

# Epistemic Foundations of Equilibria under Ambiguity

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## Abstract

In this paper, we develop an interactive epistemology perspective justifying strategic ambiguity and various equilibrium concepts for games with non-additive beliefs. To accommodate strategic ambiguity in games, we introduce an extended version of interactive belief systems in which some types might not know the action they play. Yet, each type knows his theory, i.e., his probability distribution over the entire state space. It is shown that player's beliefs about his opponents' behavior are non-additive if he considers possible that his opponents are undetermined (i.e., his theory assigns a positive probability to opponents' types who do not know what actions they play). In this framework, we establish epistemic conditions under which beliefs constitute an equilibrium under ambiguity for games with two and more than two players, respectively. Our epistemic conditions for Nash-Equilibrium appear as a special case and thus generalize the celebrated results of Aumann and Brandenburger (1995).

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## 1 Introduction

Since the seminal contribution of Ellsberg (1961), economists acknowledged that individuals facing decision problems under ambiguity often fail to form additive beliefs. As response to the Ellsberg paradox, many ambiguity models have been developed triggering growing interests to explore how ambiguous beliefs affect strategic behavior in games. As

a result, various generalizations of the standard Nash-Equilibrium have been suggested including Dow and Werlang (1994), Lo (1996), Marinacci (2000), Eichberger and Kelsey (2000), Haller (2000), and Eichberger and Kelsey (2014), Dominiak and Eichberger (2017). In this paper, we provide an epistemic explanation for ambiguous beliefs in strategic games and various equilibrium concepts under ambiguity studied in the economics literature.

An Equilibrium under Ambiguity (EUA), as Nash-Equilibrium, is formulated as an equilibrium in beliefs. That is, each player best responds to his beliefs (i.e., conjectures) about his opponents' behavior and these beliefs are consistent with actual behavior. In an EUA, any pure strategy of a player which his opponents believe to be played (i.e., any pure strategy in the support of the opponents' beliefs) constitutes his best response given his beliefs. In contrast to Nash-Equilibrium, however, players' beliefs are represented by capacities (i.e., not-necessarily additive set functions) instead of probability measures. If equilibrium beliefs are additive, EUA coincides with the standard Nash-Equilibrium.

An important achievement of epistemic approach to game theory was the identification of epistemic conditions leading to Nash-Equilibrium behavior. That is, conditions referring to what players know about the game, the players' conjectures and their rationality so that the conjectures constitute a Nash-Equilibrium. In the celebrated work of Aumann and Brandenburger (1995), Nash-Equilibrium is identified as a state of the world in which the game being played, the player's conjectures, and their rationality are mutually - or commonly - known.<sup>1</sup> A player is rational in the sense of choosing an action which is optimal given the expected payoff with respect to his conjectures about the opponents' behavior derived from his theory (i.e., a probability distribution) over the states of the world.

However, the epistemic foundation by Aumann and Brandenburger does not provide any explanation for the "critical" assumption underlying the Nash Equilibrium concept, namely, that players hold additive beliefs over all strategy combinations of their opponents.

The aim of this paper is develop an epistemic model which is suited to accommodate different forms of players' beliefs. For this purpose, we introduce an extended version of Aumann-Brandenburger's notion of interactive belief systems. The novelty of the *extended interactive belief system* (EIBS) is that it is general enough

- (i) to provide epistemic explanation for additive vs. non-additive beliefs in games, and
- (ii) to formulate epistemic conditions for various equilibrium notions under ambiguity.

The building block of interactive belief systems is set of types for each player. Roughly, a type describes all information that is relevant for interaction in a strategic situation. More precisely, every type is identified with a payoff-function, an action and a theory, i.e., a probability distribution over all states of the world (i.e., the Cartesian product of *all*

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<sup>1</sup>For games with two players mutual knowledge is sufficient. However, for games with more than two players, stronger epistemic conditions are required. Conjectures which are derived from a common prior and which are commonly known constitute a Nash-Equilibrium.

players' type spaces). A player's theory "codifies" his conjectures about his opponents' behavior as well as his beliefs about other players' types; that is, their payoff-function, actions, theories, rationality and their beliefs about all these matters and so on.

In our epistemic model, we allow for the possibility that a player of a particular type considers states of the world as possible where other types exist which are similar to the own type with respect to payoffs and theories but are distinct with respect to the actions they play. That is to say, we will allow for the possibility that there are some types of a player who do not know of what type they are in the sense that they do not know what actions they carry out. However, every type knows his payoff-function as well as his theory.

Our assumption is motivated by the possibility that players with the same payoff and the same theory (thus, inducing the same conjectures) may be indifferent between actions and, hence, may choose different actions. While each type of a player have to choose an action "at the end of the day", taking into account the possibility that some types do not know what actions they play might affect his opponents beliefs about the player's behavior. Hence, even if one assumes that a player chooses rationally there may be ambiguity about which among several actions he will choose.

When forming his beliefs over opponents' behavior, derived from a player's theory, we will assume that the player takes into account whether there are types of his opponents who know or do not know what actions they play. A player is said to consider possible that his opponent is *undetermined* if there are types in the support of his theory who do not know what actions they play, otherwise the player knows that his opponent is *determined*. It is then shown that each player's conjectures over his opponents' strategy combinations take the form of Möbius transform of belief functions (a special form of convex capacities).

The different equilibria concepts studied in economics literature mentioned previously, can be mainly distinguished by the support notion for capacities. Marinacci (2000) defines the support of a capacity as the set of opponents' strategy combinations to which the capacity ascribes a strictly positive value. Our reference equilibrium is an *Equilibrium under Ambiguity* with the notion of support à la Marinacci. Dow and Werlang (1994) suggested an alternative support of a capacity which defined as a smallest set whose complement has the capacity value of zero. The two support notions are nested in the sense that every Marinacci-support is a Dow-Werlang support, but not *vice versa*. Moreover, Marinacci-support might be empty while a Dow-Werlang support always exist. For this reason, we will refer to a Dow-Werlang Equilibrium (DWE) as an equilibrium under ambiguity with capacities whose Marinacci-support is an empty set.<sup>2</sup>

For games with two players, the following epistemic conditions lead to an EUA. It is shown that if there is a state of the world at which each player considers possible that his opponent is determined, and at which the game being played (i.e., the payoff-functions),

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<sup>2</sup>Thus, in our setup, a DWE cannot be depicted as an equilibrium under ambiguity with respect to Marinacci-support. This distinction allows us to disentangle between the two equilibrium notions in terms of different epistemic condition leading to each equilibrium, respectively.

the players' conjectures and their rationality are mutually known, then the conjectures constitute an Equilibrium under Ambiguity. In other words, an EUA is identified as a state of the world in which players best respond to their mutually known conjectures which are derived from players' theories by taking into account there might types of an opponent who do not know what actions that play. Yet, at this state, each player consider possible that there is at least one determined type of his opponent to guarantee that the Marinacci-support of each player's conjectures contains at least one strategy combination.

In the above epistemic foundation, the player's conjectures do not need to be additive. Therefore, an EUA might support behavior which is incompatible with Nash-Equilibrium. In the framework of extended interactive belief systems, one can clarify what players need to know about each other in order to behave as predicted by Nash-Equilibrium. For the mutually known conjecture to constitute a Nash-Equilibrium, it must be mutually known among the two players, in addition to their mutual knowledge of the game and of their rationality, that each of them is determined (i.e., each type in the support of each player's theory knows what action he plays. In the wake of this additional condition, our result can be interpreted as a generalization of the epistemic foundation of Nash-Equilibrium in games with two players derived by Aumann and Brandenburger (1995).

