Double Implementation in Dominant Strategy Equilibria and Ex Post Equilibria with Private Values*

Makoto Hagiwara†

July 13, 2018

Abstract

We consider the implementation problem under incomplete information and private values. We investigate double implementation of (single-valued) mappings in dominant strategy equilibria and ex post equilibria. We call a mapping a "rule". We show that the notion of an ex post equilibrium is weaker than the notion of a dominant strategy equilibrium. Then, the implementation notion is not trivial even under private values. We define a new strategic axiom that is stronger than "strategy-proofness", but weaker than "secure strategy-proofness". We call it "weak secure-strategy-proofness". We show that a rule is doubly implementable iff it is *weakly securely-strategy-proof*.

JEL Classification: C72, D71, D78

Key words: Double implementation, Dominant strategy equilibrium, Ex post equilibrium, Weak secure-strategy-proofness, Private values

1 Introduction

We investigate the implementation problem under incomplete information and private values. The objective of a social planner is embodied by a "rule". Mathematically, a

*I am grateful to William Thomson, Takuma Wakayama, and Takehiko Yamato for their invaluable advice and suggestions. I also thank Acelya Altuntas, Yu Awaya, Eun Jeong Heo, Yuki Tamura, and Rodrigo A. Velez, as well as participants at study groups (University of Rochester, 2017, 2018) for helpful comments. This work was partially supported by JSPS KAKENHI Grant Number 17J01520 and JSPS Overseas Challenge Program for Young Researchers Grant Number 201780041.

†Department of Industrial Engineering and Economics, School of Engineering, Tokyo Institute of Technology, 2-12-1 Ookayama, Meguro-ku, Tokyo, 152-8552, Japan; E-mail: *hagiwara.m.af@m.titech.ac.jp*; *m.hagiwara@rochester.edu*

rule is a mapping which for each possible preference profile, specifies an outcome.[1] The planner does not know the agents' preferences. Then, she specifies an message space for each agent and a mapping which, for each possible message profile, chooses an outcome. The pair consisting of the list of the message spaces and the mapping is a "game form". In the direct game form associated with a rule, the message space for each agent is the set of his possible preferences and the mapping is the rule.

"Strategy-proofness"requires that in the direct game form associated with the rule, for each agent, truth-telling is a dominant strategy. An important point concerning a dominant strategy equilibrium is that each agent needs only information about his own preference. He need not care about the other agents' preferences and strategies. However, laboratory experiments on *strategy-proof* rules reported that in some games, some subjects did not select dominant strategies.[2]

These observations raise a concern for implementation theory. Although in pivotal-mechanism experiments, some subjects did not adopt dominant strategies, they frequently selected a Nash equilibrium (Cason et al. [5]). There is an explanation for this observation. Suppose that there are only two subjects. If one of them, subject 1, finds a dominant strategy but the other, subject 2, does not, then as long as subject 2 chooses a best response to subject 1's strategy, a Nash equilibrium outcome is achieved. It should be easier to find a best response to subject 1's strategy than a dominant strategy. This observation led Saijo et al. [13] to formulate and investigate "secure implementation", namely double implementation in dominant strategy equilibria and Nash equilibria.[3] However,

[1] A rule is also called a "social choice function".

[2] For a summary of laboratory experiments on *strategy-proof* rules, see for example Cason et al. [5].

[3] The another study focuses on extensive game forms. In an ascending auction and a second-price auction, subjects were substantially more likely to play truth-telling under the former than under the latter (Kagel et al. [9]). Inspired from this observation, "obvious" *strategy-proofness* is defined and characterized as a cognitively limited agent can recognize that truth-telling is a dominant strategy (Li

in laboratory experiments in Cason et al. [5], each subject only knew his own preference, so that incomplete information games were considered. Usually, to define the notion of a Nash equilibrium, we investigate complete information games in which each agent knows the true preference profile.[4] Figure 1 illustrates this discussion.

	Cason et al. [5] (Laboratory experiments)	Saijo et al. [13] (A theoretical prediction)
Information structure	Incomplete information	Complete information
Result	Subjects frequently selected a Nash equilibrium	Characterizations for *secure implementability*

Figure 1

In an attempt to explain laboratory experiments in Cason et al. [5], we study double implementation in dominant strategy equilibria and ex post equilibria. From now on, "double implementation" is used in this sense. An "ex post equilibrium" is a strategy profile in which for each possible preference profile, the message profile for the preference profile is a Nash equilibrium. Bergemann and Morris [4] claim that "*in an environment with private values, the notion of ex post equilibrium is equivalent to the notion of dominant strategy equilibrium (pp. 532)*". Our first result is that in general, however, the former is weaker than the latter (Lemma 1, Example 1). Then, for the direct game form

[10]).

[4]For laboratory experiments in Cason et al. [5], one justification for *secure implementation* is that a Nash equilibrium can be interpreted as a rest point of the dynamic learning process (Cason et al. [5]). However, *secure implementation* is a theoretical prediction in a one-shot game. The another justifications for *secure implementation* are characterizations by robust implementation notions (Adachi [2], Saijo et al. [13]). Even though the implementation notions are under incomplete information, we might not explicitly study the observation of experiments unlike *secure implementation*.

associated with a rule, *dominant strategy implementability* is weaker than *ex post implementability* (Lemma 2, Example 2).[5] If we consider general game forms, not only direct game forms, then a rule may be implemented by some game form in ex post equilibria, but not by this game form in dominant strategy equilibria (Example 3).

For double implementation, we need to consider an implementation notion in dominant strategy equilibria. By the revelation principle for *dominant strategy implementability*, *strategy-proofness* is necessary (e.g., Gibbard [8]). Based on this result, *secure implementability* is characterized by a stronger axiom, "secure strategy-proofness" (Saijo et al. [13]).[6] *Secure strategy-proofness* requires that the rule be *strategy-proof* and for each preference profile and each Nash equilibrium in the complete information game induced by the direct game form and the preference profile, the outcome at the equilibrium be equal to the outcome chosen by the rule for the preference profile.

We define a new strategic axiom, "weak secure-strategy-proofness". The axiom requires that the rule be *strategy-proof* and if a strategy profile is an ex post equilibrium in the incomplete information game induced by the direct game form and the set of preference profiles, then for each preference profile, the outcome at the equilibrium be equal to the outcome chosen by the rule for the preference profile. The axiom is weaker than *secure strategy-proofness* (Proposition 4, Example 4).

