Information Hierarchies

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  - The distribution of Anne's posteriors $\beta(A)$ is a mean-preserving spread of Bob's distribution of posteriors $\beta(B)$
  - Equivalent: Anne's info is more valuable for any decision problem
- **Refinement** order $\trianglerighteq$: Anne sees what Bob sees, plus something else
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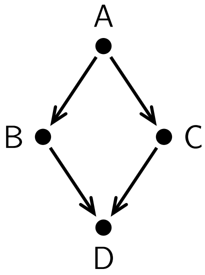
- Refinement ordering implies Blackwell ordering
- Partial converse:

  For any Blackwell ordered posterior distributions $\beta(A) \succsim \beta(B)$, there exist refinement ordered signals $\sigma(A) \trianglerighteq \sigma(B)$ inducing those posterior distributions.

Now let's add more agents...

- Fix Anne, Bob, Charlie, and David's belief dists:
  $\beta(A), \beta(B), \beta(C), \beta(D)$
- Suppose Anne has more accurate information than Bob and Charlie (Blackwell), who each have more accurate information than David
  $\beta(A) \succsim \beta(B), \beta(C)$; and $\beta(B), \beta(C) \succsim \beta(D)$



- Is it possible that Anne sees what Bob and Charlie see (refinement), who see what David sees? $\sigma(A) \trianglerighteq \sigma(B), \sigma(C)$; and $\sigma(B), \sigma(C) \trianglerighteq \sigma(D)$

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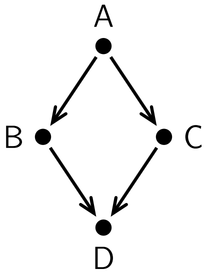

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No, not necessarily.

With ≥ 4 agents, answer depends on "network structure"

Application 1: Rationalizing reactions to unknown info

Anne obtains information from some set of sources/experiments: X, Y, Z

- We observe her reaction to each subset: X, Y, Z, XY, XZ, YZ, XYZ
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- Q: Necessary and sufficient?
- A: **No** (with ≥ 3 experiments)

Application 2: Info design with monotonicity constraints

- Designer provides private signals to multiple receivers
- Monotonicity constraints: some receivers have access to others' info
  - organizational hierarchy
  - coalition formation
  - dynamic revelation

- All information structures consistent with Blackwell order might – or might not – be possible.

  Depends on *network structure* of constraints.

The Model

States, signals, and beliefs

- Finite state space Ω with typical state ω
- Signals (experiments) – general framework
  Green and Stokey (1978), Gentzkow and Kamenica (2017)
  - A signal $\pi \in \Pi$ is a partition of $\Omega \times [0, 1]$
  - A signal realization s is a subset of $\Omega \times [0, 1]$
    (a signal realization is an element of a signal)

Think of drawing $\omega \in \Omega$ and a uniform auxiliary variable $x \in [0, 1]$
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- Refinement partial order:
  $\pi \trianglerighteq \pi'$ if every element of π is a subset of some element of π'
  (π is a finer partition than π')
- Observing both π and π' induces signal $\pi \vee \pi'$ –
  coarsest partition finer than both

States, signals, and beliefs

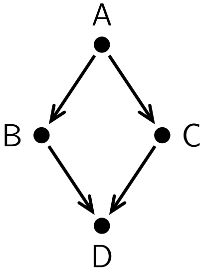
Take some state space Ω and prior belief $\mu_0 \in \Delta\Omega$

- Denote a typical distribution of beliefs by $\tau \in \Delta\Delta\Omega$
  - Any distribution of beliefs τ satisfies $\mathbb{E}_\tau[\mu] = \mu_0$
- Blackwell partial order:
  $\tau \succsim \tau'$ if τ is a mean-preserving spread of τ'
- A signal π induces a distribution of beliefs $\langle \pi \rangle$
- Refinement ordered signals imply Blackwell ordered belief dists:
  $\pi \trianglerighteq \pi' \;\Rightarrow\; \langle \pi \rangle \succsim \langle \pi' \rangle$

Information hierarchies and their graphs

An information hierarchy H is a finite partially ordered set $(N, \geq)$

- Directed graph of H:
  edge from n to n' if n covers n', i.e,
  $n > n'$ and nothing in between
- Undirected graph of H:
  erase arrows from directed graph



Beliefs and signals on a hierarchy

We have a hierarchy $H = (N, \geq)$, state space Ω, prior μ_0

- A **belief allocation** $\boldsymbol{\beta} : N \to \Delta\Delta\Omega$ maps nodes to belief distributions
- A **signal allocation** $\boldsymbol{\sigma} : N \to \Pi$ maps nodes to signals
- Signal allocation σ induces belief allocation β with $\beta(n) = \langle \sigma(n) \rangle$

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- Signal allocation σ induces belief allocation β with $\beta(n) = \langle \sigma(n) \rangle$

- Any monotone signal allocation induces a monotone belief allocation
- Given some monotone belief allocation, does there exist a monotone signal allocation that induces it?