At the other extreme, Dow-Werlang Equilibrium is identified as a state of the world in which it is mutually known that both players are undetermined in addition to their mutual knowledge of the game being played, of their conjectures and of their rationality. In contrast to EUA, conjectures constitute a DWE in states in which each player knows that all his opponent's types do not know what action he they will play. Each player best responds with an action that belongs to his opponent's Dow-Werlang support. Since every player knows that his opponent is undetermined, the Marinacci support is an empty set.

For games with  $n$ -players, additional conditions are required. In particular, one has to assume that conjectures are derived from a common prior over the states of the world. Since an EUA will be incompatible with the standard notion of stochastic independence, the common priors assumption warrants that player's conjectures are stochastically independent in the (weaker) sense of Möbius-products. Under the common prior assumption, an EUA is identified as a state of the world at which each players considers possible that *all* opponents are determined, at which the game being played, their rationality is mutually known, and at which their conjectures are commonly known. If additionally it is mutually known that all players are determined, then the EUA constitutes a Nash-Equilibrium.

The remainder of the paper is organized as follows. In Section 2, we present the extended interactive belief system. In this section, we derive player's conjectures, i.e., beliefs over the action profiles of the opponents and introduce the notion of (Choquet) rationality. In Section 3, we recall the various notions of equilibria under ambiguity studied in the literature. In Section 4, we derive epistemic conditions for equilibria under ambiguity with two players. In Section 5, we extend our epistemic analysis to games with  $n$ -players. In Section 6, we discuss the related literature and in Section 7, final remarks are provided.

## 2 Extended Interactive Belief System

Games in strategic form are considered. There is a finite set  $I = \{1, \dots, n\}$  of players indexed by  $i$ . We denote by  $-i$  the opponent players of  $i$ . There is a finite set of *actions* (pure strategies)  $A_i$  for each player  $i$ . The set of actions of all players other than  $i$  is denoted by  $A_{-i}$ . An element  $a = (a_i, a_{-i})$  of  $A = A_i \times A_{-i}$  is called the action profile. A function  $g_i : A \rightarrow \mathbb{R}$  represents player  $i$ 's payoff function. That is, a player  $i$ 's payoff  $g_i(a_i, a_{-i})$  depends on  $i$ 's action as well as the action profile of the opponent players  $-i$ . A function  $g : A \rightarrow \mathbb{R}^2$  is called a *game*. Denote by  $\mathcal{G}$  the set of all normal-form games.

In Aumann and Brandenburger (1995) (in short AB) the Nash equilibrium was viewed as a state of the world in which it is mutual - or common knowledge - that players are rational in the sense of choosing an action which is optimal given the expected payoff with respect to conjectures about the opponents' behavior derived from players' theories about the states of the world. A characteristic feature of the AB-epistemic approach is the assumption that a player's type is associated with a unique action and a notion of extended theory which identifies the player with a unique type. Hence, in any state of the world, a player not only knows the action which she chooses but also that no other action could have been chosen.

In contrast to Aumann and Brandenburger (1995), we allow for the possibility that a player of a particular type considers states of the world as possible where other types exist which are similar to the own type with respect to payoffs and theories but distinguished by the action. In particular, we will allow for the possibility that several types of player  $i$  exist which are identical with respect to their theories but distinct in the actions they choose. This assumption is motivated by the possibility that players with the same payoff and the same theory may be indifferent between actions and, hence, may choose different actions. Hence, even if one assumes that the other players choose rationally there may be ambiguity about which among several actions the opponents will choose.

If players consider it possible, however, that their opponents may choose among several actions then it is necessary to extend the notion of payoffs associated with single strategy combinations to a notion of payoff from an action in the face of several potential courses of action the other players may choose. We will assume that players will evaluate the payoff of an action by the Hurwicz (1951) criterion. According to this criterion a decision maker evaluates an action according to a weighted average of the best and worst outcome. In this case, the weight can be interpreted as the reflection of the player's attitude towards ambiguity. In our interpretation, this individual attitude towards ambiguity is an individual preference parameter, similar to a discount factor.

Given the payoff function  $g_i(a_i, a_{-i})$  of player  $i$  and a parameter  $\alpha_i \in [0, 1]$ , we define the (*ambiguity-weighted*) *payoff* of a player who chooses action  $a_i$  but considers action combinations of the opponent players in a set  $E \subseteq A_{-i}$  as possible by

$$G_i(a_i, E) := \alpha_i \min_{a_{-i} \in E} g_i(a_i, a_{-i}) + (1 - \alpha_i) \max_{a_{-i} \in E} g_i(a_i, a_{-i}). \quad (1)$$

The parameter  $\alpha_i$  is a preferential parameter which measures the degree of pessimism of player  $i$ . If there is no ambiguity about the opponents' behavior, i.e., in case of  $|E| = 1$ , we have  $G_i(a_i, \{a_{-i}\}; \alpha_i) = g_i(a_i, a_{-i})$ , irrespective of the ambiguity attitude parameter  $\alpha_i$ . The preference representation in Equation (1) is the well-known Hurwicz (1951) criterion applied to the set of the opponents' actions  $E$  which player  $i$  considers possible. Special cases include “*pure pessimism*”  $\alpha_i = 1$ , the attitude assumed most commonly in economic applications, and “*pure optimism*”  $\alpha_i = 0$ .

With this specification of (*ambiguity-weighted*) *payoff*, it is possible to stay with a notion of theories represented by a probability distribution over states.

## 2.1 Type Space and Theories

Given a strategic-form game in  $\mathcal{G}$ , we define an *extended interactive belief system*.

An *extended interactive belief system* for a game in  $\mathcal{G}$  consists of the following elements:

- (i) a finite set  $T_i$  of types for each player  $i \in I$ , and for each type  $t_i \in T_i$  of  $i$ ,
- (ii) an action  $a_i(t_i) \in A_i$ , (*t's action*)
- (iii) a function  $G_i(t_i) : A_i \times 2^{A_{-i}} \rightarrow \mathbb{R}$  (*t's weighted payoff function*, and
- (iv) a probability distribution  $p_i(t_i)$  on the set  $T$  of all types (called *t's theory*),

A type is a formal description of a player  $i$ 's action, payoff function, and theory, i.e., a probability distribution over his own types and the other players' types.

The Cartesian product of individual state spaces  $T = T_i \times T_{-i}$ , is called the *state space*. The elements  $t = (t_i, t_{-i}) \in T$  are called *states of the world* or simple *states*.

In state  $t = (t_i, t_{-i}) \in T$ , player  $i$  chooses an action  $a_i(t_i)$ , has an (ambiguity-weighted) payoff function  $G_i(t_i)$  and a theory  $p_i(t_i)$ .<sup>3</sup> A player  $i$ 's theory  $p_i(t_i)$  is the probability that  $i$ 's type  $t_i$  ascribes to his own types and the types of his opponent players.