We show that a rule is doubly implementable iff it is *weakly securely-strategy-proof* (Theorem 1). The proof involves showing that any doubly implementable rule is im-

[5]By this result, for the direct game form associated with a rule, *ex post "full"implementability* is weaker than *dominant strategy "full" implementability.* Note that under private values, *ex post "truthful" implementability* is equivalent to *"truthful" implementability* by the definitions (e.g., Bergemann and Morris [3]). In other words, "ex post incentive compatibility" is equivalent to *strategy-proofness.*

[6]In Saijo et al. [13], *secure implementability* is characterized by *strategy-proofness* and "rectangle property". *Rectangle property* is also called "rectangular property". For the definition of *rectangle property,* see Saijo et al. [13]. It is easy to show that a rule satisfies *strategy-proofness* and *rectangle property* iff it is *securely strategy-proof.*

plemented by its associated direct game form (Corollary 1). Hence, for double implementability, it suffices to focus on direct game forms.

For *secure implementation*, negative results have been established for a number of interesting rules (e.g., Fujinaka and Wakayama [6]). Even if a rule is not *securely implementable*, it may be doubly implementable (Corollary 2). Are there such rules? We provide one negative answer and one positive answer. In a school choice problem (Abdulkadiroğlu and Sönmez[1]) under incomplete information, the tentative acceptance rule is not doubly implementable (Example 5).[7] On the other hand, if the set of preference profiles is "large", then the rule may be doubly implemented (Example 6). Identifying general conditions on the set of preference profiles for double implementability of this rule is an open question.

2 Equilibrium notions

Let $N = \{1, \cdots, n\}$ be the **set of agents** and $A = \{a^1, \cdots, a^m\}$ be the **set of outcomes**. For each $i \in N$, let $R_i \in \mathcal{R}_i$ be a **preference for agent** i, where $\mathcal{R}_i$ is the **set of possible preferences for agent** i **over** A. Asymmetric and symmetric components of $R_i \in \mathcal{R}_i$ are P_i and I_i, respectively. A **preference profile** is a list $R \in \mathcal{R}$, where $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$ is the **set of preference profiles**. For each $i \in N$, let $u_i : A \times \mathcal{R}_i \to \mathbb{R}$ be a **utility representation for agent** i such that for each $a, b \in A$ and each $R_i \in \mathcal{R}_i$, $u_i(a, R_i) \geq u_i(b, R_i)$ iff $a \; R_i \; b$. Each agent's utility only depends on his own preference, so that we study private-value problems.[8]

[7] The tentative acceptance rule is usually called the deferred acceptance algorithm or the Gale-Shapley student optimal stable mechanism (Gale and Shapley [7]).

[8] If for each $i \in N$, $u_i : A \times \mathcal{R} \to \mathbb{R}$, then problems are under interdependent values. We can extend our results to interdependent-value problems. However, for double implementation, "*strategy-proofness* under interdependent values" is necessary by the revelation principle for *dominant strategy implementability*. This axiom is called "dominant strategy incentive compatibility" (Bergemann and Morris [3]). This

A **rule** is a mapping $f : \mathcal{R} \to A$ which for each preference profile $R \in \mathcal{R}$, specifies an outcome $f(R) \in A$.

A **game form** Γ is a pair (M, g) such that $M = \times_{i \in N} M_i$, where for each $i \in N$, M_i is the **message space for agent** i, and $g : M \to A$ is the **outcome mapping** which, for each message profile $m \in M$, specifies an outcome $g(m) \in A$. Let $\Gamma^f = (\mathcal{R}, f)$ be the **direct game form associated with** f.

Let $(\Gamma, \mathcal{R})$ be the **(incomplete information) game** induced by Γ and $\mathcal{R}$. A **(pure) strategy in** $(\Gamma, \mathcal{R})$ **for agent** $i \in N$ is a mapping $s_i : \mathcal{R}_i \to M_i$ which, for each preference of him $R_i \in \mathcal{R}_i$, specifies an message of him $s_i(R_i) \in M_i$. Let $s = (s_i)_{i \in N}$ be a **strategy profile** and S be a **set of strategy profiles**.

In $(\Gamma, \mathcal{R})$, let us define the two equilibrium notions which are central to our study.

Dominant strategy equilibrium: For each $i \in N$, each $R_i \in \mathcal{R}_i$, each $m_i \in M_i$, and each $m_{-i} \in M_{-i}$,

$$g(s_i(R_i), m_{-i}) \; R_i \; g(m_i, m_{-i}).$$

Let $DS(\Gamma, \mathcal{R}) \subseteq S$ be the **set of dominant strategy equilibria in** $(\Gamma, \mathcal{R})$.

Ex post equilibrium: For each $R \in \mathcal{R}$, each $i \in N$, and each $m_i \in M_i$,

$$g(s_i(R_i), s_{-i}(R_{-i})) \; R_i \; g(m_i, s_{-i}(R_{-i})).$$

Let $EP(\Gamma, \mathcal{R}) \subseteq S$ be the **set of ex post equilibria in** $(\Gamma, \mathcal{R})$.

Bergemann and Morris [4] claim that "*in an environment with private values, the notion of ex post equilibrium is equivalent to the notion of dominant strategy equilibrium*

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axiom is stronger than *ex post incentive compatibility* and it is difficult to find interesting rules satisfying this axiom.

(pp. 532)". Our first result is that in general, however, the notion of ex post equilibrium is weaker than the notion of dominant strategy equilibrium (Lemma 1, Example 1).

Lemma 1. For each game $(\Gamma, \mathcal{R})$, $DS(\Gamma, \mathcal{R}) \subseteq EP(\Gamma, \mathcal{R})$.

