Composition Independence in Compound Games: a Characterization of the Banzhaf Power Index and the Banzhaf Value[*,†]

Ori Haimanko[‡]

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Abstract

We introduce the axiom of composition independence for power indices and value maps. In the context of compound (two-tier) voting, the axiom requires the power attributed to a voter to be independent of the second-tier voting games played in all constituencies other than that of the voter. We show that the Banzhaf power index is uniquely characterized by the combination of composition independence, four semivalue axioms (transfer, positivity, symmetry, and dummy), and a mild efficiency-related requirement. A similar characterization is obtained as a corollary for the Banzhaf value on the space of all finite games (with transfer replaced by additivity).

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[†]This work was motivated by a conversation the author had with Sergiu Hart on the composition property of the Banzhaf index in compound games, in which Sergiu observed that each voter's power is independent of the games played in other constituencies. This fact is the basis for the new axiom of composition independence.

[‡]Department of Economics, Ben-Gurion University of the Negev, Beer Sheva 84105, Israel. E-mail: orih@exchange.bgu.ac.il

1 Introduction

The Banzhaf power index is arguably the most adequate mechanism for measuring the a priori *influence* of voters in a voting situation; see, e.g., the extensive discussion in Section 3.1 of Felsenthal and Machover (1998). The idea behind the index is clear and simple. If a voter, or an outside observer, stand behind the standard "veil of ignorance," the best they can do is to assume that Yes and No votes by the electorate members constitute outcomes of Bernoulli trials with $p = \frac{1}{2}$ (i.e., each voter is a priori equally likely to choose Yes or No). In this setting, it is natural to define the power of a voter to influence the voting outcome as the *probability* that his vote is decisive, namely, that the election would be lost without that voter's support but won with his support.

The Banzhaf power index has a long history. A version of it was initially suggested by Penrose (1946), followed by two subsequent rediscoveries by Banzhaf (1965, 1966, 1968) and Coleman (1971).[1] The much-used probabilistic version described above[2] has its origin in the work of Dubey and Shapley (1979), who initiated the study of the Banzhaf index in the game-theoretic framework. Following the approach of Shapley and Shubik (1954), they model a voting situation as a simple cooperative game (or voting game); the Banzhaf index of a player (voter) is then the probability that he is a *swinger* for a random coalition of other players (which each player joins with probability $\frac{1}{2}$, independently of anyone else), meaning that he turns that coalition from losing to winning by joining it.[3]

The simple probabilistic model upon which the Banzhaf index is based has a fascinating implication for measuring voting power in *compound voting*. The latter

[1] The name of John F. Banzhaf III has been the one most associated with that power index, due to the number of works he authored on the subject and the legal repercussions of his findings and recommendations. Thus, siding with most of the literature, we will use the term "Banzhaf power index" for brevity, although, and perhaps more appropriately, the index is sometimes referred to as the Penrose-Banzhaf-Coleman power index.

[2] Felsenthal and Machover (1998) call this version "the Banzhaf measure."

[3] Our notion of swinger is a slight adaptation of the term used in Dubey and Shapley (1979, p. 103), who defined it in relation to a random coalition that may include the swinger, in which case the effect of his departure from that coalition on its winning status is also considered. The probability of being a swinger is the same under both definitions, and hence both swinger notions may be used in defining the Banzhaf index.

term refers to two-tier voting systems, of which there are numerous examples ranging from high-profile ones such as the US Electoral College and the Council of the European Union to the legislative organization of a local government. The common game-theoretic model underlying these systems is the one in which the player set N is partitioned into k disjoint "constituencies" $C_1, ..., C_k$, and the outcome of the vote in a constituency C_j is described by a simple game w_j with player set C_j (w_j is, in most cases, the simple majority game). Following the vote in $N = \cup_{j=1}^{k} C_j$, decision-making moves into the "council of representatives," where players-representatives from the set $R = \{1, ..., k\}$ vote in accordance with the voting outcomes in their respective constituencies; the outcome of the council vote is in turn determined by a simple game v (which is, in most cases, a weighted majority game[4]).

In the compound game $v[w_1, ..., w_k]$ thus described, player $i_0 \in C_{j_0}$ is a swinger with respect to a coalition $S \subset N \setminus \{i_0\}$ if and only if he is a swinger in the game w_{j_0} (for $S \cap C_{j_0}$) *and* his representative j_0 is a swinger in v (for the coalition T of representatives $j \neq j_0$ whose vote is sanctioned by S, i.e., $w_j(S \cap C_j) = 1$). A little reflection reveals that when players in $N \setminus \{i_0\}$ form S by joining randomly and independently (each with probability $\frac{1}{2}$), the event that i is a swinger in w_{j_0} and the event that j_0 is a swinger in v are themselves *independent*; moreover, the induced distribution of $T \subset R$ is such that each representative participates in it with probability $\frac{1}{2}$,[5] independently of other representatives. The definition of the Banzhaf index of a player as his probability of being a swinger – for a coalition joined by each other player with probability $\frac{1}{2}$ and independently of the rest – thus implies a well-known property of the index: the power of player i_0 in the compound game $v[w_1, ..., w_k]$ is equal to the *product* of his power in his second-tier game w_{j_0} and the power of his representative j_0 in the first-tier game v.

