EXTENDED ABSTRACT: STATIONARY DISTRIBUTIONS OF EVOLUTIONARY DYNAMICS - A SPECTRAL APPROACH

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1. Introduction

Equilibrium selection in binary choice games was studied by Kandori et al. (1), Blume (2), and Binmore and Samuelson (3), among many others. The analysis usually relied on a characterization of the stationary distribution of the emerging Markov process. In many cases, an explicit formula for the stationary distribution and its asymptotic have been devised, relying on the simple observation that the process is a simple birth-death process. Recently, Sandholm and Staudigl (4) have offered an analysis in 3 strategy potential game, relying on an indirect large deviation approach. Our approach here is to offer a general framework for explicitly calculating the stationary distribution and its asymptotics. The method relies on an analysis of the process' Master equation together with technics from spectral analysis.

2. Model

N agents choose strategies from $S = 0, 1, ..., d-1$, $d \geq 2$. The population state $x \in \mathcal{X}^N = \{x \in \frac{1}{N}{}_+^{d-1} : \sum_{j=1}^{d-1} x_j \leq 1\}$ describes the fraction of agents currently choosing strategies $1, ..., d-1$. The payoff function is $F^N : \mathcal{X}^N \to R^d$, where $F_j^N(x)$ is the payoff to strategy $j \in S$ at population state $x \in \mathcal{X}^N$.

Following Blume (2), agents receive revision opportunities via independent Poisson process, switching from strategy k to $j \neq k$ with probability $\rho^\eta(a) \in (01)$, where $a \in R$ is the current payoff advantage of strategy k over strategy j.

The emerging stochastic evolutionary process is $\{X_t^{N,\eta}\}_{t\geq 0}$ is assumed to have transition density $m_x(v)$ for a transit from state x to $x + v$.

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3. COMPUTING THE STATIONARY DENSITY

3.1. **The Stationary Distribution.** The *Master equation* is given (in the interior of $\mathcal{X}^N$) by

$$\frac{\partial P}{\partial t}(t|x) = \sum_v P\left(t\middle|x + \frac{v}{N}\right) \mu_{x+v/N}(-v/N). \tag{1}$$

If $F(s|x)$ is the *Laplace transform* of $P(t|x)$ w.r.t. t, then the Master equation transforms into

$$sF(s|x) - P(0|x) = N \sum_v F\left(s\middle|x + \frac{v}{N}\right) \mu_{x+v/N}(-v). \tag{2}$$

Key ingredient in our analysis will be the following Lemma:

Lemma 1. *Let P^N be the stationary distribution of the process. Then*

$$P^N(x) = \lim_{s \to 0^+} sF(s|x). \tag{3}$$

Equation (2) together with Lemma 1 allows us to represent the ratio $\frac{P^N(x)}{P^N(y)}$ in terms of minors of the $M = \left(\mu_x(x' - x)\right)_{x,x'}$, whose rows and columns are ordered w.r.t. a linear order $\succeq$ on $\mathcal{X}^N$ s.t. $\sum_{i=1}^{d-1} x_j > \sum_{i=1}^{d-1} x'_j$ implies $x \succ x$: If M_x is the x principle minor of M and by M^x the minor obtained from M by deleting the first rows and columns $\preceq x$ then

$$\frac{P^N(x)}{P^N(y)} = \left| \frac{\det M_x \det M^x}{\det M_y \det M^y} \right|. \tag{4}$$

4. THE LIMITS

We employ methods from spectral theory to calculate the asymptotic of the minors M_x and M^x as $N \to \infty$. We essentially show that there is a $r = r(d) \in \mathbb{N}$ s.t. if $Q(z^1, z^2|a) = \sum_{i,j=1}^{r} a_{ij} z_i^1 z_j^2$ is a symmetric quadratic form in $2r$ variables and $\lambda_0(a)$ denotes the largest eigenvalue (in absolute value) of the integral transform $(Tf)(x) = \int e^{-Q(x,y|a)} f(y) dy$, then M^x and M_x are asymptotically given by an integral formula involving $\log |\lambda_0(\cdot)|$. We present methods for the calculation of λ_0.

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