Proof: Let $s \in DS(\Gamma, \mathcal{R})$. Suppose that $s \notin EP(\Gamma, \mathcal{R})$. Then, there are $R \in \mathcal{R}$, $i \in N$, and $m_i \in M_i$ such that $g(m_i, s_{-i}(R_{-i}))\ P_i\ g(s_i(R_i), s_{-i}(R_{-i}))$. Therefore, there are $i \in N$, $R_i \in \mathcal{R}_i$, $m_i \in M_i$, and $m_{-i} = s_{-i}(R_{-i}) \in M_{-i}$ such that $g(m_i, m_{-i})\ P_i\ g(s_i(R_i), m_{-i})$, which contradicts $s \in DS(\Gamma, \mathcal{R})$.■

The following example states that the converse of Lemma 1 does not hold by showing that there is a game in which a strategy profile is an ex post equilibrium but not a dominant strategy equilibrium.

Example 1: There is a game $(\Gamma, \mathcal{R})$ such that $DS(\Gamma, \mathcal{R}) \subset EP(\Gamma, \mathcal{R})$.

Let $N = \{1, 2\}$, $A = \{a^1, a^2, a^3, a^4\}$, $\mathcal{R}_1 = \{R_1, R_1'\}$, $\mathcal{R}_2 = \{R_2, R_2'\}$, and $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Preferences are as follows:

R_1	R_1'	R_2	R_2'
a^1, a^2	a^2, a^3, a^4	a^1, a^2, a^3	a^2, a^4
a^3, a^4	a^1	a^4	a^1, a^3

Then, let (u_1, u_2) be a pair of representations for each preference profile such that for each agent, the utility of the most preferred outcome is 2 and the utility of the least preferred outcome is 1.

Let f be defined as follows[9]:

[9] In this example, the rule seems artificial. However, in a specific model, we can find an interesting rule f such that $DS(\Gamma^f, \mathcal{R}) \subset EP(\Gamma^f, \mathcal{R})$. See Example 5 in Section 6.

f	R_2	R_2'
R_1	a^1	a^2
R_1'	a^3	a^4

The game induced by Γ^f and $\mathcal{R}$ has the following utilities:

	true preference	R_2		R_2'	
true preference	message	R_2	R_2'	R_2	R_2'
R_1	R_1	$\mathbf{\underline{2},\underline{2}}$	$\mathbf{\underline{2},\underline{2}}$	$\underline{2},1$	$\mathbf{\underline{2},\underline{2}}$
	R_1'	$1,\underline{2}$	$1,1$	$1,1$	$1,\underline{2}$
R_1'	R_1	$1,\underline{2}$	$\mathbf{\underline{2},\underline{2}}$	$1,1$	$\mathbf{\underline{2},\underline{2}}$
	R_1'	$\mathbf{\underline{2},\underline{2}}$	$\underline{2},1$	$\underline{2},1$	$\mathbf{\underline{2},\underline{2}}$

Let $(s_1, s_2) \equiv ((s_1(R_1), s_1(R_1')), (s_2(R_2), s_2(R_2'))$.[10] Then, $DS(\Gamma^f, \mathcal{R}) = \{((R_1, R_1'), (R_2, R_2'))\}$, and $EP(\Gamma^f, \mathcal{R}) = \{((R_1, R_1'), (R_2, R_2'))\ ,\ ((R_1, R_1), (R_2', R_2'))\}$. Hence, the strategy profile $((R_1, R_1), (R_2', R_2'))$ is an ex post equilibrium in $(\Gamma^f, \mathcal{R})$, but not a dominant strategy equilibrium in $(\Gamma^f, \mathcal{R})$. Then, $DS(\Gamma^f, \mathcal{R}) \subset EP(\Gamma^f, \mathcal{R})$.$\diamondsuit$

3 Implementation notions

For a rule f, let us define the two implementability notions which are central to our study.

Dominant strategy implementability: There is a game form $\Gamma = (M, g)$ such that

[10] Formally, let s_1 be the mapping such that for $R_1 \in \mathcal{R}_1$, agent 1 selects $s_1(R_1)$ and for $R_1' \in \mathcal{R}_1$, agent 1 selects $s_1(R_1')$, and s_2 be the mapping such that for $R_2 \in \mathcal{R}_2$, agent 2 selects $s_2(R_2)$ and for $R_2' \in \mathcal{R}_2$, agent 2 selects $s_2(R_2')$.

for each $s \in DS(\Gamma, \mathcal{R}) \neq \emptyset$,

$$g \circ s = f.^{11}$$

Ex post implementability: There is a game form $\Gamma = (M, g)$ such that for each $s \in EP(\Gamma, \mathcal{R}) \neq \emptyset$,

$$g \circ s = f.$$

For the direct game form associated with a rule, *dominant strategy implementability* is weaker than *ex post implementability* (Lemma 2, Example 2).

Lemma 2. *If a rule f is implemented by Γ^f in ex post equilibria, then it is implemented by Γ^f in dominant strategy equilibria.*

The proof of Lemma 2 is in Appendix.

The next example states that the converse of Lemma 2 does not hold by showing that the rule in Example 1 is not implementable in ex post equilibria. To prove this, we show that it does not satisfy the following property of a rule f, *ex post invariance.*[12]

First, we define notation. For each $i \in N$, a **deception for agent** i is a mapping $d_i : \mathcal{R}_i \to \mathcal{R}_i$ which, for each preference of him $R_i \in \mathcal{R}_i$, specifies a preference of him $R_i \in \mathcal{R}_i$. We can interpret it as a strategy for agent i in the game induced by a game form in which for each agent $i \in N$, $M_i = \mathcal{R}_i$ and the set of preference profiles. Let $d = (d_i)_{i \in N}$ be a **deception profile** and $\mathcal{D}$ be the **set of deception profiles**.

Ex post invariance: For each $d \in \mathcal{D}$ with $f \circ d \neq f$, there are $R \in \mathcal{R}$, $i \in N$, and $a \in A$ such that $a \; P_i \; f(d(R))$, and for each $R'_i \in \mathcal{R}_i$, $f(R'_i, d_{-i}(R_{-i})) \; R'_i \; a$.

The following result is used in the next example.

[11] $g \circ s = f$ means that for each $R \in \mathcal{R}$, $g(s(R)) = f(R)$.

[12] *Ex post invariance* is also called *ex post monotonicity.*

Proposition 1. (Bergemann and Morris [4]). *If a rule is not* ex post invariant, *then it is not* ex post implementable.

In the following example, we consider the same situation as in Example 1.

Example 2: The rule in Example 1 is not implementable in ex post equilibria.

