Fair Competition Design*

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Abstract

We study the impact of two basic principles of fairness on the structure of different sport competition systems. The first principle requires that if all players are equally strong then each player should have the same probability of being the final winner, while the second one says that a better player should not have a lower probability of being the final winner than a weaker player. We apply these principles to a class of competition systems which includes, but is not limited to, the sport tournament systems mostly used in practice such as round-robin tournaments and different kinds of knockout competitions, and completely characterize the competition structures satisfying them.

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In these characterizations a new competition structure that we call an *antler* turns out to play a referential role and allows us to single out balanced competitions and extended stepladder tournaments as having the most conspicuous structure from a theoretical point of view. We finally show that the class of fair competition systems becomes rather small when both fairness principles are jointly applied.

JEL Classification: D00, D02, D60, D63, D70, Z20
Keywords: equal treatment, monotonicity, seeding, sport competition

1 Introduction

Numerous decision problems require the selection of an alternative from a set of options, the selection process being guided by the information obtained from pairwise comparisons among the available alternatives. Examples of such problems can be found in voting theory (see, for example, Brams and Fishburn 2002, Laslier 1997, Levin and Nalebuff 1995, Moulin 1986), multi-criteria decision making (Larichev 2001, Olson 1996), and promotion mechanisms implemented in firms (Rosen 1986). However, the most popular problem of this type is probably the one consisting of selecting a winner in a sport competition, where the alternatives are the competing "players" and the pairwise comparisons take the form of "matches".

In this work we present an axiomatic approach to the fairness aspect of such kind of selection problems. Though the analysis is potentially applicable to different contexts, we set our study in the framework of sport competitions. Our interest in this field, however, goes beyond its use as a mere theoretical parable. Nowadays the sport industry has an unquestionable economic and social relevance. Single events such as the Olympic Games or the Football World Cup generate world-wide audiences of hundreds millions of people. Some national teams and leagues manage huge budgets, and sports constitute a multi-millionaire industry that concerns media, betting, advertising, sponsoring and other indirect activities related with transportation, security, merchandising, etc. Even at an amateur level, sports occupies a far from negligible component of the time, expenditures and welfare of people who practice sports, consume sports goods, enroll their children in teams, etc.

Every sport competition needs a well-defined and pre-established set of basic rules that determines the "competition system": who plays against whom, at which stage of the competition, and how it is decided who is the final winner. A competition organizer can plausibly consider different objectives when choosing, or designing the competition system: the intensity of the matches, attracting the interest of the spectators, optimizing organizational costs, etc. However, fairness is of top priority within the goals of any competition designer.

What do we exactly mean by a “fair competition design” is the main question to be answered in this work. Discussions about whether one or another system is more or less fair are often made at an intuitive and informal level. In our work we provide a structured analysis to such discussions and present a rigorous concept of “fair competition design”. In doing so, we formally define two neat principles of fairness and analyze to what extent different competition systems perform in relation to them, trying to give formal support to such informal debates. The systems we consider can be roughly partitioned into two major classes - elimination-type competitions and league-type competitions.

Elimination-type competitions

Competitions of this type are also called “knockout tournaments”, “playoff tournaments”, or “cup tournaments”. The competition is organized in rounds or “stages”. Losers are eliminated and players progress as they win their corresponding matches in the round, being paired off in the next round, so that the final winner is the player who wins all the rounds. They can be represented by binary trees as exemplified in Figure 1, where eight players participate in an elimination-type competition.

There are many well-known examples of sport competitions that use this system, such as the ATP and WTA World Tour, the NBA Finals, the FIFA World cup, many Olympic competitions and the Football Cup tournaments played across Europe.

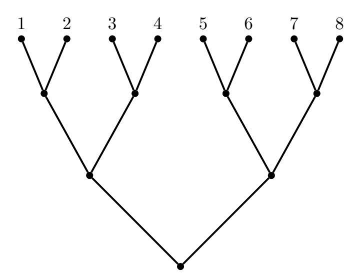


Figure 1: A balanced elimination-type competition

Elimination-type competitions as the one displayed in Figure 1 are called *balanced* since every payer is required to win the same number of matches in order to become the final winner. Variants of this structure are, for example, *double-elimination* tournaments, where two defeats are necessary for a player to be eliminated (as in the NCAA baseball tournament and many wrestling competitions), or *McIn-*

tyre systems, rather prevalent in Australian competitions, where players with some ex-ante qualification are allowed to lose a match without being eliminated. In other cases, like in the American National Football League, players have the right of "byes", meaning that, on the basis of a previous qualification rating, they have the privilege to skip the initial round (or rounds) without the need of playing. In fact, "byes" becomes necessary if the number of players is not a power of 2.