An *event*  $E$  is a subset of  $T$ . Denote by  $p_i(t_i)(E)$  the probability that player  $i$  ascribes to  $E$  at state  $t$ . If  $p_i(t_i)(E) > 0$ , we say that player  $i$  considers  $E$  as a *possible event* at  $t$ . If player  $i$  ascribes probability 1 to  $E$  at state  $t$  (i.e.,  $p_i(t_i)(E) = 1$ ), he is said to *know*  $E$  at  $t$ , otherwise he does not know the event. The event that  $i$  knows  $E$  is denoted by  $K_i E = \{t \in T : p_i(t_i)(E) = 1\}$ . Thus,  $t \in K_i E$  means that  $i$  knows  $E$  at  $t$ .

In general, a player  $i$ 's theory may be an arbitrary probability distributions on  $T$ . For instance, a player  $i$  might have a theory according to which he does not know of what type he is, what action he plays and so on. In this regard, there is a substantial difference to the version of interactive belief system suggested by Aumann and Brandenburger (1995).

**Remark 1** *In the epistemic approach of Aumann and Brandenburger (1995), a player  $i$ 's*

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<sup>3</sup>For notational convenience, when referring to player  $i$ ; at “at state  $t = (t_i, t_{-i}) \in T$ ” means “at  $t_i$ ”. That is, player “ $i$ 's theory, action, and payoff-function at  $t$ ” means “ $t_i$ 's theory, action and payoff-function”, is denoted by  $p_i(t_i)$ ,  $a_i(t_i)$  and  $G_i(t_i)$ , respectively.

theory,  $p_i(t_i)$ , is defined as a probability distribution on  $T_{-i}$ , the set types of the opponent players  $-i$ . Theories are then extended to a probability distribution  $\tilde{p}(t_i)$  over the full state space  $T$  as follows. For an event  $E \subseteq T$ ,  $\tilde{p}_i(t_i)(E)$  is the probability that a type  $t_i$ 's theory assigns to the event  $\{t_i \in T_i : (t_i, t_{-i}) \in E\}$ . In our setup, theories are probability distributions on  $T$ . Hence, we do not need to extend theories to the full state space.

In other words, Aumann and Brandenburger assume that every type  $t_i$  of player  $i$  knows to be the only (possible) type and, hence, all other types of player  $i$  have a probability of zero. This implies that player  $i$  knows  $a_i(t_i)$  to be the only (possible) action at state  $t \in T$ .

In the extended version of an interactive belief system, we will allow for a set of types which hold the same theories but will differ in regard to the actions they choose. For this, we will tighten the player's theories by imposing the following two assumptions.

**Assumption 1** For every player  $i \in I$  and every type  $t_i \in T_i$ ,  $p_i(t_i)(\{t_i\} \times T_{-i}) > 0$ .

The first assumption requires that every type of a player  $i$  regards himself as possible. Denote by  $Q(t_i) := \{\tilde{t}_i \in T_i : p_i(t_{-i})(\{\tilde{t}_i\} \times T_{-i}) > 0\}$  the set player  $i$ 's types that are considered possible by type  $t_i$  at state  $t = (t_i, t_{-i})$ . By Assumption 1, we have  $t_i \in Q(t_i)$  for every  $t_i \in T_i$  and every player  $i \in I$ .

Since types of a player who choose different strategies should hold identical beliefs, the second assumption states that every type  $\tilde{t}_i$  that is regarded as possible by another type  $t_i$  has the same theory as  $t_i$  but plays a different action.

**Assumption 2** For every player  $i \in I$  and all types  $\tilde{t}_i, t_i \in T_i$ , if  $p_i(t_i)(\{\tilde{t}_i\} \times T_{-i}) > 0$  then  $p_i(\tilde{t}_i) = p_i(t_i)$  and  $a(\tilde{t}_i) \neq a_i(t_i)$ .

Assumption 2 implies that, for all types  $t_i, \tilde{t}_i \in Q(t_i)$ , we have  $p(t_i)(\{Q(t_i) \times T_{-i}\}) = 1$ . Assumptions 1 and 2 serve as ‘‘consistency’’ assumptions on players' theories and the induced knowledge.<sup>4</sup>

In the extended interactive belief system, we can define the set of actions of player  $i$  which, from the point of view of type  $t_i$ , are possible:

$$c_i(t_i) := \{a_i \in A_i : p_i(t_i)(\{\tilde{t}_i\} \times T_{-i}) > 0 \text{ and } a(\tilde{t}_i) = a_i\} \subseteq A_i. \quad (2)$$

The set  $c_i(t_i)$  is referred to as the set of type  $t_i$ 's *conceivable actions* at state  $t \in T$ . Thus, in our setup, we will distinguish between the set of  $t_i$ 's conceivable actions  $c_i(t_i)$  and  $a_i(t_i)$ , the action (actually) played by type  $t_i$  in state  $t$ .

Notice that Assumption 1 implies that  $a_i(t_i) \subseteq c_i(t_i)$  for all types  $t_i \in T_i$ . By Assumption 2, we have that the set of types that consider each other as possible (i.e., all types in  $Q(t_i)$ ) have the same set of conceivable actions.

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<sup>4</sup>These assumptions are necessary for  $T$  to be well-defined and for the knowledge operator to satisfy the standard assumptions (such as the axioms of positive and negative introspection; the so-called S5-system of knowledge (see Kripke, Halpern and Aumann)). See Appendix for more details

**Lemma 2.1** Consider a type  $t_i \in T_i$  of a player  $i$ . Then,  $c_i(t_i) = c_i(\tilde{t}_i)$  for all  $t_i, \tilde{t}_i \in Q(t_i)$ .

**Remark 2** In the setup of Aumann and Brandenburger, the extended theories  $\tilde{p}(t_i)$  on  $T$  satisfy stronger condition than Assumption 1, i.e.,  $\tilde{p}_i(t_i)(\{t_i\} \times T_{-i}) = 1$  for every  $t_i \in T_i$ . The set of types that are considered possible by type  $t_i$  is a singleton set, i.e.,  $Q(t_i) = \{t_i\}$ . In our setup, Aumann and Brandenburger's notion of extended theories corresponds to the special case where the set of type  $t_i$ 's conceivable actions is always a singleton set and it coincides with his action  $a_i(t_i) = c_i(t_i)$ .

At state  $t$ , a player  $i$  might be ignorant about his action  $a_i(t_i)$ . However, he always knows his theory  $p_i(t_i)$  and he also knows his conceivable set of action  $c_i(t_i)$ .

**Lemma 2.2** Consider a state  $t \in T$  and a player  $i$ . Let  $p_i(t_i) = p$  be the player  $i$ 's theory and  $c_i(t_i) = H$  be his conceivable set of actions at state  $t$ . Then,  $t \in K_i[p]$  and  $t \in K_i[H]$ .

It should also be remarked that under Assumptions 1 and 2, player  $i$ 's theories induce a partition of  $i$ 's type space  $T_i$ . Let  $Q_i^1, \dots, Q_i^k, \dots, Q_i^K$  be a collection of subsets of  $T_i$  such that for each  $Q_i^k$  and all types  $t_i, \tilde{t}_i \in Q_i^k$ ,  $p_i(t_i) = p_i(\tilde{t}_i)$ . Thus,  $Q_i^1, \dots, Q_i^k, \dots, Q_i^K$  is a partition of  $T_i$ . All types in the information cell  $Q_i^k$  are "indistinguishable" with respect to their theories (and thus with respect to their conceivable action sets) but are distinguishable with respect to the action each of them carries out (i.e., for all types  $t_i, \tilde{t}_i \in Q_i^k$ ,  $a_i(t_i) \neq a_i(\tilde{t}_i)$ ).<sup>5</sup> Further (epistemic) implications from Assumptions 1 and 2 are discussed in the Appendix.