Let $d \in \mathcal{D}$ be such that for each $\tilde{R}_1 \in \mathcal{R}_1$, $d(\tilde{R}_1) = R_1$ and for each $\tilde{R}_2 \in \mathcal{R}_2$, $d(\tilde{R}_2) = R_2'$. Then, $f \circ d \neq f$:

$f \circ d$	R_2	R_2'
R_1	a^2	a^2
R_1'	a^2	a^2

Then, for each $R \in \mathcal{R}$, $f(d(R)) = a^2$ and for each $i \in N$, each $R_i \in \mathcal{R}_i$, and each $a \in A$, $a^2 \; R_i \; a$. That is, for each $R \in \mathcal{R}$, each $i \in N$, and each $a \in A$, $f(d(R)) \; R_i \; a$. Therefore, f is not *ex post invariant*. By Proposition 1, f is not *ex post implementable*, while f is implemented by Γ^f in dominant strategy equilibria by the logic of Example 1.$\diamondsuit$

Figure 2 illustrates the relationship between *dominant strategy implementability of a rule f by* Γ^f and *ex post implementability of f by* Γ^f.

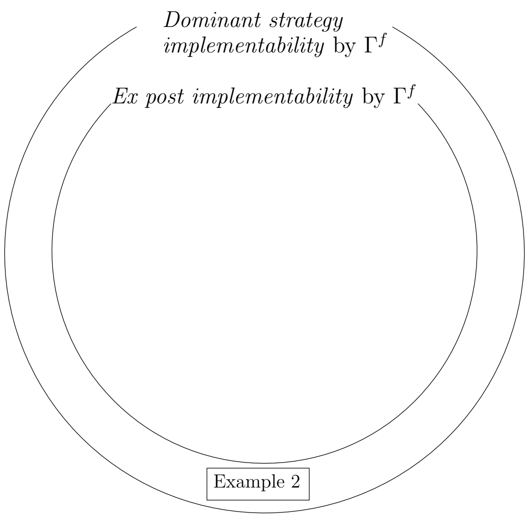

Figure 2

If we consider general game forms, not only direct game forms, then a similar result to Lemma 2 does not hold.

Example 3: A rule that is implemented by a game form in ex post equilibria, but not by this game form in dominant strategy equilibria.

Let $N = \{1, 2\}$, $A = \{a^1, a^2, a^3, a^4, a^5, a^6\}$, $\mathcal{R}_1 = \{R_1, R_1'\}$, $\mathcal{R}_2 = \{R_2, R_2'\}$, and $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Preferences are as follows:

R_1	R_1'
a^1, a^2	a^3, a^4
a^3, a^4	a^5, a^6
a^5, a^6	a^1, a^2

R_2	R_2'
a^1, a^3, a^6	a^2, a^4, a^5
a^2, a^4, a^5	a^1, a^3, a^6

Then, let (u_1, u_2) be a pair of representations for each preference profile such that for each agent, the utility of the most preferred outcome is 3, the utility of the second preferred outcome is 2, and the utility of the third preferred outcome is 1.

Let $\Gamma = (M, g)$ be the game form such that $M_1 = \{m_1, m_1', m_1''\}$, $M_2 = \{m_2, m_2'\}$, and g is as follows:

g	m_2	m_2'
m_1	a^1	a^2
m_1'	a^3	a^4
m_1''	a^5	a^6

Let f be defined as follows:

f	R_2	R_2'
R_1	a^1	a^2
R_1'	a^3	a^4

The game induced by Γ and $\mathcal{R}$ has the following utilities:

	true preference	R_2		R_2'	
true preference	message	m_2	m_2'	m_2	m_2'
R_1	m_1	$\mathbf{\underline{3}, \underline{3}}$	$\underline{3}, 2$	$\underline{3}, 2$	$\mathbf{\underline{3}, \underline{3}}$
	m_1'	$2, \underline{3}$	$2, 2$	$2, 2$	$2, \underline{3}$
	m_1''	$1, 2$	$1, \underline{3}$	$1, \underline{3}$	$1, 2$
R_1'	m_1	$1, \underline{3}$	$1, 2$	$1, 2$	$1, \underline{3}$
	m_1'	$\mathbf{\underline{3}, \underline{3}}$	$2, 2$	$\underline{3}, \underline{2}$	$\mathbf{\underline{3}, \underline{3}}$
	m_1''	$2, 2$	$2, \underline{3}$	$2, \underline{3}$	$2, 2$

Since agent 2 does not have any dominant strategy, $DS(\Gamma, \mathcal{R}) = \emptyset$. On the other hand, $EP(\Gamma, \mathcal{R}) = \{((m_1, m_1'), (m_2, m_2'))\}$. Hence, f is implemented by Γ in ex post equilibria, but not by Γ in dominant strategy equilibria.$\diamondsuit$

4 Strategy-proofness and related properties

The following axiom requires that in the direct game form associated with the rule, for each agent, truth-telling be a dominant strategy.

Strategy-proofness: For each $R \in \mathcal{R}$, each $i \in N$, and each $R_i' \in \mathcal{R}_i$,

$$f(R_i, R_{-i}) \; R_i \; f(R_i', R_{-i}).$$

The following results are the revelation principle for *dominant strategy implementability* and *ex post implementability.*

Proposition 2. (1) (e.g., Gibbard [8]) *If a rule is* dominant strategy implementable, *it is* strategy-proof.

(2) (Bergemann and Morris [4]) *If a rule is* ex post implementable, *it is* strategy-proof.

Figure 3 illustrates the relationships shown by Proposition 2.

Dominant strategy implementability $\Rightarrow$ *Dominant strategy implementability by* Γ^f $\Rightarrow$

$\Uparrow$ *Strategy-proofness*

Ex post implementability $\Rightarrow$ *Ex post implementability by* Γ^f $\Rightarrow$

Figure 3

The following property for a rule f is a sufficient and necessary condition for *secure implementation*, namely double implementation in dominant strategy equilibria and Nash equilibria (Saijo et al. [13]).

Secure strategy-proofness: (1) f is *strategy-proof*, and (2) for each $R, \tilde{R} \in \mathcal{R}$, if for each $i \in N$ and each $R'_i \in \mathcal{R}_i$, $f(\tilde{R}_i, \tilde{R}_{-i})\ R_i\ f(R'_i, \tilde{R}_{-i})$, then $f(\tilde{R}) = f(R)$.

