_{\text{conditional prob. of } \hat{\theta}} + \int_{\theta_2}^{\theta_1} \theta \cdot \underbrace{f(k_* + d - \theta) \frac{h(\hat{\theta}) g(\theta)}{Q(\hat{\theta})}}_{\text{conditional density of } \theta} \mathrm{d}\theta = \theta_1,$$

or equivalently,

$$(\hat{\theta} - \theta_1) f(k_* - \hat{\theta}) g(\hat{\theta}) = h(\hat{\theta}) \int_{\theta_2}^{\theta_1} (\theta_1 - \theta) f(k_* + d - \theta) g(\theta)\,\mathrm{d}\theta. \tag{6}$$

[22] Such an equilibrium always exists, but many other equilibria also exist. See the proof of Lemma 8 for the general case.

This condition guarantees the optimality of cutoff k for each *individual* report $\hat{\theta}$.

By integrating this individual-level condition (6) with respect to $\hat{\theta}$, we obtain an *aggregate* necessary condition for optimality:

$$\int_{\theta_1}^{\overline{\theta}} (\theta - \theta_1) f(k_* - \theta) g(\theta)\, \mathrm{d}\theta = \int_{\theta_2}^{\theta_1} (\theta_1 - \theta) f(k_* + d - \theta) g(\theta)\, \mathrm{d}\theta. \tag{7}$$

The left-hand side represents the aggregate positive effect from keeping the current manager across all types above θ_1. The right-hand side is the corresponding effect from the misreporting types. When θ_2 is given, the equilibrium value of the uniform cutoff k_* must satisfy the necessary condition (7). Indeed, we uniquely find the value of k_* that solves (7) due to the monotone likelihood property. The following lemma summarizes the above argument and provides additional results.

Lemma 8. *In any equilibrium that survives the D1 criterion, the uniform cutoff k_* for reports above θ_1 satisfies condition (7). For each value of $\theta_2 \in [-\infty, \theta_1)$, there uniquely exists $k_*(\theta_2) \in \mathbb{R}$ that solves condition (7). The function $k_*(\theta_2)$ is continuous, non-increasing, and goes to $-\infty$ as $\theta_2 \to \theta_1$.*

To construct an equilibrium, we aim to find a pair $(\theta_2^{\mathrm{f}}, k^{\mathrm{f}})$ that simultaneously satisfies $\theta_2^{\mathrm{f}} = \theta_2(k^{\mathrm{f}})$ and $k^{\mathrm{f}} = k_*(\theta_2^{\mathrm{f}})$. To this end, we consider a function

$$\Gamma(k) = \begin{cases} k_*(\theta_2(k)) & \text{if } \theta_2(k) < \theta_1 \\ -\infty & \text{otherwise,} \end{cases}$$

and its fixed point. The function Γ is non-increasing because k_* is non-increasing and θ_2 is increasing. The maximum $\Gamma(-\infty) = k_*(\underline{\theta})$ is finite and the minimum $\Gamma(+\infty) = k_*(\theta_1)$ goes to $-\infty$. Since Γ is continuous, we can find a unique fixed point k^{f}. The fixed point k^{f} is finite and thus θ_2^{f} is smaller than θ_1.

Proposition 3. *The function $\Gamma(k)$ has a unique, finite fixed point k^{f}. Define $\theta_2^{\mathrm{f}} = \theta_2(k^{\mathrm{f}})$. Then, $\theta_2^{\mathrm{f}} < \theta_1$ and the pair $(\theta_2^{\mathrm{f}}, k^{\mathrm{f}})$ satisfies $k^{\mathrm{f}} = k_*(\theta_2^{\mathrm{f}})$ as well as $\theta_2^{\mathrm{f}} = \theta_2(k^{\mathrm{f}})$.*

Due to Proposition 3, any equilibrium that survives the D1 criterion must use k^{f} as the uniform cutoff and θ_2^{f} as the threshold θ_2 of misreporting. Conversely, we can also construct an equilibrium from these two parameters. (Equation (6) constructs an equilibrium by determining the equilibrium value of $h(\hat{\theta})$.) We have fully characterized (except on the set of probability 0) the behavior of the board and the manager on the equilibrium path, but it remains unknown whether the board sets $k(\hat{\theta}) = +\infty$ after out-of-equilibrium messages

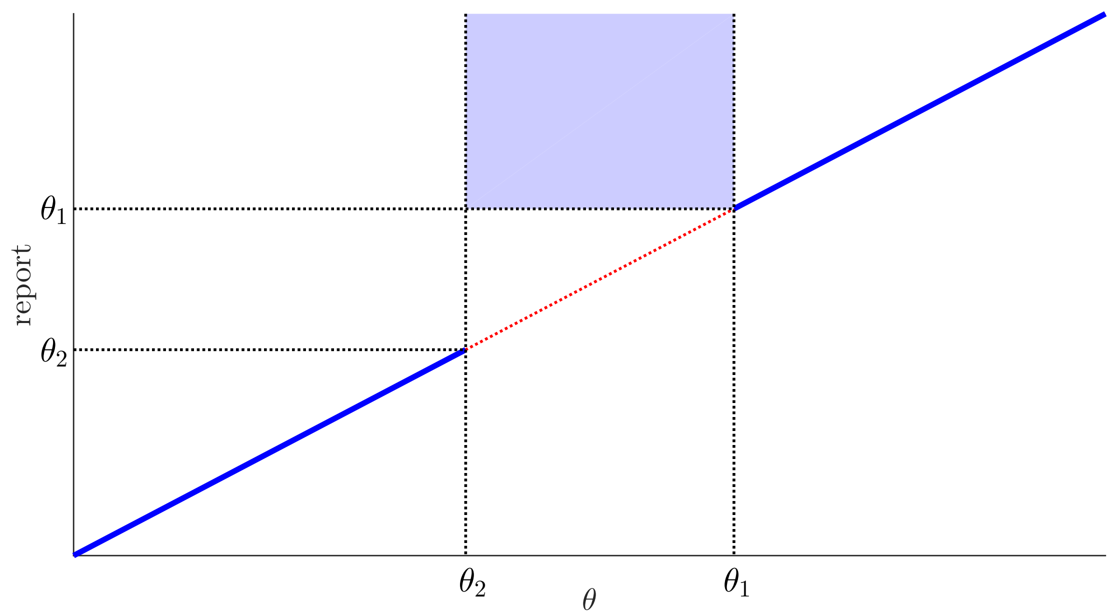

Figure 5: The manager's reporting strategy. The manager with type $\theta \in (\theta_2, \theta_1)$ uses a report in the shaded area.

between θ_2^{f} and θ_1. Although this out-of-equilibrium behavior is now irrelevant in characterizing what happens on the equilibrium path, we can obtain the desired result—an infinite cutoff for bad messages—by imposing the D2 criterion.[23]

Theorem 3. *Consider the model with exogenous $c \in [\underline{\theta} - \mathbb{E}(\theta), \overline{\theta} - \mathbb{E}(\theta)]$. There exists a perfect Bayesian equilibrium that survives both the D1 and D2 criteria. Any equilibrium that survives the D1 criterion almost surely satisfies the following properties in the first period:*

(i) *The manager truthfully reports her type if $\theta > \theta_1$ or $\theta < \theta_2^{\mathrm{f}}$. The manager with type $\theta \in (\theta_2^{\mathrm{f}}, \theta_1)$ chooses some report above θ_1.*

(ii) *The board replaces the manager when the manager reports $\hat{\theta} < \theta_1$ or the cash flow y is less than k^{f}. The board retains the manager if $\hat{\theta} > \theta_1$ and $k > k^{\mathrm{f}}$.*

Here, θ_2^{f} and k^{f} are as in Proposition 3. Furthermore, in any equilibrium that survives the D2 criterion, the board sets cutoff $+\infty$ after almost every report $\hat{\theta} \in (\underline{\theta}, \theta_1)$.

We make three remarks before proceeding to endogenize the parameter c. First, the CEO's reporting behavior is non-monotonic as depicted in Figure 5. This non-monotonicity

[23]The D2 criterion imposes more restrictions than the D1 criterion. Thus, if an equilibrium survives the D2 criterion, then this equilibrium also survives the D1 criterion. See fn. 31 regarding why the D1 criterion is insufficient.

occurs because the CEO with low θ is unable to get a reasonably high chance of retention when she mimics some type. Hence, types below θ_2 truthfully reveal their ability, learn the true state, and then are subsequently replaced. The threshold type θ_2's retention probability from mimicking is high enough that she is indifferent between truthful reporting and misreporting. The types above θ_2 but below θ_1 overstate their types, sometimes far above θ_1. The types above θ_1 report truthfully.

Second, the board employs a two-step retention policy. As the first step, the manager asks the CEO to report her current situation. If the CEO reports something too pessimistic, the board helps her to revive the firm during her tenure but the removal of the CEO is unchangeable. This corresponds to the findings of Cornelli et al. (2013), who show that boards often utilize "soft" (nonverifiable) information regarding the CEO's ability when making replacement decisions. Having passed the first step, to stay in the firm, the CEO needs to achieve the target k_* set by the board, as the second step. This equilibrium replacement behavior helps to explain the inverse relationship between performance and CEO turnover found in the empirical literature (see, e.g., Jenter and Lewellen (2017)).

Third, mangers with intermediate ability $\theta \in (\mathbb{E}[\theta], \theta_1)$ are sometimes removed due to poor communication with the board (i.e., misreporting). Because truthful reporting terminates her tenure, these CEOs are urged to overstate their situation. This miscommunication reduces the effectiveness of the advice from the board and, consequently, CEOs with ability in this range tend to have worse performance due to the lack of information.

4.2 Shareholders' decision

Due to the complexity of the model with a continuous type space, we must employ additional distributional assumptions in order to obtain analytic results with endogenously determined c. Specifically, we assume that type θ and noise ε are uniformly distributed on supports $[\underline{\theta}, \overline{\theta}]$ and $[-q, q]$, respectively. Although the uniform distribution $F(\varepsilon)$ does not fully satisfy the monotone likelihood ratio property, the distribution can be seen as a limit of distributions with this property.[24] We note that the results are not qualitatively sensitive to these assumptions, as shown in the simulations following this section.

One benefit of using the uniform distribution is that we can calculate closed-form characterizations of the board's cutoff strategy and the shareholders' optimal board policy. One drawback, however, of the uniform setting is that certain pathological cases arise which

[24] Here is an example of such a sequence. Let ϕ denote the density of the standard normal distribution and define $z_n(x)$ by $z_n = |x|/n$ for $|x| < q$ and by $z_n = n|x| - q(n - 1/n)$ for other x. Then, density $f_n(x) = \phi(z_n(x))/\int_{-\infty}^{\infty} \phi(z_n(y))\,\mathrm{d}y$ satisfies the monotone likelihood ratio property, because z_n is convex, and the corresponding distribution $F_n(x) = \int_{-\infty}^{x} f_n(x)\,\mathrm{d}x$ weakly converges to the uniform distribution.

confound the analysis. We thus impose the following regularity conditions:

$$2q/\chi > \frac{4q-\Delta}{2\Delta}, \tag{8}$$

where $\Delta = \overline{\theta} - \underline{\theta}$. This condition ensures that the distortion effect in the cutoff (i.e., too strict cutoff) is not too strong. We later see that $2q/\chi$ has a negative effect on $k - \theta_2$. When θ_2 is fully determined by other parameters—this is the case in Proposition 4 (i)—an increase in $2q/\chi$ lowers the level of equilibrium cutoff and thus mitigates the distortion in the retention decision. We also impose the following condition:

$$\underline{\theta} + q - (1 + 8q/\chi)^2 d \geq \overline{\theta} - q. \tag{9}$$

This inequality is a technical condition that significantly simplifies the analysis by eliminating subtle pathological cases that arise in the dual uniform setting. To interpret this condition, note that the inequality (9) implies $\underline{\theta} + q > \overline{\theta} - q$; that is, the support of ε is sufficiently large such that low-type CEOs can potentially mimic up to the highest type.

We first characterize the equilibria with exogenously given c for this parameterization. We focus on equilibria consistent with the equilibria found in the previous section: *The board employs a uniform cutoff k for messages above θ_1 and replaces the manager for sure with messages below θ_1.* To simplify the notation, let $\theta_{2,0} = d + F^{-1}(d/\chi) \equiv d - q + 2dq/\chi$ denote the intercept of the threshold $\theta_2 = k + \theta_{2,0}$ (see equation (5)).

Proposition 4. *Consider the game described above with exogenously given c, and we focus on the class of equilibria described above. Assume the regularity conditions* (8) *and* (9)*.*

(i) *Suppose $c \in (0, \Delta/2)$. In any equilibrium, the board sets a uniform cutoff $k = 2\theta_1 - \overline{\theta} - \theta_{2,0}$ and the threshold $\theta_2 = 2\theta_1 - \overline{\theta}$ is greater than the worst type $\underline{\theta}$. The cutoff is high enough to replace even the best type with positive probability (i.e., $\overline{\theta} - q \leq k$).*

(ii) *Suppose $c \in (-\Delta/2, 0)$. In any equilibrium, the board sets a uniform cutoff $k = 2\theta_1 - q - \underline{\theta}$ and no type below θ_1 chooses a truthful message (i.e., $\theta_2 \leq \underline{\theta}$). The cutoff is low enough such that the best type is never replaced (i.e., $\overline{\theta} - q \geq k$).*

(iii) *Suppose $c = 0$. In any equilibrium, no type below θ_1 chooses a truthful message (i.e., $\theta_2 \leq \underline{\theta}$). The neutral board has continuously many optimal cutoffs and the set of optimal cutoffs is the interval between the cutoffs given in (i) and (ii); i.e., $[\overline{\theta} - q, \underline{\theta} - \theta_{2,0}]$.*

(iv) *Suppose $|c| \geq \Delta/2$. In any equilibrium, the manager truthfully reports her type for sure. The board replaces the manager with probability 1 if $c \geq \Delta/2$. The manager is* *retained for sure if $c \leq -\Delta/2$.*

For any value of c, an equilibrium exists.

The first two cases are especially important. In case (i), types above θ_1 are rare so that the board needs to bring the cutoff k high enough to discourage lower types from mimicking. As a result, the worst types report truthfully but even the best type faces the risk of replacement. In equilibrium, the misreporting interval between θ_1 and θ_2 needs to be perfectly balanced with the types above θ_1—due to the uniform specification—so that $\overline{\theta} - \theta_1 = \theta_1 - \theta_2$. This value of θ_2, in turn, determines the value of $k = \theta_2 - \theta_{2,0}$.