The latter attribute of the Banzhaf index, to which we will refer as the *composition*

[4] The weights given to different representatives may be (roughly) proportional to the population sizes of the counties they represent; but that is often not the case, either by necessity or by design. (See Chapter 4 of Felsenthal and Machover (1998)) for many examples of weighted voting in the US.)

[5] To be precise, in order for the this property to hold all second-tier games w_j need to be decisive, i.e., constant-sum (as in the scenario where all w_j are simple majority games with an odd number of players).

property, was first noticed by Owen (1975). Owen also showed that the composition property is satisfied by the Banzhaf value – the natural extension of the Banzhaf index to all games – for properly generalized compound games whose components are not necessarily simple. This strikingly simple behavior of the Banzhaf index has an immediate computational upshot, as the complexity of calculating the power of a voter in real-life instances of compound voting (where constituency sizes can run into millions) is immensely reduced. Indeed, the first-tier games are mostly too small to pose a serious problem in computing power,[6] while the (possibly huge) second-tier games are usually the simple majority ones, where the power can be approximated with a high degree of precision by applying the standard Stirling's formula. This computational simplicity is what stands behind the derivation of the famous "square-root rule" of Penrose, as rendered in Theorem 3.4.3 of Felsenthal and Machover (1998), whereby all voters enjoy (approximately) equal voting power only when the Banzhaf power indices of their representatives are proportional to the square root of the size of their respective constituencies.[7]

The appeal of the composition property leads to the natural question of whether there are other sensible power indices sharing this property. It turns out that the composition property is too powerful to allow any significant freedom of choice. Owen (1978) showed that, on the space of all games,[8] the Banzhaf value is essentially the only value map that satisfies standard axioms in conjunction with the composition property.[9] The strength of the composition property is particularly noticeable when attention is restricted to simple games: Dubey et al. (2005) showed that imposing just two axioms together with the composition property yields the Banzhaf power index. (The two axioms are *transfer* (or *valuation*), which has been the standard substitute

[6] For instance, when the first-tier game is a weighted majority one, the generating functions method suggested in Owen (1982, Chapter X, pp. 226-227) provides an effective way to compute the Banzhaf index of each representative if the number of representatives is not too large (which is typically the case).

[7] The underlying assumption behind this principle is that all second-tier voting games are simple majority games, which is the case in most real-life instances of compound voting.

[8] The claim is also true for the space of all constant-sum games.

[9] See Theorems 7 and 8 in Owen (1978). Although some other indices, such as the useless null index and the simplistic "dictatorial" index, also emerge from his axiomatization, they are easily removed by adding the dummy and strict positivity axioms; see Section 5 in Owen (1978).

for the additivity axiom in the context of simple games since its introduction in Dubey (1975), and strict *positivity* (or *monotonicity*), which requires the power measure to be non-negative and non-zero.)

The composition property may, as an axiom, be criticized on the grounds of being a technical or computational requirement, lacking a compelling conceptual basis. As a very strong condition, however, the composition property can be weakened in various ways, which may provide a conceptually sounder axiom. That is the path we intend to follow, starting with the following observation. According to the composition property, the ingredients for computing the power of player $i_0 \in C_{j_0}$ in the compound game $v[w_1, ..., w_k]$ are his power in his second-tier game w_{j_0} and the power of his representative j_0 in first-tier game v. In particular, in order to compute his power, all i_0 needs to be aware of is his own game w_{j_0} and the game v played on the first tier. We shall state that partial aspect of the composition property as an axiom, calling it *composition independence*. The axiom will require, for any compound game[10] $v[w_1, ..., w_k]$, the power of any player i_0 who belongs to a constituency C_{j_0} to be independent of the second-tier games w_j played in all other constituencies (i.e., for $j \neq j_0$).

On the conceptual level, the justification of composition independence lies in understandable limitations of the knowledge that a player may possess about the overall voting structure. The new axiom says that a huge deal of seemingly important details are, in fact, of no relevance to the player when he assesses his voting power. Composition independence implies that he does not need to have any detailed idea of the structure of the voting process (i.e., the simple game played) in any of the constituencies other than his own. In particular, he does not need to know the number of voters in other constituencies, or even what these constituencies are. The only characteristic of the other constituencies the player is expected to know is their total number, as it pertains to the knowledge of the first-tier game v between the representatives, which remains necessary.