Beliefs and signals on a hierarchy

- Fix H, Ω, μ_0. Monotone belief allocation β is constructible if there exists a monotone signal allocation σ inducing it
- H is Ω-universally constructible if for all μ_0, every monotone belief allocation is constructible
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Which hierachies are universally constructible?

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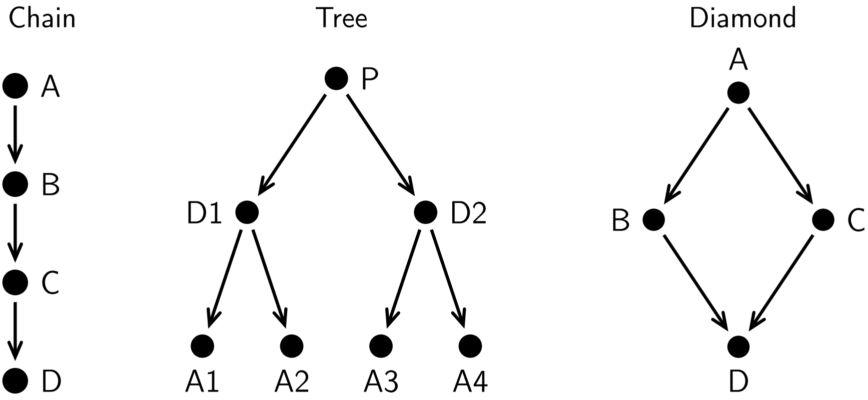
Which hierachies are universally constructible?

Note:

- Our results about universal constructibility do not depend on priors μ_0
- Universal constructibility may depend on the cardinality of Ω: may be universally constructible for small but not large state spaces

Examples

Examples of information hierarchies



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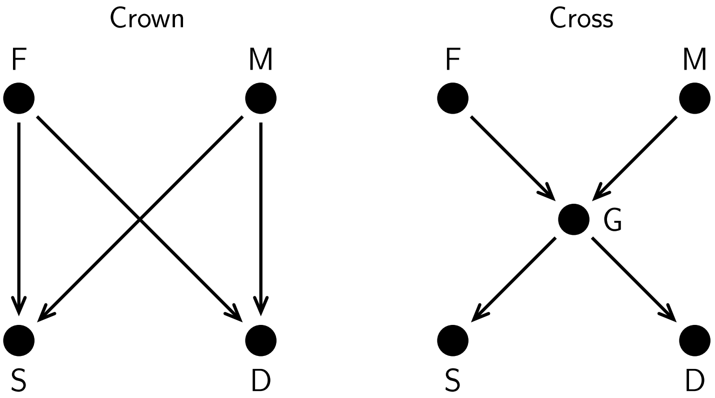
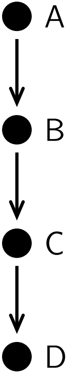


Note that adding a node can remove edges

Universal constructibility of example hierarchies

A chain is universally constructible

Consider the chain hierarchy H.



Take any monotone belief allocation β on H.

We wish to find a monotone signal allocation σ inducing β.

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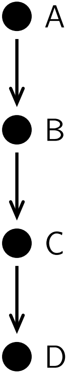


Take any monotone belief allocation β on H.
We wish to find a monotone signal allocation σ inducing β.

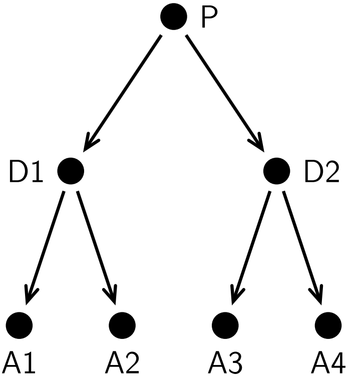
Start at the bottom and go up:

- Take some signal $\sigma(D)$ inducing $\beta(D)$
- Standard result: for any $\beta(C) \succsim \langle\sigma(D)\rangle$, there exists a signal $\sigma(C)$ satisfying $\sigma(C) \trianglerighteq \sigma(D)$ with $\langle\sigma(C)\rangle = \beta(C)$
- Repeat to refine $\sigma(C)$ into $\sigma(B)$, refine $\sigma(B)$ into $\sigma(A)$

A tree is universally constructible

In the tree, we can't start at some node and only "go up."