In contrast, the board in case (ii) faces a large mass of types above θ_1 so that the cutoff is too low to discourage imitation. Consequently, some types above θ_1 are never replaced with the friendly choice of a cutoff. The threshold is type $\theta = k + q$; types above the threshold always have output higher than the cutoff. The equilibrium condition in this case is thus $(k + q) - \theta_1 = \theta_1 - \underline{\theta}$.

Cases (iii) and (iv) are less important in two different senses. Case (iv) is a trivial case where the board employs an extremely aggressive or friendly retention policy. The equilibrium multiplicity in case (iii) is apparently problematic, but the choice of k does not affect the payoff for the shareholders because the neutral board perfectly represents the shareholders' interest. Therefore, we only need to analyze the first two cases, keeping in mind that the extreme board (case (iv)) could be optimal. Indeed, we ultimately show that the optimal choice of c always lies in case (i).

We now aim to find the optimal level of c for shareholders. As in the two-type model, the shareholders' payoff can be divided into the two periods. In the first period, shareholders receive the payoff

$$V_1 = -d \left\{ G(\theta_1) - G(\theta_2) \right\},$$

plus $\mathbb{E}[\theta]$. Here, the difference $G(\theta_1) - G(\theta_2) = \Pr\{\theta \in [\theta_2, \theta_1]\}$ is the probability of misreporting. The CEO in period 2 never misreports her type, but the board's retention policy in the first period affects the expected value of θ in the second period. The expected second-period value is

$$V_2 = \int_{\theta_1}^{\overline{\theta}} (\theta - \mathbb{E}[\theta]) F(\theta - k) g(\theta) \, \mathrm{d}\theta + \int_{\theta_2}^{\theta_1} (\theta - \mathbb{E}[\theta]) F(\theta - k - d) g(\theta) \, \mathrm{d}\theta, \tag{10}$$

plus $\mathbb{E}[\theta]$. The shareholders maximize the sum $V = V_1 + V_2$ by controlling c.

We can analytically calculate the values of V_1 and V_2 in this dual uniform environment due to Proposition 4. In the exposition, we focus on the relevant case of $c \in (0, \Delta/2)$. (See the Appendix for the case of the friendly board.) Since $\theta_2 = 2\theta_1 - \overline{\theta} = \mathbb{E}[\theta] + 2c - \Delta/2$, the period-1 payoff V_1 is

$$V_1 = -d\left(\frac{1}{2} - \frac{c}{\Delta}\right). \tag{11}$$

The period-2 payoff V_2 is similarly given as:

$$V_2 = \int_{\theta_1}^{\overline{\theta}} (\theta - \mathbb{E}[\theta]) \left(\frac{\theta - k + q}{2q}\right) \frac{\mathrm{d}\theta}{\Delta} + \int_{\theta_2}^{\theta_1} (\theta - \mathbb{E}[\theta]) \left(\frac{\theta - d - k + q}{2q}\right) \frac{\mathrm{d}\theta}{\Delta}. \tag{12}$$

By $k = 2\theta_1 - \overline{\theta} - \theta_{2,0} = \mathbb{E}[\theta] + 2c - \Delta/2 - \theta_{2,0}$, we obtain the following cubic function:

$$V_2 = \frac{1}{48q}\left(1 - \frac{2c}{\Delta}\right)\left\{24c \cdot \theta_{2,0} + 4c(\Delta + 6q) + 2\Delta^2 - d(18c - 3\Delta) - 16c^2\right\}. \tag{13}$$

Note that $1 - 2c/\Delta > 0$ because $c < \Delta/2$.

Since $V = V_1 + V_2$ is a cubic function with a positive coefficient on c^3, the function V will increase, attain a local maximum, decrease, and then increase again if V behaves regularly enough. Such non-monotonicity occurs due to the countervailing disciplinary and distortion effects. Recall that these two effects first emerged in the analysis of the two-type setting in Section 3. The disciplinary effect from raising c results in low-type CEOs reporting truthfully to the board. In this continuous setting, the distortion effect not only appears in the form of an excessively strict *cutoff*, but also emerges as an inefficiently high standard in *reporting*. That is, the board demands highly optimistic reports for retention considerations, and consequently, intermediate types $\theta \in (\mathbb{E}[\theta], \theta_1)$ must leave their position after truthful reporting.

After a few calculations, we obtain that the local maximum is attained at

$$c_A^* = \frac{1}{8}\left\{d\left(1 + \frac{8q}{\chi}\right) + 2\Delta - \sqrt{D}\right\}, \tag{14}$$

where $D = d^2(1 + 8q/\chi)^2 + 4d\Delta - 32dq + 4\Delta^2 > 0$. In the Appendix, we show that this local optimum is indeed the global optimum.

Theorem 4. *Assume the regularity conditions* (8) *and* (9)*. It is optimal for shareholders to choose an aggressive board with* $c \in (0, \Delta/2)$*. The optimal value of* c *is uniquely given by equation (14).*

Theorem 4 states that shareholders set the board to be aggressive, and provides a closed-form characterization of the optimal policy c. We note that, in contrast to the two-type case, the optimal policy is not maximal aggression (i.e., $c \in (0, \Delta/2)$). We find that a moderately aggressive board is optimal since the shareholders' disutility from the distorted retention decision eventually exceeds the benefit of the disciplinary effect as the degree of aggressiveness, c, increases.

This result is in stark contrast to several theoretical studies which have found that a management-friendly board (excessive retention) is optimal for shareholders. By considering the interrelationship between advising and replacement, we find that aggressive boards (excessive replacement) can be optimal for shareholders, which upends the results of models that separately examine advising (e.g., Adams and Ferreira (2007)) or replacement (e.g., Almazan and Suarez (2003), Laux (2008), Casamatta and Guembel (2010), Inderst and Mueller (2010), Dow (2013)).

Furthermore, the explicit solution c_A^* allows for comparative statics analysis. Note that the variables q and Δ are interchangeable with the variances of noise ε and type θ, respectively, in the comparative statics below, because $\text{Var}(\varepsilon) = q^2/3$ and $\text{Var}(\theta) = \Delta^2/12$.

Proposition 5. *Assume the regularity conditions* (8) *and* (9)*. The optimal aggressiveness c_A^* is increasing in the cost of miscommunication, d, and in the variance of noise ε, and decreasing in the variance of type θ. An increase in the private benefits, χ, decreases (increases) c_A^* if $\Delta - d(8q - \Delta)$ is positive (negative).*

An increase in d, the value loss from uninformative communication, results in shareholders setting a more aggressive board. Intuitively, this occurs since shareholders prefer to elicit greater truthful reporting, and thus informative communication, from the CEO. Indeed, the coefficient on c in V_1 increases (see equation (11)). On the other hand, an increase in d also *decreases* the cutoff k, as seen in Proposition 4 (i), by making truthful communication more attractive and misreporting less likely. Consequently, to compensate for this milder replacement threshold by the board and due to their strengthened preference for truthful reporting, shareholders push to make the board *more* aggressive in their replacement of the CEO as d increases.

Similarly, shareholders prefer a more aggressive board as the variance of noise (and thus q) increase. In this case, the support of ε expands and more low-type managers are potentially able to mimic a higher type in their observed output. Likewise, there is less room for very high types to meet the cutoff with certainty. As a result, the cutoff-based retention policy becomes less effective and the choice of cutoff k becomes less important (indeed, V_2 shrinks as q increases; see (13)). In other words, the increase in the noise dilutes the distortion effect of aggressiveness, while the disciplinary effect, represented by V_1, is unchanged. With the disciplinary effect intact but the distortion effect weakened, shareholders face an incentive to make the board *more* aggressive in response to an increase in q.

Proposition 5 also shows that the shareholder's optimal aggressiveness c_A^* is decreasing in the variance of the CEO's productivity θ (and thus Δ). As the variance increases, the shareholders face an increased risk in replacing a highly talented CEO. We find that the increase in variance amplifies the disutility from the distortion effect of a high cutoff k, and leads shareholders to prefer a comparatively less aggressive board.

Lastly, the effect of an increase in the private benefits χ is negative when Δ (or equivalently, the variance of productivity θ) is sufficiently large compared to d and q (or equivalently, the variance of noise ε). As the private benefit χ increases, misreporting becomes more appealing for a low-type manager. The board, in turn, responds to the CEO's increased incentive for mimicry by increasing the retention standard k. The shareholders do not favor this decision of raising the bar when Δ is large enough. As already seen in the previous paragraph, an increase in Δ worsens the distortion effect of an aggressive board. Hence, the board's response to increasing the cutoff k is an *overreaction* from the shareholders' perspective, thus resulting in a *decrease* in c_A^*. Conversely, the distortion effect becomes relatively unimportant in a highly noisy situation (when q is high relative to Δ), which is also already seen two paragraphs above. This induces the shareholders to make the board more aggressive in response to the increased misreporting incentive from a higher χ.

5 Empirical implications

The results in the above sections provide a number of novel empirical predictions. First, Theorem 2 and Proposition 5 predict that there should be comparatively higher CEO turnover in firms or industries where CEO communication with the board is more valuable or when the board's advisory capacity is more salient. In terms of more concretely classifying firms/industries where the board's advisory role is prominent, Klein (1998) and Coles et al. (2008, 2012) suggest that this is true for firms which are comparatively more complex, such as firms which are more diversified, larger, or more highly leveraged. For example, multi-segmented firms which have operations in different industries or segments may rely more on the board's advising capacity, as the board generally includes directors who are experts in different industries (Hermalin and Weisbach (1988), Yermack (1996)). Moreover, Markarian and Parbonetti (2007), Coles et al. (2008, 2012), and Linck et al. (2008) find evidence consistent with the notion that complexity of the industry or firm is also met with a greater advisory role of the board. In addition, the board's advisory role is more salient for firms which have a higher incidence of acquisitions or other major corporate actions (Paul (2007)). Hence, we predict that the board replaces the CEO relatively more often when informative communication with the board is more valuable (high d), such as for firms with greater internal or external complexity, or for firms who engage more frequently in major corporate actions, such as acquisitions or divestitures.

Taylor (2010) finds that large firms—a proxy used for complexity by Coles et al. (2008)—exhibit excessive replacement of the CEO, consistent with the prediction above. Moreover, Weisbach (1988), Dahya et al. (2002), and Guo and Masulis (2015) have found a positive association between outside director presence on the board and CEO turnover. A number of studies provide evidence which suggests that outside directors offer better advice (e.g., Dalton et al. (1999), Agrawal and Knoeber (2001), Fich (2005)) and that complex firms appoint more outside directors (e.g., Coles et al. (2008, 2012)).[25] To the extent that outside directors are appointed due to firm complexity, these empirical findings are consistent with our first empirical prediction.

Our second prediction is that the board is *more* aggressive in replacing the CEO when the CEO's ability is more difficult to assess given the performance of the firm, or when there is greater uncertainty regarding the CEO's ability (high q). Existing research related to this prediction has found evidence which is consistent. For example, in a sample of leveraged buyouts, Cornelli and Karakaş (2015) find that CEO turnover decreases in the second phase of acquisition and is lower than the corresponding rate for public firms. This is consistent with the above prediction as private equity firms are very closely involved in the firm's operations, resulting in less uncertainty in their inferences of the CEO's ability. Additionally, a number of studies have found that private firms replace their CEO less often than public firms (e.g., Lel et al. (2014), Gao et al. (2017)). Private firms generally have more concentrated, closely held, and illiquid ownership, which results in stronger monitoring incentives and hence less uncertainty over assessments of CEO ability (Kahn and Winton (1998)).

Recall that the parameter Δ denotes the variance of CEO talent. Proposition 5 predicts that the board is less aggressive in their replacement of the CEO when the variance of the replacement is high. This can be interpreted as industries with CEO markets which entail greater uncertainty over candidates. For example, the results predict that firms in industries which require significant firm-specific information face greater uncertainty in their CEO hiring (i.e., hiring a top executive from another firm may not lead to similar performance as the polices and knowledge are very firm-specific), and thus these industries will be met with relatively less CEO turnover.

The variance parameter Δ in CEO ability can be interpreted to reflect the degree of ho-

[25] Hermalin and Weisbach (1988) argue that "the CEO may choose an outside director who will give good advice and counsel, who can bring valuable experience and expertise to the board" (p. 590).

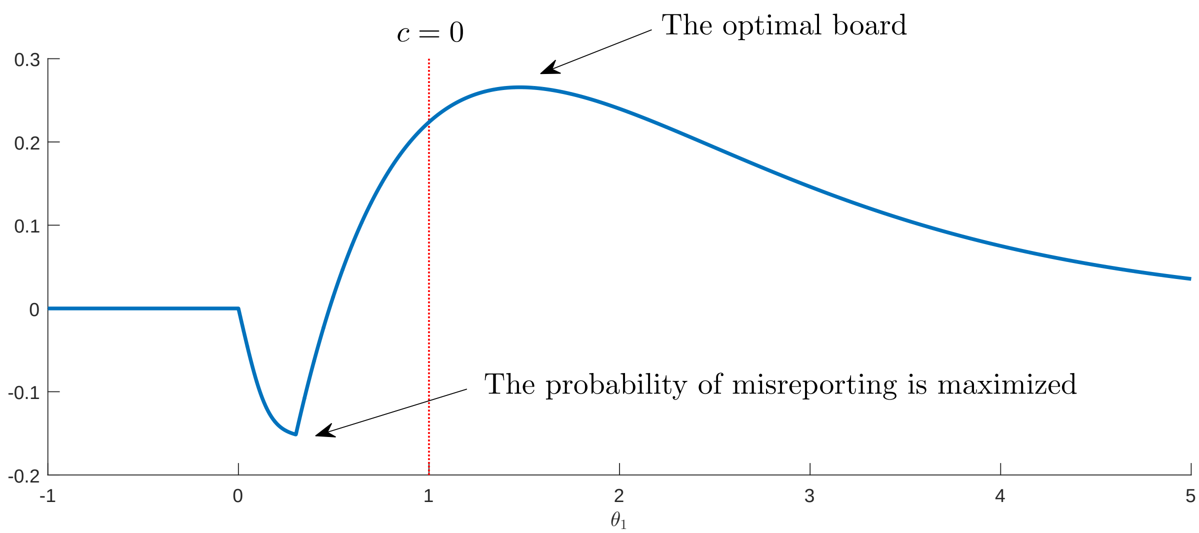

Figure 6: The objective function of the board.

mogeneity among firms in an industry. As Parrino (1997) argues, there is less variation in the replacement CEO when the new CEO is hired from a peer firm in a relatively homogeneous industry. Parrino (1997) has found some evidence consistent with the above prediction; he documents that CEO turnover is higher in industries which are relatively more homogeneous.