To interpret this axiom, let us define the following notions. For each $R \in \mathcal{R}$, let (Γ, R) be the **complete information game** induced by Γ and R. An message profile $m \in M$ is a **dominant strategy equilibrium in** (Γ, R) if for each $i \in N$, each $m'_i \in M_i$, and each $m'_{-i} \in M_{-i}$, $g(m_i, m'_{-i})\ R_i\ g(m'_i, m'_{-i})$. Let $DS(\Gamma, R)$ be the **set of dominant strategy equilibria in** (Γ, R). An message profile $m \in M$ is a **Nash equilibrium in** (Γ, R) if for each $i \in N$ and $m'_i \in M_i$, $g(m_i, m_{-i})\ R_i\ g(m'_i, m_{-i})$. Let $NE(\Gamma, R)$ be the **set of Nash equilibria in** (Γ, R).

A rule f is **securely implementable** if there is a game form $\Gamma = (M, g)$ such that for each $R \in \mathcal{R}$, $\{f(R)\} = g(DS(\Gamma, R)) = g(NE(\Gamma, R))$.

Secure strategy-proofness requires that the rule f be *strategy-proof*, and for each $R \in \mathcal{R}$ and each Nash equilibrium in (Γ^f, R), the outcome at the Nash equilibrium be equal to the outcome chosen by the rule for R. By this axiom, *secure implementability* is characterized.

Proposition 3. (Saijo et al. [13]) *A rule is* securely implementable *iff it is* securely strategy-proof.

The following axiom is weaker than *secure strategy-proofness* (Saijo et al. [13]).

Non-bossiness (in welfare\outcome): For each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, if $f(R_i, R_{-i})\ I_i\ f(R'_i, R_{-i})$, then $f(R_i, R_{-i}) = f(R'_i, R_{-i})$.

The following property for a rule f requires that f be *strategy-proof*, and if a strategy profile is an ex post equilibrium in $(\Gamma^f, \mathcal{R})$, then for each preference profile, the outcome at the ex post equilibrium be equal to the outcome chosen by the rule for the preference profile.

Weak secure-strategy-proofness: (1) f is *strategy-proof*, and (2) for each $d \in \mathcal{D}$, if for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $f(d_i(R_i), d_{-i}(R_{-i}))\ R_i\ f(R'_i, d_{-i}(R_{-i}))$, then $f \circ d = f$.

Weak secure-strategy-proofness is weaker than *secure strategy-proofness* (Proposition 4, Example 4). Note that an ex post equilibrium $s \in S$ is a strategy profile in which for each preference profile $R \in \mathcal{R}$, the message profile $s(R) \in M$ is a Nash equilibrium.

Proposition 4. *If a rule is* securely strategy-proof, *then it is* weakly securely-strategy-proof.

Proof. Let a rule f be *securely strategy-proof*. It suffices to show that it satisfies (2) of *weak secure-strategy-proofness*.

Let $d \in \mathcal{D}$. The proof is by contradiction. For each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, suppose that $f(d_i(R_i), d_{-i}(R_{-i}))\ R_i\ f(R'_i, d_{-i}(R_{-i}))$. Suppose also that there is $R'' \in \mathcal{R}$ such that $f(d(R'')) \neq f(R'')$. Let $\tilde{R} = d(R'')$. We have that for each $i \in N$ and each $R'_i \in \mathcal{R}_i$, $f(\tilde{R}_i, \tilde{R}_{-i})\ R''_i\ f(R'_i, \tilde{R}_{-i})$, but $f(\tilde{R}) \neq f(R'')$, which contradicts (2) of *secure strategy-proofness*.■

The following example shows that the converse of Lemma 4 does not hold.

Example 4: A rule that is *weakly securely-strategy-proof*, but not *securely strategy-proof.*

Let $N = \{1, 2\}$, $A = \{a^1, a^2, a^3, a^4\}$, $\mathcal{R}_1 = \{R_1, R'_1\}$, $\mathcal{R}_2 = \{R_2, R'_2\}$, and $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Preferences are as follows:

R_1	R'_1	R_2	R'_2
a^1, a^2	a^3, a^4	a^1, a^2, a^3	a^2, a^4
a^3, a^4	a^1, a^2	a^4	a^1, a^3

Then, let (u_1, u_2) be a pair of representations for each preference profile such that for each agent, the utility of the most preferred outcome is 2 and the utility of the least preferred outcome is 1.

Let f be defined as follows:

f	R_2	R_2'
R_1	a^1	a^2
R_1'	a^3	a^4

The game induced by Γ^f and $\mathcal{R}$ has the following utilities:

	true preference	R_2		R_2'	
true preference	message	R_2	R_2'	R_2	R_2'
R_1	R_1	$\mathbf{\underline{2},\underline{2}}$	$\mathbf{\underline{2},\underline{2}}$	$\underline{2},1$	$\mathbf{\underline{2},\underline{2}}$
	R_1'	$1,\underline{2}$	$1,1$	$1,1$	$1,\underline{2}$
R_1'	R_1	$1,\underline{2}$	$1,\underline{2}$	$1,1$	$1,\underline{2}$
	R_1'	$\mathbf{\underline{2},\underline{2}}$	$\underline{2},1$	$\underline{2},1$	$\mathbf{\underline{2},\underline{2}}$

Let $(d_1, d_2) \equiv ((d_1(R_1), d_1(R_1')), (d_2(R_2), d_2(R_2'))) = ((R_1, R_1'), (R_2, R_2'))$. Then, $DS(\Gamma^f, \mathcal{R}) = EP(\Gamma^f, \mathcal{R}) = \{(d_1, d_2)\}$ and $NE(\Gamma^f, R) = \{(d_1(R_1), d_2(R_2)), (R_1, R_2')\}$. The rule f is *strategy-proof* and $f \circ d = f$. Therefore, it is *weakly securely-strategy-proof.* On the other hand, for $(R_1, R_2) \in \mathcal{R}$, $(R_1, R_2') \in NE(\Gamma^f, (R_1, R_2))$, but $f(R_1, R_2') \neq f(R_1, R_2)$. Hence, it is not *secure strategy-proof.* The rule does not satisfy *non-bossiness* either. For (R_1, R_2), agent 2, and $R_2' \in \mathcal{R}_2$, $f(R_1, R_2) = a^1 \; I_2 \; a^2 = f(R_1, R_2')$, but $a^1 \neq a^2$.$\diamondsuit$

