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**Dipti Dubey, S. K. Neogy & Debasish Ghorui**

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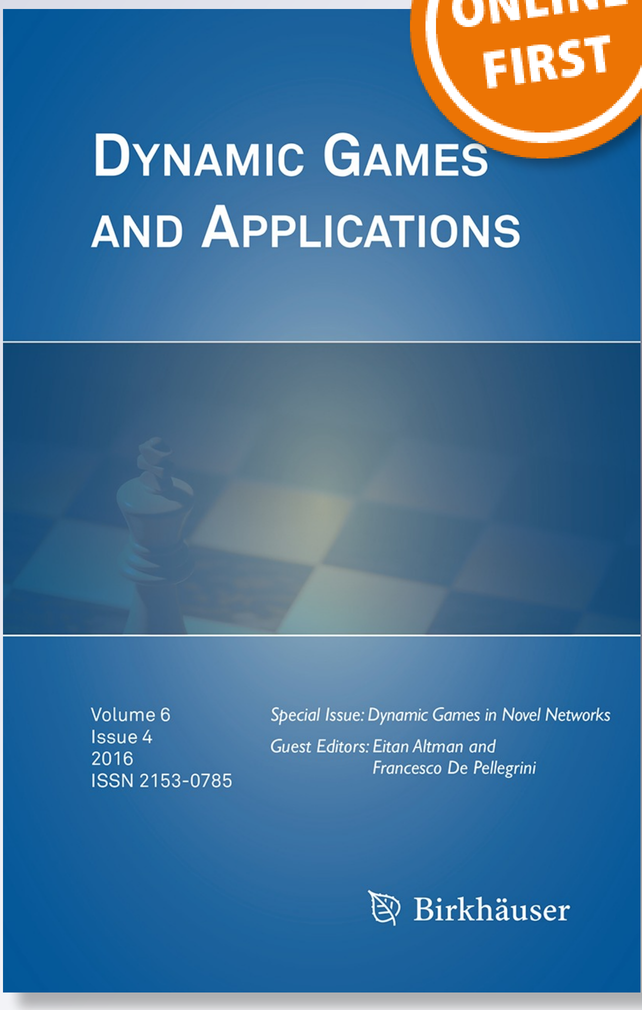


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# Completely Mixed Strategies for Generalized Bimatrix and Switching Controller Stochastic Game

**Dipti Dubey**[1] · **S. K. Neogy**[1] · **Debasish Ghorui**[2]



**Abstract** In this paper, we revisit a result by Jurg et al. (Linear Algebra Appl 141:61–74, 1990) where the necessary and sufficient condition for a bimatrix game to be weakly completely mixed is given. We present an alternate proof of this result using linear complementarity approach. We extend this result to a generalization of bimatrix game introduced by Gowda and Sznajder (Int J Game Theory 25:1–12, 1996) via a generalization of linear complementarity problem introduced by Cottle and Dantzig (J Comb Theory 8:79–90, 1970). We further study completely mixed switching controller stochastic game (in which transition structure is a natural generalization of the single controller games) and extend the results obtained by Filar (Proc Am Math Soc 95:585–594, 1985) for completely mixed single controller stochastic game to completely mixed switching controller stochastic game. A numerical method is proposed to compute a completely mixed strategy for a switching controller stochastic game.

**Keywords** Generalized bimatrix game · Completely mixed Strategies · Switching controller stochastic game · Vertical linear complementarity problem

## 1 Introduction

Kaplansky [10] considered the class of completely mixed matrix games (games in which all optimal strategies are strictly positive in every component) and showed that in games where

✉ S. K. Neogy
skn@isid.ac.in

Dipti Dubey
diptidubey@isid.ac.in

Debasish Ghorui
deba.jumath@gmail.com

1 Indian Statistical Institute, 7, S.J.S. Sansanwal Marg, New Delhi 110016, India

2 Jadavpur University, Kolkata 700032, India

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both players have only completely mixed optimal strategies, the payoff matrix is square and each player has a unique optimal strategy. Kaplansky [10] also gave a necessary and sufficient condition on the payoff matrix for a game of value zero to be completely mixed. Raghavan [21] extended this result for bimatrix games. A bimatrix game is a noncooperative nonzero-sum two person game with a finite number of pure strategies. Let player I has $m$ pure strategies and player II, $n$ pure strategies. In a game if player I chooses strategy $i$ and player II chooses strategy $j$ they incur the costs $a_{ij}$ and $b_{ij}$, respectively, where $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B = [b_{ij}] \in \mathbb{R}^{m\times n}$ are given cost matrices. We denote a bimatrix game with payoff matrices $A$ and $B$ by $(A, B)$.

A mixed strategy for player I is a probability vector $x \in \mathbb{R}^m$ whose $i$th component $x_i$ represents the probability of choosing pure strategy $i$ where $x_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^{m} x_i = 1$. Similarly, a mixed strategy for player II is a probability vector $y \in \mathbb{R}^n$. If player I adopts a mixed strategy $x$ and player II adopts a mixed strategy $y$ then their *expected costs* are given by $x^{\mathrm{T}}Ay$ and $x^{\mathrm{T}}By$, respectively. We say that the game is completely mixed if every optimal strategy of either player is completely mixed. In this paper, we visualize all game problems considered as a linear complementarity problem (LCP) and look at the completely mixed strategies as a solution of LCP.

The notion of weakly completely mixed bimatrix game was introduced and studied by Jurg et al. [9]. They gave the necessary and sufficient condition for a bimatrix game to be weakly completely mixed. We give an alternate proof of this result using LCP approach and extend this result to a generalization of bimatrix game (called as generalized bimatrix game) introduced by Gowda and Sznajder [8] using vertical linear complementarity approach. We then extend the results by Filar [5] for completely mixed single controller stochastic game to completely mixed switching controller stochastic game.

Paper is organized as follows. In Sect. 2, we present the preliminaries and an alternative proof of necessary and sufficient condition for a bimatrix game to be weakly completely mixed observed by Jurg et al. [9]. We extend this result to a generalized bimatrix game in Sect. 3. In Sect. 4, we obtain sufficient conditions for a switching controller stochastic game to be completely mixed. Finally in Sect. 6, we present a numerical method for computation of completely mixed strategies for switching controller stochastic game.

## 2 Preliminaries

We consider matrices and vectors with real entries. Any vector $x \in \mathbb{R}^n$ is a column vector unless otherwise specified, and $x^{\mathrm{T}}$ denotes the transpose of $x$. For any matrix $A \in \mathbb{R}^{m\times n}$, $A_{i\cdot}$ denotes the $i$th row of $A$ and $A_{\cdot j}$ denotes the $j$th column of $A$.

The linear complementarity problem is defined as follows. Given a real square matrix $A$ of order $n$ and a vector $q \in \mathbb{R}^n$, the linear complementarity problem is to find $w \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$ such that

$$\begin{aligned} w - Az = q, \quad w \geq 0, \quad z \geq 0, \\ w^{\mathrm{T}} z = 0. \end{aligned}$$

This problem is denoted as LCP $(q, A)$. For further details and applications of this problem, see Cottle et al. [2] and Murty [19].

Let $(A, B)$ be a bimatrix game. A pair of mixed strategies $(x^*, y^*)$ with $x^* \in \mathbb{R}^m$ and $y^* \in \mathbb{R}^n$ is said to be a *Nash equilibrium pair* if

$$(x^*)^{\mathrm{T}} A y^* \le x^{\mathrm{T}} A y^* \quad \text{for all mixed strategies } x \in \mathbb{R}^m \quad \text{and}$$
$$(x^*)^{\mathrm{T}} B y^* \le (x^*)^{\mathrm{T}} B y \quad \text{for all mixed strategies } y \in \mathbb{R}^n.$$

Let $\mathscr{E}(A, B)$ be the set of all pairs of equilibrium strategies. Let $v_1 = (x^*)^{\mathrm{T}} A y^*$ and $v_2 = (x^*)^{\mathrm{T}} B y^*$. We say that $\mathscr{E}(A, B)$ is completely mixed if the elements of $\mathscr{E}(A, B)$ are completely mixed pairs.

Note that the addition of a constant to all entries of $A$ or $B$ leaves the set of equilibrium points of the game $(A, B)$ invariant. Without loss of generality we assume that all entries of the matrices $A$ and $B$ are positive.

Consider the following LCP:

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -e_m \\ -e_n \end{bmatrix} + \begin{bmatrix} 0 & A \\ B^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \; \begin{bmatrix} u \\ v \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x \\ y \end{bmatrix} = 0, \; \begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} \ge 0, \tag{1}$$

where $e_m$ and $e_n$ are $m$ vectors and $n$ vectors whose components are all 1's.