The idea of an extended interactive belief system is illustrated in the example below.

**Example 1** There are two players called Alice (Player 1) and Bob (Player 2) with sets of actions  $A_1 = \{L, R\}$  and  $A_2 = \{U, D\}$ . The players face the following payoff structure:

		Player 2	
		L	R
Player 1	U	1, 2	1, 1
	D	0, 0	2, 1

There are three types of each player, i.e.,  $T_i = \{t_i^1, t_i^2, t_i^3\}$  where  $i \in \{1, 2\}$ . Suppose that each type plays the game above. That is, the game being played is commonly known. Every table below represents a type's theory, his action and his conceivable set of actions:

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<sup>5</sup>This argument can be shifted to Section 3.3. where the extended probability measure  $p_i(\cdot; t_i)$  on  $T$  is defined. The equivalent definition would be the following: Let  $Q_i^1, \dots, Q_i^k, \dots, Q_i^K$  be a partition of  $T_i$  so that for each  $t_i \in Q_i^k$ ,  $p_i(Q_i^k; t_i) = 1$ .



Player 1 (Alice):

$p_1(t_1^1)$	$t_2^1$	$t_2^2$	$t_2^3$
$t_1^1$	$\frac{1}{3}$	0	$\frac{2}{3}$
$t_1^2$	0	0	0
$t_1^3$	0	0	0

$a_1(t_1^1) = \{U\}$   
 $c_1(t_1^1) = \{U\}$

$p_1(t_1^2)$	$t_2^1$	$t_2^2$	$t_2^3$
$t_1^1$	0	0	0
$t_1^2$	$\frac{1}{6}$	0	$\frac{1}{3}$
$t_1^3$	$\frac{1}{6}$	0	$\frac{1}{3}$

$a_1(t_1^2) = \{D\}$   
 $c_1(t_1^2) = \{D, U\}$

$p_1(t_1^3)$	$t_2^1$	$t_2^2$	$t_2^3$
$t_1^1$	0	0	0
$t_1^2$	$\frac{1}{6}$	0	$\frac{1}{3}$
$t_1^3$	$\frac{1}{6}$	0	$\frac{1}{3}$

$a_1(t_1^3) = \{U\}$   
 $c_1(t_1^3) = \{D, U\}$

Player 2 (Bob):

$p_2(t_2^1)$	$t_2^1$	$t_2^2$	$t_2^3$
$t_1^1$	$\frac{1}{3}$	0	0
$t_1^2$	0	0	0
$t_1^3$	$\frac{2}{3}$	0	0

$a_2(t_2^1) = \{R\}$   
 $c_2(t_2^1) = \{R\}$

$p_2(t_2^2)$	$t_2^1$	$t_2^2$	$t_2^3$
$t_1^1$	0	$\frac{1}{6}$	$\frac{1}{6}$
$t_1^2$	0	0	0
$t_1^3$	0	$\frac{1}{3}$	$\frac{1}{3}$

$a_2(t_2^2) = \{L\}$   
 $c_2(t_2^2) = \{L, R\}$

$p_2(t_2^3)$	$t_2^1$	$t_2^2$	$t_2^3$
$t_1^1$	0	$\frac{1}{6}$	$\frac{1}{6}$
$t_1^2$	0	0	0
$t_1^3$	0	$\frac{1}{3}$	$\frac{1}{3}$

$a_2(t_2^3) = \{R\}$   
 $c_2(t_2^3) = \{L, R\}$

Consider state  $(t_1^1, t_2^1)$ . In this state, Alice plays  $U$  while Bob's chooses  $R$ . Also, Alice assigns probability 1 to her type  $t_1^1$  and Bob assigns probability 1 to his type  $t_2^1$ . Thus, the actions Alice and Bob play at state  $(t_1^1, t_2^1)$  coincide with her and his conceivable action set (i.e.,  $U$  and  $R$ , respectively).

Consider state  $(t_1^3, t_2^3)$ . Every players chooses the same action as previously. However, at  $(t_1^3, t_2^3)$ , Alice's theory assigns a positive probability to both types  $t_1^2$  and  $t_1^3$ . By Assumption 2, both types have the same theory and thus the same conceivable choice set,  $\{U, D\}$ . The same argument applies to Bob whose conceivable choice set at state  $(t_1^3, t_2^3)$  is  $\{R, L\}$ .

In other words, at state  $(t_1^3, t_2^3)$ , Alice thinks that she might be of another type,  $t_1^2$ , who actually plays  $D$  instead of  $U$ . Both types have the same theory but each of them carries out a different action. That is, "at the beginning of the day", Alice cannot distinguish whether she is of type  $t_1^2$  or  $t_1^3$  according to her theory but she can distinguish her type by the action she plays "at the end of the day". Notice that Alice's theories induce partition  $Q_1^1 = \{t_1^1\}$  and  $Q_1^2 = \{t_1^2, t_1^3\}$  and Bob's theories induce partition  $Q_2^1 = \{t_2^1\}$  and  $Q_2^2 = \{t_2^2, t_2^3\}$ .

**Notational conventions.** For any state  $t \in T$ , we write  $a(t)$  to denote the tuple  $(a_i(t_i), a_i(t_{-i}))$  of actions at  $t$ ,  $g(t)$  for the tuple  $(g_i(t_i), g_i(t_{-i}))$  of payoff-functions at  $t$ , and  $p(t)$  for the tuple  $(p_i(t_i), p_i(t_{-i}))$  of theories at  $t$ . We will refer to  $g(t)(a_i, a_{-i})$  as "the game played at  $t$ " (i.e.,  $g(t)(a_i(t_i), a_{-i}(t_{-i}))$ ).

Functions on  $T$ , such as  $a_i, a, g_i, g$ , and  $p_i$  define various sorts of events. For any function  $f$  on  $T$  and any  $y \in f(T)$ ,  $[f = y]$ , or simply  $[y]$ , denotes the event  $\{t \in T : f(t) = y\}$ . For example,  $[p_i] := \{t \in T : p_i(t) = p_i\}$  is the event that  $i$ 's theory is  $p_i$ ,  $[a_i] := \{t \in T : a_i(t) = a_i\}$  is the event that  $i$  chooses  $a_i$  and so on.

The event that  $i$  knows that he knows  $E$  is denoted by  $K_i K_i E := \{t \in T : p(t_i)(K_i E) = 1\}$ . Set  $K^1 := K_1 \cap \dots \cap K_n$ , the event that all players know event  $E$ . If  $t \in K^1 E$ , we say that  $E$  is *mutually known* at  $t$ . Set  $C K E := K^1 E \cap K^1 K^1 E \cap \dots$ , the event that all players know  $E$  and that all players know that they know  $E$  and so on at infinitum. If  $t \in C K E$ ,  $E$  is said to be *commonly known* at  $t$ .

## 2.2 Conjectures and Rationality

In this section, we derive the notion of players' conjectures and rationality in an extended interactive belief system.