Building on this, Proposition 5 provides additional predictions regarding turnover rates within homogeneous and heterogeneous industries. The results show that, when the variance Δ is sufficiently high, a high level of private benefits of control, χ, should be met with a less aggressive board and less turnover. The converse is true when Δ is sufficiently low. Hence, the results of the model predict that, when considering only homogeneous industries, there should be a less aggressive board and hence less turnover when the CEO has relatively more control over the firm. Conversely, in heterogeneous industries, we expect greater turnover in firms where the CEO has more control over the firm.

6 Numerical Results

In this section, we present numerical exercises for additional economic implications and to show that the results of the model are robust to alternative distributions of θ and ε. We first assume that the noise ε is normally distributed and type θ is exponentially distributed. More specifically, the distribution of ε has mean 0 and variance 1 and the exponential distribution has intensity $\lambda = 1$. Also, we use $d = 1$ and $\chi = 2$ throughout the numerical analyses.

As shown in Figure 6, the optimal board is moderately aggressive (note that $\theta_1 = 1 + c$); the optimal value of c is numerically given as $c \approx 0.4775$. The bottom of the negative

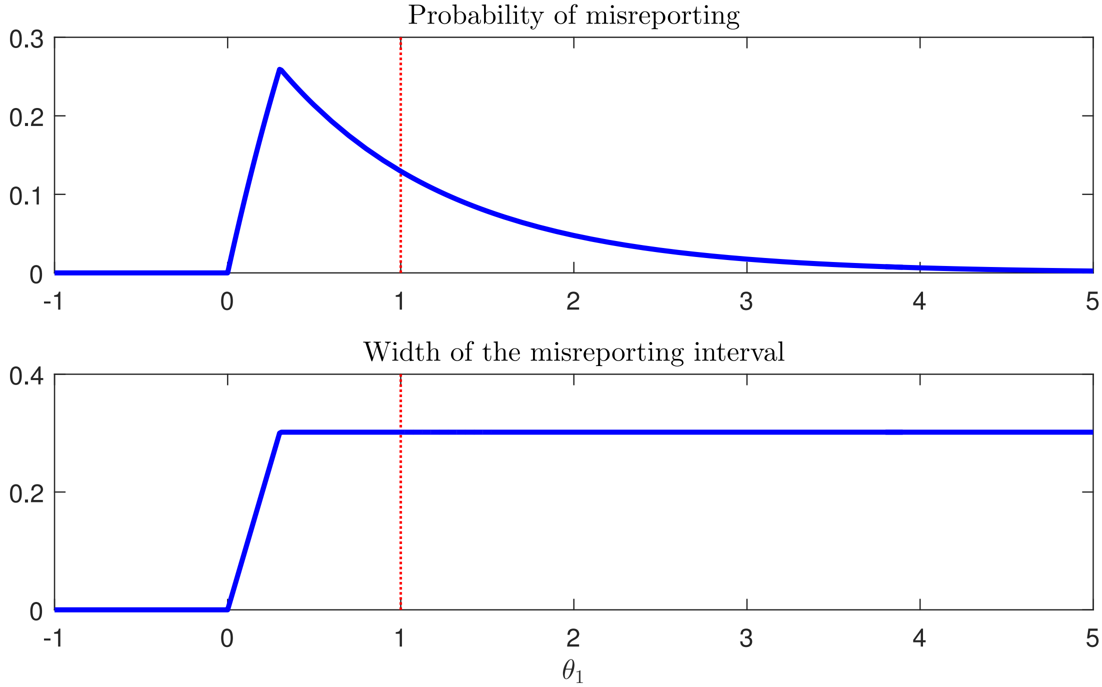

Figure 7: Two variables that capture the misreporting behavior of the manger: the probability of misreporting (i.e., $G(\theta_1) - G(\theta_2)$) and the width of the misreporting interval $[\max\{\theta_2, \underline{\theta}\}, \theta_1]$.

spike in Figure 6 represents the point where θ_2 reaches $\underline{\theta}$ $(= 0)$. This sharp drop in the shareholders' payoff occurs due to frequent misreporting triggered by the board's excessively friendly retention policy. Figure 7 elucidates this point by describing how misreporting behavior changes in response to a change in $\theta_1 = \mathbb{E}[\theta] + c$. The bottom of the negative spike in Figure 6 corresponds to the peak of the spike in the top panel of Figure 7 (i.e., when the misreporting probability is greatest). The misreporting region and the interval initially increase as θ_1 increases, and is maximized when θ_2 reaches $\underline{\theta}$ $(= 0)$. After this point, the misreporting interval $[\theta_2, \theta_1]$ is pushed leftward and exponentially reduces its probability while keeping the width $\theta_2 - \theta_1$ constant.[26] This disciplinary effect—the reduction of misreports due to the board's aggressive retention policy—creates the hump after the negative spike in Figure 6 and induces a moderately aggressive board to be optimal for shareholders.

To see how the other effect—distortion in retention decisions—hurts the value of the firm for shareholders, see Figure 8 which depicts the contribution of the equilibrium retention pol-

[26]Interestingly, Figure 7 shows that, while the probability of misreporting declines as θ_1 increases, the misreporting interval $[\theta_2, \theta_1]$ remains constant after the maximum is reached. This occurs because the board is shifting the misreporting interval $[\theta_2, \theta_1]$ further away from the high-density regions populated with low types.

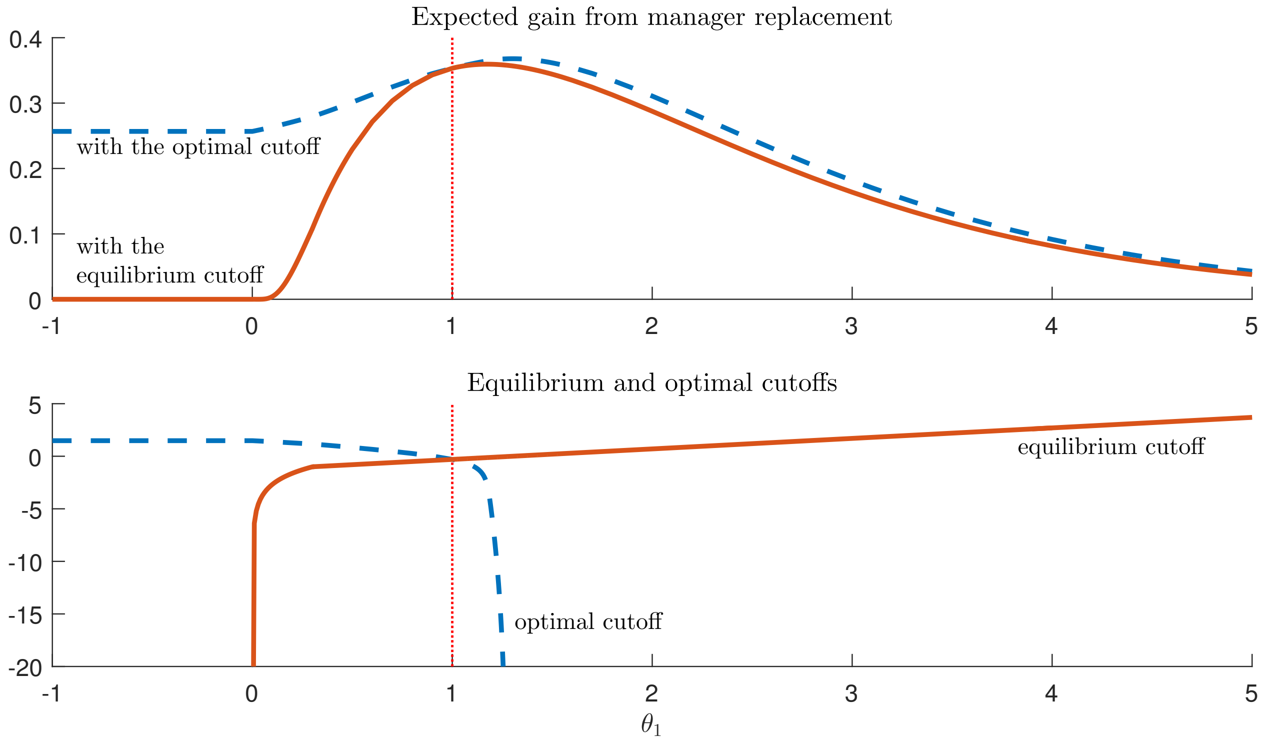

Figure 8: The gain from the manager selection ($V_2 - \mathbb{E}[\theta]$) and the choice of cutoff in the equilibrium retention policy. The solid curves represent the actual values in equilibrium. The dashed curves correspond to the optimal cutoff $k^{\text{opt}}(\theta_1, \theta_2)$ with θ_1 and θ_2 fixed.

icy to the shareholders' objective function. To gauge the level of distortion in the equilibrium retention policy, we introduce the *optimal cutoff* $k^{\text{opt}}(\theta_1, \theta_2)$ with the misreporting interval $[\theta_2, \theta_1]$ as given. Formally, the optimal cutoff $k^{\text{opt}}(\theta_1, \theta_2)$ is the solution of the maximization problem

$$\int_{\theta_1}^{\overline{\theta}} (\theta - \mathbb{E}[\theta]) F(\theta - k) g(\theta) \, \mathrm{d}\theta + \int_{\theta_2}^{\theta_1} (\theta - \mathbb{E}[\theta]) F(\theta - k - d) g(\theta) \, \mathrm{d}\theta.$$

The corresponding first-order condition

$$\int_{\theta_1}^{\overline{\theta}} (\theta - \mathbb{E}[\theta]) f(\theta - k^{\text{opt}}) g(\theta) \, \mathrm{d}\theta = \int_{\theta_2}^{\theta_1} (\mathbb{E}[\theta] - \theta) f(\theta - k^{\text{opt}} - d) g(\theta) \, \mathrm{d}\theta,$$

resembles but differs from the equilibrium condition (7), in which the board uses $\theta_1 = \mathbb{E}[\theta] + c$ in place of $\mathbb{E}[\theta]$. As shown in the lower half of Figure 8, the equilibrium cutoff is suboptimal unless $c = 0$: the board is too strict in setting k when aggressive (i.e., $c > 0$) and too lenient when friendly (i.e., $c < 0$).

Although Figure 8 shows that the equilibrium cutoff significantly deviates from the optimal level, this figure also suggests that the effect to the value of the firm is limited on the aggressive side (i.e., $c > 0$). To see why this is the case, we examine how the board's aggressive cutoff impacts the second-period value V_2 of the firm, defined in equation (10). We decompose this effect by considering three groups: the truth-telling top group $(\theta_1, \overline{\theta})$, misreporting intermediate group $(\mathbb{E}[\theta], \theta_1)$, and misreporting bottom group $(\theta_2, \mathbb{E}[\theta])$. (The types below θ_2 are the real bottom group, but they are replaced for sure anyway and thus not affected by the uniform cutoff.)

According to Figure 9, the suboptimality of the equilibrium cutoff slightly affects the bottom group $(\theta_2, \mathbb{E}[\theta])$ but has virtually no effect on the top and middle groups when $\theta_1 > \mathbb{E}[\theta]$ is close enough to $\mathbb{E}[\theta]$. Indeed, the effect on the bottom group is *positive* because a high cutoff helps to remove unwanted types in this group. Instead, the suboptimally high cutoff decreases the period-2 values from the top two groups, but the top group appears almost unaffected. The effect to the middle group is also minute (although it appears to be large due to the scaling of the graph). This observation does not change even if we replace the optimal cutoff with the first-best, but infeasible, retention policy: $k = +\infty$ for the bottom group and $k = -\infty$ for the top two groups. This observation implies that the types in the top group can easily pass the equilibrium cutoff, which is inflated upwards due to $c > 0$. Also, the contribution from the middle group is negligibly small even with the most favorable cutoff $k = -\infty$. This negligibility is partly because the upward bias of the equilibrium cutoff increases θ_2 and significantly lowers the probability of misreporting, as we have also seen in Figure 7.

We obtain similar results when θ is normally distributed. Figure 10 indicates that the optimal board is, once again, aggressive. In this figure, we assume that θ is normally distributed with mean 10 and standard deviation 1. All of the other parameters are the same. Although this case is more smooth than the exponential case—there is no longer a negative spike in the payoff or a discontinuity in the type distribution—we still observe similar patterns in Figure 11. The probability of misreporting is decreasing in θ_1 in the aggressive region $\theta_1 > \mathbb{E}[\theta]$ and the distortive impact of aggressiveness in the cutoff is quite limited.

These two numerical results exemplify the robustness of the results. Here, we present only two numerical results, but an aggressive board easily turns out to be optimal as long as the parameters are not too extreme. Even though analytic calculations are intractable, except for the dual uniform environment we have studied in Section 4.2, the numerical results presented in this section prove how commonly aggressive boards emerge in our setting.