It is easy to see that if $(x^*, y^*)$ is a Nash equilibrium pair then $(\bar{x}, \bar{y})$ is a solution to (1) where

$$\bar{x} = x^*/(x^*)^{\mathrm{T}} B y^* \;\text{ and }\; \bar{y} = y^*/(x^*)^{\mathrm{T}} A y^*. \tag{2}$$

Note that $\bar{x} \neq 0$ and $\bar{y} \neq 0$ is ensured from the positivity of the payoff matrices $A$ and $B$. Conversely, if $(\bar{x}, \bar{y})$ is a solution of (1) then $\bar{x} \neq 0$ and $\bar{y} \neq 0$. Therefore, $(x^*, y^*)$ is a Nash equilibrium pair where

$$x^* = \bar{x}/e_m^{\mathrm{T}} \bar{x} \;\text{ and }\; y^* = \bar{y}/e_n^{\mathrm{T}} \bar{y}.$$

Lemke and Howson [11] gave an efficient and constructive procedure for obtaining an equilibrium pair by solving LCP$(q, M)$ where $M = \begin{bmatrix} 0 & A \\ B^{\mathrm{T}} & 0 \end{bmatrix}$ and $q = \begin{bmatrix} -e_m \\ -e_n \end{bmatrix}$.

Let $\mathscr{S} \subset \mathscr{E}(A, B)$. Two points $(x, y), (x', y') \in \mathscr{S}$ are called $\mathscr{S}$-interchangeable if $(x, y'), (x', y) \in \mathscr{S}$. We call $\mathscr{S}$ a Nash subset for the game $(A, B)$ if every pair of equilibria in $\mathscr{S}$ is $\mathscr{S}$-interchangeable. A Nash subset $\mathscr{S}$ is called maximal if there does not exist a Nash subset $\mathscr{T} \subset \mathscr{E}(A, B)$ such that $\mathscr{S}$ is properly contained in $\mathscr{T}$. A game $(A, B)$ is said to be weakly completely mixed if there exists a maximal Nash subset $\mathscr{S} \subset \mathscr{E}(A, B)$ such that all equilibria in $\mathscr{S}$ are completely mixed. The set of equilibria of a completely mixed bimatrix game contain only one (completely mixed) equilibrium. Therefore, a completely mixed bimatrix game is *weakly completely mixed*. For details see Jurg et al. [9].

We need the following definition in the sequel.

**Definition 1** We call $B$, a submatrix of order $m \times m$ of $(I, -A)$ a proper basis matrix if

(i) $-A_{\cdot j}$ is a column of $B \Rightarrow I_{\cdot j}$ is not a column of $B$.
(ii) The columns of $B$ form a linearly independent set.

We now present an alternative proof of the following result by Jurg et al. [9].

**Theorem 1** *Let $A > 0$ and $B > 0$. Then $(A, B)$ is a weakly completely mixed bimatrix game if and only if*

(i) *$A$ and $B$ are square matrices of order $n$;*
(ii) *$A$ and $B$ have full rank;*
(iii) *all entries of both $(B^{\mathrm{T}})^{-1} e_n$ and $A^{-1} e_n$ are positive.*

*Proof* Assume that (i), (ii) and (iii) hold. Consider the equivalent LCP $(q, M)$ of $(A, B)$ where $M = \begin{bmatrix} 0 & A \\ B^{\mathrm{T}} & 0 \end{bmatrix}$ and $q = \begin{bmatrix} -e_n \\ -e_n \end{bmatrix}$. Note that a proper basis matrix $\tilde{B} = \begin{bmatrix} 0 & -A \\ -B^{\mathrm{T}} & 0 \end{bmatrix}$ of $(I, -M)$ is nonsingular by (ii) and (iii). This implies $\tilde{z} = \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \tilde{B}^{-1}q > 0$ is a solution of LCP $(q, M)$. Hence in the corresponding Nash equilibrium pair $(x^*, y^*)$ of $(A, B)$, strategies $x^*$ and $y^*$ are completely mixed. Further, note that in any other Nash equilibrium pair $(x, y)$ of $(A, B)$ corresponding to a solution of LCP $(q, M)$ other than $(\tilde{x}, \tilde{y})$, strategies $x$ and $y$ cannot be completely mixed. Therefore, $\mathscr{S} = \{(x^*, y^*)\} \subset \mathscr{E}(A, B)$ is a maximal Nash subset and hence $(A, B)$ is a weakly completely mixed bimatrix game.

Conversely, suppose that that the game $(A, B)$ is weakly completely mixed. This implies that there exists a maximal Nash subset $\mathscr{S} \subset \mathscr{E}(A, B)$ such that all equilibria in $\mathscr{S}$ are completely mixed. Let $(x^*, y^*) \in \mathscr{S}(A, B)$. Since $(x^*, y^*)$ is completely mixed, in the corresponding solution $(x', y')$ of LCP $(q, M)$ all the coordinates will be positive. Therefore, proper feasible basis of $(I, -M)$ corresponding to the solution $(x', y')$ is $\tilde{B} = \begin{bmatrix} 0 & -A \\ -B^{\mathrm{T}} & 0 \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix} = \tilde{B}^{-1}q = \begin{bmatrix} 0 & -(B^{\mathrm{T}})^{-1} \\ -A^{-1} & 0 \end{bmatrix} \begin{bmatrix} -e_m \\ -e_n \end{bmatrix} > 0$. For this to hold, $A$ and $B$ have to be square matrices and nonsingular. Further,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -(B^{\mathrm{T}})^{-1} \\ -A^{-1} & 0 \end{bmatrix} \begin{bmatrix} -e_n \\ -e_n \end{bmatrix} = \begin{bmatrix} (B^{\mathrm{T}})^{-1}e_n \\ A^{-1}e_n \end{bmatrix} > 0, \text{ implies (iii).}$$ □

## 3 Generalized Bimatrix game and Its Vertical Linear Complementarity Formulation

In this section, we aim to extend Theorem 1 to a generalized bimatrix game framework using the vertical linear complementarity approach. First we recall the concept of a vertical block matrix introduced by Cottle and Dantzig [1] in connection with the generalization of the linear complementarity problem. Consider a rectangular matrix $A$ of order $m \times k$ with $m \geq k$. Suppose $A$ is partitioned row-wise into $k$ blocks in the form

$$A = \begin{bmatrix} A^1 \\ A^2 \\ \vdots \\ A^k \end{bmatrix}$$

where each $A^j = (a^j_{rs}) \in \mathbb{R}^{m_j \times k}$ with $\sum_{j=1}^{k} m_j = m$. Then $A$ is called a *vertical block matrix of type* $(m_1, \ldots, m_k)$. The vertical block matrix arises naturally in the literature of stochastic games where the states are represented by the columns and actions in each state are represented by rows in a particular block (see [14–17]).

We shall now present the generalization of the linear complementarity problem by Cottle and Dantzig [1] involving a vertical block matrix known as vertical linear complementarity problem and it is stated as follows:

Given a vertical block matrix $A$ of type $(m_1, \ldots, m_k)$ and a vector $q \in \mathbb{R}^m$, the *vertical linear complementarity problem* (VLCP) is to find $w \in \mathbb{R}^m$ and $z \in \mathbb{R}^k$ such that

$$w - Az = q, \ w \geq 0, \ z \geq 0$$
$$z_j \prod_{i=1}^{m_j} w_i^j = 0, \ \text{for } j = 1, 2, \ldots, k.$$

This problem is denoted as VLCP $(q, A)$. For details on VLCP see [12,18,26] and references therein.

A submatrix of size $k$ of $A$ is called a *representative submatrix* if its $i$th row is drawn from the $i$th block $A^i$ of $A$. Clearly, a vertical block matrix of type $(m_1, m_2, \ldots, m_k)$ has at most $\prod_{j=1}^{k} m_j$ distinct representative submatrices.

The following theorem presents a relationship between solution of VLCP $(q, A)$ and the LCP obtained from representative submatrix.