A player  $i$ 's *conjectures* represent his beliefs over the other players' behavior/actions. The conjectures are encoded in a theory that player  $i$  holds at state  $t \in T$ . More formally, we will denote by  $\gamma_i(t_i)$  player  $i$ 's conjectures on  $2^{A_{-i}}$  induced by  $i$ 's theory  $p_i(t_i)$  at  $t$ .

When forming beliefs about his opponents' behavior, player  $i$  takes into account the sets of conceivable actions of his opponent players. That is, a player  $i$ 's conjectures about the behavior of players  $-i$  will be affected by his opponents' knowledge about themselves.

At state  $t = (t_i, t_{-i}) \in T$ , denote by

$$c_{-i}(t_{-i}) := \times_{j \neq i \in I} c_j(t_j) \subseteq A_{-i} \quad (3)$$

the set of conceivable actions of the opponent players where  $c_j(t_j) \subseteq A_j$  for every  $i \neq j$ . For a single strategy profile  $a_{-i} \in A_{-i}$ , a player  $i$ 's conjecture  $\gamma_i(t_i)([a_{-i}])$  at state  $t$  that his

opponents play the action profile  $a_{-i}$  is the probability that he ascribes to all the opponent types whose sets of conceivable actions are singletons, i.e.,  $c_j(t_j) = \{a_j\}$  for every  $i \neq j$ . In other words, the player  $i$ 's conjecture that his opponents play action profile  $a_{-i}$  is the probability that his theory assigns to all the opponents' types who know that they play  $a_j$ . More generally, player  $i$ 's conjectures over all subsets of  $A_{-i}$  are defined as follows.

**Definition 2.1** *Let  $t = (t_i, t_{-i}) \in T$  be a state and  $p(t_i)$  be a player  $i$ 's theory. The player  $i$ 's conjectures  $\gamma_i(t_i)$  on  $2^{A_{-i}}$  in state  $t$  are defined as follows. For any subset  $H \subseteq A_{-i}$ :*

$$\gamma_i(t_i)(H) := \begin{cases} \sum_{t_{-i} \in T_{-i}} p_i(t_i)(T_i \times \{t_{-i}\}) & , \text{ if } c_{-i}(t_{-i}) = H, \\ 0 & , \text{ otherwise.} \end{cases} \quad (4)$$

We denote by  $\gamma_i(t) := (\gamma_i(t_i), \gamma_i(t_{-i}))$  the tuple of all players' conjectures at state  $t \in T$ .

In the extended interactive belief system, a player  $i$ 's conjectures, as defined in Definition 2.1, are represented by a probability distribution over the power set of his opponents' action profiles in  $A_{-i}$ . Hence, one can view the conjectures as the Möbius transform of a belief function, i.e., a totally monotone capacity on  $2^{A_{-i}}$ .<sup>6</sup>

**Lemma 2.3** *Let  $t \in T$  be a state. A player  $i$ 's conjecture  $\gamma_i(t_i)$  at state  $t$  is the Möbius transform of the belief function  $\phi_i^\gamma(t_i)$  on  $2^{A_{-i}}$  defined as*

$$\phi_i^\gamma(t_i)(H) = \sum_{G \subseteq H} \gamma_i(t_i)(G) \quad \text{for any } H \subseteq A_{-i}. \quad (6)$$

Given the complete description of a player  $i$ 's type at state  $t$ , one can define his expected payoff from playing an action  $a_i$  given his conjectures  $\gamma_i$  in that state.

**Definition 2.2** *Given a player  $i$ 's action  $a_i(t_i) = a_i$ , conjecture  $\gamma_i(t_i) = \gamma_i$ , and payoff-function  $G(t_i)$  at state  $t = (t_i, t_{-i}) \in T$ , the player  $i$ 's expected payoff  $V_i(a_i, \gamma_i)$  from playing action  $a_i$  with respect to  $\gamma_i$  is defined as*

$$V_i(a_i, \gamma_i) := \sum_{E \subseteq A_{-i}} \gamma_i(t_i)(E) G_i(t_i)(a_i, E). \quad (7)$$

---

<sup>6</sup>A real-valued set function  $\phi : 2^{A_{-i}} \rightarrow \mathbb{R}$  is called a *capacity* on  $2^{A_{-i}}$  if it is *normalized* (i.e.,  $\phi(\emptyset) = 0$  and  $\phi(S) = 1$ ) and *monotone* (i.e.,  $\phi(F) \leq \phi(E)$  for all  $F \subseteq E$ ). A capacity  $\phi$  is called *totally monotone* if for any  $K \geq 2$  and any collection of events  $E_1, \dots, E_K \in 2^{A_{-i}}$ , it satisfies the following condition:

$$\phi\left(\bigcup_{k=1}^K E_k\right) \geq \sum_{\substack{J \subseteq \{1, \dots, K\} \\ I \neq \emptyset}} (-1)^{|I|+1} \phi\left(\bigcap_{k \in I} E_k\right), \quad (5)$$

where  $|J|$  is the cardinality of the index set  $J$  (see Dempster (1967) and Shafer (1976)).

Since a player's conjecture is the Möbius transform of his belief function  $\phi_i^\gamma$ , defined on the algebra of his opponents' action profiles, his expected payoff can be equivalently seen as the weighted mean of the Choquet integral taken with respect to  $\phi_i^\gamma$  the Choquet integral taken with respect to  $\bar{\phi}_i$ , the dual of  $\phi_i^\gamma$ . The dual capacity of  $\phi_i^\gamma$  is defined as  $\bar{\phi}_i(E) := 1 - \phi_i^\gamma(E)$  for all  $E \subseteq A_{-i}$ .

**Lemma 2.4** *Let  $t \in T$  be a state. The expected payoff function  $V_i(a_i, \gamma_i)$  is the Choquet integral with respect to a JP-capacity  $\nu^{JP}(\alpha_i, \phi_i^\gamma) := \alpha_i \phi_i^\gamma + (1 - \alpha_i) \bar{\phi}_i$  as defined in Eichberger and Kelsey (2014):*

$$\begin{aligned} V_i(a_i, \gamma_i) &= \int g_i(a_i, a_{-i}) d\nu^{JP}(\alpha_i, \phi_i^\gamma) \\ &= \alpha_i \int g_i(a_i, a_{-i}) d\phi_i^\gamma + (1 - \alpha_i) \int g_i(a_i, a_{-i}) d\bar{\phi}_i. \end{aligned} \tag{8}$$

Given a player  $i$ 's conjecture  $\gamma_i(t_i) = \gamma_i$  and his payoff function  $G(t_i)$  at state  $t \in T$ , one can define the set of his best responses. That is,

$$BR_i(\gamma_i) := \arg \max_{a_i \in A_i} V_i(a_i, \gamma_i).$$

A player  $i$  is said to be *rational* at state  $t \in T$  if, the action  $a_i(t_i)$  he chooses at  $t$  is a best response given his conjectures about the opponents' behavior.<sup>7</sup>

**Definition 2.3** *Let  $g(t)$  be a game and  $a(t)$  an action profile played at state  $t \in T$ . A player  $i$  with conjectures  $\gamma_i(t)$  is said to be rational at state  $t$  if,  $a_i(t_i) \in BR_i(\gamma_i)$ .*

Below we derive conjecture induced by the players' theories presented in Example 1.