Figure 4 illustrates the relationship between *secure strategy-proofness* and *weak secure-strategy-proofness.*

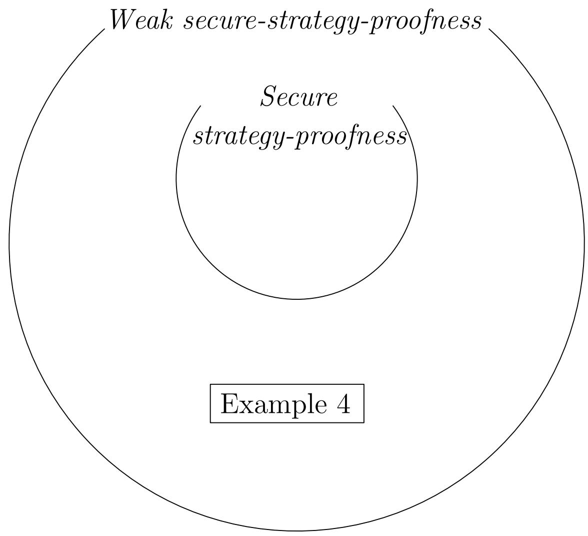

Figure 4

5 Results

As we have discussed in Section 1, we would like to investigate double implementation in dominant strategy equilibria and ex post equilibria.

Double implementability: There is a game form $\Gamma = (M, g)$ such that:

(1) for each $s \in DS(\Gamma, \mathcal{R}) \neq \emptyset$,

$$g \circ s = f,$$

(2) for each $s \in EP(\Gamma, \mathcal{R}) \neq \emptyset$,

$$g \circ s = f.$$

Our main result is as follows:

Theorem 1. *A rule is doubly implementable iff it is* weakly securely-strategy-proof.

The proof is in Appendix. It involves showing that any doubly implementable rule is also doubly implemented by the direct game form associated with it. Hence, for double implementability, it suffices to focus on direct game forms.

Corollary 1. *A rule is doubly implementable iff it is doubly implemented by the direct game form associated with* f.

By Proposition 4 and Theorem 1, *secure strategy-proofness* is sufficient for double implementation.

Corollary 2. *If a rule is* securely strategy-proof, *then it is doubly implementable.*

Figure 5 illustrates the relationship between *secure implementability* and double implementability.

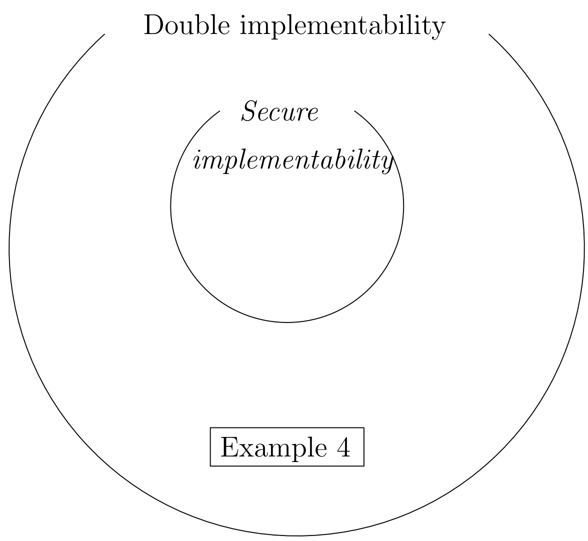

Figure 5

6 Discussion

In restricted public-good-provision problems, the Groves-Clarke rules are *securely implementable* (Saijo et al. [12][13]).[13] Also, in direct game forms, whether the rules work well in laboratory experiments or not has been investigated and one of the rules worked better than a rule that is *dominant strategy implementable*, but not *securely implementable* (Cason et al. [5]). By Corollary 2, the rules are also doubly implementable.

[13] For the definition of the Groves-Clarke rules, see for example Saijo et al. [12].

For *secure implementation*, negative results have been established for a number of interesting rules (e.g., Fujinaka and Wakayama [6]). Even if a rule is not *securely implementable*, it may be doubly implementable (Corollary 2). Are there such rules? We provide one negative answer and one positive answer.

We consider school choice problems (Abdulkadiroğlu and Sönmez[1]) under incomplete information. Let $(N, X, \mathcal{R}, c, \succsim)$ be a school choice problem under incomplete information. Let N be a **set of students**, X be a **set of schools**, and ϕ mean that for each student, he does not have any school and for each school, it gets an empty seat. Let $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$ be a **set of strict preference profiles**. Let $c = (c_x)_{x \in X}$ be a **profile of capacities** such that for each $x \in X$, $c_x \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers.[14] A capacity for a school is the maximum number of students whom the school can accept. Let $\succsim = (\succsim_x)_{x \in X}$ be a **profile of priorities** such that for each $x \in X$, $\succsim_x$ is a strict ordering over $N \cup \{\phi\}$. Let $(a_1, \cdots, a_n) \in A = (X \cup \{\phi\})^N$ be an outcomes such that for each $x \in X$, $|\{i \in N : a_i = x\}| \leq c_x$.

The following example is one negative result on double implementation.

Example 5: The tentative acceptance rule is not doubly implementable.[15]

Let $(N, X, \mathcal{R}, c, \succsim)$ be such that $N = \{1, 2\}$, $X = \{a, b\}$, for each $i \in N$, $\mathcal{R}_i = \{R_i, R'_i\}$, $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$, and preferences and $(c, \succsim)$ are as follows: for each $i \in N$,

[14] A capacity for a school is also called its "quota".

[15] For the definition of the tentative acceptance rule, see for example Abdulkadiroğlu and Sönmez[1].

R_i	R'_i
a	b
b	a
ϕ	ϕ

$c_a = 1$	$c_b = 1$
$\succsim_a$	$\succsim_b$
2	1
1	2
ϕ	ϕ

Let (u_1, u_2) be a pair of representations for each preference profile such that for each agent, the utility of the most preferred school is 2, the utility of the second preferred school is 1, and the utility of the third preferred school is 0.