**Theorem 2** (Proposition 4.3 in [3, p.167]) *VLCP $(q, A)$ has a complementary feasible solution if and only if there exists a representative submatrix $A_G$ and a corresponding subvector $q_G$ of $q$ so that LCP $(q_G, A_G)$ is solvable with a solution $z$ and $Az + q \geq 0$.*

Gowda and Sznajder [8] introduced a generalization of the bimatrix game. This generalized version of the bimatrix game is described as follows:

Let $\mathscr{A}$ and $\mathscr{B}$ be two given finite sets of matrices, $\mathscr{A}$ containing $s$ matrices and $\mathscr{B}$ containing $r$ matrices, each of order $m \times n$. Note that we consider the case when sets $\mathscr{A}$ and $\mathscr{B}$ are finite because effective computation of generalized Nash equilibria is feasible only when these sets are finite. Player I (the row player) forms his payoff matrix (also called as a *row representative* of $\mathscr{A}$) whose $i$th row is chosen as the $i$th row of some $A \in \mathscr{A}$ and then plays his choice of a mixed strategy over $\{1, \ldots, m\}$. Similarly, Player II (the column player) forms his payoff matrix (also called as a *column representative of* $\mathscr{B}$) whose $j$th column is chosen by him as the $j$th column of some $B \in \mathscr{B}$ and then plays his choice of mixed strategy over $\{1, \ldots, n\}$. The rest of the description of the game is the same as that of a bimatrix game. We denote a generalized bimatrix game as $(\mathscr{A}, \mathscr{B})$. In a generalized bimatrix game, we aim to compute a generalized Nash equilibrium. Concept of generalized Nash equilibrium is proposed by Gowda and Sznajder [8].

**Definition 2** A pair of mixed strategies $(x^*, y^*)_{(M,N)}$ is said to be a *generalized Nash equilibrium pair* if there exists a there exists a row representative submatrix $M$ and a column representative submatrix $N$ such that

$$(x^*)^{\mathrm{T}} M y^* \leq x^{\mathrm{T}} R y^*$$

and

$$(x^*)^{\mathrm{T}} N y^* \leq (x^*)^{\mathrm{T}} S y,$$

for all mixed strategies $x$ and $y$, all row representative submatrices $R$ of $\mathscr{A}$ and all column representative submatrices $S$ of $\mathscr{B}$.

We denote a generalized Nash equilibrium pair by $(x^*, y^*)_{(M,N)}$. Let $\mathscr{E}(\mathscr{A}, \mathscr{B})$ be the set of all pairs of equilibrium strategies. We say that $\mathscr{E}(\mathscr{A}, \mathscr{B})$ is completely mixed if the elements of $\mathscr{E}(\mathscr{A}, \mathscr{B})$ are completely mixed pairs.

Along the similar lines to Jurg et al. [9], we define a weakly completely mixed generalized bimatrix game. Let $\mathscr{S}_{(M,N)} \subset \mathscr{E}(\mathscr{A}, \mathscr{B})$, where subscript $(M, N)$ represents that $\mathscr{S}_{(M,N)}$ contains all generalized Nash equilibrium pairs corresponding to the row representative matrix $M$ and the column representative matrix $N$. Two points $(x, y)_{(M,N)}, (x', y')_{(M,N)} \in \mathscr{S}_{(M,N)}$

are called $\mathscr{S}_{(M,N)}$-interchangeable if $(x, y')_{(M,N)},\ (x', y)_{(M,N)} \in \mathscr{S}(M,N)$. We call $\mathscr{S}_{(M,N)}$ a *generalized Nash subset* for the game $(\mathscr{A}, \mathscr{B})$ if every pair of equilibria in $\mathscr{S}_{(M,N)}$ is $\mathscr{S}_{(M,N)}$-interchangeable. A generalized Nash subset $\mathscr{S}_{(M,N)}$ is called maximal if there does not exist a generalized Nash subset $\mathscr{T}_{(M,N)} \subset \mathscr{E}(\mathscr{A}, \mathscr{B})$ such that $\mathscr{S}_{(M,N)}$ is properly contained in $\mathscr{T}_{(M,N)}$. A generalized bimatrix game $(\mathscr{A}, \mathscr{B})$ is said to be *weakly completely mixed* if there exists a maximal generalized Nash subset $\mathscr{S}_{(M,N)} \subset \mathscr{E}(\mathscr{A}, \mathscr{B})$ such that all equilibria in $\mathscr{S}_{(M,N)}$ are completely mixed.

Mohan et al. [18] considered the question of computing a generalized Nash equilibrium point for the generalized bimatrix game as a vertical linear complementarity model discussed below.

Suppose, $\mathscr{A} = \{A^p \mid p = 1, \ldots, s\}$ and $\mathscr{B} = \{B^q \mid q = 1, \ldots, r\}$. Consider the matrices $C^j,\ j = 1, \ldots, m$ and $D^j,\ j = 1, \ldots, n$ defined as follows:

$$\begin{aligned} C^j_{i\cdot} &= A^i_{j\cdot},\ 1 \le i \le s \\ D^j_{i\cdot} &= (B^i)^{\mathrm{T}}_{j\cdot},\ 1 \le i \le r. \end{aligned}$$

Without loss of generality, we may assume that each $A^p,\ p = 1, \ldots, s$ and each $B^q,\ q = 1, \ldots, r$ are positive matrices. Hence each $C^j,\ j = 1, \ldots, m$ and each $D^j,\ j = 1, \ldots, n$ are positive matrices.

Let $C = \begin{bmatrix} C^1 \\ C^2 \\ \vdots \\ C^m \end{bmatrix}$ and $D = \begin{bmatrix} D^1 \\ D^2 \\ \vdots \\ D^n \end{bmatrix}$ where each $C^j$ is of order $s \times n$ and each $D^j$ is of order $r \times m$ and by our assumption $C > 0,\ \ D > 0$.

Consider the matrix

$$A = \begin{bmatrix} 0 & C^1 \\ 0 & C^2 \\ \vdots & \vdots \\ 0 & C^m \\ D^1 & 0 \\ D^2 & 0 \\ \vdots & \vdots \\ D^n & 0 \end{bmatrix} = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix} \tag{3}$$

where the '0' blocks are of appropriate orders. This is a vertical block matrix of type $(s, \ldots, s, r, \ldots, r)$ where the number of blocks is $m + n$.

Note that a generalized Nash equilibrium point of the game $(\mathscr{A}, \mathscr{B})$ can be computed by obtaining a solution to the VLCP $(-e, A)$ where $-e$ is the column vector of order $(ms + nr)$, each of whose coordinate is $-1$ and vice versa (see [8,18]).

Also the representative submatrix

$$A_G = \begin{bmatrix} 0 & C_G \\ D_G & 0 \end{bmatrix} \tag{4}$$

obtained from the vertical block matrix $A$ of type $(s, \ldots, s, r, \ldots, r)$ will be of order $(m + n) \times (m + n)$.

We now prove the following result.

**Theorem 3** *Let* $(\mathscr{A}, \mathscr{B})$ *be a generalized bimatrix game with* $\mathscr{A} = \{A^p > 0 \mid p = 1, \dots, s\}$ *and* $\mathscr{B} = \{B^q > 0 \mid q = 1, \dots, r\}$, *where* $A^p$, $B^p$ *are of order* $m \times n$. *Then* $(\mathscr{A}, \mathscr{B})$ *is a weakly completely mixed generalized bimatrix game if and only if all matrices* $A^p$, $p = 1, \dots, s$, $B^q$, $q = 1, \dots, r$ *are square matrices of order n and there exists a representative submatrix* $A_G$ *of vertical block matrix* $A$ *(defined in (3) and (4)) for which the following holds:*

(i) $C_G$ *and* $D_G$ *are nonsingular;*
(ii) *all entries of both* $e_n^{\mathrm{T}} C_G^{-1}$ *and* $D_G^{-1} e_n$ *are positive;*
(iii) *There exists a solution* $z^T = (x, y)$ *of LCP* $(-e_{2n}, A_G)$ *such that* $-e_{ns+nr} + Az \geq 0$.