We will show that both the Banzhaf power index and the Banzhaf value can be uniquely characterized using the composition independence axiom. As composition

[10] As in the premise for the composition property, it will be assumed that the second-tier games $w_1, ..., w_k$ are constant-sum.

independence is relatively mild on its own, the result of the type of Dubey et al. (2005) – where the composition property was accompanied by just two extra requirements – should not be expected. And, indeed, in our results a total of five (logically independent) requirements will accompany composition independence. Four of them are the *semivalue axioms* (the term comes from the works of Dubey et al. (1981) and Einy (1987) who considered maps satisfying the conjunction of these axioms in the context of general games, and simple games, respectively). The semivalue axioms for power indices on the domain of simple games – the already mentioned *transfer* and *positivity* axioms, *symmetry*, and *dummy* – are quite standard, and all four axioms or their subsets figure prominently in the literature on axiomatizations. (In particular, the transfer axiom, which is the "heaviest" of the four, has been a backbone of most axiomatic approaches to the Banzhaf index; see, e.g., Dubey and Shapley (1979), Lehrer (1988), Albizuri and Ruiz (2001), Dubey et al. (2005), Casajus (2012), Haimanko (2017)). For general games, linearity replaces transfer in the set of semivalue axioms, but its weaker form – *additivity* – will suffice for our needs. (Additivity is the most frequently used axiom in the treatment of value maps ever since its introduction in Shapley (1953).)

The extra non-semivalue axiom that we impose is new, and contains a requirement that is significantly weaker than efficiency. The efficiency axiom, whereby the total power of all players is equal to 1 (or the total value is equal to the worth of the grand coalition, in the case of general games) is almost invariably assumed in axiomatizations of the Shapley-Shubik power index and the Shapley value, but it is flagrantly violated by the Banzhaf index. However, we will show that the Banzhaf index has the following flavor of efficiency: Consider a sequence of simple games where the size of the player set tends to infinity; then it cannot be the case that, in the limit, *every* player's power is above some positive constant that is common to all players.[11] This will be postulated for general power indices by our *vanishing power* axiom. Stated slightly more generally, the same axiom will be assumed on value maps in the context of general games.

The paper is organized as follows. In Section 2 we recall the basic definitions

[11] We will establish this claim in Remark 2. Any efficient power index would satisfy such a claim, but it is not entirely obvious in the case of the non-efficient Banzhaf index.

pertaining to finite and simple games, and the notions of the Banzhaf index and value. Section 3 states our axioms for power indices on the domain of simple games and Section 4 proves the characterization result for the Banzhaf index. Section 5 does likewise for the Banzhaf value on the space of all finite games.

2 Preliminaries

2.1 Finite games and simple games

Let U be an infinite universe of *players*; it may be assumed w.l.o.g. that U includes the set $\mathbb{N}$ of positive integers. Denote the collection of all *coalitions* (subsets of U) by 2^U, and the empty coalition by $\emptyset$. A *game* on U is given by a map $v : 2^U \to \mathbb{R}$ with $v(\emptyset) = 0$. A coalition $N \subset U$ is called a *carrier* of v if $v(S) = v(S \cap N)$ for any $S \in 2^U$. We say that v is a *finite game* if it has a finite carrier; the minimal carrier of such v is, in effect, its true player set. The space of all finite games on U is denoted by $\mathcal{G}$. A game $v \in \mathcal{G}$ is said to be *constant-sum*, or of constant sum c, if $v(S) + v(U \setminus S) = c\ (= v(U))$ for every $S \in 2^U$.

The domain $\mathcal{SG} \subset \mathcal{G}$ of *simple games* on U consists of all $v \in \mathcal{G}$ such that: (i) $v(S) \in \{0, 1\}$ for all $S \in 2^U$; (ii) $v(U) = 1$; and (iii) v is *monotonic*, i.e., if $S \subset T$ then $v(S) \leq v(T)$. If $v \in \mathcal{SG}$, a coalition S is *winning* if $v(S) = 1$, and *losing* otherwise. If $v \in \mathcal{SG}$ is constant-sum, i.e., S is winning in v if and only if $U \backslash S$ is losing in v, then v is called *decisive*.

The space $\mathcal{AG} \subset \mathcal{G}$ of *additive games* consists of all $v \in \mathcal{G}$ satisfying $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$. Any $w \in \mathcal{AG}$ with (finite) carrier N is identifiable with the vector[12] $\{w(i) \mid i \in N\}$, and thus may be thought of as a payoff vector to the players in N.

2.2 Power indices and value maps

A *power index* φ is a map $\varphi : \mathcal{SG} \to \mathcal{AG}$, where $\varphi(v)(i)$ is interpreted as the voting power of player i in a simple game v. A map $\varphi : \mathcal{G} \to \mathcal{AG}$, defined on the full domain of finite games, is called a *value map*; for $v \in \mathcal{G}$, $\varphi(v)(i)$ may be viewed as an

[12]We shall henceforth omit braces when indicating one-element sets.

evaluation of player i's "utility of playing the game" (see Roth 1988).[13]

The *Banzhaf value* is a value map β that is given, for any $v \in \mathcal{G}$ with a finite carrier N, by

$$\beta(v)(i) = \frac{1}{2^{n-1}} \sum_{S \subset N \setminus i} v(S \cup i) - v(S)$$

if $i \in N$, where $n = |N|$; and

$$\beta(v)(i) = 0$$

if $i \in U \setminus N$. It is easy to see that $\beta(v)$ is well defined, being independent of the choice of the carrier N of v. The restriction of β to $\mathcal{SG}$ is the *Banzhaf (power) index.*

3 Axioms for Power Indices

This section introduces our axioms – plausible requirements that a general power index φ may be expected to obey – whose combination will later be shown to uniquely characterize the Banzhaf power index. We begin with four familiar *semivalue* axioms that are quite routinely assumed in dealing with power indices, either in their entirety or in part.[14]

In order to state the first axiom, given $v, w \in \mathcal{SG}$ let $v \vee w, v \wedge w \in \mathcal{SG}$ be defined by

$$(v \vee w)(S) = \max\{v(S), w(S)\}, \quad (v \wedge w)(S) = \min\{v(S), w(S)\}$$

for all $S \in 2^U$.