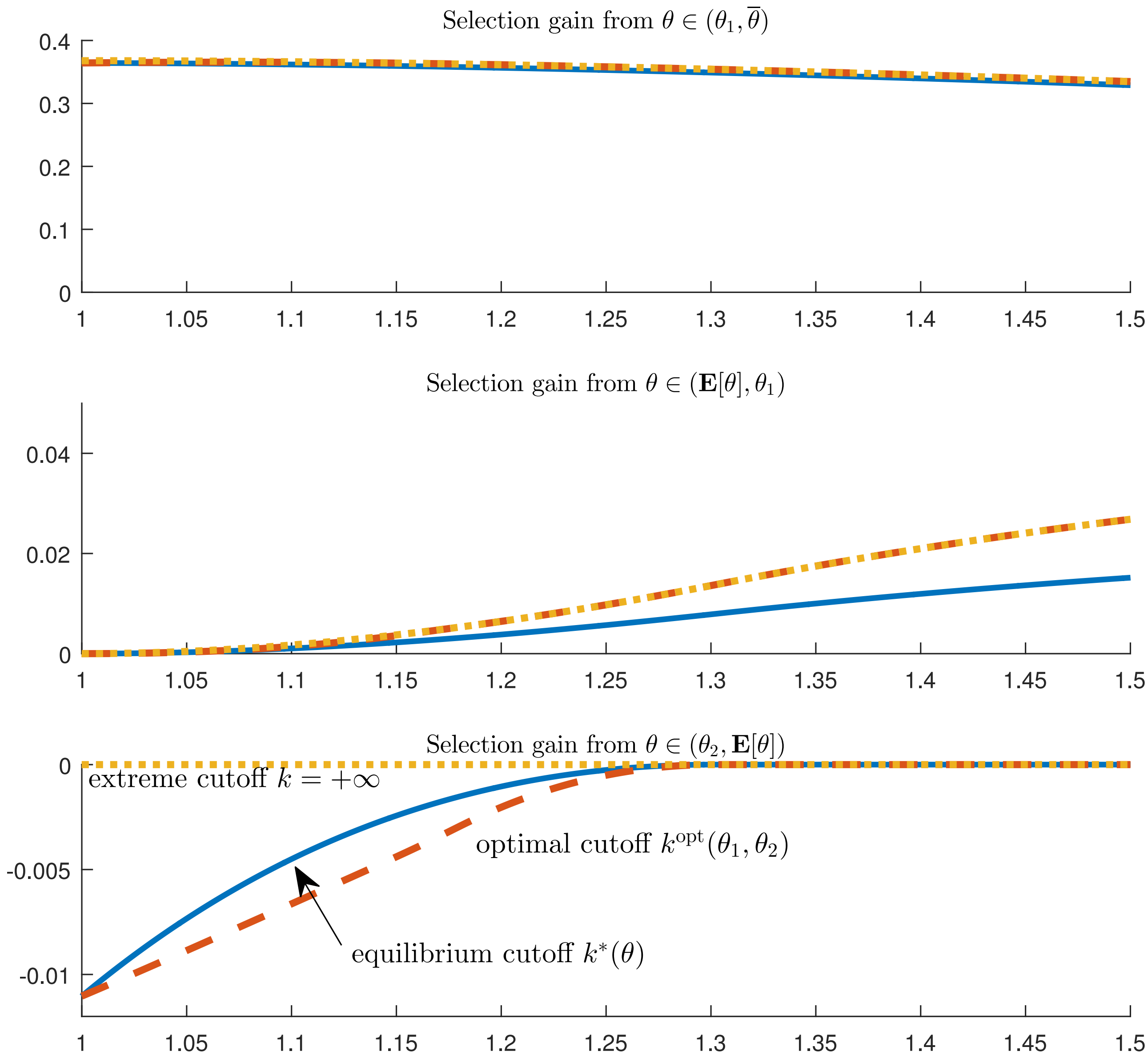

Figure 9: The contribution of groups $(\theta_1, \overline{\theta})$, $(\mathbb{E}[\theta], \theta_1)$, $(\theta_2, \mathbb{E}[\theta])$ to the normalized period-2 value of the firm. The solid curves represent the actual equilibrium values. The dashed curves represent the values with the optimal cutoff $k^{\text{opt}}(\theta_1, \theta_2)$. The dotted curves represent those with the infeasible, first-best retention policy (i.e., $k = -\infty$ for the top two groups and $k = +\infty$ for the bottom group).

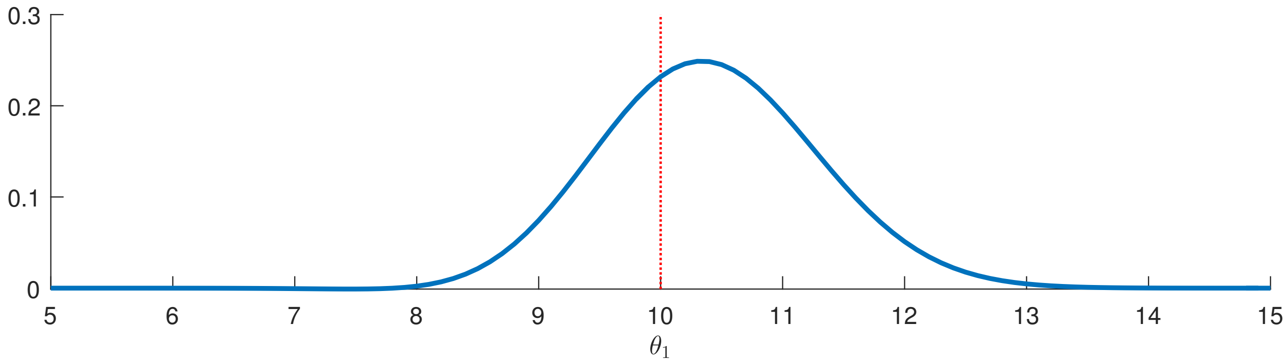

Figure 10: The objective function when $\theta \sim N(10, 1)$.

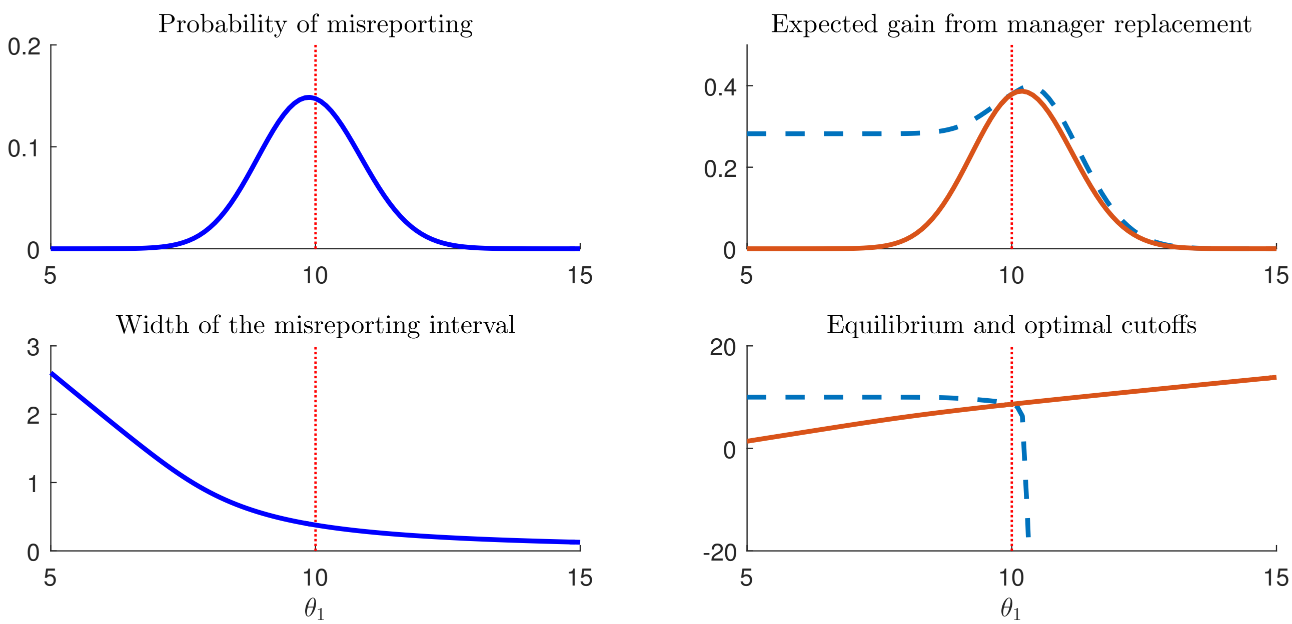

Figure 11: The counterparts of Figures 7 and 8 when $\theta \sim N(10, 1)$.

7 Conclusion

This study investigates CEO replacement when informative communication is beneficial for all parties. The CEO must communicate information to the board, but due to her career concerns, she may be inclined to misreport her private information. There are two notable features in the unique equilibrium we have found. First, the board demands high standards in *both* reports and outputs. The board sets a minimally acceptable optimism θ_1 and output k, whereby the CEO can retain her position only when *both* report $\hat{\theta}$ and output y meet these standards (i.e., $\hat{\theta} > \theta_1$ and $y > k$). Second, the manager employs a non-monotonic reporting strategy. The CEO truthfully reports when her type θ is either satisfactory for the board (i.e., $\theta > \theta_1$) or too low for worthwhile mimicry (i.e., $\theta < \theta_2$). The CEO with a type between these two thresholds, θ_1 and θ_2, overstates her type, seeking a reasonable chance of continuation.

This equilibrium policy of aggressive boards (excessive replacement) occurs when the concentration of low type managers is sufficiently high or when the private benefits of control for the manager in the second period are sufficiently low. This is in stark contrast to the numerous studies which find that management-friendly boards are optimal for shareholders. The results capture several features that are prevalent in practice or in the empirical literature, such as misreporting by managers along the equilibrium path, increased likelihood of removal following poor performance, and excessive replacement (as documented in Taylor (2010)).

The baseline model can be extended in several directions. To focus on the relationship between advising and replacement, we have assumed that the CEO cannot make an effort decision. Including an effort decision by the manager may lead to additional interesting results. CEOs who will be fired for sure will have their effort incentive reduced; however, board aggression can induce greater effort by managers in the misreporting region, as they may exert more effort to counteract the loss in informative communication. This can improve their output measure and make retention more likely. Hence, it is unclear how imposing an effort decision affects the optimality of board aggressiveness. Another direction the model can be explored is through an endogenous entrenchment mechanism that the CEO can impose, such as a long-lived project that is tied to the CEO's presence in the firm. This may lead to more or less aggression depending on how costly it is for the CEO to entrench herself, as well as the additional opportunity loss of the project to shareholders from removal.

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Internet Appendix (For online publication)

A Technical issues on equilibrium and signaling

A.1 Strategies and equilibrium

We impose a standard innocuous measure-theoretic restriction on strategies. After observing $\hat{\theta}^1$, the board must use a Borel-measurable *retention policy* $z : \mathbb{R} \to [0,1]$. The value $z(y)$ represents the probability that the board retains the manager after observing output y (in addition to the message $\hat{\theta}^1$). This condition ensures that the manager can properly evaluate the expected value of each message for her (i.e., if the board used a non-measurable policy, the CEO cannot calculate expected values).[27] Also note that we allow the board to randomize retention and replacement for each value of y (in a measurable manner).

In defining the perfect Bayesian equilibrium, we impose sequential rationality for *all* realizations of history. In contrast, due to the measure theoretic problem associated with conditioning, we require the posterior belief $\beta(\cdot|\hat{\theta})$ after message $\hat{\theta}$ to have the properties that define conditional probability only on a set $\hat{\Theta} \subseteq \Theta$ of messages that is reached with probability 1. We can specify the posterior belief after observing output y more naturally because y has a density $f(y - \theta + d \cdot \mathbf{1}_{\{\theta \neq \hat{\theta}\}})$. Specifically, after seeing y, the board must assign probability

$$\beta(S|\hat{\theta}, y) = \frac{\int_S f(y - \theta + d \cdot \mathbf{1}_{\{\theta \neq \hat{\theta}\}})\, \mathrm{d}\beta(\theta|\hat{\theta})}{\int_\Theta f(y - \theta + d \cdot \mathbf{1}_{\{\theta \neq \hat{\theta}\}})\, \mathrm{d}\beta(\theta|\hat{\theta})}$$

on a Borel-measurable set $S \subseteq \Theta$. We do not have to specify beliefs for period 2 because the board never makes a decision.

A.2 Signaling and equilibrium selection criteria

We interpret the present paper's model as a signaling game as follows. The CEO is the sender and the board is the receiver. The type and message spaces are identical and give by a set Θ, which is either $\{\theta_L, \theta_H\}$ or $(\underline{\theta}, \overline{\theta})$. The action space for the receiver is the set of all the Borel-measurable retention policies $z : \mathbb{R} \to [0,1]$. Recall that the action space includes all the behavioral strategies for the board (after observing some message).

When Θ contains continuously many types, it is an obstacle in defining the D1 criterion that the definition of out-of-equilibrium messages is not self-evident because most (if not all) of the messages are chosen with probability 0. Nevertheless, we often encounter messages that we can naturally endorse as on-equilibrium messages. In particular, if some single type chooses some message with positive probability, then this message should be on equilibrium even if the chance that this message is chosen is 0 in the entire game. Motivated by the above argument, we define a *clearly on-equilibrium message* as a message that is chosen by

[27] As we will see later, on the equilibrium path, optimal retention policies will be in the form of a *cutoff rule*; i.e., there is some *cutoff* $k \in [-\infty, +\infty]$ such that $z(y) = 1$ for $y > k$ and $z(y) = 0$ for $y < k$.

some type with positive probability. We define an *out-of-equilibrium message* as a message that is not clearly on equilibrium.

After defining out-of-equilibrium messages, we can define the D1 and D2 criteria (Cho and Kreps (1987)). Let $Z^*(\hat{\theta})$ be the set of retention policies that are a best response for the board with some non-degenerate belief (i.e., a probability distribution where no single point has probability 1) over type θ after observing message $\hat{\theta}$.[28] A perfect Bayesian equilibrium *survives the D1 criterion* if for all out-of-equilibrium message $\hat{\theta}$, the posterior belief $\beta(\cdot|\hat{\theta})$ assigns no probability on types $\theta \in \Theta$ that satisfy the following condition: There exists a type $\theta^* \in \Theta$ such that $u_s(\theta, \hat{\theta}, z) \geq u_s^*(\theta)$ implies $u_s(\theta^*, \hat{\theta}, z) > u_s^*(\theta^*)$ for all $z \in Z^*(\hat{\theta})$. Here, $u_s(\theta, \hat{\theta}, z)$ is the payoff that the CEO with type θ and message $\hat{\theta}$ receives when the board uses retention policy z. Also, $u_s^*(\theta)$ is the equilibrium payoff for type θ. A perfect Bayesian equilibrium *survives the D2 criterion* if for all out-of-equilibrium message $\hat{\theta}$, the posterior belief $\beta(\cdot|\hat{\theta})$assigns no probability on types $\theta \in \Theta$ that satisfy the following condition: For all $z \in Z^*(\hat{\theta})$, there exists a type $\theta_z^* \in \Theta$ such that $u_s(\theta, \hat{\theta}, z) \geq u_s^*(\theta)$ implies $u_s(\theta_z^*, \hat{\theta}, z) > u_s^*(\theta_z^*)$.