*Proof* Suppose $A^p$, $p = 1, \dots, s$, $B^q$, $q = 1, \dots, r$ are square matrices of order $n$ and there exists a representative submatrix $A_G$ of vertical block matrix $A$ such that (i), (ii) and (iii) holds. Then there exists a solution $\tilde{z}^T = (\tilde{x}, \tilde{y})$ of LCP $(-e_{2n}, A_G)$ such that $-e_{ns+nr} + Az \geq 0$. By Theorem 2, it follows that the solution $(\tilde{x}, \tilde{y})$ of LCP$(-e_{2n}, A_G)$ produces a solution corresponding to VLCP$(-e_{ns+nr}, A)$. Following the same arguments as in the proof of Theorem 1, it is easy to see that in the corresponding generalized Nash equilibrium pair $(x^*, y^*)_{(C_G, D_G)}$ of $(\mathscr{A}, \mathscr{B})$, strategies $x^*$ and $y^*$ are completely mixed. Further, note that in any other generalized Nash equilibrium pair $(x, y)_{(C_G, D_G)}$ of $(\mathscr{A}, \mathscr{B})$ corresponding to a solution of LCP $(-e_{2n}, A_G)$ other than $(\tilde{x}, \tilde{y})$, strategies $x$ and $y$ cannot be completely mixed. Therefore, $\mathscr{S}_{(C_G, D_G)} = \{(x^*, y^*)_{(C_G, D_G)}\} \subset \mathscr{E}(\mathscr{A}, \mathscr{B})$ is a maximal generalized Nash subset and hence $(\mathscr{A}, \mathscr{B})$ is a weakly completely mixed generalized bimatrix game.

Conversely, suppose $(\mathscr{A}, \mathscr{B})$ is a weakly completely mixed generalized bimatrix game. This implies that there exists a generalized maximal Nash subset $\mathscr{S}_{(C_G, D_G)} \subset \mathscr{E}(\mathscr{A}, \mathscr{B})$ such that all equilibria in $\mathscr{S}_{(C_G, D_G)}$ are completely mixed. This implies corresponding to $(x^*, y^*)_{(C_G, D_G)} \in \mathscr{S}_{(C_G, D_G)}$ there exists a solution $\tilde{z}^T = (\tilde{x}, \tilde{y}) > 0$ of LCP $(-e_{m+n}, A_G)$ such that $-e_{ms+nr} + A\tilde{z} \geq 0$ holds. Since $(x^*, y^*)$ is completely mixed, in the corresponding solution $(\tilde{x}, \tilde{y})$ of LCP $(-e_{m+n}, A_G)$ where $A_G = \begin{bmatrix} 0 & C_G \\ D_G & 0 \end{bmatrix}$ all the coordinates will be positive. Therefore, proper feasible basis of $(I, -A_G)$ corresponding to the solution $(\tilde{x}, \tilde{y})$ is $\tilde{B} = \begin{bmatrix} 0 & -C_G \\ -D_G & 0 \end{bmatrix}$ and $\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \tilde{B}^{-1} q = \begin{bmatrix} 0 & -(D_G)^{-1} \\ -(C_G)^{-1} & 0 \end{bmatrix} \begin{bmatrix} -e_m \\ -e_n \end{bmatrix} > 0$. For this to hold, $C_G$ and $D_G$ have to be nonsingular square matrices of order $n$. By construction of $C_G$ and $D_G$ it follows that $A^p$, $p = 1, \dots, s$, $B^p$, $p = 1, \dots, r$ are square matrices of order $n$. Further, $\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} 0 & -(D_G)^{-1} \\ -(C_G)^{-1} & 0 \end{bmatrix} \begin{bmatrix} -e_n \\ -e_n \end{bmatrix} = \begin{bmatrix} (D_G)^{-1} e_n \\ (C_G)^{-1} e_n \end{bmatrix} > 0$, implies (ii). □

The following example illustrates the construction of the equivalent VLCP for a given generalized generalized bimatrix game.

*Example 1* Consider the generalized bimatrix game $(\mathscr{A}, \mathscr{B})$ where the sets $\mathscr{A}$ and $\mathscr{B}$ are as follows: $\mathscr{A} = \left\{ \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 6 \\ 2 & 8 \end{bmatrix} \right\}$,

$\mathscr{B} = \left\{ \begin{bmatrix} 3 & 4 \\ 8 & 3 \end{bmatrix}, \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \right\}$. The related matrices $C^j$ and $D^j$ are as follows:

$C^1 = \begin{bmatrix} 3 & 1 \\ 1 & 6 \end{bmatrix}$, $C^2 = \begin{bmatrix} 1 & 5 \\ 2 & 8 \end{bmatrix}$, $D^1 = \begin{bmatrix} 3 & 8 \\ 5 & 7 \end{bmatrix}$, $D^2 = \begin{bmatrix} 4 & 3 \\ 6 & 8 \end{bmatrix}$.

Note that $A = \begin{bmatrix} 0 & C^1 \\ 0 & C^2 \\ D^1 & 0 \\ D^2 & 0 \end{bmatrix}$ is a vertical block matrix of type (2, 2, 2, 2).

A solution obtained by solving VLCP$(-e_8, A)$ is $z_1 = 0.2174$, $z_2 = 0.0435$ $z_3 = 0.2857$, $z_4 = 0.1429$, and the corresponding generalized Nash equilibrium is $x_1 = 0.83$, $x_2 = 0.17$ $y_1 = 0.67$, $y_2 = 0.33$. The corresponding row representative matrix is $\begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$ and column representative matrix is $\begin{bmatrix} 3 & 8 \\ 4 & 3 \end{bmatrix}$.

## 4 Completely Mixed Strategies in Switching Controller Stochastic Game

Stochastic games were first formulated by Shapley [25]. A stochastic game is played in stages. In this section, we consider the existence of completely mixed strategies for a special class of stochastic games. The theory of stochastic games has been applied to study many practical problems like search problems, military applications, advertising problems, the traveling inspector model and various economic applications. For details see [7]. We first present some preliminaries and notations on stochastic game which will be used in this section.

A two-player finite state/action space zero-sum stochastic game is defined by the following objects.

1. A state space $S = \{1, 2, \ldots, N\}$.
2. For each $s \in S$, finite action sets $A(s) = \{1, 2, \ldots, m_s\}$ for Player I and $B(s) = \{1, 2, \ldots, n_s\}$ for Player II.
3. A reward law $R(s)$ for $s \in S$ where $R(s) = [r(s, i, j)]$ is an $m_s \times n_s$ matrix whose $(i, j)$th entry denotes the payoff from Player II to Player I corresponding to the choices of action $i \in A(s)$, $j \in B(s)$ by Player I and Player II, respectively.
4. A transition law $q = \{q_{ij}(s, s') : (s, s') \in S \times S, i \in A(s), j \in B(s)\}$, where $q_{ij}(s, s')$ denotes the probability of a transition from state $s$ to state $s'$ given that Player I and Player II choose actions $i \in A(s)$, $j \in B(s)$, respectively.

The game is played in stages $t = 0, 1, 2, \ldots$. At some stage $t$, the players find themselves in a state $s \in S$ and independently choose actions $i \in A(s)$, $j \in B(s)$. Player II pays Player I an amount $r(s, i, j)$ and at stage $(t + 1)$, the new state is $s'$ with probability $q_{ij}(s, s')$. Play continues at this new state.

The players guide the game via strategies. In general, strategies can depend on complete histories of the game until the current stage. We are, however, concerned with the simpler class of *stationary strategies* which depend only on the current state $s$ and not on stages. So for Player I, a stationary strategy

$$f \in F_S = \{f_i(s) \ : \ s \in S, i \in A(s), f_i(s) \geq 0, \sum_{i \in A(s)} f_i(s) = 1\}$$

indicates that the action $i \in A(s)$ should be chosen by Player I with probability $f_i(s)$ when the game is in state $s$. Similarly for Player II, a stationary strategy

$$g \in G_S = \{g_j(s) \ : \ s \in S, j \in B(s), g_j(s) \geq 0, \sum_{j \in B(s)} g_j(s) = 1\}$$

indicates that the action $j \in B(s)$ should be chosen with probability $g_j(s)$ by Player II when the game is in state $s$. Here, $F_S$ and $G_S$ denote the set of all stationary strategies for Player I and Player II, respectively, in the state $s$. Let $f(s)$ and $g(s)$ be the corresponding vectors of dimension $m_s$ and $n_s$, respectively.