By computing the tentative acceptance rule f^{TA}, for each preference profile, the outcome is as follows:

f^{TA}	R_2	R'_2
R_1	(b, a)	(a, b)
R'_1	(b, a)	(b, a)

The game induced by $\Gamma^{f^{TA}}$ and $\mathcal{R}$ has the following utilities:

	true preference	R_2		R'_2	
true preference	message	R_2	R'_2	R_2	R'_2
R_1	R_1	$\underline{\mathbf{1}}, \underline{\mathbf{2}}$	$\underline{2}, 1$	$\underline{1}, 1$	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$
	R'_1	$\underline{\mathbf{1}}, \underline{\mathbf{2}}$	$1, \underline{2}$	$\underline{\mathbf{1}}, \underline{\mathbf{1}}$	$1, \underline{1}$
R'_1	R_1	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$	$1, 1$	$\underline{2}, 1$	$1, \underline{2}$
	R'_1	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$	$\underline{\mathbf{2}}, \underline{\mathbf{2}}$	$\underline{\mathbf{2}}, \underline{\mathbf{1}}$	$\underline{\mathbf{2}}, \underline{\mathbf{1}}$

Let $(s_1, s_2) \equiv ((R_1, R'_1), (R_2, R'_2))$. Then, $DS(\Gamma^{f^{TA}}, \mathcal{R}) = \{(s_1, s_2)\}$, and $EP(\Gamma^{f^{TA}}, \mathcal{R}) = \{(s_1, s_2), ((R'_1, R'_1), (R_2, R_2))\}$. Hence, for the preference profile $(R_1, R'_2) \in \mathcal{R}$, the ex post equilibrium $((R'_1, R'_1), (R_2, R_2))$ does not induces the outcome chosen by f^{TA} for (R_1, R'_2), while for each preference profile, the outcome of the dominant strategy equilibrium (s_1, s_2) is equal to the outcome chosen by the rule for the preference profile.[16] $\diamondsuit$

For the other models, some interesting rules are not doubly implementable: (1) For auctions with an indivisible good and quasi-linear preferences (Vickrey [14]), the second-price-auction rule is not doubly implementable. (2) In location problems with single-peaked preferences (Moulin [11]), the median rule is not doubly implementable.

In contrast to Example 5, if the set of preferences for agent 1 includes a preference at which the ordering of school a is first, the ordering of ϕ is second, and the ordering of school b is third, then the tentative acceptance rule is doubly implemented. Therefore, if the set of preference profiles is "large", then the rule may be doubly implemented. Identifying general conditions on the set of preference profiles for double implementability is an open question.

Example 6: The tentative acceptance rule is doubly implemented under some condition on the set of preference profiles.

Let $(N, X, \mathcal{R}, c, \succsim)$ be the same problem as in Example 5 except for that $\mathcal{R}_1 = \{R_1, R'_1, R''_1\}$ and preferences are as follows:

R_1	R'_1	R''_1	R_2	R'_2
a	b	a	a	b
b	a	ϕ	b	a
ϕ	ϕ	b	ϕ	ϕ

[16] In the same example, the top-trade-cycle rule is not doubly implementable. For the definition of the top-trade-cycle rule, see for example Abdulkadiroğlu and Sönmez[1].

Let (u_1, u_2) be a pair of representations for each preference profile such that for each agent, the utility of the most preferred school is 3, the utility of the second preferred school is 2, and the utility of the third preferred school is 1.

By computing the tentative acceptance rule f^{TA}, for each preference profile, the outcome is as follows:

f^{TA}	R_2	R_2'
R_1	(b, a)	(a, b)
R_1'	(b, a)	(b, a)
R_1''	(ϕ, a)	(a, b)

The game induced by $\Gamma^{f^{TA}}$ and $\mathcal{R}$ has the following utilities:

	true preference	R_2		R_2'	
true preference	message	R_2	R_2'	R_2	R_2'
R_1	R_1	$\mathbf{\underline{2}, \underline{3}}$	$\underline{3}, 2$	$\underline{2}, 2$	$\mathbf{\underline{3}, \underline{3}}$
	R_1'	$\mathbf{\underline{2}, \underline{3}}$	$2, \underline{2}$	$\mathbf{\underline{2}, \underline{2}}$	$2, 2$
	R_1''	$1, \underline{3}$	$\underline{3}, 2$	$1, 2$	$\mathbf{\underline{3}, \underline{3}}$
R_1'	R_1	$\mathbf{\underline{3}, \underline{3}}$	$2, 2$	$\underline{3}, 2$	$2, \underline{3}$
	R_1'	$\mathbf{\underline{3}, \underline{3}}$	$\mathbf{\underline{3}, \underline{2}}$	$\mathbf{\underline{3}, \underline{2}}$	$\mathbf{\underline{3}, \underline{2}}$
	R_1''	$1, \underline{3}$	$2, 2$	$1, 2$	$2, \underline{3}$
R_1''	R_1	$1, \underline{3}$	$\underline{3}, 2$	$1, 2$	$\mathbf{\underline{3}, \underline{3}}$
	R_1'	$1, \underline{3}$	$1, \underline{2}$	$1, \underline{2}$	$1, \underline{2}$
	R_1''	$\mathbf{\underline{2}, \underline{3}}$	$\underline{3}, 2$	$\underline{2}, 2$	$\mathbf{\underline{3}, \underline{3}}$

Let $(s_1, s_2) \equiv ((R_1, R_1', R_1''), (R_2, R_2'))$. Then, $DS(\Gamma^{f^{TA}}, \mathcal{R}) = EP(\Gamma^{f^{TA}}, \mathcal{R}) = \{(s_1, s_2)\}$.