A special type of elimination competition with byes has the so-called "stepladder" structure (see Figure 2), where players are ranked according to some prior qualification criterion and byes are assigned on the basis of such ranking, thus, the better ranked the player is the later the round he [1] enters in the competition. This system and variants of it are used in ten-pin bowling, squash and Basque pelota, for example.

Any elimination-type competition requires to solve a problem of "seeding", that is, assigning players' names to the "leaves" of the competition's tree. In other words, deciding the pairing in the initial matches and, if that is the case, which player(s) deserve(s) the byes. Clearly, the seeding will have a crucial impact on the development of the tournament and on the chances for a player to become the final winner.

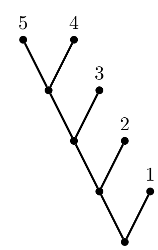


Figure 2: A stepladder competition

League-type competitions

In a league-type competition every player plays a given number of matches against other players. A certain number of points are assigned to the winner of each match, and the final winner is the player with the highest score of points. The most widely used league-type competitions are "single round-robin tournaments" where each player plays against every other player once and "double round-robin tournaments" when each player plays twice against every other player. For example, most European national football and basketball leagues are double round-robin tournaments. Also, numerous Olympic team competitions have a qualification stage consisting of a single round-robin tournament. "Swiss-system" competitions are leagues where players do not play against every other player. It is widely used in chess, curling, cricket competitions as well as in amateur or inferior

[1]The gender of the player has been chosen by tossing a coin.

categories in order to save costs and simplify the logistics that a round-robin tournament involves.

While the organization of an elimination-type competition faces the problem of seeding, the organization of round-robin competitions has to solve the problem of "scheduling" the rounds, that is, deciding the sequence of the matches throughout the rounds, which for some authors might affect the final result of the competition (see Briskorn and Knust 2010 and Krumer et al. 2017).

The two basic competition systems described above can be combined in different ways (see Scarf et al. 2009). The best known formula takes the form of a multi-stage tournament where, in a first stage players are partitioned into groups and each group plays a round-robin tournament. The results of each of those leagues determine the players that qualify for the second stage, where participants play an elimination-type tournament. This is applied, among many others, in soccer's UEFA Champions League, FIFA World cup, Eurobasket, and in numerous competitions in the Olympic Games.

Each type of competition system has its pros and cons, related for instance with the number of matches needed as to have a final winner, organizational costs, profitability for the organizer, the possibility for inconsequential matches to be played, or the manipulability by the players. All those issues are of big interest, but the attention of this work is exclusively devoted to the analysis and comparison of the main competition systems from the point of view of the fairness intrinsically associated to their structure.

Literature overview

The literature about fairness in competition systems is rather disseminated. Usually, related works consist of the study of particular competition systems and fairness aspects related with their specificities. Most of the attention in this respect has been paid to study how alternative seeding procedures in an elimination-type competition perform according to different properties, which in general are related with the idea of favoring the stronger players. For example, Horen and Riezman (1985) analyses knockout tournaments with 4 players, Hwang (1982) considers the 8-players case, Prince et al. (2013) analyses the 8-players and 16-players cases, and Schwenk (2000) looks at the general case under a special form of random seeding.

In the case of round-robin tournaments Briskorn and Knust (2010) study properties with fairness connotations in relation of the schedule of the rounds, Moon and Pullman (1970) study "equalizing" handicapping methods, Rubinstein (1980) shows axiomatically that the point system used in round-robin competitions is the only one that satisfies three axioms inspired from social choice theory, and Levin and Nalebuff (1995) makes an analogy between round-robin punctuation systems and voting systems.

As for other competition systems, Csató (2013) proposes a method to order players in a Swiss-system competition and Ely et al. (2015) study the performance of a stepladder competition with three players in terms of suspense and surprise.

Fairness in sports has also been analyzed with respect to tie-breaking mechanisms (see, for example, Apesteguía and Palacios-Huerta 2010 and Che and Hendershott 2008).

There is a line of research concerned with manipulability by players, whose exertion of effort is introduced as a strategic variable (cf. Rosen 1986, Groh et al. 2012, Krumer et al. 2017, and Pauly 2014). The analysis has been always made for a very reduced number of players (usually 4), since it is generally accepted that the extension to more players involves an excessive complexity due to the highly complicated combinatorial structure of the problem.

As for the comparison among different competition systems, studies usually consist of applying statistical simulation technics in order to check the fulfillment of particular properties, or how the competition systems perform according to particular metrics (cf. Appleton 1995, McGarry and Schutz 1997, Searls 1963, Glenn 1960, Scarf et al. 2009, Ryvkin and Ortmann 2008 and Ryvkin 2010).