**Example 2** *Consider state  $(t_1^1, t_2^1)$ . At this state, Alice assigns probability  $\frac{1}{3}$  to Bob being of type  $t_2^1$  and probability  $\frac{2}{3}$  to his type  $t_2^3$ . Both types play  $R$ . However, only type  $t_2^1$  knows (i.e., assigns probability 1 to) his own type and thus knows that he plays  $R$ . When Bob is of type  $t_2^3$ , he also considers possible that he is of type  $t_2^2$  who plays  $L$ . Thus, when Bob is of type  $t_2^3$ , his set of conceivable actions is  $\{R, L\}$ . Hence, Alice's theory at state  $(t_1^1, t_2^1)$  induces the following conjecture  $\gamma_1(t_1^1)$  (resp., belief function  $\phi_1(t_1^1)$ ) on  $2^{A_2}$ :*

·	{R}	{L}	{R, L}
γ <sub>1</sub> (·)	1/3	0	2/3
φ <sub>1</sub> <sup>γ</sup> (·)	1/3	0	1

---

<sup>7</sup>Alternatively, one could require a player  $i$  to be *strongly rational* at state  $t \in T$ , if every action in his conceivable action set  $c_i(t)$  constitutes a best response with respect to his conjecture  $\gamma_i(t)$ , i.e.,  $c_i(t) \subseteq BR_i(\gamma_i(t))$ .

Notice that any of Alice's types holds the same conjectures and thus she has the same conjectures at every state  $t \in T$ .

Moreover, at state  $(t_1^1, t_2^1)$ , Alice's best response given her conjecture is playing  $R$  (i.e.,  $BR(\gamma_1) = \{R\}$ ). Since Alice plays  $R$  in this state, she behaves rational (and she also behaves rational when her type is  $t_1^3$ ). However, when Alice's type is  $t_1^2$ , she plays  $L$  and thus she behaves irrational with respect to her conjectures  $\phi_1^\gamma(t_1^2)$ .

Similar for Bob, his conjectures at every state are represented by

$$\begin{array}{c|ccc} \cdot & \{U\} & \{D\} & \{U, D\} \\ \hline \gamma_2(\cdot) & \frac{1}{3} & 0 & \frac{2}{3} \\ \phi_2^\gamma(\cdot) & \frac{1}{3} & 0 & 1 \end{array}$$

### 3 Equilibrium Notions under Ambiguity

In this section, we recall notions of equilibrium under ambiguity. We focus on equilibrium notions in which

- ambiguity concerns the strategy choice of opponent players, i.e.  $a_{-i} \in A_{-i}$ , and
- equilibrium is defined in terms of beliefs represented by capacities (i.e., not necessarily additive probabilities) over  $2^{A_{-i}}$ , an algebra of pure strategy combinations.

The equilibrium concepts suggested in Dow and Werlang (1994), Marinacci (2000), Eichberger and Kelsey (2000), and Eichberger and Kelsey (2014) satisfy these two conditions. Dow and Werlang (1994), Marinacci (2000), and Eichberger and Kelsey (2000) consider the special case of *pure pessimism*, while Eichberger and Kelsey (2014) allow for *optimism* and *pessimism* as attitudes towards ambiguity. Since the treatment of ambiguity attitude in the latter article covers the other models as special cases, we will follow the general equilibrium notion presented by Eichberger and Kelsey (2014).

Suppose ambiguous beliefs about the opponents' strategy choice can be represented by a convex capacity  $\mu$  on all subsets of  $A_{-i}$ . Denote by  $\bar{\mu}$  the dual capacity which is defined by  $\bar{\mu}(E) = 1 - \mu(A_{-i} \setminus E)$  for any  $E \subseteq A_{-i}$ . For  $\alpha \in [0, 1]$ , a *Jaffray-Philippe (JP) capacity* is defined as a convex combination of the capacity  $\mu$  and its dual  $\bar{\mu}$ , i.e.,

$$\nu^{JP}(\alpha, \mu) := \alpha\mu + (1 - \alpha)\bar{\mu}. \quad (9)$$

It follows immediately from fundamental properties of the Choquet integral, that the

Choquet expected payoff of  $g_i(a_i, a_{-i})$  with respect to  $\nu^{JP}(\alpha, \mu)$  is

$$\begin{aligned} \int g_i(a_i, a_{-i}) d\nu^{JP}(\alpha, \mu)(a_{-i}) &= \alpha \int g_i(a_i, a_{-i}) d\mu(a_{-i}) + (1 - \alpha) \int g_i(a_i, a_{-i}) d\bar{\mu}(a_{-i}) \\ &= \alpha \min_{p \in \text{core}(\mu)} \int g_i(a_i, a_{-i}) dp(a_{-i}) \\ &\quad + (1 - \alpha) \max_{p \in \text{core}(\mu)} \int g_i(a_i, a_{-i}) dp(a_{-i}). \end{aligned}$$

This justifies to take  $\alpha$  as an ambiguity attitude parameter. When  $\alpha = 1$ , the JP-capacity is convex and one obtains the case of pure pessimism (or ambiguity aversion) axiomatized by Schmeidler (1989) and used in Dow and Werlang (1994), Marinacci (2000), and Eichberger and Kelsey (2000). The other extreme is when  $\alpha = 0$  and the JP-capacity is concave. This case refers to pure optimism (or ambiguity seeking) as shown by Wakker (2001).

Given beliefs  $\mu_i$  on  $A_{-i}$  and ambiguity attitude  $\alpha_i$ , the best-reply correspondence of a player  $i$  given his beliefs is defined as

$$BR_i(\mu_i) := \arg \max_{a_i \in A_i} \int g_i(a_i, a_{-i}) d\nu^{JP}(\alpha, \mu)(a_{-i}) \subseteq A_i.$$

Any equilibrium notion needs to relate equilibrium beliefs  $\mu^* = (\mu_1^*, \dots, \mu_n^*)$  to the best replies  $BR(\mu^*) = (BR_1(\mu_1^*), \dots, BR_n(\mu_n^*))$ . Similarly, to the definition of a Nash equilibrium this consistency is achieved by the requirement that the support of a capacity  $\mu_i$ ,  $\text{supp}(\mu_i)$ , contains only best replies of the opponents.

**Definition 3.1** *An  $n$ -tuple of capacities  $\mu^* = (\mu_1^*, \dots, \mu_n^*)$  constitutes an Equilibrium under Ambiguity (EUA), if for all players  $i \in I$ ,*

$$\text{supp}(\mu_i^*) \subseteq \times_{j \neq i} BR_j(\mu_j^*). \quad (10)$$

There are different notions of support for capacities. Apart from differences in ambiguity attitudes, the models by Dow and Werlang (1994), Marinacci (2000), Eichberger and Kelsey (2000), and Eichberger and Kelsey (2014) are distinct mainly by their respective support notions. For a comparison of various support notions for convex capacities in the context of game theory see Dominiak and Eichberger (2016b).

First, we focus on the support notion introduced by Marinacci (2000). The *Marinacci-support* (*M-support*) of a capacity  $\mu_i$ , denoted by  $\text{supp}_M(\mu_i)$ , is defined as the set of all action profiles for which the capacity is strictly larger than zero.