We also need to go down.



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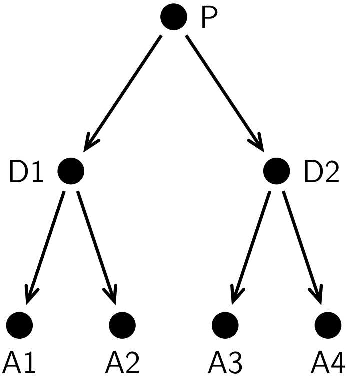

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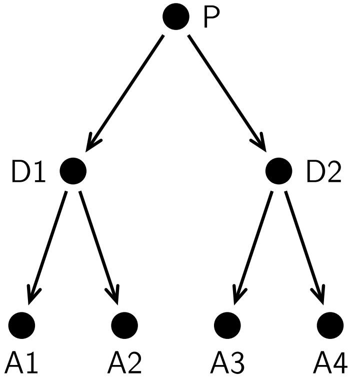


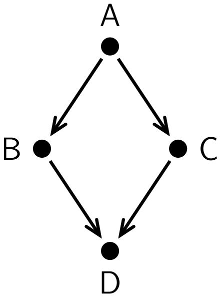
Problem: If we have some $\sigma(D1)$ inducing $\beta(D1)$, it may not be possible to coarsen $\sigma(D1)$ into any signal inducing $\beta(A2)$.

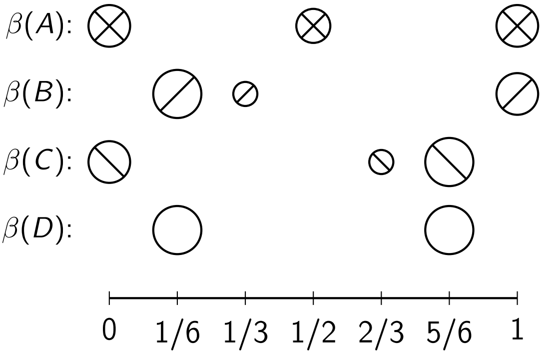
Solution: We show how to go down by reassigning new signals to all previously assigned higher nodes, then coarsening

So – we can start at arbitrary node and refine to go up, reassign and coarsen to go down, and fill out tree (or cross)

The diamond is not universally constructible

Take $\Omega = \{0, 1\}$, with beliefs in $[0, 1]$ interval





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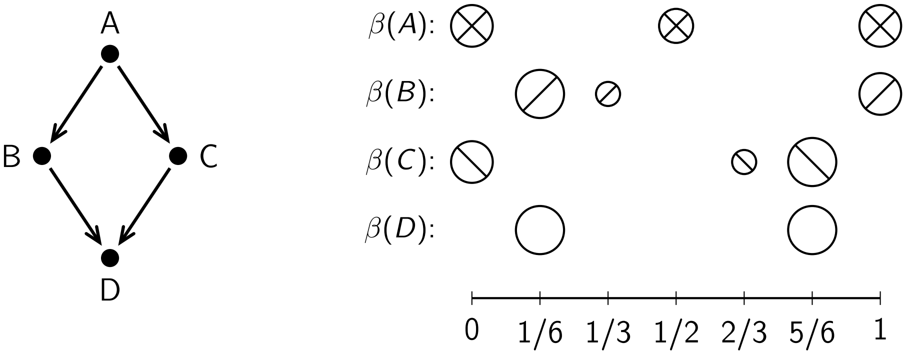


Suppose beliefs β are induced by some monotone σ

Refinement implies beliefs are a martingale (preserve mean) as we go up:

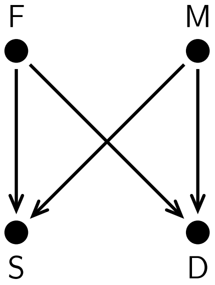
- $D \to B \to A$ path: $\frac{1}{6} \to \frac{1}{6} \to \{0, \frac{1}{2}\}$
- $D \to C \to A$ path: positive prob on $\frac{1}{6} \to \frac{2}{3} \to 1$