Finally, there is a considerable body of literature in Operations Research related with sports. The OR analysis of sports includes many aspects, apart of fairness, that are out of the scope of this paper. The interested reader is referred to Wright (2014) and Kendall et al. (2010) for surveys in the field.

Basic principles of fairness and our contribution

Our aim is to provide a comprehensive axiomatic analysis of the main competition systems which will allow us to make comparative judgements among them. The following two basic fairness principles shape the core of our analysis. First, avoiding to give arbitrary advantages to players, that is, those that are not justified by the performance of the players in the competition or by their merits. Second, avoiding that players who are *a priori* "better" are punished due to bad luck or other spurious reasons. In fact, very often the structure of competitions is precisely adjusted with this aim. Examples are matches that consist of multiple legs (like in many basket competitions), the use of byes in order to benefit better players, or particular seeding rules in knockout competitions that pair players so that the better the player is the worse his opponent is.

We formalize these two basic ideas of fairness by means of two simple axioms that have a "rank-preserving" flavor: Not giving arbitrary advantages can be viewed as an "equal treatment" requirement and can be stated as "if two players are equally strong, the structure of the competition system should give to them the same probability to become the final winner". Second, not favoring worse players can be viewed as a "monotonicity in strength" requirement that can be stated as "if a player j is weaker than i, the competition system should not give to j a higher probability to be the final winner than the one given to i". In fact, these two ideas are the translation to our context of the Aristotelian Justice Principle of "treating equals equally and unequals unequally".

Each of these two requirements is presented in a strong form and in a weak form. The corresponding strong versions impose that a competition system should fulfill the property for every possible assignment (or seeding) of the players in the system, while the corresponding weak forms just require the fulfillment of the properties for at least one assignment of the players.

Our results include characterizations of the competition systems satisfying the different fairness properties. In the case of the weak versions of the axioms, the characterization results not only determine the structures that satisfy them, but also specify the class of seeding rules that let the structure satisfy the axioms. Generally speaking, *equal treatment* leads to balanced competitions in which every player participates in the same number of matches (Theorem 1, Theorem 4, and Theorem 5), while *monotonicity in strength* drastically restricts the number of players in the competition to two (Theorem 2 and Theorem 6). When *weak monotonicity in strength* is under consideration, then the class of competition systems fulfilling it increases. In the particular case of elimination-type competitions it turns out that they are weakly monotonic in strength only if the tree structure that represents the competition does not contain a special substructure that we name as "antler" (Theorem 3). This structure combines the characteristics of balanced elimination competitions with the "byes" spirit of stepladders. Moreover, it is shown that the seeding rule for which an antler-free competition satisfies weak monotonicity in strength is unique. Finally, we note that the class of competition systems becomes rather small when both fairness principles are jointly applied.

It is worth mentioning that the variety of structures we consider includes the usual competition systems, such as round-robin, stepladder, Swiss-system or balanced knock-out competitions. Additionally, more sophisticated structures, that we call "hybrid structures" (see for example Figures 4 and 8 in the next sections), or league-type competitions where players do not necessarily play the same number of matches are considered as well. Some of these types of competitions have been applied in sports and others not. For this reason, beyond its theoretical interest, this work is likewise of potential interest for practitioners in the sports industry, including competitions design, betting, etc.

The rest of the paper is organized as follows. Section 2 presents the basic elements of the formal model, in particular the representation of competitions by means of binary trees, the winning probability matrix as a tool to capture the strength of the players, the players' assignment issue and some preliminary lemmas. Section 3 discusses four axioms that express the two aforementioned fairness principles. Sections 4 and 5 are devoted to the characterization results with respect to elimination-type and league-type competition systems, respectively. Section 6 contains the concluding remarks and addresses posible extensions.

2 The model

The main ingredients of our model are the graph representation of a competition system, the description of players' strength in terms of winning probability matrices, the notion of a seeding rule and the probability for each player to be the final winner as a consequence of all those elements.

Graph representation of competition systems

We assume that matches always take place between two players in such a way that ties are not possible[2], and represent a competition system by means of a graph.[3] In the case of elimination-type competitions each match is rep-

[2]Sometimes a match, or "fixture", can consist of multiple "legs". For example, in the UEFA Champions League of soccer ties are possible in any of the two legs of the match but not in the whole match (fixture) because, as in many other sports, there are certain pre-established tie-breaking mechanisms, like extra time or penalty shooting.