Axiom I: Transfer (Tran) For any $v, w \in \mathcal{SG}$, $\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w)$.

[13] We do not impose the usual efficiency requirement on a value map φ (whereby the equality $\varphi(v)(U) = v(U)$ should hold for any $v \in \mathcal{G}$), and, indeed, the objects of our investigation (namely, the Banzhaf power index and the Banzhaf value, defined next) do not satisfy efficiency. The interpretation of $\varphi(v)(i)$ as i's "utility of playing the game" is still valid, however, as the framework of Roth allows inefficient subjective valuations when players are averse to strategic risk (see Roth (1988, p. 61)).

[14] Variants of these axioms have been present in the original axiomatizations of the Shapley-Shubik and the Banzhaf power indices (see Dubey (1975) and Dubey and Shapley (1979)). The term "semivalue" comes from Einy (1987); see Remark 1 below.

As was shown in Dubey et al. (2005, p. 24), **Tran** can be restated in an equivalent but conceptually clearer form, amounting to a requirement that the change in power depends only on the change in the voting game.[15]

Next, denote by Π the set of all permutations of U. For any $\pi \in \Pi$ and $v \in \mathcal{G}$, define a game $\pi v \in \mathcal{G}$ by $(\pi v)(S) = v(\pi(S))$ for all $S \in 2^U$. The game πv is the same as v except that players are relabeled according to π^{-1}.[16]

Axiom II: Symmetry (Sym). For any $v \in \mathcal{SG}$, $i \in U$, and $\pi \in \Pi$, $\varphi(\pi v)(i) = \varphi(v)(\pi(i))$.

According to **Sym**, if players are relabeled in a game, their power indices will be relabeled accordingly. Thus, irrelevant characteristics of the players, outside of their role in the game v, have no influence on the power index.

Axiom III: Positivity (Pos). For any $v \in \mathcal{SG}$ and $i \in U$, $\varphi(v)(i) \geq 0$.

The positivity requirement is natural, as every $v \in \mathcal{SG}$ is monotonic by assumption and hence the influence of any player joining a coalition is always non-negative.

Axiom IV: Dummy (Dum). If $v \in \mathcal{SG}$ and i is a dummy player in v, i.e. $v(S \cup i) = v(S) + v(i)$ for every $S \subset U \setminus i$, then $\varphi(v)(i) = v(i)$.

A dummy player in a simple game can be either a dictator (if $v(i) = 1$), in which case a coalition is winning if and only if it contains i, or a null player (if $v(i) = 0$), that does not belong to the minimal carrier of v. Accordingly, **Dum** can be viewed as a normalization requirement, assigning power 1 to a dictator and power 0 to a null player.

Remark 1 (Semivalues). Einy (1987) referred to power indices satisfying **Tran, Sym, Pos,** and **Dum** as *semivalues*, and we adopt this term. He showed that a power index φ is a semivalue if and only if it has the following representation: there exists

[15] The possibility of such a restatement has been mentioned in Dubey and Shapley (1979, p. 106). A special version of the restatement also appeared in Laruelle and Valenciano (2001).

[16] That is, player $\pi^{-1}(i)$ has the same role in πv as player i in v.

a (uniquely determined) probability measure ξ on $[0,1]$ such that, for every $v \in \mathcal{SG}$ with some finite carrier N,

$$\varphi(v)(i) = \sum_{S \subset N \setminus i} p_{|S|}^{|N|} [v(S \cup i) - v(S)] \tag{1}$$

if $i \in N$, where

$$p_s^n = \int_0^1 x^s (1-x)^{n-s-1} d\xi(x); \tag{2}$$

and $\varphi(v)(i) = 0$ if $i \in U \setminus N$. When ξ is concentrated on $\frac{1}{2}$, the Banzhaf index β is obtained; in particular, β is a semivalue. □

Now recall the notion of a compound game (see Shapley (1964)). Consider $v, w_1, ..., w_k \in \mathcal{SG}$ such that $R = \{1, ..., k\}$ (the set of *representatives*) is a carrier for v, and $w_1, ..., w_k$ have disjoint finite carriers (*constituencies*) $C_1, ..., C_k$. (Note that no assumption is made on the relation between the set of representatives and the constituencies; we think of player $j \in R$ as the representative of constituency C_j, but he need not be a member of C_j.) The game $u \in \mathcal{SG}$ is said to be the *compounding* of v with $w_1, ..., w_k$, written $u = v[w_1, ..., w_k]$, if

$$u(S) = v(\{j \mid w_j(S) = 1\})$$

for all $S \in 2^U$. Notice that $\bigcup_{j=1}^k C_j$ is a carrier for u.