Fixed stationary strategies $f$ and $g$ induce a Markov chain on $S$ with transition matrix $P(f, g)$ whose $(s, s')$th entry is given by

$$P_{ss'}(f, g) = \sum_{i \in A(s)} \sum_{j \in B(s)} q_{ij}(s, s') f_i(s) g_j(s),$$

and the expected current reward vector $r(f, g)$ has entries defined by

$$r_s(f, g) = \sum_{i \in A(s)} \sum_{j \in B(s)} r(s, i, j) f_i(s) g_j(s) = f^{\mathrm{T}}(s) R(s) g(s)$$

With fixed general strategies $f, g$ and an initial state $s$, the stream of expected payoff to Player I at stage $t$, denoted by $v_s^t(f, g)$, $t = 0, 1, 2, \ldots$ is well defined and the resulting discounted and undiscounted payoffs are

$$\phi_s^{\beta}(f, g) = \sum_{t=0}^{\infty} \beta^t v_s^t(f, g) \text{ for a } \beta \in (0, 1)$$

and

$$\phi_s(f, g) = \lim_{T \uparrow \infty} \inf \frac{1}{T+1} \sum_{t=0}^{\mathrm{T}} v_s^t(f, g).$$

A pair of strategies $(f^*, g^*)$ is optimal for Player I and Player II in the undiscounted game if for all $s \in S$

$$\phi_s(f, g^*) \leq \phi_s(f^*, g^*) = v_s^* \leq \phi_s(f^*, g),$$

for any strategies $f$ and $g$ of Player I and Player II respectively. The number $v_s^*$ is called the *value of the game* starting in state $s$, and $v^* = (v_1^*, v_2^*, \ldots, v_N^*)$ is called the *value vector.* The definition for discounted case is similar.

Shapley [25] proved the existence of value vector and optimal stationary strategies for discounted case and gave a method for iterative computation of the value of a stochastic game with discounted payoff. In the literature of stochastic game, many authors have considered stochastic games with special structures on payoff and transition matrix for which one can hope for finite step algorithms. These special classes of stochastic games are listed below.

(i) Stochastic games with perfect information
(ii) Single controller stochastic games
(iii) Switching controller stochastic games
(iv) Separable reward and state independent transitions (Ser-Sit) games
(v) Additive reward-additive transition games (ARAT games).

These games of special structures possess ordered field property and for more details one can refer to an excellent survey paper by Raghavan and Filar [22]. Among these structured classes, in this paper we consider switching controller stochastic game and obtain sufficient condition for completely mixed stationary strategies.

In single controller stochastic games, player II is *single controller* which means $q_{ij}(s, s') = q_j(s, s')$, $\forall\, i, j, s, s'$ i.e., the transition probabilities depend on the actions of only one player (player II, in our case). Filar [5] considered completely mixed single controller stochastic games and showed that the properties analogous to those derived by Kaplansky [10] for the completely mixed matrix games hold. Filar [5] showed that for a completely mixed discounted single controller game, the payoff matrix $R(s)$ is square and nonsingular for each $s \in S$. For a completely mixed discounted single controller game, Filar also showed that each player possesses a unique optimal stationary strategy which follows from the fact that $R(s)$ is square and nonsingular for each $s \in S$, and the rest follows from Kaplansky's result [10].

The class of switching control stochastic games is introduced by Filar [4]. In a switching control stochastic game the law of motion is controlled by Player I alone when the game is played in a certain subset of states and Player II alone when the game is played in other states. In other words, a switching control game is a stochastic game in which the set of states are partitioned into sets $S_1$ and $S_2$ where the transition function is given by

$$q_{ij}(s, s') = \begin{cases} q_i^1(s, s'), \text{ for } s' \in S, s \in S_1, i \in A(s) \text{ and } \forall j \in B(s) \\ q_j^2(s, s'), \text{ for } s' \in S, s \in S_2, j \in B(s) \text{ and } \forall i \in A(s) \end{cases}$$

Let $\Gamma = [\Gamma(s), s = 1, \ldots, N]$ be a discounted switching controller game with payoff matrix $R(s) > 0$ for each state $s \in S$, where $F^*(s)$ and $G^*(s)$ denote the sets of optimal stationary strageties in state $s$ for players I and II, respectively. We shall say that $\Gamma$ is completely mixed if $(f^*, g^*) \in F^* \times G^*$ implies that $f_i^*(s)$ and $g_j^*(s)$ are strictly positive for all $i$, $j$ and $s$.

Many researchers have attempted to formulate the problem of computing value vector and optimal strategies for structured stochastic games as complementarity models and obtained finite step methods, for more deatils see [13,17,22,27]. Schultz [23,24] formulated the discounted switching controller game as a linear complementarity problem as a continuation of work done by Filar and Schultz [6] and used the following result to establish the complementarity conditions presented in Theorem 5.

**Theorem 4** *A $\beta$-discounted zero-sum stochastic game possesses values $v_s^\beta$ for $s \in S$ and optimal stationary strategies $f$ and $g$ for player I and II, respectively, if and only if $(v^\beta, f, g)$ solves the following nonlinear system. Find $(v^\beta, f, g)$ where $v_s^\beta \in \mathbb{R}^{|S|}$ such that*

$$v_s^\beta - \beta \sum_{s' \in S} v_{s'}^\beta \sum_{j=1}^{n_s} q_{ij}(s, s') g_j(s) - [R(s)g(s)]_i \geq 0, \; i \in A(s), s \in S \tag{5}$$

$$-v_s^\beta + \beta \sum_{s' \in S} v_{s'}^\beta \sum_{i=1}^{m_s} q_{ij}(s, s') f_i(s) + [f(s)R(s)]_j \geq 0, \; j \in B(s), s \in S \tag{6}$$

$$f \in F_s, g \in G_s \tag{7}$$

*Remark 1* Note that if $v_s^\beta$, $f(s)$, $g(s)$ satisfy (5), (6) and (7) then

$$v_s^\beta = [\beta P(f, g) v^\beta]_s + r_s(f, g) \tag{8}$$

In (8), the quadratic form $f(s)R(s)g(s)$ has notation $r_s(f, g)$.

We extend Filar's results [5] for single controller stochastic game to switching controller stochastic game. However, we follow a different approach and make use of the LCP framework to obtain these results. First, we provide a sufficient condition for existence of completely mixed optimal stationary strategies $f(s)$ and $g(s)$.

**Theorem 5** *Consider the following LCP$(q, M)$ obtained from the inequalities of $\beta$-discounted switching control stochastic game where $R(s)$ is square and nonsingular for each $s = 1, 2, \ldots, N$. The game has completely mixed optimal stationary strategies $f(s)$ and $g(s)$ if the components of $\mathscr{B}^{-1}q$ are positive corresponding to $f_i(s)$; $g_j(s)$ for each*

*$s \in S$, $i \in A(s)$ $j \in B(s)$, where $\mathscr{B}$ is the proper basis matrix obtained from columns corresponding to variables $w, f, g, v, \theta$ of $[I : -M]$ obtained from the LCP$(q, M)$ given by the following inequalities:*

$$f \in F_S, \ g \in G_S \tag{9}$$

$$v_s^\beta - \beta \sum_{s' \in S} v_{s'}^\beta q_i^1(s, s') - [R(s)g(s)]_i \geq 0, \ i \in A(s), s \in S_1 \tag{10}$$

$$v_s^\beta - \theta_s - [R(s)g(s)]_i \geq 0, \ i \in A(s), s \in S_2 \tag{11}$$

$$-v_s^\beta + \theta_s + [f(s)R(s)]_j \geq 0, \ j \in B(s), s \in S_1 \tag{12}$$

$$-v_s^\beta + \beta \sum_{s' \in S} v_{s'}^\beta q_j^2(s, s') + [f(s)R(s)]_j \geq 0, \ j \in B(s), s \in S_2 \tag{13}$$

$$f(s) \text{ is complementary in } (10) \text{ and } (11) \tag{14}$$

$$g(s) \text{ is complementary in } (12) \text{ and } (13) \tag{15}$$

*Proof* Following Schultz [24], it is easy to establish the above result by showing that (9) through (15) satisfy the conditions stated in Theorem 4 and therefore $(f, g)$ are optimal with values $v^\beta(s)$. Similarly if $v_s^\beta$, $f(s)$, $g(s)$ satisfy (5), (6) and (7), then defining

$$\theta_s = \begin{cases} \beta \sum_{s' \in S} \sum_{i=1}^{m_s} v_{s'}^\beta q_i^1(s, s') f_i(s), & s \in S_1 \\ \beta \sum_{s' \in S} \sum_{j=1}^{n_s} v_{s'}^\beta q_j^2(s, s') g_j(s), & s \in S_2 \end{cases} \tag{16}$$

forces (9) through (15) to be satisfied. Let $\mathscr{B}$ is the proper basis matrix obtained from columns corresponding to variables $w, f, g, v, \theta$ of $[I : -M]$. If the components of $\mathscr{B}^{-1}q$ are positive corresponding to $f_i(s)$; $g_j(s)$ for each $s \in S$, $i \in A(s)$ $j \in B(s)$ in the solution of the LCP$(q, M)$ then by definition, the solution is completely mixed. □

*Remark 2* Schultz [24] attempted to solve discounted switching controller game by solving a linear complementarity problem using Lemkes algorithm over 100 randomly generated problem. Schultz observed that Lemkes algorithm with a special $d$ works for these randomly generated problems and made a remark that "the ad hoc formulation and initialization have worked in practice but have not been proven that it always find a complementarity solution". However, it is possible that Lemke's algorithm may not always compute a solution in completely mixed strategies even if it exists. We note that for the proposed LCP$(q, M)$ in Theorem 5 one can take a principal pivot transform of $M$ with respect to $v_s^\beta$, $f(s)$, $g(s)$ and compute a solution in completely mixed strategies provided $B^{-1}q > 0$ corresponding to $f(s)$ and $g(s)$.