[3]What we call competition in this paper is usually called "tournament" in colloquial language and in sports literature. We prefer to avoid such term because in graph theory and in social choice theory tournaments are rather different mathematical objects (see, for

resented by an *elementary binary tree*, that is, a graph with three nodes $\{a, b, w\}$ and two links $\{aw, bw\}$ with, let us say, player i being assigned to node a, player j being assigned to node b, and the winner of the match between i and j being assigned to node w. In this case we say that i *is matched with* j.

Elimination-type competitions can be then represented by a graph connecting in a specific way such elementary binary trees, forming a binary tree with a finite number of nodes like those in the examples of the previous section.

In order to perform an adequate comparative study we represent league-type competitions consistently with the above described representation of elimination-type competitions by means of binary trees. In particular, league-type competitions will be represented by a collection (*forest*) of disconnected elementary binary trees, each of them representing a match of the league (see Figure 3).[4]

Given a graph G of any of the two types described above, we denote by $t \in G$ a particular binary tree of the graph, and by $\#G$ the number of trees that G contains. The set of nodes (or *vertices*) of a binary tree $t \in G$ is denoted by $V(t)$. The set of leaves (or terminal nodes) of $t \in G$ is denoted by $\Lambda(t)$ (note that it is a strict subset of $V(t)$). The set of leaves of a graph G is denoted by $\Lambda(G)$ (obviously $\Lambda(t) = \Lambda(G)$ when $G = \{t\}$ holds).

The distance between two nodes of $t \in G$ is defined by the minimal number of edges necessary to connect them. The *level* $\ell(v(t))$ of a node $v(t) \in V(t)$ is defined by the distance between it and the root of the binary tree t. The k-th level of a tree t is the set of all nodes of the tree of level k.[5] The *height* $h(t)$ of a binary tree t is the maximal level of its leaves, that is $h(t) = max_{\lambda \in \Lambda(t)}\{\ell(\lambda)\}$. By $\Lambda^k(t)$ we denote the set of leaves of t whose level is k.

We say that a binary tree $t \in G$ is *balanced* (or that it *represents a balanced competition*) if the level of all its leaves is the same. For example,

example, Harary and Moser 1966, Laslier 1997 or Moulin 1996).

[4]The figure represents a single round-robin competition among four players. However, as we will see later, our definition of a league-type competition includes the possibility for players not to play the same number of matches.

[5]In sport competitions the different levels of a tree take the interpretation of *rounds*, but they are usually numbered in inverse terms, that is, the last level of the tree constitutes the first round of the competition, the second last level constitutes the second round and so on.

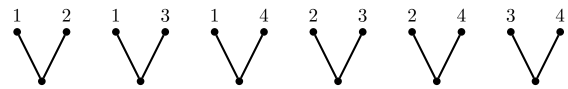


Figure 3: A league-type competition

simple elimination competitions like those of Figure 1 are represented by a balanced binary tree, or any match of a league-type competition is also represented by an elementary balanced binary tree (see Figure 3). But the representations of elimination competitions with byes (cf. Figure 2) are not balanced.

On the other hand, a *stepladder* competition can be represented by a binary tree t that has two leaves at level $h(t)$ and a unique leaf at each level ℓ for all $\ell < h(t)$ (Figure 2).

When a binary tree is neither balanced nor a stepladder, we will say that it represents a *hybrid* competition system (see, for example, Figure 4).

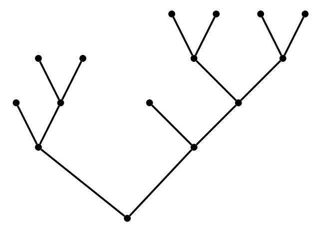


Figure 4: A hybrid elimination-type competition

Players' strength and winning probabilities

Let N be the finite set of competing players. We assume that the elements of N are completely ordered according to a binary relation R of *strengh* so that, for all $i, j \in N$, iRj is interpreted as *"player i is at least as strong as player j"*. The corresponding asymmetric and symmetric factors of R are denoted, respectively, by P and I, so that iPj is means that *"player i is strictly stronger than j* and iIj as *"players i and j are equally strong"*.

We attach a probabilistic meaning to the binary relation of strength in the sense that iRj presupposes that "the probability with which player i defeats in a match player j is greater or equal than 0.5". We denote this by $p_{ij} \geq 0.5$. Given that R is complete, we have that iPj is accordingly interpreted as $p_{ij} > 0.5$ and iIj is interpreted as $p_{ij} = 0.5$. Throughout the next sections we take the non-deterministic view that $0 < p_{ij} < 1$ for all $i, j \in N$. This assumption is taken just to show that the presence of deterministic values is not what makes the different theorems and lemmas hold in a trivial way. However it is easy to check that those results also hold for the case $p_{ij} \in [0,1]$. For notational convenience, we adopt the convention that players are ordered in N according to R, that is, if iPj then $i < j$ (if iIj then either $i < j$ or $j < i$).