Axiom V: Composition Independence (CompInd). Let $v[w_1, ..., w_k]$ be a compound simple game in which $w_1, ..., w_k \in \mathcal{SG}$ are decisive, with corresponding constituencies $C_1, ..., C_k$. Given $1 \leq j \leq k$, let $w'_1, ..., w'_{j-1}, w'_{j+1}, ..., w'_k \in \mathcal{SG}$ be another collection of decisive games with corresponding disjoint carriers $C'_1, ..., C'_{j-1}, C'_{j+1}, ..., C'_k \subset U \setminus C_j$. Then, for every $i \in C_j$,

$$\varphi(v[w_1, w_2, ..., w_k])(i) = \varphi\left(v[w'_1, ..., w'_{j-1}, w_j, w'_{j+1}, ..., w'_k]\right)(i).$$

The axiom requires the voting power of any player i in any constituency C_j to be *independent* of the voting games that played in all other constituencies (and even of the composition of the other constituencies). Thus, i should be able to determine his power based only on the knowledge of the voting game of his constituency, and of the second-tier voting game played between the representatives $1, ..., k$.

The Banzhaf index β satisfies **CompInd** because it adheres to a much stronger requirement, the *composition property.* The latter means that the index is multiplicatively separable for any compound game: it is well known (see Theorem 2 in Owen (1975)) that if $v[w_1, ..., w_k]$ is a compound game in which $w_1, ..., w_k$ are decisive, then

$$\beta\left(v[w_1, ..., w_k]\right)(i) = \beta\left(v\right)(j) \cdot \beta\left(w_j\right)(i) \tag{3}$$

for every $j = 1, ..., k$ and every $i \in C_j$. It follows that $\beta\left(v[w_1, ..., w_k]\right)(i)$ does not depend on the games $w_{j'}$ for $j' \neq j$, and hence β satisfies **CompInd**.

Axiom VI: Vanishing Power (**VanPow**). Let $\{v_k\}_{k=1}^{\infty} \subset \mathcal{SG}$ be a sequence of games with corresponding (nonempty) carriers $\{N_k\}_{k=1}^{\infty}$, and assume that $\lim_{k\to\infty} |N_k| = \infty$. Then

$$\lim\inf_{k\to\infty} \min_{i\in N_k} \varphi\left(v_k\right)(i) \leq 0.$$

The axiom embodies a mild aspect of efficiency (that would require the total power of all players to be equal to 1), by stipulating that, when the size of the player set of a game tends to infinity, it cannot be the case that *every* player's power is above some positive constant that is common to all players.[17]

Remark 2. (The Banzhaf index satisfies VanPow). While the Banzhaf index is not efficient, it does satisfy the significantly weaker **VanPow**. Indeed, by Theorem 2 of Dubey and Shapley (1979),

$$\sum_{i\in N_k} \beta\left(v_k\right)(i) \leq \frac{|N_k|}{2^{|N_k|-1}} \binom{|N_k|-1}{\left[\frac{|N_k|}{2}\right]}, \tag{4}$$

where $[n]$ denotes the integer part of n. This implies that

$$(0 \leq)\ \min_{i\in N_k} \beta\left(v_k\right)(i) \leq \frac{1}{2^{|N_k|-1}} \binom{|N_k|-1}{\left[\frac{|N_k|}{2}\right]}.$$

A direct application of the Stirling's formula $n! \sim n^n e^{-n}\sqrt{2\pi n}$ readily yields the asymptotic relation $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$, which in turn implies that $\binom{n-1}{\left[\frac{n}{2}\right]} \sim \frac{2^n}{\sqrt{2\pi n}}$. Therefore

$$\lim_{k\to\infty} \min_{i\in N_k} \beta\left(v_k\right)(i) = \lim_{k\to\infty} \frac{1}{2^{|N_k|-1}} \binom{|N_k|-1}{\left[\frac{|N_k|}{2}\right]} = 0. \ \square$$

[17] When **Pos** is also assumed to hold, **VanPow** implies that $\lim\inf_{k\to\infty} \min_{i\in N_k} \varphi\left(v_k\right)(i) = 0$, as power cannot be negative.

4 Uniqueness of the Banzhaf Index

The six axioms of the previous section uniquely characterize the Banzhaf power index:

Theorem 1. The Banzhaf index β is the only power index on $\mathcal{SG}$ that satisfies **Tran, Sym, Pos, Dum, CompInd,** and **VanPow.**

Proof. The fact that β satisfies the axioms has been established in the previous section. Now assume that φ is a power index on $\mathcal{SG}$ that satisfies the six axioms. In particular, φ is a semivalue (see Remark 1), and hence there exists a probability measure ξ on $[0,1]$ for which (1), (2) hold. Denote by u_i the unanimity game with carrier $\{i\} \subset U$, i.e., a dictator game, where

$$u_i(S) = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{otherwise} \end{cases}$$

for all $S \in 2^U$, and by $m_{i,j,k}$ the simple two-player majority game on $\{i,j,k\} \subset U$, where

$$m_{i,j,k}(S) = \begin{cases} 1, & \text{if } |S \cap \{i,j,k\}| \geq 2, \\ 0, & \text{otherwise} \end{cases}$$

for all $S \in 2^U$. Such u_i and $m_{i,j,k}$ are decisive.