B Proofs

B.1 Proof of Theorem 1

We first provide the value of $\hat{c}$, which is uniquely given by $\mu^*(\hat{c}) = \mu(k_0; 1)$. Since $\mu^*(c) = \{\mathbb{E}[\theta] + c - \theta_L\}/\Delta\theta$, we obtain $\mathbb{E}[\theta] + \hat{c} = \Delta\theta \cdot \mu(k_0; 1) + \theta_L \in (\theta_L, \theta_H)$.

We now show the uniqueness of equilibrium, assuming that (A1) the high type θ_H always report the truthful message; and (A2) after the low message θ_L, the board believes that the manager is a low type for sure. We later show these two assumptions must hold in any D1 equilibrium (Lemmas 10 and 11). First suppose $\mu^*(c) \leq \mu(k_0, 1)$. If the mimicking probability σ^* is less than 1, the equilibrium cutoff k^* must be k_0 or greater by (3) and consequently, $\mu(k^*; \sigma^*) > \mu(k_0, 1) \geq \mu^*(c)$. That is, this value of k is suboptimal and thus σ^* must be 1. With $\sigma^* = 1$, the optimal cutoff for the board is given by $\mu(k; 1) = \mu^*(c)$ and the equilibrium cutoff k^* must be identical to the unique solution of this equality condition. Second suppose $\mu^*(c) > \mu(k_0, 1)$. If $\sigma^* = 1$, then the equilibrium cutoff k^* is given by $\mu(k^*; 1) = \mu^*(c)$. Since $\mu^*(c) > \mu(k_0, 1)$, we have $k^* > k_0$. This means $\sigma^* = 0$ by (3). This is a contradiction. Therefore, $\sigma^* < 1$ and $k^* = k_0$ hold in equilibrium. To make k^* optimal for the board, σ^* must satisfy $\mu(k_0; \sigma^*) = \mu^*(c)$ and this condition uniquely determines the value of σ^*.

[28]The restriction to non-degenerate beliefs is needed only to obtain Lemma 3, which eliminates the possibility that the board assigns probability one on type $\theta_1 = \mathbb{E}[\theta] + c$ and employs some pathological retention policy. This possibility significantly complicates our application of the D1 and D2 criteria and lets the proof of Lemma 5 fail.

There are several alternative approaches that can eliminate the above problem. We can achieve the desired result by imposing a behavioral assumption that the board chooses either choice with probability 1 whenever indifferent. Alternatively, we can treat type θ_1 as simply nonexistent. It is also possible to restrict the board to cutoff rules. It also works to discretize the type space such that type θ_1 is genuinely nonexistent. In any way, non-cutoff retention policies (after report $\hat{\theta} > \theta_1$) cannot survive minor fluctuations of the model and thus we should naturally eliminate them as in Lemma 3.

Now we apply the D1 criterion to eliminate implausible equilibrium. In particular, we need to preclude an equilibrium where both types choose message θ_L. In such an equilibrium, the pecking order of the productivity may be reversed; i.e., $\theta_H - d$ can be less than θ_L. In this case, as shown in the next lemma, the board employs a *reversed cutoff rule*: the board replaces (retains) the manager if $y > k_L$ ($y < k_L$, respectively) with some cutoff k_L.

Lemma 9. *Consider the board's optimal retention policy for message θ_L. When $\theta_H - d > \theta_L$, the optimal policy is a cutoff rule with a unique optimal cutoff $k_L \in [-\infty, +\infty]$. When $\theta_H - d < \theta_L$, the optimal policy is a reversed cutoff rule with a unique optimal cutoff $k_L \in [-\infty, +\infty]$.*

Proof. If the board assigns probability p on θ_H after observing message θ_L, then the posterior probability on θ_H after observing output y in addition is

$$\mu_L(y) = \frac{pf(y - \theta_H - d)}{pf(y - (\theta_H - d)) + (1-p)f(y - \theta_L)} = \frac{p}{p + (1-p)Q(y)}$$

with $Q(y) = f(y - \theta_L)/f(y - (\theta_H - d))$.

First suppose $p \in (0, 1)$. When $\theta_H - d > \theta_L$, the likelihood ratio Q is increasing and the posterior belief μ_L is decreasing. Thus, a cutoff rule is optimal. When $\theta_H - d < \theta - L$, the likelihood ratio Q is decreasing and the posterior belief μ_L is increasing. In this case, the optimal strategy needs to be a reversed cutoff rule. In either case, the optimal cutoff k_L is uniquely determined by $\mu_L(k_L)\theta_H + (1 - \mu_L(k_L))\theta_L = \mathbb{E}[\theta] + c$, or equivalently, $\mu_L(k_L) = \mu^*(c)$.

When $p \in \{0, 1\}$, the board either always replaces the manager (when $p = 0$) or always retains her (when $p = 1$). We can represent these policies as reversed and regular cutoff strategies with an extreme cutoff $k_L \in \{-\infty, +\infty\}$. □

The next lemma proves that the the D1 criterion implies the assumption (A1). In particular, there is no D1 equilibrium where both types pool on the bad message θ_L.

Lemma 10. *The manager with type θ_H truthfully reports her type in any equilibrium that survives the D1 criterion.*

Proof. We first eliminate the case that type θ_H sends message θ_L with probability $p \in (0, 1)$. We show that type θ_L strictly prefers message θ_L to θ_H. If this is the case, the board retains the manager with report θ_H regardless of the output because type θ_H is only the type that chooses message θ_H. With this retention policy, type θ_H strictly prefers message θ_H; a contradiction. (In this case, we do not need the D1 criterion because both messages are used with positive probability.)

First suppose $\theta_H - d > \theta_L$. In this case, the board uses a cutoff rule with cutoff k_L for message θ_L. The type θ_H is indifferent between the two messages only if the retention probability $F(\theta_H - d - k_L)$ for θ_L equals the one $F(\theta_H - k_H)$ for θ_H; i.e., $k_L = k_H - d$. Since k_L is the lower, type θ_L clearly prefers the truthful report.

Second suppose $\theta_H - d < \theta_L$. The indifference condition for type θ_H is $k_L + d - \theta_H = \theta_H - k_H$ due to the reversal. Here, k_L is the cutoff for the equilibrium reversed cutoff rule for message θ_L. The retention probability for type θ_L is $F(k_L - \theta_L)$, which is greater than $F(\theta_L - d - k_H)$ because $k_L - \theta_L = 2\theta_H - \theta_L - d - k_H > \theta_L - d - k_H$. Again, type θ_L prefers message θ_L.

Finally, when $\theta_H - d = \theta_L$, both types face the same retention probability with message θ_L. When the message is θ_H, type θ_L has a lower retention rate than type θ_H. Hence, type θ_L prefers the truthful message in this case as well.

We turn to the case that both types choose θ_L for sure. To apply the D1 criterion, we show that, whenever the board employs a retention policy for message θ_H that is a best response with *some* belief, type θ_L weakly prefers the message θ_H only if type θ_H strictly prefer that message. If this is the case, the D1 criterion prunes the possibility that type θ_L sends message θ_H and the board assigns probability 1 on type θ_H after observing message θ_L. With this belief, the board always retains the manager after observing message θ_H and thus the manager with type θ_H truthfully reports her type. This is a contradiction.

We consider the three cases once again. First suppose $\theta_L < \theta_H - d$. Type θ_L weakly prefers message θ_H only if $k_H > k_L + d$ and type θ_H strictly prefers message θ_H. Second, when $\theta_L > \theta - d$, type θ_L weakly prefers message θ_H only if $\theta_L - k_H > k_L + d - \theta_L$. The retention probability for type θ_H is $F(\theta_H - k_H)$ with message θ_H and $F(k_L + d - \theta_H)$ with message θ_L. The former is the grater because $\theta_H - k_H > k_L + d + \theta_H - 2\theta_L > k_L + d - \theta_H$. Thus, type θ_H prefers the truthful report. Finally, when $\theta_L = \theta_H - d$, both types face the same retention probability with message θ_L. Hence, whenever type θ_L weakly prefers message θ_H, type θ_H strictly prefers that message. □

We then show that the the assumption (A2) holds even when nobody chooses message θ_L.

Lemma 11. *Suppose that the manager sends message θ_H regardless of her type in an equilibrium that survives the D1 criterion. Then, the board assigns probability 1 on type θ_L after observing message θ_L.*

Proof. To apply the D1 criterion, we show that, whenever the board employs a retention policy for message θ_L that is a best response with *some* belief, type θ_H weakly prefers the message θ_L only if type θ_L strictly prefer that message. If this is the case, the D1 criterion prunes the possibility that type θ_H sends message θ_L and the board assigns probability 1 on type θ_L after observing message θ_L.

We consider the three cases once again. First suppose $\theta_H - d > \theta_L$. Suppose that the board uses cutoff $k_L \in [-\infty, +\infty]$ for message θ_L. Type θ_H weakly prefers message θ_L only if the retention probability is higher with message θ_L than with the truthful message; i.e., $k_L + d > k_H$ in this case. If this is the case, type θ_L strictly prefers the truthful message.

Second, consider the case of $\theta_H - d < \theta_L$ and let k_L be the cutoff (of a reversed cutoff rule) for message θ_L. Type θ_H weakly prefers message θ_L only if $F(k_L - \theta_H + d) > F(\theta_H - k_H)$, or equivalently, $k_L - \theta_H + d > \theta_H - k_H$. The retention probability for type θ_L is $F(k_L - \theta_L)$ with message θ_L and $F(\theta_L - d - k_H)$ with message θ_H. The former is the greater because $k_L - \theta_L > 2\theta_H - \theta_L - d - k_H > \theta_L - d - k_H$.

Finally, when $\theta_H - d = \theta_L$, the two types face the same retention probability with message θ_L. Since type θ_H has a higher equilibrium retention rate than type θ_L, type θ_L strictly prefers message θ_L whenever type θ_H weakly prefers message θ_L. □

Finally, we show the existence of a D1 equilibrium. Let σ^* and k^* as in the statement of this theorem. We show that the following strategies constitute an equilibrium: the manager with type θ uses report θ_H with probability 1 when $\theta = \theta_H$ and with probability σ^* when $\theta = \theta_L$; the board uses cutoff k^* after observing report θ_H and cutoff $+\infty$ after report θ_L; and the board believes that the manger is a low type after observing message θ_L. We have already shown that this equilibrium survives the D1 criterion by Lemma 11: the message θ_L is the only message that can be out of equilibrium, and if it is the case, the type that survives for this message is θ_L (the D1 criterion does not eliminate all the types). The optimality of (σ^*, k^*) follows from the best response conditions. After message θ_L, the board optimally choose cutoff $+\infty$ because of the most pessimistic belief. The high types optimally reports the true type for this extreme cutoff following the low message θ_L.

B.2 Proof of Proposition 1

Recall that the board receive the expected payoff of $\mathbb{E}[\theta] + c$ after replacing a CEO, whereas the payoff is at most θ_H and at least θ_L if the CEO is retained. Thus, the board always replaces the CEO when $\mathbb{E}[\theta] + c > \theta_H$ and, similarly, always retain the CEO when $\mathbb{E}[\theta] + c < \theta_L$.

B.3 Proof of Lemma 1

On the interval $[\hat{c}, \theta_H - \mathbb{E}[\theta]]$, the equilibrium cutoff k is a constant k_0 and the probability σ is decreasing in c. Since the normalized objective function $V(\sigma, k)$ is decreasing in σ with $k = k_0$ fixed, the objective function is increasing in c. The maximum value on this domain is achieved at the maximum value $c = \theta_H - \mathbb{E}[\theta]$.

B.4 Proof of Lemma 2

On the interval $(\theta_L - \mathbb{E}[\theta], \hat{c}]$, the equilibrium cutoff, implicitly given by $\mu(k; 1) = \mu^*(c)$, is increasing in c whereas σ is always equal to 1. When $\sigma = 1$, the objective function for the shareholders is $V(\sigma, k) = \pi(1-\pi)\Delta\theta \cdot W(k) - (1-\pi)d$, where $W(k) = F(\theta_H - k) - F(\theta_L - k - d)$. Note that

$$W'(k) = f(\theta_L - k - d) - f(\theta_H - k) = f(\theta_L - k - d)\left\{1 - \frac{f(\theta_H - k)}{f(\theta_L - k - d)}\right\}$$

changes the sign, from positive to negative, only once because of the monotone likelihood ratio property. In other words, the functions $W(k)$ and $V(1, k)$ has a single peak. Also, the first-order derivative hits 0 when the cutoff k is given as the equilibrium value with $c = 0$ as explained in the text. Therefore, $c = 0$ is optimal if it is included in the domain; if not, $c = \hat{c}$ is optimal.

B.5 Proof of Theorem 2

We first show that $\hat{c} \leq 0$ is equivalent to $m \leq m_*$. The condition $\hat{c} \leq 0$ is equivalent to $\mu(k_0; 1) \leq \mu^*(\hat{c})$, which in turn is equivalent to $R(k_0) \geq 1$ since $\mu^*(\hat{c}) = \pi$. Note that $R(k_0) \equiv f(m)/f(m - 2m_*) = 1$ occurs only when $m = m_*$. Since the ratio $f(m)/f(m - 2m_*)$ is decreasing in m, the condition $R(k_0) \geq 1$ is equivalent to $m \leq m_*$.

When $m \leq m_*$, or equivalently $\hat{c} \leq 0$, the objective function V is increasing in c on $(\theta_L - \mathbb{E}[\theta], \theta_H - \mathbb{E}[\theta])$ and thus the optimal value of c is $\theta_H - \mathbb{E}[\theta]$. This result is part (i) of this theorem.

To show part (ii), suppose $m > m_*$, or equivalently $\hat{c} > 0$. As shown in the text, the value of the normalized objective function V at the first peak $c = 0$ is V_N and that at the second-peak $c = \theta_H - \mathbb{E}[\theta]$ is V_A. Their difference

$$V_A - V_N = \Delta \cdot \pi(1-\pi) \left\{ F(2m_* - m) - \left[2F(m_*) - 1 - \frac{d}{\pi \cdot \Delta\theta} \right] \right\},$$

is non-negative if and only if the condition in part (ii) of this theorem holds.

B.6 Proof of Proposition 2

When $\theta_1 \equiv \mathbb{E}[\theta] + c < \underline{\theta}$, the board unconditionally retains the CEO. In contrast, the board always replace the CEO if $\theta_1 > \overline{\theta}$. In either case, the CEO has no incentive to misreport her type and choose a truthful message as the unique optimal choice.