According to this interpretation, every binary relation of strength R induces a set of *winning probability matrices*, p_R, defined on $N \times N$ that *support* (or are compatible with) R. More precisely, p_R is the set of all probability matrices such that, for $p \in p_R$, we have that $p_{ij} \geq 0.5$ if and only if iRj. We further assume that R is known but the particular values of the supporting probability matrix $p \in p_R$ is not necessarily known. Following the related models (for example, David (1963), Hwang (1982) Horen and Riezmann (1985) and Scwenck (2000)) we also assume that, given R, every probability matrix $p \in p_R$ satisfies the following two conditions:

$$\forall i, j \in N, p_{ij} + p_{ji} = 1. \tag{1}$$

$$p_{ij} \geq 0.5 \text{ implies } p_{ik} \geq p_{jk} \text{ for each } k \in N \setminus \{i, j\}. \tag{2}$$

The interpretation of (1) is straightforward. Condition (2) just expresses the fact that any player defeats with higher probability a weaker player than a stronger player. It also implies that if two players are equally strong ($p_{ij} = 0.5$) then they should defeat with equal probability any third player. Conditions (1) and (2) are equivalent to what is sometimes referred as "strong stochastic transitivity" of the representing probability matrix (cf. David 1963). Assuming that players are ordered according to their strength, strongly stochastically transitive matrices are nondecreasing in rows, nonincreasing in columns, and whenever $p_{ij} = 0.5$ the corresponding rows and columns of i and j are equal.

The next lemma shows that if $p \in p_R$ satisfies (1) and (2) then transitivity of R is guaranteed.

Lemma 1 *Let R be a complete binary relation of strength defined on N and let $p \in p_R$. If p satisfies (2), then R is transitive.*

Proof. Suppose that p satisfies (2) and take $i, j, k \in N$ such that iRj and jRk. Then, given that $p \in p_R$, $p_{ij} \geq 0.5$ and $p_{jk} \geq 0.5$. If follows then by condition (2) that $p_{ik} \geq p_{jk}$ should hold. Given that $p_{jk} \geq 0.5$, we have $p_{ik} \geq 0.5$, and since $p \in p_R$, iRk follows as required to show that R is transitive. ■

Moreover, the following lemma will turn out to be useful for later proofs.

Lemma 2 *Let $a, b, c, d \in N$ be such that $aRbRcRd$. Then, for all $p \in p_R$ such that p satisfies (1) and (2), we have $p_{ad} \geq p_{bc}$.*

Proof. Let $p \in p_R$. Then aRb implies $p_{ab} \geq 0.5$, and given that p satisfies (2), $p_{ac} \geq p_{bc}$. Similarly, cRd implies $p_{ca} \geq p_{da}$ and thus, by condition (1), $p_{ad} \geq p_{ac}$ follows. Finally, by $p_{ac} \geq p_{bc}$, we conclude that $p_{ad} \geq p_{bc}$ holds. ■

Seeding and the probability of being the final winner

Given a finite set N of competing players and a graph G of the type discussed above, a *seeding rule* $s : \Lambda(G) \to N$ assigns players of N to the leaves of G. When $s(\lambda) = i$ holds for $\lambda \in \Lambda(G)$ and $i \in N$, we say that *"player i is assigned, or "seeded", to leaf λ"*. We assume that any such rule satisfies the following two properties: (1) s is a surjective function, that is, every player is seeded to at least one leaf in G, and (2) the restriction of s to any binary tree $t \in G$ is an injective function, that is, no player from N is seeded to more than one leaf of t, but there could be players that are not seeded to a leaf of t when $\#G > 1$.

When a seeding rule s satisfies these conditions we will say that s is a *feasible* seeding for G. As a consequence of the two assumptions above, we have that if s is feasible for $G = \{t\}$, then each player in N is seeded to exactly one leaf of G and $|\Lambda(G)| = |N|$. The two conditions also imply that no player plays against himself and that, once the seeding is made, no reseeding is needed for the competition to "run". On the other hand, these conditions imply $2 \leq |N| \leq |\Lambda(G)|$ whenever $\#G > 1$ holds.[6]

At this point we can define a *competition system* as a pair (G, N) consisting of the graph that represents the structure of the matches of the competition and the set N of players to be seeded to the leaves of the graph. Given the conditions imposed on s not any pair (G, N) will be admissible. As already pointed out, if G contains a single binary tree, then the number of players and the number of leaves of G are the same and, if G is a collection of elementary binary trees, then the number of players should be at least two and at most the number of leaves of the forest. Thus, we will say that a competition system (G, N) is *admissible* when it is possible to define a feasible seeding rule for it. The set of all feasible seeding rules for a competition system (G, N) will be denoted by $\mathcal{S}^{(G,N)}$.