The above formulation in the vertical linear complementarity formulation setting is given below.

$$v_s^\beta - \beta \sum_{s' \in S} v_{s'}^\beta q_i(s, s') - [R(s)g(s)]_i \geq 0, \quad i \in A(s), s \in S_1 \tag{17}$$

$$v_s^\beta - \theta_s - [R(s)g(s)]_i \geq 0, \quad i \in A(s), s \in S_2 \tag{18}$$

$$-v_s^\beta + \theta_s + [f(s)R(s)]_j \geq 0, \quad j \in B(s), s \in S_1 \tag{19}$$

$$-v_s^{\beta} + \beta \sum_{s' \in S} v_{s'}^{\beta} q_j(s, s') + [f(s)R(s)]_j \geq 0, \quad j \in B(s), s \in S_2 \tag{20}$$

$$1 - \sum_{i \in A(s)} f_i(s) \geq 0, \quad s \in S_1 \tag{21}$$

$$-1 + \sum_{i \in A(s)} f_i(s) \geq 0, \quad s \in S_1 \tag{22}$$

$$1 - \sum_{j \in B(s)} g_j(s) \geq 0, \quad s \in S_2 \tag{23}$$

$$-1 + \sum_{j \in B(s)} g_j(s) \geq 0, \quad s \in S_2 \tag{24}$$

$$1 - \sum_{i \in A(s)} f_i(s) \geq 0, \quad s \in S_2 \tag{25}$$

$$-1 + \sum_{i \in A(s)} f_i(s) \geq 0, \quad s \in S_2 \tag{26}$$

$$-1 + \sum_{j \in B(s)} g_j(s) \geq 0, \quad s \in S_1 \tag{27}$$

$$1 - \sum_{j \in B(s)} g_j(s) \geq 0, \quad s \in S_1 \tag{28}$$

$$f_i(s)[v_s^{\beta} - \beta \sum_{s' \in S} v_{s'}^{\beta} q_i(s, s') - [R(s)g(s)]_i] = 0, \quad i \in A(s), s \in S_1 \tag{29}$$

$$f_i(s) \left[v_s^{\beta} - \theta_s - [R(s)g(s)]_i\right] = 0, \quad i \in A(s), s \in S_2 \tag{30}$$

$$g_j(s) \left[-v_s^{\beta} + \theta_s + [f(s)R(s)]_j\right] = 0, \quad j \in B(s), s \in S_1 \tag{31}$$

$$g_j(s) \left[-v_s^{\beta} + \beta \sum_{s' \in S} v_{s'}^{\beta} q_j(s, s') + [f(s)R(s)]_j\right] = 0, \quad j \in B(s), s \in S_2 \tag{32}$$

$$v_s^{\beta} \left[1 - \sum_{i \in A(s)} f_i(s)\right] \left[-1 + \sum_{i \in A(s)} f_i(s)\right] = 0, \quad s \in S_1 \tag{33}$$

$$v_s^{\beta} \left[1 - \sum_{j \in B(s)} g_j(s)\right] \left[-1 + \sum_{j \in B(s)} g_j(s)\right] = 0, \quad s \in S_2 \tag{34}$$

$$\hat{\theta}^{\beta}(s) \left[1 - \sum_{i \in A(s)} f_i(s)\right] = 0, \quad s \in S_2 \tag{35}$$

$$\tilde{\theta}^{\beta}(s) \left[-1 + \sum_{i \in A(s)} f_i(s)\right] = 0, \quad s \in S_2 \tag{36}$$

$$\hat{\theta}^{\beta}(s) \left[-1 + \sum_{j \in B(s)} g_j(s)\right] = 0, \quad s \in S_1 \tag{37}$$

$$\tilde{\theta}^{\beta}(s) \left[1 - \sum_{j \in B(s)} g_j(s)\right] = 0, \quad s \in S_1 \tag{38}$$

$$f_i(s), g_j(s), v_s^\beta,\ \hat{\theta}^\beta(s), \tilde{\theta}^\beta(s) \geq 0,\ i = 1, \ldots, m_s, j = 1, \ldots, n_s, s = 1, \ldots, N.$$

Note that in the above formulation $\theta_s = (\hat{\theta}^\beta(s) - \tilde{\theta}^\beta(s))$. Observe that the above is a vertical linear complementarity problem. We call this formulation as *Formulation A*. We will make use of *Formulation A* for numerical computation in Sect. 6.

Nearly, all the results for completely mixed single controller stochastic game obtained by Filar [5] can be generalized and we may derive similar results for a completely mixed discounted switching controller game.

**Lemma 1** *Consider a completely mixed discounted switching controller game. Then $R(s)$ is square and nonsingular for each $s \in S$.*

*Proof* For a $\beta$-discounted switching controller game, we can associate with each state payoff matrix $R(s)$, a Shapley matrix

$$R_\beta(s) = \begin{cases} \left[ r(s,i,j) + \beta \sum_{s' \in S} q_i^1(s,s')v(s') \right] & \text{for } s' \in S,\ \ s \in S_1, i \in A(s),\ \text{and } \forall\ j \in B(s) \\ \left[ r(s,i,j) + \beta \sum_{s' \in S} q_j^2(s,s')v(s') \right], & \text{for } s' \in S,\ \ s \in S_2, j \in B(s),\ \text{and } \forall\ i \in A(s) \end{cases}$$

We then use similar arguments as in [5, p. 588] to obtain the desired result. □

**Lemma 2** *For a completely mixed discounted switching controller game, each player possesses a unique optimal stationary strategy.*

*Proof* We make use of the approach used in [5] to obtain this result. □

The following theorem demonstrates that the existence of a solution of a set of linear inequalities satisfying complementarity condition given by (39) to (64) ensures that the strategies are completely mixed for a $\beta$-discounted switching control stochastic game.

**Theorem 6** *Let $R(s)$ is square and nonsingular for each $s \in S$. A $\beta$-discounted switching control stochastic game has completely mixed optimal stationary strategies $f(s)$ and $g(s)$ if $(v^\beta, f, g)$ solves the following VLCP.*

$$v_s^\beta - \beta \sum_{s' \in S} v_{s'}^\beta q_i^1(s,s') - [R(s)g(s)]_i \geq 0,\ \ i \in A(s), s \in S_1 \tag{39}$$

$$v_s^\beta - \theta_s - [R(s)g(s)]_i \geq 0,\ \ i \in A(s), s \in S_2 \tag{40}$$

$$-v_s^\beta + \theta_s + [f(s)R(s)]_j \geq 0,\ \ j \in B(s), s \in S_1 \tag{41}$$

$$-v_s^\beta + \beta \sum_{s' \in S} v_{s'}^\beta q_j^2(s,s') + [f(s)R(s)]_j \geq 0,\ \ j \in B(s), s \in S_2 \tag{42}$$

$$1 - \sum_{i \in A(s)} f_i(s) \geq 0,\ \ s \in S_1 \tag{43}$$

$$-1 + \sum_{i \in A(s)} f_i(s) \geq 0,\ \ s \in S_1 \tag{44}$$

$$1 - \sum_{j \in B(s)} g_j(s) \geq 0,\ \ s \in S_2 \tag{45}$$

$$-1 + \sum_{j \in B(s)} g_j(s) \geq 0,\ \ s \in S_2 \tag{46}$$

$$1 - \sum_{i \in A(s)} f_i(s) \geq 0, \;\; s \in S_2 \tag{47}$$

$$-1 + \sum_{i \in A(s)} f_i(s) \geq 0, \;\; s \in S_2 \tag{48}$$

$$-\varepsilon + f_i(s) \geq 0, \;\; i \in A(s), s \in S_2 \tag{49}$$

$$-\varepsilon + g_j(s) \geq 0, \;\; j \in B(s), s \in S_2 \tag{50}$$

$$-1 + \sum_{j \in B(s)} g_j(s) \geq 0, \;\; s \in S_1 \tag{51}$$

$$1 - \sum_{j \in B(s)} g_j(s) \geq 0, \;\; s \in S_1 \tag{52}$$

$$-\varepsilon + f_i(s) \geq 0, \;\; i \in A(s), s \in S_1 \tag{53}$$

$$-\varepsilon + g_j(s) \geq 0, \;\; j \in B(s), s \in S_1 \tag{54}$$

$$f_i(s)[v_s^\beta - \beta \sum_{s' \in S} v_{s'}^\beta q_i^1(s, s') - [R(s)g(s)]_i] = 0, \;\; i \in A(s), s \in S_1 \tag{55}$$