Next consider the game $v = m_{1,2,3}[u_1, u_2, m_{3,4,5}]$. By (1) and (2) in Remark 1,

$$\varphi(v)(1) = \int_0^1 \left[x \cdot ((1-x)^3 + 3(1-x)^2 x) + (1-x) \cdot \left(x^3 + 3x^2(1-x)\right)\right] d\xi(x). \tag{5}$$

On the other hand,

$$\varphi(m_{1,2,3})(1) = \int_0^1 2x(1-x)\, d\xi(x). \tag{6}$$

It follows from **CompInd**, applied to the compound games $v = m_{1,2,3}[u_1, u_2, m_{3,4,5}]$ and $m_{1,2,3} = m_{1,2,3}[u_1, u_2, u_3]$, that the expressions in (5) and (6) are equal. Thus

$$\int_0^1 p(x) d\xi(x) = 0, \tag{7}$$

where

$$\begin{aligned} p(x) &= \left[x \cdot ((1-x)^3 + 3(1-x)^2 x) + (1-x) \cdot \left(x^3 + 3x^2(1-x)\right)\right] - 2x(1-x) \\ &= -x(1-x)(2x-1)^2. \end{aligned}$$

Notice that $p(0) = p(\frac{1}{2}) = p(1) = 0$, and that the polynomial p is *negative* on $[0,1] \setminus \{0, \frac{1}{2}, 1\}$. Thus (7) implies that ξ is supported on $\{0, \frac{1}{2}, 1\}$.

Finally, for each $k \geq 2$ consider the games $\overline{v}_k = u_1 \vee ... \vee u_k$ and $\overline{\overline{v}}_k = u_1 \wedge ... \wedge u_k$ with carrier $N_k = \{1, ..., k\}$. By applying (1) and (2), and using the fact that ξ is supported on $\{0, \frac{1}{2}, 1\}$, we obtain

$$\varphi\left(\overline{v}_k\right)(i) = \xi(0) + \frac{1}{2^{k-1}}\xi\left(\frac{1}{2}\right)$$

and

$$\varphi\left(\overline{\overline{v}}_k\right)(i) = \xi(1) + \frac{1}{2^{k-1}}\xi\left(\frac{1}{2}\right)$$

for every $i \in N_k$. The conjunction of **Pos** and **VanPow** mandates that $\liminf_{k\to\infty} \min_{i\in N_k} \varphi\left(\overline{v}_k\right)(i)$ $= 0$ and $\liminf_{k\to\infty} \min_{i\in N_k} \varphi\left(\overline{\overline{v}}_k\right)(i) = 0$, which is only possible if $\xi(0) = 0$ and $\xi(1) = 0$. We conclude that ξ is, in fact, supported on $\frac{1}{2}$, which means that $\varphi = \beta$. ■

Remark 3 (Logical independence of the axioms). No single axiom in the statement of Theorem 1 may be omitted, as for any axiom there are power indices other than β that satisfy the other five axioms. Indeed:

1. Let $\varphi(v)(i) = 0$ for every $v \in \mathcal{SG}$ and $i \in U$ who is *not* a dummy player in v, and $\varphi(v)(i) = v(i)$ for i who is a dummy in v. The power index φ satisfies all the axioms except **Tran**.

2. Fix $i_0 \in U$, and let φ be given, for every $v \in \mathcal{SG}$, by $\varphi(v)(i) = \beta(v)(i)$ if $i \neq i_0$, and $\varphi(v)(i_0) = v(i_0)$. The power index φ satisfies all the axioms except **Sym**.

3. For every $v \in \mathcal{SG}$ and $i \in U$, let $\varphi(v)(i) = 2\beta(v)(i) - v(i)$. The power index φ satisfies all the axioms except **Pos**.

4. The null index, $\varphi \equiv 0$, satisfies all the axioms except **Dum**.

5. The Shapley-Shubik power index, given by

$$\varphi(v)(i) = \sum_{S\subset N\setminus i} \frac{|S|!(n-|S|-1)!}{n!}\left[v(S\cup i) - v(S)\right]$$

   for every $v \in \mathcal{SG}$ with a finite carrier N and $i \in N$, and $\varphi(v)(i) = 0$ for every $i \in U \setminus N$, satisfies all the axioms except **CompInd.**

6. Let $\varphi(v)(i) = v(i)$ for every $v \in \mathcal{SG}$ and $i \in U$. The power index φ satisfies all the axioms except **VanPow**. As **VanPow** only has "bite" because the universe of players U is infinite, this example also shows that our axioms would not uniquely characterize β for a finite U. $\square$

5 Axioms for Value Maps and the Banzhaf Value

In this section we will extend and modify our axioms in order to fit the setting of value maps. Of the four semivalue axioms, **Tran** changes the most, returning to its original form of the *additivity axiom* that has been immensely popular in cooperative game theory since its introduction in Shapley (1953). (**Tran**, first suggested and used in Dubey (1975), was a necessary adaptation of additivity in the context of simple games, as their set is not closed under addition.)