[6]This leaves out, for example, double elimination tournaments, where losers in previous matches could be reseeded in later rounds, or two-stage competitions where in the first stage several parallel leagues are played, and in the second one players are again seeded in an elimination competition depending on their performance in the league-phase.

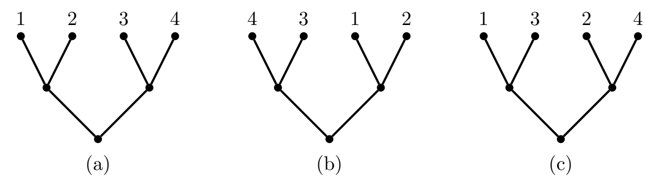


Figure 5: The seedings in (a) and (b) are equivalent seedings, while those in (a) and (c), and in (b) and (c) are not

Let (G, N) be a competition system and $s \in \mathcal{S}^{(G,N)}$. Notice then that s determines the set of potential matches that can be played at each round. Moreover, if a probability matrix p is given, the set of potential matches at each round is endowed with a probability distribution. Then, given (G, N) and two feasible seeding rules s and s', we say that these two seeding rules are *equivalent* with respect to p if the probability distribution associated to the set of potential matches at each round for s and for s' are equal. For instance, Figure 5 represents, for a balanced binary tree of height 2, a situation where the two left seedings are equivalent, but none of these two seeings is equivalent to the right one.[7]

Given a competition system (G, N), we denote by w_t the player who reaches the root of the binary tree $t \in G$ and we say that $i \in N$ is the *winner of the competition* (G, N) if $|\{t \in G : w_t = i\}| \geq |\{t \in G : w_t = j\}|$ holds for all $j \in N$. Obviously, if G consists of a single binary tree, then the player that reaches the root of the tree is the winner of the competition.

Given a competition system (G, N), a seeding rule $s \in \mathcal{S}^{(G,N)}$, and a probability matrix p, the *winning function* $\varphi : N \to [0, 1]$ attaches to each player $i \in N$ the probability $\varphi_i(G, s, p)$ with which he will be the final winner of the competition. The probability with which a player reaches the root of a particular tree $t \in G$ is analogously denoted by $\varphi_i(t, s, p)$. Obviously, $G = \{t\}$ implies $\varphi_i(G, s, p) = \varphi_i(t, s, p)$ for each $i \in N$. Finally, if s and s' are equivalent seedings, then $\varphi_i(G, s, p) = \varphi_i(G, s', p)$ holds for each $i \in N$.

[7]If t is balanced and $h(t) = 1$, each seeding rule s admits a unique set of equivalent seedings with two seedings: (1,2) and (2,1). If $h(t) = 2$, there are 3 possible non-equivalent seedings, each of which having 8 equivalent seedings. Prince et al. (2013) prove that the number of non-equivalent seedings when there are n players is $\frac{n!}{2^{n-1}}$. For example, if $h(t) = 3$ there are 315 possible non-equivalent seedings, each of which with 128 equivalent seedings. A stepladder admits as equivalent only the two seedings where the two players at the highest level are permuted.

3 Fairness axioms

We propose the following properties, which express the two principles of fairness already advanced in the Introduction. Assuming that a binary relation R of strength is defined on the player set, we state each of these properties in a strong and in a weak version.

Equal Treatment (ET) A competition system (G, N) satisfies Equal Treatment if for all $s \in \mathcal{S}^{(G,N)}$, iIj for all $i, j \in N$ implies $\varphi_i(G, s, p) = \varphi_j(G, s, p)$ for all $i, j \in N$.

Weak Equal Treatment (WET) A competition system (G, N) satisfies Weak Equal Treatment if there exists $s \in \mathcal{S}^{(G,N)}$ such that iIj for all $i, j \in N$ implies $\varphi_i(G, s, p) = \varphi_j(G, s, p)$ for all $i, j \in N$.

Monotonicity in Strength (MS) A competition system (G, N) satisfies Monotonicity in Strength if for all $s \in \mathcal{S}^{(G,N)}$, for all $i, j \in N$, and for all $p \in p_R$ such that $p_{ij} > 0.5$, $\varphi_i(G, s, p) \geq \varphi_j(G, s, p)$ holds.