**Definition 3.2** *The M-support of  $\mu_i$  on  $A_{-i}$ ,  $\text{supp}_M(\mu_i)$ , is the set  $\{a_{-i} \in A_{-i} : \mu_i(a_{-i}) > 0\}$ .*

When defining the notion of EUA under JP-capacities, Eichberger and Kelsey (2014) suggest as definition of the support of  $\nu^{JP}(\alpha, \mu)$  to consider the support of the convex

capacity  $\mu$  defined as the intersection of the supports of all probability distributions in the core of  $\mu$ , i.e.,

$$\text{supp}(\mu) = \bigcap_{p \in \text{core}(\mu)} \text{supp}(p), \quad (11)$$

where  $\text{core}(\mu) = \{p \in \Delta(A_{-i}) \mid p(E) \geq \mu(E) \text{ for all } E \in 2^{A_{-i}}\}$ .

For the case of belief functions  $\phi_i^\gamma$ ,  $\text{supp}(\phi_i^\gamma)$  and the M-support of  $\phi_i^\gamma$  coincide with the set of all action profiles  $a_{-i} \in A_{-i}$  to which the Möbius transform assigns strictly positive values (see Proposition 2.1 and 3.3 in Dominiak and Eichberger (2016b)). That is,

$$\text{supp}(\phi_i^\gamma) = \text{supp}_M(\phi_i^\gamma) = \{a_{-i} \in A_{-i} \mid \gamma_i(\{a_i\}) > 0\}. \quad (12)$$

For convex capacities and pure pessimism, the existence of EUA has been established by Marinacci (2000) and Eichberger and Kelsey (2000). For JP-capacities defined as a convex combination of a belief function  $\phi_i^\gamma$  and its dual  $\bar{\phi}_i^\gamma$ , the existence is proven by Dominiak and Eichberger (2016a). For existence of EUA under more general JP-capacities see Eichberger and Kelsey (2014).

An equilibrium under ambiguity is illustrated below.

**Example 3** Consider again the normal-form game presented in Example 1.

When beliefs are additive, there are two pure Nash Equilibria  $(U, L)$  and  $(D, R)$ , and one Nash Equilibrium with non-degenerate beliefs, i.e.,  $\phi_1(\{U\}) = \phi_1(\{D\}) = \frac{1}{2}$  and  $\phi_2(\{L\}) = \phi_2(\{R\}) = \frac{1}{2}$ . There five types of equilibria under ambiguity:

$$\begin{aligned} (1) \ (\phi_1^*, \phi_2^*) &= \begin{cases} \phi_1^*(L) > \frac{1}{2}, \ \phi_1^*(r) = 0, \\ \phi_2^*(U) > 0, \ \phi_2^*(d) = 0, \end{cases} & \begin{aligned} \text{supp}_M(\phi_1^*) &= \{L\}, & BR_1(\phi_1^*) &= \{U\}, \\ \text{supp}_M(\phi_2^*) &= \{U\}, & BR_2(\phi_2^*) &= \{L\}, \end{aligned} \\ (2) \ (\phi_1^*, \phi_2^*) &= \begin{cases} \phi_1^*(l) = 0, \ \phi_1^*(r) > 0, \\ \phi_2^*(u) = 0, \ \phi_2^*(d) > \frac{1}{2}, \end{cases} & \begin{aligned} \text{supp}_M(\phi_1^*) &= \{R\}, & BR_1(\phi_1^*) &= \{D\}, \\ \text{supp}_M(\phi_2^*) &= \{D\}, & BR_2(\phi_2^*) &= \{R\}, \end{aligned} \\ (3) \ (\phi_1^*, \phi_2^*) &= \begin{cases} \phi_1^*(l) > 0, \ \phi_1^*(r) = \frac{1}{2}, \\ \phi_2^*(u) = \frac{1}{2}, \ \phi_2^*(d) > 0, \end{cases} & \begin{aligned} \text{supp}_M(\phi_1^*) &= \{L, R\}, & BR_1(\phi_1^*) &= \{U, D\}, \\ \text{supp}_M(\phi_2^*) &= \{U, D\}, & BR_2(\phi_2^*) &= \{L, R\}, \end{aligned} \\ (4) \ (\phi_1^*, \phi_2^*) &= \begin{cases} \phi_1^*(l) = 0, \ \phi_1^*(r) \in [0, \frac{1}{2}), \\ \phi_2^*(u) \in [0, \frac{1}{2}), \ \phi_2^*(d) = 0, \end{cases} & \begin{aligned} \text{supp}(\phi_1^*) &= \{R\}, & BR_1(\phi_1^*) &= \{U\}, \\ \text{supp}(\phi_2^*) &= \{U\}, & BR_2(\phi_2^*) &= \{R\}. \end{aligned} \\ (5) \ (\phi_1^*, \phi_2^*) &= \begin{cases} \phi_1^*(l) = 0, \ \phi_1^*(r) = \frac{1}{2}, \\ \phi_2^*(u) = \frac{1}{2}, \ \phi_2^*(d) = 0, \end{cases} & \begin{aligned} \text{supp}_M(\phi_1^*) &= \{R\}, & BR_1(\phi_1^*) &= \{U, D\}, \\ \text{supp}_M(\phi_2^*) &= \{U\}, & BR_2(\phi_2^*) &= \{R, L\}. \end{aligned} \end{aligned}$$

The behavior supported by EUA (1) and (2) is the same as the pure Nash Equilibrium. The EUA (3) coincides with the Nash Equilibrium behavior where players uniformly randomize

between their pure strategies. In the mixed-strategy Nash-equilibrium, as well as in (3), both players are indifferent between any pure strategy in the support of the equilibrium capacities. The behavior captured by EUA (4) and (5) cannot be accommodated by the Nash Equilibrium. In these equilibria, players do not believe that coordination is possible. In EUA (4), Player 1 chooses  $U$  and Player 2 chooses  $R$  yielding them a certain payoff of 1. This strategy insures each player against the ambiguity about his/her opponent behavior. In equilibrium (5), the only difference is that both players are indifferent between playing the two pure strategies, while in the equilibrium 4 playing  $U$  and  $R$  will be always preferred.

## 4 Equilibrium under Ambiguity with Two Players.

In this section, we will derive epistemic conditions for various notions of equilibrium under ambiguity. We first focus on strategic games with two players.

As it will be shown below, an important aspect that disentangles the various equilibrium notions is how players reason about their opponents. A player  $i$  is said to be *determined* in state  $t \in T$  if his conceivable choice set in  $t$  is a singleton set, i.e.,  $c_i(t_i) = \{a_i\}$  for some  $a_i \in A_i$ , otherwise the player is *undetermined*. Notice that, by Lemma 2.2, a player  $i$  who is determined in state  $t$  knows  $a_i(t_i)$ , the action he plays in state  $t$ .

**Theorem 4.1** *Let  $g$  be a game and  $\gamma = (\gamma_i, \gamma_j)$  be a pair of conjectures. Suppose that at some state  $t \in T$ , every player considers possible that his opponent is determined, and it is mutually known that  $g$  is played, that the players' conjectures are  $\gamma$ , and that the players are rational. Then,  $\gamma = (\gamma_i, \gamma_j)$  constitutes an Equilibrium under Ambiguity for game  $g$ .*

For a pair of conjectures to constitute an EUA, every player has to consider his opponent player as being determined. Yet, it is not required that player knows that his opponent is determined. That is, the player  $i$  might assign a strictly positive probability to the event that his opponent  $j$  is undetermined. This implies that his beliefs over  $j$ 's actions are represented by a belief function  $\phi_i^\gamma$  in  $2^{A_j}$  with a non-empty M-support. If, however, every player knows that his opponent is determined, then his beliefs are represented by a standard additive probability measure. In the wake of this observation, the epistemic conditions for the Nash Equilibrium derived by Aumann and Brandenburger (1995, Theorem A), can be re-formulated as a corollary of Theorem 4.1. In the extended interactive beliefs system, the sufficient condition for the Nash Equilibrium is the fact that players mutually known that all players are determined, i.e., the players' conceivably choice sets are singleton sets.