B.7 Proof of Lemma 3

Let β denote the posterior distribution after observing $\hat{\theta}$. The expected value of type θ after observing report $\hat{\theta}$ and output y is

$$M(y) = \frac{\int_{(\underline{\theta}, \theta_1]} \theta f(y - \theta + d) \, \mathrm{d}\beta + \int_{(\theta_1, \overline{\theta})} \theta f(y - \theta + d \cdot \mathbf{1}_{\{\theta \neq \hat{\theta}\}}) \, \mathrm{d}\beta}{\int_{\underline{\theta}}^{\overline{\theta}} f(y - \theta + d \cdot \mathbf{1}_{\{\theta \neq \hat{\theta}\}}) \, \mathrm{d}\beta}.$$

The sign of $M(y) - \theta_1$ is identical to that of

$$\begin{aligned} L(y) &= \frac{\int_{\underline{\theta}}^{\overline{\theta}} f(y - \theta + d \cdot \mathbf{1}_{\{\theta \neq \hat{\theta}\}}) \, \mathrm{d}\beta}{f(y - \theta_1 + d)} \cdot \big(M(y) - \theta_1 \big) \\ &= \int_{(\underline{\theta}, \theta_1]} (\theta_1 - \theta) \frac{f(y - \theta + d)}{f(y - \theta_1 + d)} \, \mathrm{d}\beta - \int_{(\theta_1, \overline{\theta})} (\theta - \theta_1) \frac{f(y - \theta + d \cdot \mathbf{1}_{\{\theta \neq \hat{\theta}\}})}{f(y - \theta_1 + d)} \, \mathrm{d}\beta. \end{aligned}$$

Unless the posterior β assigns probability 1 on type θ_1, the function L is continuous and decreasing in y due to the monotone likelihood ratio property: the first integral is decreasing and the second is increasing. Therefore, it is optimal for the board to replace the manager when $y < L^{-1}(0)$ and to retain her when $y > L^{-1}(0)$. That is, a cutoff rule with $k = L^{-1}(0)$ is optimal. Here, $L^{-1}(0)$ is well-defined after continuously extending the domain of L to $[-\infty, +\infty]$.

B.8 Proof of Lemma 4

The manager's payoff depends only on the cutoff k when she misreports her type. Therefore, she chooses a report with the minimum cutoff and no type uses any report with a higher cutoff.

B.9 Proof of Lemmas 5 and 6

We repeatedly apply the following lemma in this proof.

Lemma 12. *Let $k \in [-\infty, +\infty]$ and $z : \mathbb{R} \to [0,1]$ be a (measurable) retention policy. When $t > t'$, the following four implications hold:*

$$\begin{aligned}
\int_k^\infty f(y-t')\,\mathrm{d}y \geq \int_{-\infty}^\infty z(y) f(y-t')\,\mathrm{d}y \;&\Rightarrow\; \int_k^\infty f(y-t)\,\mathrm{d}y \geq \int_{-\infty}^\infty z(y) f(y-t)\,\mathrm{d}y\\
\int_k^\infty f(y-t)\,\mathrm{d}y \leq \int_{-\infty}^\infty z(y) f(y-t)\,\mathrm{d}y \;&\Rightarrow\; \int_k^\infty f(y-t')\,\mathrm{d}y \leq \int_{-\infty}^\infty z(y) f(y-t')\,\mathrm{d}y\\
\int_{-\infty}^k f(y-t)\,\mathrm{d}y \geq \int_{-\infty}^\infty z(y) f(y-t)\,\mathrm{d}y \;&\Rightarrow\; \int_{-\infty}^k f(y-t')\,\mathrm{d}y \geq \int_{-\infty}^\infty z(y) f(y-t')\,\mathrm{d}y\\
\int_{-\infty}^k f(y-t')\,\mathrm{d}y \leq \int_{-\infty}^\infty z(y) f(y-t')\,\mathrm{d}y \;&\Rightarrow\; \int_{-\infty}^k f(y-t)\,\mathrm{d}y \leq \int_{-\infty}^\infty z(y) f(y-t)\,\mathrm{d}y.
\end{aligned}$$

Moreover, the inequalities in the first two consequents are strict if $z(y) \neq \mathbf{1}_{\{y>k\}}$ on a set with a positive Lebesgue measure. The inequalities in the last two consequents are strict if $z(y) \neq \mathbf{1}_{\{y<k\}}$ on a set with a positive Lebesgue measure.

Proof. The first two implications follows from

$$\begin{aligned}
&\int_k^\infty f(y-t')\,\mathrm{d}y - \int_{-\infty}^\infty z(y) f(y-t')\,\mathrm{d}y\\
&= f(k-t') \left\{ \int_k^\infty \big(1-z(y)\big) \frac{f(y-t')}{f(k-t')}\,\mathrm{d}y - \int_{-\infty}^k z(y) \frac{f(y-t')}{f(k-t')}\,\mathrm{d}y \right\}\\
&\leq f(k-t') \left\{ \int_k^\infty \big(1-z(y)\big) \frac{f(y-t)}{f(k-t)}\,\mathrm{d}y - \int_{-\infty}^k z(y) \frac{f(y-t)}{f(k-t)}\,\mathrm{d}y \right\}\\
&= \frac{f(k-t')}{f(k-t)} \left\{ \int_k^\infty f(y-t)\,\mathrm{d}y - \int_{-\infty}^\infty z(y) f(y-t)\,\mathrm{d}y \right\}.
\end{aligned}$$

Here, the inequality is due to the monotone likelihood ration property. Similarly,

$$\begin{aligned}
&\int_{-\infty}^k f(y-t)\,\mathrm{d}y - \int_{-\infty}^\infty z(y) f(y-t)\,\mathrm{d}y\\
&\geq \frac{f(k-t)}{f(k-t')} \left\{ \int_{-\infty}^k f(y-t')\,\mathrm{d}y - \int_{-\infty}^\infty z(y) f(y-t')\,\mathrm{d}y \right\}.
\end{aligned}$$

proves the latter half. In either case, the inequality is strict when the condition in the statement is satisfied. □

The following lemma constitutes an essential part of this proof.

Lemma 13. *Consider an equilibrium that survives the D1 criterion. If message $\theta \in (\theta_1, \overline{\theta})$ is out of equilibrium, then the board assigns probability 1 on types above θ_1 after this message. Consequently, no message above θ_1 is out of equilibrium.*

Proof. Let $\hat{\theta}$ be a message that type θ uses in equilibrium and let $\hat{z}(y)$ be the equilibrium retention policy for message $\hat{\theta}$.

We first prune the possibility that type $\theta' \in (\underline{\theta}, \theta_1] \setminus \{\hat{\theta}\}$ chooses message θ. Suppose that the type θ' weakly prefers message θ to $\hat{\theta}$ if the board uses a cutoff rule with cutoff k for message θ. Since type θ' is neither θ nor $\hat{\theta}$, we have

$$\int_k^{\infty} f(y - (\theta' - d))\,\mathrm{d}y \geq \int_{-\infty}^{\infty} \hat{z}(y) f(y - (\theta' - d))\,\mathrm{d}y$$

and thus, by Lemma 12,

$$\int_k^{\infty} f(y - \theta)\,\mathrm{d}y \geq \int_{-\infty}^{\infty} \hat{z}(y) f(y - \theta)\,\mathrm{d}y.$$

That is, type θ has a higher retention rate with the truthful message θ than the equilibrium message $\hat{\theta}$. Since type θ gains d in addition by truth-telling, this type strictly prefers the truthful message. Therefore, after observing message θ, the board assigns no probability on the set $(\underline{\theta}, \theta_1] \setminus \{\hat{\theta}\}$.

We then consider type $\hat{\theta}$. Once again suppose the board uses cutoff k for message θ. The type $\hat{\theta}$ weakly prefers message θ to $\hat{\theta}$ only if

$$\int_k^{\infty} f(y - (\hat{\theta} - d))\,\mathrm{d}y > \int_{-\infty}^{\infty} \hat{h}(y) f(y - \hat{\theta})\,\mathrm{d}y. \tag{B.1}$$

First suppose $\theta - d \geq \hat{\theta}$. In this case, define $\hat{k}$ by

$$\int_{-\infty}^{\infty} \hat{z}(y) f(y - \hat{\theta})\,\mathrm{d}y = \int_{-\infty}^{\hat{k}} f(y - \hat{\theta})\,\mathrm{d}y. \tag{B.2}$$

By combining (B.1) and (B.2), we obtain $F(\hat{\theta} - d - k) > F(\hat{k} - \hat{\theta})$, or equivalently, $\hat{k} < 2\hat{\theta} - d - k$. Also, from (B.2), by Lemma 12,

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{z}(y) f(y - (\theta - d))\,\mathrm{d}y &\leq \int_{-\infty}^{\hat{k}} f(y - (\theta - d))\,\mathrm{d}y = F(\hat{k} - \theta + d) \\ &< F(2\hat{\theta} - \theta - k) < F(\theta - k) = \int_k^{\infty} f(y - \theta)\,\mathrm{d}y.\end{aligned}$$

That is, type θ have a higher retention rate with the truthful message than message $\hat{\theta}$. Therefore, in this case, we prune the possibility that type $\hat{\theta}$ chooses message θ.

Now suppose $\theta - d < \hat{\theta}$. This time, we define $\hat{k}$ by

$$\int_{-\infty}^{\infty} \hat{h}(y) f(y - \hat{\theta})\,\mathrm{d}y = \int_{\hat{k}}^{\infty} f(y - \hat{\theta})\,\mathrm{d}y. \tag{B.3}$$

From (B.1) and (B.3), we have $\hat{\theta} - d - k > \hat{\theta} - \hat{k}$ and thus $\hat{k} > k + d$. By Lemma 12, equation (B.3) implies

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{z}(y) f(y - (\hat{\theta} - d))\,\mathrm{d}y \le \int_{\hat{k}}^{\infty} f(y - (\hat{\theta} - d))\,\mathrm{d}y = F(\hat{\theta} - d - \hat{k})\\ < F(\hat{\theta} - d - (k - d)) < F(\theta - k) = \int_{k}^{\infty} f(y - \theta)\,\mathrm{d}y.\end{aligned}$$

Hence, again, type θ strictly prefers the truthful message to the equilibrium message. In either case, the board assigns no probability on $(\underline{\theta}, \theta_1]$ after observing message θ in any D1 equilibrium.

To show the second part of the statement, suppose $\hat{\theta}$ is out of equilibrium. By the first part of this lemma, the board optimally retains the manager for sure after observing that message. If this is the case, the type $\hat{\theta}$ should choose the truthful message $\hat{\theta}$, which contradicts the assumption that message $\hat{\theta}$ is out of equilibrium. □

We first show Lemma 6 by combining Lemmas 4 and 13.

Proof of Lemma 6

Let $k(\hat{\theta})$ denote the cutoff for message $\hat{\theta} \in (\theta_1, \overline{\theta})$. Suppose to the contrary $k(\hat{\theta}) > k(\hat{\theta}')$ for some $\hat{\theta}, \hat{\theta}' \in (\theta_1, \overline{\theta})$. Then, no type uses message $\hat{\theta}$ as a misreport by Lemma 4. By Lemma 4, the message $\hat{\theta}$ needs to be used by type $\hat{\theta}$. In this case, the board retains the manager for sure, i.e., $k(\hat{\theta}) = -\infty$. This contradicts with $k(\hat{\theta}) > k(\hat{\theta}')$.

Proof of Lemma 5

First observe that, by Lemma 6, the type θ^* prefers the truthful message to any other message above θ_1 because all of these messages use the same cutoff k_*. We show that the retention rate for any message $\hat{\theta} \in (\underline{\theta}, \theta_1]$ does not exceed the truthful counterpart. That is,

$$\int_{-\infty}^{\infty} \hat{z}(y) f(y - \hat{\theta}^* - d)\,\mathrm{d}y \le \int_{k_*}^{\infty} f(y - \theta^*)\,\mathrm{d}y, \tag{B.4}$$

where $\hat{z}(y)$ is the retention policy for message $\hat{\theta}$. If this is the case, the message θ^* is the unique optimal choice for the type θ^*.

The condition (B.4) is clearly satisfied when $k_* = -\infty$. We thus assume $k_* > -\infty$. In this case, there must be a set of types with positive Lebesgue measure that report some message above θ_1 with positive probability because otherwise the board assigns probability 1 on $(\theta_1, \overline{\theta})$ and sets $k(\theta) = -\infty$ for some message θ above θ_1. At least one of such types differs from $\hat{\theta}$ and let θ_* denote this type. Since type θ_* weakly prefers the cutoff k_* to the retention policy for $\hat{\theta}$,

$$\int_{-\infty}^{\infty} \hat{z}(y) f(y - (\theta_* - d))\, \mathrm{d}y \leq \int_{k_*}^{\infty} f(y - (\theta_* - d))\, \mathrm{d}y. \tag{B.5}$$

By Lemma 12, the inequality (B.5) implies

$$\int_{-\infty}^{\infty} \hat{z}(y) f(y - (\theta^* - d))\, \mathrm{d}y \leq \int_{k_*}^{\infty} f(y - (\theta^* - d))\, \mathrm{d}y = F(\theta^* - d - k_*) \leq F(\theta^* - k_*).$$

This is the condition (B.4) and thus the truthful message is uniquely optimal. Therefore, any type above θ_1 reports the truthful message with probability 1.

B.10 Proof of Lemma 7

First note that, on the equilibrium path, the CEO is replaced if her type is below θ_1 and her message is θ_1 or below because all the types above θ_1 report truthful messages (Lemma 5).[29] Thus, for the types below θ_1, it is optimal to choose either (a) truthful messages or (b) messages above θ_1 accompanied with the uniform cutoff. The threshold θ_2 is defined, by (eq:theta2), as the type that makes the CEO indifferent between these two choices. As explained in the text, types more than the threshold θ_2 have a higher chance of survival than the indifferent type θ_2 and thus prefer (b); types below θ_2 prefer (a). Therefore, in equilibrium, types below θ_2 (and below θ_1) report truthful messages, whereas types above θ_2 (but below θ_1) choose messages above θ_1.