Weak Monotonicity in Strength (WMS) A competition system (G, N) satisfies Weak Monotonicity in Strength if there exists $s \in \mathcal{S}^{(G,N)}$ such that, for all $i, j \in N$ and for all $p \in p_R$ such that $p_{ij} > 0.5$, $\varphi_i(G, s, p) \geq \varphi_j(G, s, p)$ holds.

As already explained, the design of competitions often seems to be guided by two basic fairness principles: treating equals equally and not favouring weaker players. The four axioms above provide a particular formal expression to those principles, with the aim of analyzing whether different competition systems satisfy them or not.

The equal treatment requirements express the idea that, as for the final probability of winning, the competition system should not be biased towards any particular player if all of them are equally skilled. For example, Che and Hendershott (2008) use this kind of principle when analyzing fair tie-breaking mechanisms.

The two monotonicity axioms require the competition system not to benefit weaker players under any of the possible probability matrices compatible with the strength of the players. We have already pointed out that many competitions are precisely designed in order to avoid that the worse team wins by pure luck: for example, round-robin tournaments and even double round-robin tournaments minimize such an effect with a high number of matches, stepladder competitions seem precisely to be aimed to benefit better players, best players are matched with the worst ones in knockout competitions, or sometimes matches (usually finals) consist of a higher number of legs at the better competitor's home, like in basket. These principle seems also to guide many tie-breaking rules, such as in tennis, where a difference of more than two points is necessary to break the tie, or in soccer, where at least five penalties should be shot.

Apart of the fact that WET is logically weaker than ET and WMS is logically weaker than MS, the normative power of the weaker versions and the strong versions may depend on the concrete intended application, and in particular on the conjectures about the benevolence of the competition designer. On the one hand, ET and MS avoid the possibility of manipulation by a potentially corrupted competition designer because they ensure that there is no possibility of finding any particular seeding rule that benefits a particular player in relation with another one who is more or equally skilled. On the other hand, WET and WMS rely on the confidence of the benevolence of the competition designer, in the sense that the focus is on the competition systems where the seeder can always find a seeding rule that is "adequate", both regarding their structure and the corresponding seeding, independently of the values of the probability matrix that supports the strength relation.

4 Elimination-type competitions

Given that an elimination-type competition is represented by a unique tree, we will denote it by the pair (t, N) with N being the player set and t being the single binary tree.

4.1 Balanced competitions and equal treatment

Our first result connects the stronger version the equal treatment axiom with balanced competition systems.

Theorem 1 *An elimination-type competition system (t, N) satisfies ET if and only if t is balanced.*

Proof. Let (t, N) be an elimination-type competition system with t being balanced and let $s \in \mathcal{S}^{(t,N)}$ be an arbitrary but fixed seeding rule. Given the balancedness of t, any of its leaves has the same level coinciding with $h(t)$. Then, by $p_{ij} = 0.5$ for all $i, j \in N$, $\varphi_i(t, s, p) = (0.5)^{h(t)}$ holds for each $i \in N$. Thus, (t, N) satisfies ET.

Suppose now that (t, N) is an elimination-type competition system satisfying ET. Let $s \in \mathcal{S}^{(G,N)}$ be an arbitrary but fixed seeding rule. Suppose that iIj for all $i, j \in N$ but t is not balanced. We have then that $p_{ij} = 0.5$ holds for all $i, j \in N$. Given that t is not balanced there are leaves $\lambda, \lambda' \in \Lambda(t)$ with $\ell(\lambda) \neq \ell(\lambda')$. It follows then that $\varphi_{s(\lambda)}(t, s, p) = (0.5)^{\ell(\lambda)} \neq (0.5)^{\ell(\lambda')} = \varphi_{s(\lambda')}(t, s, p)$ in contradiction to (t, N) satisfying ET. ■

Notice that the argument used in the proof of Theorem 1 is independent of the characteristics of the seeding rule. Hence, we can immediately conclude that no additional competition systems emerge when ET is replaced by its weak version WET.

Corollary 1 *An elimination-type competition system* (t, N) *satisfies WET if and only if* t *is balanced.*

4.2 Minimal competitions and monotonicity in strength

We call a competition system *minimal* if there are only two participants in it. In particular, and for the case of elimination-type competitions, this implies that the competition consists of a unique match. As we show next, minimal competitions turn out to be directly related to the monotonicity in strength requirement.

Theorem 2 *An elimination-type competition system* (t, N) *satisfies MS if and only if it is minimal.*

Proof. We first show that if $N = \{1, 2\}$, then (t, N) satisfies MS. Let p be an arbitrary but fixed probability matrix and notice that $s \in \mathcal{S}^{(t,N)}$ implies $\varphi_1(t, s, p) = p_{12}$ and $\varphi_2(t, s, p) = p_{21}$. Hence, $p_{12} > 0.5$ implies $\varphi_1(t, s, p) > \varphi_2(t, s, p)$ as required for MS to be satisfied.