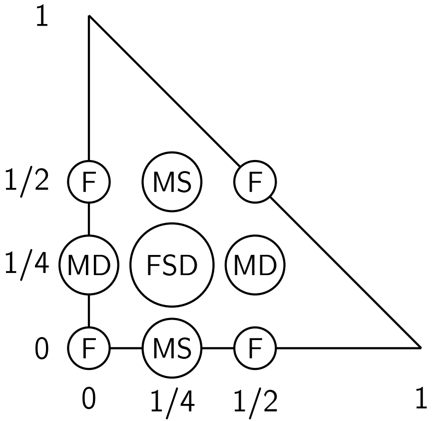
Contradiction: Given belief $\frac{1}{6}$ at D, both 0 and > 0 prob of belief 1 at A

The crown is not universally constructible

Take $\Omega = \{0, 1, 2\}$, with beliefs in triangle

There does not exist a counterexample with binary states





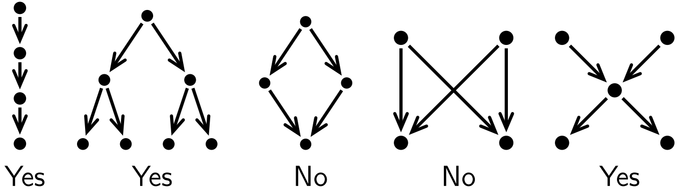
- Given belief at S, conditional beliefs at D inconsistent across $S \to F \to D$ and $S \to M \to D$

Main theorem

Universal Constructibility

Theorem

An information hierarchy is universally constructible if and only if its undirected graph is a forest (no cycles).



Universal Constructibility

Theorem

An information hierarchy is universally constructible under refinement if and only if its undirected graph is a forest.

If direction:

Proposition

An information hierarchy is universally constructible if its undirected graph is a forest.

Already shown for trees; forest is just a disjoint union of trees

Universal Constructibility

Theorem

An information hierarchy is universally constructible if and only if its undirected graph is a forest.

Only if direction:

Proposition

Suppose $|\Omega| \geq 3$*. An information hierarchy is* Ω*-universally constructible only if its undirected graph is a forest.*

We show that in any hierarchy where the undirected graph has a cycle, we can "embed" a version of either the diamond or crown counterexample

Extension: other notions of "knowing more than"

In paper, we extend results from refinement to other, weaker notions of "Anne knows more than Bob" that also imply Blackwell

Examples:

- Bob's signal is statistically redundant with Anne's:
  Anne's belief doesn't change if she observes Bob's signal
- Anne knows Bob's (first-order) belief about the state

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- Any notion of "knowing more than" that implies martingale property:
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Say a relation on signals is *proper* if it is implied by refinement order, and it implies martingale property (which in turn implies Blackwell)

Theorem

Fix any proper relation $\mathcal{R}$. An information hierarchy is universally constructible under $\mathcal{R}$ if and only if its undirected graph is a forest.

Discussion

Rationalizing reactions to unknown info sources

- Anne has access to a set of experiments
- Dataset: Anne's belief distribution at
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- When is a data set rationalizable?

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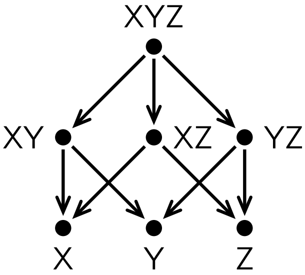
Key observation: Anne's signal allocation is (refinement-)monotone



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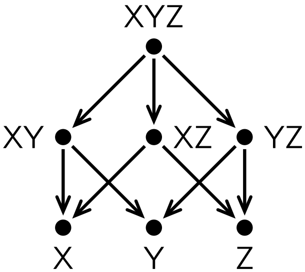

Necessary conditions for rationalizability:

1. Bayes plausibility: same mean belief
2. Blackwell monotonicity: more experiments $\Rightarrow$ mean-preserving spread

These conditions are **not** sufficient. (If ≥ 3 experiments.)

Open Question: full characterization of rationalizable datasets

Open questions

- Big question
  Fix a non-forest H. There exist non-constructible belief allocations.
  Is a given belief allocation constructible or not?

- Small question
  Universal constructibility with binary state space:
    Diamond, no. Crown, yes.
  Full characterization of hierarchies?

Thank you