$$f_i(s)[v_s^\beta - \theta_s - [R(s)g(s)]_i] = 0, \;\; i \in A(s), s \in S_2 \tag{56}$$

$$g_j(s)[-v_s^\beta + \theta_s + [f(s)R(s)]_j] = 0, \;\; j \in B(s), s \in S_1 \tag{57}$$

$$g_j(s)\left[-v_s^\beta + \beta \sum_{s' \in S} v_{s'}^\beta q_j^2(s, s') + [f(s)R(s)]_j\right] = 0, \;\; j \in B(s), s \in S_2 \tag{58}$$

$$v_s^\beta \left[1 - \sum_{i \in A(s)} f_i(s)\right]\left[-1 + \sum_{i \in A(s)} f_i(s)\right] = 0, \;\; s \in S_1 \tag{59}$$

$$v_s^\beta \left[1 - \sum_{j \in B(s)} g_j(s)\right]\left[-1 + \sum_{j \in B(s)} g_j(s)\right], \;\; s \in S_2 \tag{60}$$

$$\hat{\theta}^\beta(s)\left[1 - \sum_{i \in A(s)} f_i(s)\right]\left[-1 + \sum_{i \in A(s)} f_i(s)\right] = 0, \;\; s \in S_2 \tag{61}$$

$$\tilde{\theta}^\beta(s)\Pi_{i \in A(s)}[-\varepsilon + f_i(s)]\Pi_{j \in B(s)}[-\varepsilon + g_j(s)] = 0, \;\; s \in S_2 \tag{62}$$

$$\hat{\theta}^\beta(s)\left[-1 + \sum_{j \in B(s)} g_j(s)\right]\left[1 - \sum_{j \in B(s)} g_j(s)\right] = 0, \;\; s \in S_1 \tag{63}$$

$$\tilde{\theta}^\beta(s)\Pi_{i \in A(s)}[-\varepsilon + f_i(s)]\Pi_{j \in B(s)}[-\varepsilon + g_j(s)] = 0, \;\; s \in S_1 \tag{64}$$

$$\varepsilon > 0, \; f_i(s), g_j(s), v_s^\beta, \; \hat{\theta}^\beta(s), \tilde{\theta}^\beta(s) \geq 0, \;\; i \in A(s), j \in B(s).$$

*Proof* Following Schultz [24], it is easy to show that (39) through (64) satisfy the conditions stated in Theorem 4 and therefore $(f, g)$ are optimal with completely mixed strategies $f^*(s), g^*(s)$ and values $v^\beta(s)$ $s = 1, \ldots, N$. □

*Remark 3* Note that in the above formulation $\theta_s = (\hat{\theta}^\beta(s) - \tilde{\theta}^\beta(s))$. We call this formulation as *Formulation B*. Observe that the above formulation is a vertical linear complementarity problem. It is well known that one may use Cottle–Dantzig's algorithm to compute completely mixed strategies and the value vector if the algorithm does not terminate with a secondary ray.

However, one may use modified version of generalized PPT algorithm by Neogy et al. [20] described in the next section that consider only a partial number of PPTs to obtain a solution that is completely mixed. Another strategy is to initiate Cottle–Dantzig's algorithm with a suitable $d$ so that Cottle–Dantzig algorithm's moves in a path and successfully computes a solution. We need to choose $\varepsilon$ very small to compute completely mixed strategies while using Cottle–Dantzig's algorithm or modified generalized PPT algorithm for Formulation B.

## 5 Numerical Method for Computation of Completely Mixed Strategies for Switching Controller Stochastic Game

In this section, we consider computational aspect of switching controller stochastic games and illustrate an approach for computation of numerical solutions. For developing the solution methods we make use of the fact that in completely mixed stationary strategies $f(s), g(s)$ and value vector $v(s)$ always appear in the basis at the positive level. We use the vertical linear complementarity framework and use a modified version of stepwise generalized principal pivoting algorithm developed by Neogy et al. [20] for computing the solution of a vertical linear complementarity problem. We require the following definitions to describe the modified generalized principal pivoting algorithm.

For a square matrix $M$ suppose $B$ is a nonsingular complementary matrix of $[I, -M]$. Let $\bar{B}$ be the matrix of columns of $[I, -M]$ not in $B$. We say that $F$ is a principal pivot transform (PPT) of $M$ with respect to $B$ if $F = -B^{-1}\bar{B}$. We note that LCP$(q, M)$ is equivalent to LCP$(B^{-1}q, F)$. In [18], the concept of a generalized principal pivot transform was introduced for a rectangular matrix $A \in \mathbb{R}^{m\times k}$ where rows are partitioned into $k$ blocks. Suppose $\mathscr{B}$ is a nonsingular proper matrix. We say that $\mathscr{F}$ is a generalized PPT of $A$ with respect to $\mathscr{B}$ if $\mathscr{F} = -\mathscr{B}^{-1}\bar{\mathscr{B}}$, where $\bar{\mathscr{B}}$ is the matrix of columns not in $\mathscr{B}$. Note that $\bar{\mathscr{B}}$ contains $k$ columns and that $\mathscr{F}$ is an $m \times k$ vertical block matrix of type $(m_1, \ldots, m_k)$. The generalized principal pivoting method for solving VLCP's transforms the original problem into its equivalent problem by taking a generalized PPT.

### 5.1 Generalized Principal Pivoting Algorithm

In this section, we present a method to obtain a solution of a completely mixed switching controller game problem that consist of finding completely mixed stationary strategies $f(s), g(s)$ for Player I and Player II and the value vector $v_s$. We will use the VLCP formulation (Formulation A) to describe the modified version of generalized PPT algorithm.

**Algorithm (Modified Scheme):**

**Step 0** Input: A vertical block matrix $A$ of type $(m_1, \ldots, m_k)$ and $q \in \mathbb{R}^m$.
**Step 1** Set $\nu = 0$. Let $\mathscr{B}_\nu$ be a proper basis matrix involving columns corresponding to $f_i(s), g_j(s), v^\beta(s)$ and some columns of $[I : -A]$.
**Step 2** Compute $z = \mathscr{B}_\nu^{-1}q$. If $\mathscr{B}_\nu^{-1}q \geq 0$, then $z$ is a solution of VLCP$(q, A)$. Further if $z_i > 0$ corresponding to $f_i(s), g_j(s)$, then $z$ is a completely mixed solution for switching controller stochastic game. Stop, otherwise go to Step 3.
**Step 3** If $\mathscr{B}_\nu^{-1}q \ngeq 0$ then go to Step 4.
**Step 4** $\nu = \nu + 1$. Find a proper basis matrix $\mathscr{B}_\nu$ involving subset of the columns such that basis matrix consists of $f(s), g(s)$, the set of related variable corresponding to $v^\beta(s)$ and some columns of $[I : -A]$ (i.e., some columns corresponding to $\theta^\beta(s), w^\theta(s)$ (For example if the set of related variable corresponding to $v^\beta(s)$ is

$(v^\beta(s), w_1^v(s), w_2^v(s))$ then exactly two variable from this set will be in the basis). Go to Step 2.

**Theorem 7** *If the generalized PPT Algorithm ends in Step 2 with the component of* $\mathscr{B}^{-1}q$ *being positive corresponding to* $f_i(s)$; $g_j(s)$ *and* $v(s)$ *for each* $s \in S$, $i \in A(s)$ $j \in B(s)$, *then the switching controller game is completely mixed.*

We use the above algorithm to obtain a solution of the following switching controller stochastic game.