B.11 Proof of Lemma 8

We first show the condition (7) is necessary when $\theta_2 \in (-\infty, \theta_1)$. Since the board needs to uniformly choose a single cutoff k_* for almost every messages $\hat{\theta}$ above θ_1, the uniform cutoff k_* satisfies the first-order condition

$$\mathbb{E}\left[(\theta - \theta_1) f(k_* + d \cdot \mathbf{1}_{\{\theta \neq \hat{\theta}\}} - \theta) \,\middle|\, \hat{\theta}\right] = 0$$

for such messages. By the law of total expectation,

$$\begin{aligned} 0 &= \mathbb{E}\left[\mathbf{1}_{\{\hat{\theta} > \theta_1\}} \cdot (\theta - \theta_1) f(k_* + d \cdot \mathbf{1}_{\{\theta \neq \hat{\theta}\}} - \theta)\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\{\theta > \theta_1\}} \cdot (\theta - \theta_1) f(k_* - \theta)\right] - \mathbb{E}\left[\mathbf{1}_{\{\theta \in (\theta_1, \theta_2)\}} \cdot (\theta - \theta_1) f(k_* + d - \theta)\right]. \end{aligned}$$

The last two expectations represent the left-hand and right-hand sides of the desired condition (7).

[29] The entire proof should be interpreted as a measure theoretic statements. For example, we allow some types below θ_1 is retained even with messages below θ_1 as long as the set of such types has Lebesgue measure 0.

The condition (7) is equivalent to $J(k_*;\theta_2) = 0$, where

$$J(k_*;\theta_2) = \int_{\theta_1}^{\overline{\theta}} (\theta - \theta_1)\frac{f(k_* - \theta)}{f(k_* - \theta_1)} g(\theta)\,\mathrm{d}\theta - \int_{\theta_2}^{\theta_1} (\theta_1 - \theta)\frac{f(k_* + d - \theta)}{f(k_* - \theta_1)} g(\theta)\,\mathrm{d}\theta. \tag{B.6}$$

By the monotone likelihood ratio property, the condition $J(k_*;\theta_2) = 0$ has a unique solution $k_*(\theta_2)$ given θ_2. The solution is decreasing in θ_2 because $J(k_*;\theta_2)$ is increasing in k_* and non-decreasing in θ_2. Also, $k_*(\theta_2)$ is a continuous function because $J(k_*;\theta_2)$ is jointly continuous (and increasing in k_*). As θ_2 approaches θ_1, the solution $k_*(\theta_2)$ decreases to $-\infty$ because $J(k_*;\theta_2)$ converges to a positive value as $\theta_2 \nearrow \theta_1$, whenever cutoff k_* is finite.

B.12 Proof of Theorem 3

Let (θ_2^*, k^*) be the unique fixed point. We first construct an equilibrium that survives the D2 (and thus D1) criteria.

Existence

We start by specifying how misreporting types mix their reports. Define $h(\hat{\theta})$ for each $\hat{\theta} \in (\theta_1, \overline{\theta})$ by

$$h(\hat{\theta}) = \frac{(\hat{\theta} - \theta_1) f(k^* - \hat{\theta}) g(\hat{\theta})}{\int_{\theta_2^*}^{\theta_1} (\theta_1 - \theta) f(k^* + d - \theta) g(\theta)\,\mathrm{d}\theta}.$$

This function h works as a density function:

$$\int_{\theta_1}^{\overline{\theta}} h(\hat{\theta})\,\mathrm{d}\hat{\theta} = \frac{\int_{\theta_1}^{\overline{\theta}} (\hat{\theta} - \theta_1) f(k^* - \hat{\theta}) g(\hat{\theta})\,\mathrm{d}\hat{\theta}}{\int_{\theta_2^*}^{\theta_1} (\theta_1 - \theta) f(k^* + d - \theta) g(\theta)\,\mathrm{d}\theta}$$

is equal to 1 because k^* satisfies the condition (7).

We consider the following strategies and beliefs. The manager with type $\theta \in [\theta_2^*, \theta_1]$ uses the density $h(\hat{\theta})$ to randomize the messages $\hat{\theta} \in (\theta_1, \overline{\theta})$. The manager with some other type reports a truthful message. After observing message $\hat{\theta} \in (\theta_1, \overline{\theta})$, the board calculates a posterior belief by the Bayes' rule and employs the uniform cutoff k^*. After message $\hat{\theta} \in (\underline{\theta}, \theta_1]$, the board sets cutoff $k = +\infty$ (i.e., replacement for sure).

We briefly discuss the optimality of the strategies. The optimality of k^* follows from the fact that the density function $h(\hat{\theta})$ satisfies the first-order condition (6) for all $\hat{\theta} \in (\theta_1, \overline{\theta})$. The extreme cutoff $k = +\infty$ is a best response for the lower messages because after these messages the board assigns no probability on types above θ_1. The manager with type above θ_1 chooses truthful messages as a unique optimum to get the highest retention rate and the additional productivity d. By the definition of θ_2, the remaining types also choose optimal messages for them.

It remains to show that this belief systems survives the D1 and D2 criteria. Consider a cutoff $k_\varepsilon = k^* - \varepsilon$ slightly lower than k^*. Suppose that this cutoff k_ε is accompanied with an out-of-equilibrium message $\hat{\theta} \in [\theta_2, \theta_1]$. Then, when ε is sufficiently small, type $\theta = \hat{\theta}$ strictly prefer the truthful message $\hat{\theta}$ for this type to the equilibrium misreporting, whereas all the other types get worse off with this message than their equilibrium messages. Note that the cutoff k_ε becomes a best response for the board by controlling the belief for this message; when the board assigns probability p on $(\overline{\theta} + \theta_1)/2$ and $1 - p$ on $(\underline{\theta} + \theta_1)/2$, we can make any level of cutoff a best response by correctly adjusting p. Therefore, neither the D1 or D2 criterion can eliminate such a belief.

D2 Criterion

The necessary conditions for D1 equilibria are already shown by Lemmas 5–7. It remains to show that, in any D2 equilibrium, the board must replace the CEO after (almost) every message $\hat{\theta} \in (\theta_2, \theta_1)$.

We claim that the board never assign probability on types above $\hat{\theta}$ for all out-of-equilibrium messages $\hat{\theta}$. To this end, we assume that the manager with type $\theta^* > \hat{\theta}$ weakly prefers the message $\hat{\theta}$ with a retention policy z to the equilibrium message for type θ^* and show that some type θ_z strictly prefers the message $\hat{\theta}$ to the equilibrium message for type θ_z.

First consider the case that the retention policy z differs from the cutoff rule with cutoff k^* in a measure-theoretic sense; i.e., $\{y : z(y) \neq \mathbf{1}_{\{y>k^*\}}\}$ has a positive Lebesgue measure. The manager with type $\theta^* > \hat{\theta}$ weakly prefers the message $\hat{\theta}$ with the above retention policy to the equilibrium message only if the retention rate with message $\hat{\theta}$ and retention policy z is at least as high as with some misreport above θ_1 and cutoff k^*:

$$\int_{-\infty}^{\infty} z(y) f(y - (\theta^* - d))\, \mathrm{d}y \geq \int_{k_*}^{\infty} f(y - (\theta^* - d))\, \mathrm{d}y.$$

Let $\theta_* \in (\theta_2, \hat{\theta})$ be a type that chooses a message above θ_1 and faces cutoff k^* in equilibrium. By Lemma 12, the type θ_* has a higher chance of retention with message $\hat{\theta}$ and retention policy z than with the equilibrium message and the cutoff k^*. That is, when such retention policy z is accompanied with message $\hat{\theta}$, type θ_* strictly prefers the message $\hat{\theta}$ to the equilibrium message for this type whenever type θ^* weakly prefers the message $\hat{\theta}$ to the equilibrium message for type θ.

Now consider the case that the retention policy z is identical to the cutoff rule with k^*. (We need the D2 criterion, instead of D1, just for this part.)[30] In this case, the type $\hat{\theta}$ prefers the truthful message $\hat{\theta}$ to its equilibrium choice, accompanied with the uniform cutoff k^*, whereas the type θ is indifferent between $\hat{\theta}$ and the equilibrium choice (or prefers the latter).[31] That is, even when z is identical to the cutoff rule with k^*, we can find a type (in this case, type $\hat{\theta}$) strictly prefer message $\hat{\theta}$ to the equilibrium message.

[30] The type $\hat{\theta}$ does not necessarily eliminate all the other types through the D1 criterion. Consider a type $\theta \in (\theta_2, \hat{\theta} + d/2)$. We can easily show that when the board uses a reversed cutoff rule with a cutoff level k that makes type θ indifferent (i.e., $F(k - \theta) = F(\theta - k^*)$), the retention rate for type $\hat{\theta}$ is lower with the truthful message $\hat{\theta}$ than with the equilibrium misreports (i.e., $F(k - \hat{\theta} - d) < F(k - \hat{\theta} - d)$). In particular, the type θ can be more than θ_1 when $\hat{\theta}$ is close enough to θ_1; that is, the D1 criterion may not be able to eliminate some types above θ_1.

[31] The type θ may get a better deal than cutoff k^* in equilibrium because the posterior belief is indeterminate on a set of measure 0. If it is the case, we do not need the argument for the D2 criterion; the D1 criterion suffices. In general, of course, we need the D2 criterion to obtain the desired result.

We have shown that whenever type $\theta^* > \hat{\theta}$ weakly prefers message $\hat{\theta}$ to its equilibrium message, there exists some type that strictly prefers message $\hat{\theta}$ to the equilibrium message for that type. Therefore, after observing $\hat{\theta}$, the board assigns probability 1 on types below $\hat{\theta}$ ($< \theta_1$) and replaces the CEO regardless of the realization of y.

B.13 Proof of Proposition 4

Case (iv) is obvious. We focus on cases (i)–(iii); i.e., $|c| < \Delta/2$. We first provide a necessary condition for equilibrium. Let $\theta_* = \max\{\underline{\theta}, \theta_2\}$ and $\theta^* = \min\{\overline{\theta}, k + q\}$.

Lemma 14. *Suppose* $|c| < \Delta/2$. *In equilibrium,* $\theta_1 - \theta_* = \theta^* - \theta_1$ *and* $k \in [\theta_1 - q, \theta_* - d + q]$.

Proof. Given $\theta_2 < \theta_1$, the objective of the board is to maximize

$$B(k;\theta_2) = \int_{\theta_1}^{\overline{\theta}} (\theta - \theta_1) F(k - \theta)\, \mathrm{d}\theta + \int_{\theta_*}^{\theta_1} (\theta - \theta_1) F(k + d - \theta)\, \mathrm{d}\theta. \tag{B.7}$$

The function $B(k;\theta_2)$ is zero when no type survives (i.e., $k \geq \overline{\theta} + q$). Also, $B(k;\theta_2)$ is positive when $k \in [\theta_1 - d + q, \overline{\theta} + q)$ because no type below θ_1 survives. Among these values of k, the lowest value $k = \theta_1 - d + q$ gives the highest rate of retention and the highest value of $B(k)$ within the interval. Thus, the values of $k > \theta_1 - d + q$ are all suboptimal. Similarly, it is suboptimal to choose $k < \theta_1 - q$ because the retention rate of types below θ_1 increases as k decreases on that region. Therefore, we can focus on $k \in I = [\theta_1 - q, \theta_1 - d + q]$.

We further partition the interval I into three intervals: $I_1 = [\theta_1 - q, \overline{\theta} - q]$, $I_2 = (\overline{\theta} - q, \theta_* - d + q)$, and $I_3 = [\theta_* - d + q, \theta_1 - d + q]$. The second interval I_2 is nonempty because of the second regularity condition (9). Actually, $k \in I_3$ never occurs in equilibrium because $k \in I_3$ implies that the threshold type θ_2 has no chance of survival. This contradicts the definition of θ_2: the manager with type θ_2 needs to be indifferent between truth-telling and misreporting.

We investigate the optimality condition for k. If $k \in I_1$, then $\theta^* = k + q$ and the function $B(k;\theta_2)$ becomes

$$\begin{aligned} B(k;\theta_2) = &\int_{\theta^*}^{\overline{\theta}} (\theta - \theta_1) \cdot 1\, \mathrm{d}\theta + \int_{\theta_1}^{\theta^*} (\theta - \theta_1) \frac{k - \theta + q}{2q}\, \mathrm{d}\theta \\ &+ \int_{\theta_*}^{\theta_1} (\theta - \theta_1) \frac{k + d - \theta + q}{2q}\, \mathrm{d}\theta. \end{aligned} \tag{B.8}$$

On the second interval I_2, we obtain $\theta^* = \overline{\theta}$ and

$$B(k;\theta_2) = \int_{\theta_1}^{\theta^*} (\theta - \theta_1) \frac{k - \theta + q}{2q}\, \mathrm{d}\theta + \int_{\theta_*}^{\theta_1} (\theta - \theta_1) \frac{k + d - \theta + q}{2q}\, \mathrm{d}\theta, \tag{B.9}$$

and (B.10). In either case,

$$\frac{\partial B}{\partial k} = \frac{1}{4q} \left\{ (\theta_1 - \theta_*)^2 - (\theta^* - \theta_1)^2 \right\} \tag{B.10}$$

and thus in equilibrium $(\theta_1 - \theta_*)^2 = (\theta^* - \theta_1)^2$, or equivalently, $\theta_1 - \theta_* = \theta^* - \theta_1$ must be satisfied. $\square$

Consider case (i) of this proposition. By Lemma 14, $\theta_* = \theta_2 > \underline{\theta}$ must hold because otherwise $\theta^* - \theta_1 < \Delta/2 < \theta_1 - \underline{\theta} = \theta_1 - \theta_*$. Also, $\theta^* = \overline{\theta} < k + q$ because the second regularity condition (9) implies

$$\underline{\theta} - \theta_{2,0} > \overline{\theta} - q \tag{B.11}$$

and thus $k + q \geq \underline{\theta} + q - \theta_{2,0} > \overline{\theta}$. Therefore, an equilibrium candidate is uniquely given by the equilibrium condition $\overline{\theta} - \theta_1 = \theta_1 - \theta_2$. To verify it is indeed an equilibrium, we simply need to confirm $\theta_2 = 2\theta_1 - \overline{\theta} > \underline{\theta}$ (by $c > 0$) and $k = 2\theta_1 - \overline{\theta} - \theta_{2,0} > \overline{\theta} - q$ (by (B.11)).