**Corollary 4.2** *Let  $g$  be a game and  $\gamma = (\gamma_i, \gamma_j)$  be a pair of players' conjectures. Suppose at some state  $t \in T$ , it is mutually known that the players are determined, that  $g$  is played, that  $\gamma$  are the players' conjectures, and that the players are rational. Then,  $\gamma = (\gamma_i, \gamma_j)$  constitutes a Nash Equilibrium for game  $g$ .*



In the case of pure pessimism (i.e.,  $\alpha = 1$ ), Dow and Werlang (1994) suggested an alternative notion of equilibrium behavior under a different notion of support. A *Dow-Werlang-support* (*DW-support*) of a capacity  $\mu_i$ , denoted by  $\text{supp}_{DW}(\mu_i)$ , is the smallest set of action profiles whose complement has the capacity value zero. Formally, the support is defined as follows.

**Definition 4.1** *A DW-support of  $\mu_i$  on  $A_{-1}$ ,  $\text{supp}_{DW}(\mu_i)$ , is a set  $E \subseteq A_{-1}$  such that  $v_i(A_{-1} \setminus E) = 0$  and  $v_i(A_{-1} \setminus F) > 0$  for any  $F \subset E$ .*

It is important to remark that a DW-support of a capacity  $\mu_i$  always exists while the M-support might be an empty set. Moreover, both support notions are “nested” in the sense that the M-support, provided it exists, is always a DW-support.<sup>8</sup> For this reason, we will consider conjectures that constitute an equilibrium under ambiguity for which the M-support is an empty set. That is, we are interested in equilibrium beliefs under ambiguity for which the DW-support is not the M-support. We refer to such an equilibrium as Dow-Werlang Equilibrium (DWE).

For a pair conjecture to constitute a DWE, one has to assume that it is mutually known that all players are undetermined. Formally, the following epistemic conditions lead to DWE.

**Proposition 4.3** *Let  $g$  be a game and  $\gamma = (\gamma_i, \gamma_j)$  a pair of conjectures. Suppose that at some state  $t \in T$ , it is mutually known that the players are undetermined, that  $g$  is played, that the players’ conjectures are  $\gamma$ , and that the players are rational. Then,  $\gamma = (\gamma_i, \gamma_j)$  constitutes a Dow-Werlang Equilibrium for game  $g$ .*

The notion of DWE includes as special type of equilibrium under ambiguity where the players’ belief functions take the form of “complete” ignorance capacities. That is, if  $v_i^\gamma(E) = 0$  for all  $E \subset A_i$  and  $v_i^\gamma(E) = 1$  for  $E = A_i$ . If  $v_i^\gamma$  on  $2^{A_{-i}}$  is a complete ignorance capacity, then every singleton set  $\{a_{-i}\}$ , where  $a_{-i} \in A_{-i}$ , is a DW-support.

However, the epistemic condition provided in Theorem 4.3, do not encompass the complete ignorance case. Or, putting it differently, the conditions are too strong; the mutual knowledge of rationality rules out the complete ignorance capacities. The reason is that if a player has a theory inducing an incomplete ignorance capacity, then he might not know that he is rational (unless he is indifferent between all actions).

An epistemic justification for DWE under incomplete ignorance requires a separate treatment. We refer to a DWE under complete ignorance capacities as a Complete Ignorance Equilibrium (for short, CIE).

One epistemic condition from Theorem 4.3 has to be weakened. Namely, it is assumed that player  $i$  knows that his opponent player  $j$  is rational play, but this rationality is not mutually known. That is to say, player  $i$  does not know that his is actually rational (unless he is indifferent between all his actions, i.e.  $BR_i(v_i) = A_i$ ).

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<sup>8</sup>See Lemma 1 in Eichberger and Kelsey (2014).

**Proposition 4.4** *Let  $g$  be a game and  $\gamma = (\gamma_i, \gamma_j)$  be a pair of conjectures. Suppose that at some state  $t \in T$ , every player knows that his opponent is rational, and it is mutually known that the players are “completely” undetermined, that the conjectures are  $\gamma$ . Then,  $\gamma = (\gamma_i, \gamma_j)$  constitutes a Complete Ignorance Equilibrium for game  $g$ .*

## 5 Equilibria under Ambiguity with $N$ Players.

In this section, we extend our analysis to strategic games with more than two players.

For strategic game whose number of players exceeds two, epistemic foundation of equilibria under ambiguity requires a special treatment. In particular, one has to assume that players theories (and thus their conjectures) are derived from a common prior over the entire state space.

A probability distribution  $p$  on  $T$  is called a *common prior* if for each player  $i \in I$  and all their types  $t_i \in T_i$ , the conditional distribution of  $p$  given  $t_i$  is  $p_i(t_i)$ , i.e.,  $i$ 's theory. In words, players are said to have a common prior on  $T$  if all differences between their probability assessments are due only to differences in their information.

In our setup, the common priors assumption have two important implications. First, it guarantees players' behavior is stochastically independent in the sense of Möbiou transforms. Second, the common prior assumption warrants that player  $j$ 's conjecture about behavior about player  $i$  agree with player  $k$ 's conjectures about  $i$ 's behavior.

Under the common prior assumption, the following conditions lead to an EUA in strategic games with more than two players.

**Theorem 5.1** *Let  $g$  be a game and  $\gamma = (\gamma_i, \dots, \gamma_N)$  be an  $N$ -tuple of conjectures. Suppose that players have a common prior  $p$  on  $T$ . Suppose that at some state  $t \in T$ , at which each player  $i$  considers possible that all his opponents  $-i$  are determined, at which it is mutually known that  $g$  is played and that all players are rational, and at which that it is commonly known that the players' conjectures are  $\gamma$ . Then, for each player  $i$ , all his opponents  $-i$  agree on the same conjecture about  $i$  and the  $N$ -tuple of conjectures  $\gamma = (\gamma_i, \dots, \gamma_N)$  constitutes an Equilibrium under Ambiguity for game  $g$ .*

At states of the world at which it is mutually known that all players are determined, an EUA reduces to the standard Nash-Equilibrium for games with more than two players.

**Corollary 5.2** *Let  $g$  be a game and  $\gamma = (\gamma_i, \dots, \gamma_N)$  be an  $N$ -tuple of conjectures. Suppose that players have a common prior  $p$  on  $T$ . Suppose that at some state  $t \in T$ , at which it is mutually known that all players are determined, that  $g$  is played, that all players are rational, and at which that it is commonly known that the players' conjectures are  $\gamma$ . Then, for each player  $i$ , all his opponents  $-i$  agree on the same conjecture about  $i$  and the  $N$ -tuple of conjectures  $\gamma = (\gamma_i, \dots, \gamma_N)$  constitutes a Nash-Equilibrium for game  $g$ .*

## 6 Related Literature

## 7 Conclusion

## A Proofs

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