Hence, for each preference profile, the outcome of the dominant strategy equilibrium and the ex post equilibrium is equal to the outcome chosen by the rule for the preference profile.$\diamondsuit$

Appendix

Proof of Lemma 2. Let f be a rule that is implemented by Γ^f in ex post equilibria. Then, for each $s \in EP(\Gamma^f, \mathcal{R}) \neq \emptyset$, $f \circ s = f$. Since by Proposition 2 (2), f is *strategy-proof*, $DS(\Gamma^f, \mathcal{R}) \neq \emptyset$. Since f is implemented by Γ^f in ex post equilibria, by Lemma 1, for each $s \in DS(\Gamma^f, \mathcal{R}) \subseteq EP(\Gamma^f, \mathcal{R})$, $f \circ s = f$. Therefore, f is implemented by Γ^f in dominant strategy equilibria.$\blacksquare$

Proof of Theorem 1. First, we consider the if part. Let a rule f be *weakly securely-strategy-proof*. We show that $\Gamma^f = (\mathcal{R}, f)$ doubly implements f. By (1) of *weak secure-strategy-proofness* and Lemma 1, $\emptyset \neq DS(\Gamma^f, \mathcal{R}) \subseteq EP(\Gamma^f, \mathcal{R})$. By Lemma 2, it suffices to show that for each $s \in EP(\Gamma^f, \mathcal{R})$, $f \circ s = f$. Note that in $(\Gamma^f, \mathcal{R})$, for each $i \in N$, $s_i : \mathcal{R}_i \to \mathcal{R}_i$ so that $s \in \mathcal{D}$. By the definition of an ex post equilibrium, for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $f(s_i(R_i), s_{-i}(R_{-i}))\ R_i\ f(R'_i, s_{-i}(R_{-i}))$. By (2) of *weak secure-strategy-proofness*, $f \circ s = f$.

Next, we prove the only if part. Let a rule f be doubly implementable. Then, let $\Gamma = (M, g)$ be a game form which doubly implements f. By Proposition 2 (1), f is *strategy-proof*. Therefore, it suffices to show that f satisfies (2) of *weak secure-strategy-proofness*.

Let $d \in \mathcal{D}$. Let the hypothesis of (2) be satisfied: for each $R \in \mathcal{R}$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $f(d_i(R_i), d_{-i}(R_{-i}))\ R_i\ f(R'_i, d_{-i}(R_{-i}))$. We show that $f \circ d = f$.

Since Γ doubly implements f, $DS(\Gamma, \mathcal{R}) \neq \emptyset$. Let $s \in DS(\Gamma, \mathcal{R})$. Since Γ implements f in dominant strategy equilibria, i.e., $g \circ s = f$, we have $g \circ s \circ d = f \circ d$. That is, for each $R \in \mathcal{R}$, $g(s(d(R))) = f(d(R))$. Similarly, since Γ implements f in dominant strategy equilibria, for each $i \in N$, each $R'_i \in \mathcal{R}_i$, and each $R_{-i} \in \mathcal{R}_{-i}$, $g(s_i(R'_i), s_{-i}(d(R_{-i}))) = f(R'_i, d(R_{-i}))$. Since $f(d_i(R_i), d_{-i}(R_{-i}))\ R_i\ f(R'_i, d_{-i}(R_{-i}))$ by the hypothesis, $g(s(d(R))) = f(d(R))$, and $g(s_i(R'_i), s_{-i}(d_{-i}(R_{-i}))) = f(R'_i, d_{-i}(R_{-i}))$, we have $g(s_i(d_i(R_i)), s_{-i}(d_{-i}(R_{-i})))\ R_i\ g(s_i(R'_i), s_{-i}(d_{-i}(R_{-i})))$. When $R'_i = R_i$, since $s \in DS(\Gamma, \mathcal{R})$, for each $m_i \in M_i$ $g(s_i(R_i), s_{-i}(d_{-i}(R_{-i})))\ R_i\ g(m_i, s_{-i}(d_{-i}(R_{-i})))$. Therefore, for each $R \in \mathcal{R}$, each $i \in N$, and each $m_i \in M_i$, $g(s_i(d_i(R_i)), s_{-i}(d_{-i}(R_{-i})))\ R_i\ g(m_i, s_{-i}(d_{-i}(R_{-i})))$. Thus, $s \circ d$ is an ex post equilibrium. Since Γ implements f in ex post equilibria, $g \circ (s \circ d) = f$. Since $f \circ d = g \circ s \circ d$ and $g \circ s \circ d = f$, we have $f \circ d = f$. Therefore, f is *weakly securely-strategy-proof.*■

References

[1] Abdulkadiroğlu A, Sönmez T (2003), "School Choice: A Mechanism Design Approach", *American Economic Review*, 93, 729-747.

[2] Adachi T (2014), "Robust and Secure Implementation: Equivalence Theorems" *Games and Economic Behavior*, 86: 96-101.

[3] Bergemann D, Morris S (2005), "Robust Mechanism Design" *Econometrica* 73: 1771–1813.

[4] Bergemann D, Morris S (2008), "Ex Post Implementation", *Games and Economic Behavior*, 63: 527–566.

[5] Cason T, Saijo T, Sjöström T, and Yamato T (2006), "Secure Implementation Experiments: Do Strategy-proof Mechanisms Really Work?", *Games and Economic Behavior*, 57: 206-235.

[6] Fujinaka Y, Wakayama T (2011), "Secure Implementation in Shapley-Scarf Housing Markets", *Economic Theory*, 48: 147-169.

[7] Gale D, Shapley L S (1962), "College Admissions and the Stability of Marriage", *American Mathematical Monthly*, 69: 9-15.

[8] Gibbard A (1973), "Manipulation of Voting Schemes: A General Result", *Econometrica*, 41: 587-601.

[9] Kagel J H, Harstad R M, and Levin D (1987), "Information Impact and Allocation Rules in Auctions with Affiliated Private Values: A Laboratory Study", *Econometrica*, 55: 1275-1304.

[10] Li S (2017), "Obviously Strategy-proof Mechanisms", *American Economic Review*, 107: 3257-87.

[11] Moulin H (1980), "On Strategy-Proofness and Single Peakedness", *Public Choice*, 35, 437–455.

[12] Saijo T, Sjöström T, and Yamato T (2003), "Secure Implementation: Strategy-proof Mechanisms Reconsidered", *RIETI Discussion Paper Series 03-E-019.*

[13] Saijo T, Sjöström T, and Yamato T (2007), "Secure Implementation", *Theoretical Economics*, 2: 203-229.

[14] Vickrey W (1961), "Counterspeculation, Auctions, and Competitive Sealed Tenders", *Journal of Finance*, 16: 8-37.