Axiom I': Additivity (Add) For any $v, w \in \mathcal{G}$, $\varphi(v+w) = \varphi(v) + \varphi(w)$.

The other three semivalue axioms undergo only two small changes: the domain of games switches from $\mathcal{SG}$ to $\mathcal{G}$, and the games in the premise of **Pos'** are assumed to be monotonic (that was not necessary for the domain $\mathcal{SG}$ of **Pos** because simple games that we consider are monotonic by definition).

Axiom II': Symmetry (Sym'). For any $v \in \mathcal{G}$, $i \in U$, and $\pi \in \Pi$, $\varphi(\pi v)(i) = \varphi(v)(\pi(i))$.

Axiom III': Positivity (Pos'). For any monotonic $v \in \mathcal{G}$ and $i \in U$, $\varphi(v)(i) \geq 0$.

Axiom IV': Dummy (Dum'). If $v \in \mathcal{G}$ and i is a dummy player in v, i.e. $v(S \cup i) = v(S) + v(i)$ for every $S \subset U \setminus i$, then $\varphi(v)(i) = v(i)$.

Although the proof of our forthcoming characterization result for the Banzhaf value does not require any change in **CompInd**, i.e., it would have sufficed to limit attention to simple first- and second-tier games in compounding, we will introduce a version of the axiom for general compound games in order to stress that the property in **CompInd** is not specific to simple games, as far as the Banzhaf value is concerned.

Following Owen (1964), we define a general compound game as follows. Consider $v, w_1, ..., w_k \in \mathcal{G}$ such that the set of representatives $R = \{1, ..., k\}$ is a carrier for v, and $w_1, ..., w_k$ have disjoint finite carrier-constituencies $C_1, ..., C_k$. Let us moreover assume that $w_j(S) \in [0, 1]$ for each $j \in R$ and $S \in 2^U$. The *compounding* of v with $w_1, ..., w_k$, $u = v[w_1, ..., w_k] \in \mathcal{G}$, is given by

$$u(S) = \sum_{T \subset R} \left(\prod_{j \in T} w_j(S) \cdot \prod_{j \in R \setminus T} (1 - w_j(S)) \right) \cdot v(T)$$

for all $S \in 2^U$. Each game w_j can be thought of as determining the *probability* that a coalition S "controls" the representative j of the constituency C_j in the first-tier game v; $u(S)$ is then the expected payoff to S in that probabilistic scenario.

Axiom V': Composition Independence (CompInd'). Let $v[w_1, ..., w_k] \in \mathcal{G}$ be a compound game in which $w_1, ..., w_k \in \mathcal{G}$ are non-negative and of constant sum 1, with corresponding constituencies $C_1, ..., C_k$. Given $1 \leq j \leq k$, let $w'_1, ..., w'_{j-1}, w'_{j+1}, ..., w'_k \in \mathcal{G}$ be another collection of non-negative games of constant sum 1, with corresponding disjoint carriers $C'_1, ..., C'_{j-1}, C'_{j+1}, ..., C'_k \subset U \setminus C_j$. Then, for every $i \in C_j$,

$$\varphi(v[w_1, w_2, ..., w_k])(i) = \varphi\left(v[w'_1, ..., w'_{j-1}, w_j, w'_{j+1}, ..., w'_k]\right)(i).$$

Since the Banzhaf value β on $\mathcal{G}$ satisfies (3) for any $v[w_1, ...w_k]$ as above (see Theorem 2 in Owen (1975)), it satisfies **CompInd'.** And, as a semivalue on $\mathcal{G}$ (see Dubey et al. (1981)), β satisfies **Add, Sym', Pos'** and **Dum'.**

The **VanPow** axiom could also have been left unchanged without affecting our characterization result. Again, we present a generalized version in order to emphasize that the Banzhaf value β satisfies such a generalization.

Axiom VI': Vanishing Power (VanPow'). Let $\{v_k\}_{k=1}^{\infty} \subset \mathcal{G}$ be a sequence of monotonic games with corresponding (nonempty) carriers $\{N_k\}_{k=1}^{\infty}$, and assume that $\lim_{k\to\infty} |N_k| = \infty$ and that $\limsup_{k\to\infty} v_k(N_k) < \infty$. Then

$$\lim\inf_{k\to\infty} \min_{i \in N_k} \varphi(v_k)(i) \leq 0.$$

Remark 4 (The Banzhaf value satisfies VanPow'). Let $A > 0$ be such that $v_k(N_k) \leq A$ for all $k \geq 1$. For each $0 < q \leq A$ and $k \geq 1$, denote by v_k^q the simple game[18] with carrier N_k that is given by $v_k^q(S) = 1$ if $v_k(S) \geq q$, and $v_k^q(S) = 0$ otherwise. Notice that

$$v_k(S) = \int_0^A v_k^q(S) dq \tag{8}$$

for all $S \in 2^U$. From the definition of β and (8) it follows that, for every $k \geq 1$ and $i \in N_k$,