*Example 2* Consider a switching controller stochastic game with 2 states in which both players have two strategies. The rewards and transitions are

$$R(1) = \begin{bmatrix} 1 & 4 \\ 2 & 1 \end{bmatrix}, \quad q(1,1) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad q(1,2) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix},$$
$$R(2) = \begin{bmatrix} 1 & 4 \\ 5 & 1 \end{bmatrix}, \quad q(2,1) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad q(2,2) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}.$$

The discount factor is taken as $\beta = 0.8$. This game can be formulated and solved as a VLCP $(q, A)$. Here

$$A = \begin{bmatrix} \mathscr{R} & \mathscr{B} \\ \mathscr{A} & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ \tilde{c} \end{bmatrix} \text{ and the decision vector } z = \begin{bmatrix} y \\ x \end{bmatrix}$$

where $\mathscr{A}$, $\mathscr{B}$, $\mathscr{R}$ and the vector $\tilde{c}$ are given below:

$$\mathscr{A}y = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_1(1) \\ f_2(1) \\ f_1(2) \\ f_2(2) \\ g_1(1) \\ g_2(1) \\ g_1(2) \\ g_2(2) \end{bmatrix},$$

$$\tilde{c} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix},$$

$$\mathscr{B}x = \begin{bmatrix} 0.6 & -0.4 & 0 & 0 & 0 & 0 \\ 1 & -0.8 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0.4 & -0.6 & 0 & 0 & 0 & 0 \\ 0.8 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v(1) \\ v(2) \\ \hat{\theta}(1) \\ \tilde{\theta}(1) \\ \hat{\theta}(2) \\ \tilde{\theta}(2) \end{bmatrix},$$

$$
\mathscr{R}y = \begin{bmatrix}
0 & 0 & 0 & 0 & -1 & -4 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & -1 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
f_1(1) \\ f_2(1) \\ f_1(2) \\ f_2(2) \\ g_1(1) \\ g_2(1) \\ g_1(2) \\ g_2(2)
\end{bmatrix}.
$$

Based on the formulation A described earlier, the matrix $A$ and the vector $q$ are obtained as given below.

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & -1 & -4 & 0 & 0 & 0.6 & -0.4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 & 1 & -0.8 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -4 & 0 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -5 & -1 & 0 & 1 & 0 & 0 & -1 & 1 \\
1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\
4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 0.4 & -0.6 & 0 & 0 & 0 & 0 \\
0 & 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0.8 & -1 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad q = \begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ -1 \\ 1
\end{bmatrix}.
$$

Note that $A$ is a vertical block matrix of type $(1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 1, 1)$.

We use generalized PPT algorithm to solve this switching controller stochastic game problem. The generalized PPT of $A$ with respect to $\alpha = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13\}$ needs to be computed. It may be noted that $-A_{\cdot 9}$ can replace either $I_{\cdot 9}$ or $I_{\cdot 10}$ and accordingly we can have two PPT's. Same is true for $-A_{\cdot 10}$. Therefore, to compute a generalized PPT $\mathscr{F}_1$ w.r.t $\alpha^* = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13\}$ we first compute

$$
\begin{aligned}
\mathscr{B}_1 &= [-A_{\cdot 1}\ -A_{\cdot 2}\ -A_{\cdot 3}\ -A_{\cdot 4}\ -A_{\cdot 5}\ -A_{\cdot 6}\ -A_{\cdot 7}\ -A_{\cdot 8} \\
&\quad -A_{\cdot 9}\ I_{\cdot 10}\ -A_{\cdot 10}\ I_{\cdot 12}\ -A_{\cdot 11}\ I_{\cdot 14}\ A_{\cdot 13}\ I_{\cdot 16}\ ]. \\
\bar{\mathscr{B}}_1 &= [I_{\cdot 1}\ I_{\cdot 2}\ I_{\cdot 3}\ I_{\cdot 4}\ I_{\cdot 5}\ I_{\cdot 6}\ I_{\cdot 7}\ I_{\cdot 8}\ I_{\cdot 9}\ I_{\cdot 11}\ -A_{\cdot 12}\ I_{\cdot 13}\ I_{\cdot 15}\ -A_{\cdot 14}\ ].
\end{aligned}
$$

From $\mathscr{B}_1$ we obtain the solution $z_1^*$ as follows.

$$z_1^* = \mathscr{B}_1{}^{-1} q = \begin{bmatrix} f_1(1) \\ f_2(1) \\ f_1(2) \\ f_2(2) \\ g_1(1) \\ g_2(1) \\ g_1(2) \\ g_2(2) \\ v(1) \\ w(10) \\ v(2) \\ w(12) \\ \hat{\theta}(1) \\ w(14) \\ \hat{\theta}(2) \\ w(16) \end{bmatrix} = \begin{bmatrix} 0.2500 \\ 0.7500 \\ 0.6075 \\ 0.3925 \\ 0.6869 \\ 0.3131 \\ 0.4286 \\ 0.5714 \\ 10.9579 \\ 0 \\ 11.5888 \\ 0 \\ 9.2079 \\ 0 \\ 8.8745 \\ 0 \end{bmatrix}.$$

Note that the generalized PPT is not unique. Now we compute the other generalized PPT, $\mathscr{F}_2$ w.r.t $\alpha^* = \{2, 3, 4, 5, 7, 8, 10, 11, 13, 15\}$. Note that

$$\begin{aligned} \mathscr{B}_2 &= [-A_{\cdot 1} \ - A_{\cdot 2} \ - A_{\cdot 3} \ - A_{\cdot 4} \ - A_{\cdot 5} \ - A_{\cdot 6} \ - A_{\cdot 7} \ - A_{\cdot 8} \ I_{\cdot 9} \\ &\quad -A_{\cdot 9} \ - A_{\cdot 10} \ I_{\cdot 12} \ - A_{\cdot 11} \ I_{\cdot 14} \ - A_{\cdot 13} \ I_{\cdot 16} \ ]. \\ \bar{\mathscr{B}}_2 &= [I_{\cdot 1} \ I_{\cdot 2} \ I_{\cdot 3} \ I_{\cdot 4} \ I_{\cdot 5} \ I_{\cdot 6} \ I_{\cdot 7} \ I_{\cdot 8} \ I_{\cdot 10} \ I_{\cdot 11} \ - A_{\cdot 12} \ I_{\cdot 13} \ I_{\cdot 15} \ - A_{\cdot 14} \ ]. \end{aligned}$$

With above selection of $\mathscr{B}_2$ we obtain the solution $z_2^*$ as follows.

$$z_2^* = \mathscr{B}_1{}^{-1} q = \begin{bmatrix} f_1(1) \\ f_2(1) \\ f_1(2) \\ f_2(2) \\ g_1(1) \\ g_2(1) \\ g_1(2) \\ g_2(2) \\ w(9) \\ v(1) \\ v(2) \\ w(12) \\ \hat{\theta}(1) \\ w(14) \\ \hat{\theta}(2) \\ w(16) \end{bmatrix} = \begin{bmatrix} 0.2500 \\ 0.7500 \\ 0.6075 \\ 0.3925 \\ 0.6869 \\ 0.3131 \\ 0.4286 \\ 0.5714 \\ 0 \\ 10.9579 \\ 11.5888 \\ 0 \\ 9.2079 \\ 0 \\ 8.8745 \\ 0 \end{bmatrix}.$$

It may be noted that above two PPT's gives solution to the problem and both the solutions are exactly same except that the position of the components of $v(1)$ is interchanged, i.e., in case of the first PPT, $v(1) = 10.9579$ and $w(10) = 0$, $v(2) = 11.5888$., whereas in case of the second PPT, $w(9) = 0$ and $v(1) = 10.9579$, $v(2) = 11.5888$. In the basis, $\mathscr{B}_i$, $-A_{\cdot 10}$ can replace either $I_{\cdot 11}$ or $I_{\cdot 12}$ and accordingly we can have another two PPT's. Therefore, we need to compute maximum four generalized PPT to obtain the solution.

**Worst Case Analysis** The dimension of VLCP is $\sum m_i(s) + \sum n_j(s) + 4N \times \sum m_i(s) + \sum n_j(s) + 3N$, $i \in A(s)$, $j \in B(s)$. For the algorithm described here we need to compute $2^N$ generalized PPT in the worst case where $N$ is the number of states rather than $2^{n^*}$ where $n^* = \sum m_i(s) + \sum n_j(s) + 4N$. Therefore, this is a partial enumeration algorithm only.

## 6 Conclusions

In this paper, linear complementarity framework is used to establish the necessary and sufficient condition given by Jurg et al. [9] for a bimatrix game to be weakly completely mixed. Apart from providing a shorter proof of the above result, linear complementarity approach formed a base to characterize a weakly completely mixed generalized bimatrix and completely mixed switching controller stochastic game using a generalized linear complementarity framework. The proposed generalized linear complementarity formulation for a switching controller stochastic game led to a computational scheme for obtaining completely mixed strategies.

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