We next consider case (ii). In this case, $\theta^* = k - q < \overline{\theta}$ holds because otherwise $\theta - \theta_1 > \Delta/2 > \theta_1 - \theta_*$. Once again by (B.11), we obtain $\theta_2 = k + \theta_{2,0} < \overline{\theta} + q + \theta_{2,0} < \underline{\theta}$ and thus $\theta_* = \underline{\theta}$. These conditions uniquely determine an equilibrium: $k = 2\theta_1 - \underline{\theta} - q$ and $\theta_2 = 2\theta_1 - \underline{\theta} - q + \theta_{2,0}$.

In case (iii), $\theta^* = \overline{\theta}$ and $\theta_* = \underline{\theta}$ hold. Suppose otherwise. Then, $\theta^* - \theta_1 = \theta_1 - \theta_* < \Delta/2$ and thus $\theta^* = k + q$ and $\theta_* = \theta_2$ hold. However, $\theta_2 = k + \theta_{2,0} < \overline{\theta} - q + \theta_{2,0} < \underline{\theta}$ by (B.11); a contradiction. In this case, any value of k works as an equilibrium cutoff as long as $k + q \geq \overline{\theta}$ and $k + \theta_{2,0} \leq \underline{\theta}$.

B.14 Proof of Theorem 4

We first calculate the payoff function for shareholders.

Lemma 15. *Assume the regularity conditions (9) and (8). The equilibrium payoff for the shareholder with exogenous c is*

$$V_A(c) = \frac{1}{\Delta q}\left\{\frac{2}{3}\cdot c^3 - \left(\frac{dA}{4} + \frac{\Delta}{2}\right)\cdot c^2 + d\left(\alpha\cdot\frac{\Delta}{2} + q\right)\cdot c + v_0\right\}$$

when $c \in (0, \Delta/2)$, and

$$V_F(c) = \frac{1}{\Delta q}\left\{-\frac{2}{3}\cdot c^3 - \left(\frac{d}{4} + \frac{\Delta}{2}\right)\cdot c^2 - dq\cdot c + v_0\right\}$$

when $c \in (-\Delta/2, 0)$, where $v_0 = [2\Delta^2 - 3d(8q - \Delta)]\Delta/48$.

Proof. First assume $c \in (0, \Delta/2)$. By Proposition 4, we have $k = 2\theta_1 - \overline{\theta} - \theta_{2,0}$, $k < \overline{\theta} - q$, and $\theta_2 = 2\theta_1 - \overline{\theta}$ in equilibrium. Thus, the payoff for shareholders, multiplied by Δq, is

$$\begin{aligned}\Delta q V_A &= q\left\{-d(\theta_1-\theta_2)+\int_{\theta_1}^{\overline{\theta}}(\theta-\mu)F(\theta-k)\,\mathrm{d}\theta+\int_{\theta_2}^{\theta_1}(\theta-\mu)F(\theta-d-k)\,\mathrm{d}\theta\right\}\\ &= -dq\left(\frac{\Delta}{2}-c\right)+\frac{1}{2}\left[\mu(k-q)\theta-(k-q+\mu)\frac{\theta^2}{2}+\frac{\theta^3}{3}\right]_{\theta=\mu+c}^{\mu+\Delta/2}\\ &\qquad +\frac{1}{2}\left[\mu(k+d-q)\theta-(k+d-q+\mu)\frac{\theta^2}{2}+\frac{\theta^3}{3}\right]_{\theta=\mu+2c-\Delta/2}^{\mu+c}\\ &= -dq\left(\frac{\Delta}{2}-c\right)+\left[\frac{c^3}{3}-\frac{2d(1+\alpha)-\Delta}{8}c^2-\frac{\Delta^2}{8}c+\frac{\Delta^2}{96}\{6d(1+\alpha)d+5\Delta\}\right]\\ &\qquad +\left[\frac{c^3}{3}-\frac{6d\alpha+3\Delta}{8}c^2+\frac{\Delta}{8}(4d\alpha+\Delta)c-\frac{\Delta^2}{96}(6d\alpha+\Delta)\right].\end{aligned}$$

We obtain the desired expression by simplifying the above expression.

We next consider the case of $c \in (-\Delta/2, 0)$. We know that $k = 2\theta_1 - q - \underline{\theta}$, $k < \overline{\theta} - q$ and $\theta_2 < \underline{\theta}$ due to Proposition 4. Hence,

$$\begin{aligned}\Delta q V_F &= q\left\{-d(\theta_1-\theta_2)+\int_{k+q}^{\overline{\theta}}(\theta-\mu)\,\mathrm{d}\theta+\int_{\theta_1}^{k+q}(\theta-\mu)F(\theta-k)\,\mathrm{d}\theta+\int_{\underline{\theta}}^{\theta_1}(\theta-\mu)F(\theta-d-k)\,\mathrm{d}\theta\right\}\\ &= -dq\left(c+\frac{\Delta}{2}\right)-q\left[2c^2+\Delta c\right]+\left[-\frac{c^3}{3}+\frac{3}{8}(4q-\Delta)c^2+\left(\Delta q-\frac{\Delta^2}{8}\right)c+\frac{\Delta^2}{96}(12q-\Delta)\right]\\ &\qquad +\left[-\frac{c^3}{3}+\frac{1}{8}(4q-\Delta-2d)c^2+\frac{\Delta^2}{8}\cdot c+\frac{\Delta^2}{96}(6d+5\Delta-12q)\right].\end{aligned}$$

Once again, we obtain the desired expression after a few more calculations. □

These two cubic functions V_A and V_F may have a local maximum at c_A^* and c_F^*, respectively. We later see that the locally maximum values (i.e., $V_A(c_A^*)$ and $V_F(c_F^*)$) are proportional to

$$\begin{aligned}\Omega(x) = &-2(1+4x)^3d^3+2(1+4x)d^2\left[(1+4x)\sqrt{D(x)}-6\Delta+48q\right]\\ &+8d(\Delta-8q)\left[\sqrt{D(x)}+3\Delta\right]+8\Delta^2\left[\sqrt{D(x)}+2\Delta\right]\end{aligned}$$

when $x = \alpha$ and 0, respectively, where $D(x) = d^2(1+4x)^2 + 4d\Delta - 32dq + 4\Delta^2$. Note that

$D(\alpha)$ is positive because

$$D(\alpha) > 16\alpha^2 d^2 + 8\alpha d^2 + d^2 + \frac{16dq}{2\alpha+1} - 32dq + \frac{64q^2}{(2\alpha+1)^2} = \frac{\left(8\alpha^2 d + 6\alpha d + d - 8q\right)^2}{(2\alpha+1)^2} > 0. \tag{B.12}$$

The inequality follows from the first regularity condition (8), or equivalently, $\Delta > 4q/(1+2\alpha)$.

Lemma 16. *If $D(0) > 0$ and (9) holds, then $\Omega'(x) > 0$ for all $x \in (0, \alpha)$. Also, $\Omega'(x) > 0$ for all positive $x > \frac{4q-\Delta}{2\Delta}$.*

Proof. The first-order derivative of Ω is given by

$$\frac{1}{24d^2}\Omega'(x) = 16q - 2\Delta - d(1+4x)^2 + (1+4x)\sqrt{D(x)}. \tag{B.13}$$

If $16q \geq 2\Delta + d(1+4x)^2$, then (B.13) is positive; this is the case when (9) holds. Suppose otherwise. Then, (B.13) is positive if and only if

$$\lambda(x) = \frac{1}{32}\left\{(1+4x)^2 D(x) - \left[d(1+4x)^2 + 2\Delta - 16q\right]^2\right\} = x(1+2x)\Delta^2 - 2q(4q-\Delta)$$

is positive. Since

$$\lambda(x) > \lambda\left(\frac{4q-\Delta}{2\Delta}\right) = 4q\left(\frac{4q-\Delta}{2}\right) - 2q(4q-\Delta) = 0,$$

we obtain $\Omega'(x) > 0$. □

Lemma 17. *Assume the regularity conditions (9) and (8). The function V_A is uniquely maximized at $c_A^* = (dA + 2\Delta - \sqrt{D(\alpha)})/8$ within the domain $(0, \Delta/2)$. The maximum value is positive.*

Proof. The first-order derivative of V_A has two roots at $c = (dA + 2\Delta \pm \sqrt{D(\alpha)})/8$. We show the smaller root $c_A^* = (dA + 2\Delta - \sqrt{D(\alpha)})/8$ is a unique local maximizer on the domain $(0, \Delta/2)$. The smaller root is a unique local minimizer on the unrestricted domain $\mathbb{R}$ as an elementary property of cubic functions. Thus, we only need to show $c_A^* \in (0, \Delta/2)$. The root c^* is always positive because

$$D(\alpha) = (dA + 2\Delta)^2 - 16d(\alpha\Delta + 2q) < (dA + 2\Delta)^2.$$

We show that the condition (8) guarantees $c_A^* < \Delta/2$, or equivalently, $\sqrt{D(\alpha)} > 2\Delta - dA$. Since (8) is equivalent to $q < \Delta(1+2\alpha)/4$, we obtain

$$D > d^2A^2 + 4d\Delta - 32d \cdot \frac{\Delta(1+2\alpha)}{4} + 4\Delta^2 = (2\Delta - dA)^2.$$

Hence, $\sqrt{D(\alpha)} > \sqrt{(2\Delta - dA)^2} = 2\Delta - dA$ if $2\Delta - dA \geq 0$; otherwise, $\sqrt{D(\alpha)} > 0 > 2\Delta - dA$. In either case, we obtain $c^* < \Delta/2$ and thus the point c_A^* is a local maximizer on the domain $(0, \Delta/2)$.

Now we show $V_A(c_A^*)$ is always positive. After several calculations, we obtain $V_A(c_A^*) = \frac{1}{768\Delta q}\Omega(\alpha)$. By Lemma 16,

$$\begin{aligned}768\Delta q V_A(c_A^*) &= \Omega(\alpha) > \Omega\left(\frac{4q-\Delta}{2\Delta}\right) \\ &= 2\Delta^{-3}\left[2\Delta^2 - d(8q-\Delta)\right]^2\left\{\left|2\Delta^2 - d(8q-\Delta)\right| + \left[2\Delta^2 - d(8q-\Delta)\right]\right\} \geq 0.\end{aligned}$$

because $\alpha > \frac{4q-\Delta}{2\Delta}$ by (8).

Lastly, observe that the local maximum attained at c_A^* is indeed a global maximum on $(0, \Delta/2)$ because $V_A'(0) > 0$ and $V_A(\Delta/2) = 0$. □

To complete the proof of Theorem 4, we show V_F never exceeds $V_A(c^*)$. First note if V_F does not have a local maximizer on its domain $(-\Delta/2, 0)$, then V_F is negative because V_F is a cubic function with $V_F'(0) < 0$ and a negative coefficient on c^3. Suppose a local maximizer c_F^* exists. Then, it must be the larger root of $V_F'(c) = 0$; that is, $c_F^* = (\sqrt{D(0)} - d - 2\Delta)/8$. The local maximum is given by $V_F(c_F^*) = \frac{1}{768\Delta q}\Omega(0)$, which is less than $V_A(c_A^*) = \frac{1}{768\Delta q}\Omega(\alpha)$ by Lemma 16 and the second regularity condition (9). In either case, V_F cannot exceed $V_A(c_A^*)$ and it is therefore optimal for shareholders to choose $c = c_A^*$.

B.15 Proof of Proposition 5

The first-order derivatives of c_A^* are

$$\begin{aligned}\frac{\partial c_A^*}{\partial d} &= \frac{1}{8\sqrt{D}}\left\{(1+4\alpha)\sqrt{D} + \left[16q - d(1+4\alpha)^2 - 2\Delta\right]\right\} \\ \frac{\partial c_A^*}{\partial \Delta} &= \frac{1}{4\sqrt{D}}\left\{\sqrt{D} - (d+2\Delta)\right\} \\ \frac{\partial c_A^*}{\partial \chi} &= \frac{\alpha^2 d}{4q\sqrt{D}}\left\{d(1+4\alpha) - \sqrt{D}\right\} \\ \frac{\partial c_A^*}{\partial q} &= \frac{d}{2q\sqrt{D}}\left\{\alpha\sqrt{D} + \left[4q - \alpha(1+4\alpha)d\right]\right\}\end{aligned}$$

where $\alpha = 2q/\chi$ and $D = d^2(1+4\alpha)^2 + 4d\Delta - 32dq + 4\Delta^2$ (> 0). First, $\partial c_A^*/\partial d > 0$ because the second regularity condition (9) implies $d \leq \frac{2(8q-\Delta)}{(1+4\alpha)^2}$ and thus

$$16q - d(1+4\alpha)^2 - 2\Delta \geq 16q - \frac{2(8q-\Delta)}{(1+4\alpha)^2}\cdot(1+4\alpha)^2 - 2\Delta = 0.$$

Second, $\partial c_A^*/\partial\Delta < 0$ because

$$\begin{aligned}D - (d+2\Delta)^2 = 8d\left\{\alpha(1+2\alpha)d - 4q\right\} &\geq 8d\left\{\alpha(1+2\alpha)\cdot\frac{2(8q-\Delta)}{(1+4\alpha)^2} - 4q\right\} \\ &= -\frac{16d}{(1+4\alpha)^2}\left\{\alpha(1+2\alpha)\Delta + 2\left(1+4\alpha+8\alpha^2\right)q\right\} < 0\end{aligned}$$

by the second regularity condition (9). Third, $\partial c_A^*/\partial \chi$ has the opposite sign of $d(\Delta - 8q) + \Delta^2$ because $D = \{d(1+4\alpha)\}^2 - 4\{d(\Delta - 8q) + \Delta^2\}$.

Lastly, we show $\partial c_A^*/\partial q > 0$. This is obvious when $4q \geq \alpha(1+4\alpha)d$. Suppose otherwise; i.e., $d > \frac{4q}{\alpha(1+4\alpha)}$. Then, a necessary and sufficient condition for $\partial c_A^*/\partial q > 0$ is

$$\alpha^2 D - \left[\alpha(1+4\alpha)d - 4q\right]^2 = 4(\alpha\Delta + 2q)\left\{\alpha(d+\Delta) - 2q\right\}$$

is positive. This condition is true because

$$\alpha(d+\Delta) - 2q > \alpha\left\{\frac{4q}{\alpha(1+4\alpha)} + \frac{4q}{2\alpha+1}\right\} - 2q = \frac{2q}{(1+2\alpha)(1+4\alpha)} > 0$$

due to the supposition $d > \frac{4q}{\alpha(1+4\alpha)}$ and the first regularity condition (8).