$$\beta(v_k)(i) = \int_0^A \beta(v_k^q)(i)\, dq,$$

and hence

$$\begin{aligned}\sum_{i \in N_k} \beta(v_k)(i) &= \sum_{i \in N_k} \int_0^A \beta(v_k^q)(i)\, dq \\ &= \int_0^A \left(\sum_{i \in N_k} \beta(v_k^q)(i) \right) dq \leq \frac{A\,|N_k|}{2^{|N_k|-1}} \binom{|N_k|-1}{\left[\frac{|N_k|}{2}\right]},\end{aligned}$$

where the last inequality is immediate from (4). Arguing as in Remark 2 from this point on, we obtain the equality $\lim_{k\to\infty} \min_{i \in N_k} \beta(v_k)(i) = 0$. □

The six modified axioms uniquely characterize the Banzhaf value, just as their original versions did in the case of the Banzhaf power index:

Theorem 2. The Banzhaf value β is the only value map on $\mathcal{G}$ that satisfies **Add, Sym', Pos', Dum', CompInd',** and **VanPow'.**

Proof. It has already been established that β satisfies the above axioms, and it only remains to be shown that any value map φ satisfying the axioms must coincide with β. Given any such φ, its restriction $\varphi\mid_{\mathcal{SG}}$ to the domain $\mathcal{SG}$ clearly satisfies the axioms **Tran,**[19] **Sym, Pos, Dum, CompInd,** and **VanPow** for power indices, and thus

$$\varphi\mid_{\mathcal{SG}} = \beta\mid_{\mathcal{SG}} \tag{9}$$

by Theorem 1.

[18] Notice that v_k^q may be the zero game, which is, in principle, excluded from the domain $\mathcal{SG}$. For technical reasons, we will admit the game $v = 0$ as part of $\mathcal{SG}$ in our forthcoming considerations, keeping in mind that $\beta(0) = 0$.

[19] **Add** implies **Tran** because $v + w = v \vee w + v \wedge w$ for any $v, w \in \mathcal{SG}$.

Next fix a finite set $\emptyset \neq N \subset U$, and, for any $\emptyset \neq T \subset N$, denote by $u_T \in \mathcal{SG}$ the unanimity game with carrier T, given by

$$u_T(S) = \begin{cases} 1, & \text{if } T \subset S, \\ 0, & \text{otherwise} \end{cases}$$

for all $S \in 2^U$. It is well known that $\{u_T\}_{\emptyset \neq T \subset N} \subset \mathcal{SG}$ forms a basis for the vector space $\mathcal{G}_N$ of games with carrier N, and thus any $v \in \mathcal{G}_N$ can be written as a unique linear combination $v = \sum_{\emptyset \neq T \subset N} a_T u_T$ of the members of this basis. It follows from **Add** that

$$\varphi(v) = \sum_{\emptyset \neq T \subset N} \varphi(a_T u_T). \tag{10}$$

Add also implies that $\varphi(a u_T) = a\varphi(u_T)$ for any rational a, and an application of **Pos'** (enabled by the fact that $a u_T$ is monotonic for any positive a) establishes the equality $\varphi(a u_T) = a\varphi(u_T)$ for any real a. Using this and (9), equation (10) yields

$$\begin{aligned} \varphi(v) &= \sum_{\emptyset \neq T \subset N} \varphi(a_T u_T) = \sum_{\emptyset \neq T \subset N} a_T \varphi(u_T) \\ &= \sum_{\emptyset \neq T \subset N} a_T \beta(u_T) = \sum_{\emptyset \neq T \subset N} \beta(a_T u_T) = \beta(v). \end{aligned}$$

Our argument therefore shows that φ and β coincide on the space $\mathcal{G}_N$ for every nonempty finite $N \subset U$. But, obviously, $\mathcal{G} = \bigcup_{\emptyset \neq N \subset U,\ |N| < \infty} \mathcal{G}_N$, and so φ and β coincide on the entire $\mathcal{G}$. ■

Remark 5. (Logical independence of the axioms in Theorem 2). None of the axioms in the statement of the theorem can be omitted. Each power index listed in Remark 3 can be extended to the domain $\mathcal{G}$ (by using $v \in \mathcal{G}$ instead of $v \in \mathcal{SG}$ in its definition), yielding a value map different from β that simultaneously satisfies five given axioms out of the six. □

Remark 6. (Equal treatment instead of symmetry). The symmetry axiom (**Sym** or **Sym'**) can be replaced by the weaker *equal treatment* (**ET**) requirement in Theorems 1 and 2. The latter stipulates that if $i, j \in U$ are substitutes in a game v (i.e., for every $S \subset U \setminus \{i, j\}$, $v(S \cup i) = v(S \cup j)$), then $\varphi(v)(i) = \varphi(v)(j)$. The replacement by **ET** is possible due to the known results that the combination of **Tran**, **Dum**, and **ET** implies **Sym** for power indices (see Proposition 3.5 in Albizuri and Ruiz (2001)), and the combination of **Add**, **Dum'**, and **ET** implies **Sym'** for value maps (see Theorem 4(b) in Malawski (2002)). □

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