Suppose now that (t, N) satisfies MS. We have to prove that $|N| = 2$ holds in such a case. We split the proof in three steps referring to the possible cases when t is not an elementary binary tree. In each of these steps we show that (t, N) violates MS contradicting our supposition.

Step 1 If t is balanced with $h(t) = 2$, then (t, N) violates MS.

Proof. Notice that by the definition of a feasible seeding rule, $h(t) = 2$ implies that N consists of four players, $N = \{1, 2, 3, 4\}$. Consider the seeding rule $s \in \mathcal{S}^{(t,N)}$ assigning the players to the leaves of t in such a way that the initial matches are between players 1 and 2, and 3 and 4, respectively. Take the probability matrix p as specified below.

$$p = \begin{pmatrix} 0.5 & 0.85 & 0.86 & 0.90 \\ & 0.5 & 0.60 & 0.70 \\ & & 0.5 & 0.60 \\ & & & 0.5 \end{pmatrix}$$

We have that $\varphi_2(t,s,p) = (p_{21} \cdot p_{34} \cdot p_{23}) + (p_{21} \cdot p_{43} \cdot p_{24}) = 0.096$ and $\varphi_3(t,s,p) = (p_{34} \cdot p_{12} \cdot p_{31}) + (p_{34} \cdot p_{21} \cdot p_{32}) = 0.1074$. Thus, MS is violated since $p_{23} = 0.60 > 0.5$ and $\varphi_3(t,s,p) > \varphi_2(t,s,p)$.

Step 2 If t is a stepladder with $h(t) = 2$, then (t, N) violates MS.

Proof. Again by the definition of a feasible seeding rule, $h(t) = 2$ implies that N consists of three players, $N = \{1, 2, 3\}$. Consider the seeding rule $s \in \mathcal{S}^{(t,N)}$ assigning the players to the leaves of t in such a way that the initial match is between player 1 and player 2. Let p be a probability matrix such that $p_{12} = 0.51$, $p_{13} = 0.53$ and $p_{23} = 0.52$. Then $\varphi_2(t,s,p) = p_{12} \cdot p_{23} = 0.2548$ and $\varphi_3(t,s,p) = (p_{12} \cdot p_{31}) + (p_{21} \cdot p_{32}) = 0.4749$. Thus, MS is violated since $p_{23} = 0.52 > 0.5$ and $\varphi_3(t,s,p) > \varphi_2(t,s,p)$.

Step 3 If t is such that $h(t) > 1$, then (t, N) violates MS.

Proof. Let $\Lambda^{h(t)}$ be the set of all leaves of t at the maximal level $h(t)$. Notice that one of the following two situations necessary happens: (i) There exists a subtree t' of t which is a stepladder with $h(t') = 2$ and $\Lambda^{h(t')} \subseteq \Lambda^{h(t)}$, or (ii) There exists a balanced subtree t'' of t with $h(t'') = 2$ and $\Lambda^{h(t'')} \subseteq \Lambda^{h(t)}$.

The reason for the above two possibilities is as follows. Since t is a binary tree, $\left|\Lambda^{h(t)}\right| \geq 2$. Take $\lambda_1, \lambda_2 \in \Lambda^{h(t)}$ to be such that they have a common immediate predecessor and denote it by a. Again by t being a binary tree, a has a unique immediate predecessor; call it b. Given that b is not a leaf, it should have an intermediate successor $a' \neq a$. There are then two possibilities. First, if a' has no successors, then the set of nodes $\{\lambda_1, \lambda_2, a, b, a'\}$ and the corresponding edges form a stepladder t' with $h(t') = 2$ and $\Lambda^{h(t')} = \{\lambda_1, \lambda_2\} \subseteq \Lambda^{h(t)}$. Second, if a' does have successors, then there are exactly two of them which we call λ_3 and λ_4. Note that λ_3 and λ_4 are indeed leaves of t as by assumption, $\lambda_1, \lambda_2 \in \Lambda^{h(t)}$. Hence, in this case, the set of nodes $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, a, a', b\}$ and the corresponding edges form a balanced tree t'' with $h(t'') = 2$ and $\Lambda^{h(t'')} = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \subseteq \Lambda^{h(t)}$.

Let us then consider the two addressed possibilities separately.

Case (i) There exists a subtree t' of t which is a stepladder with $h(t')=2$ and $\Lambda^{h(t')} \subseteq \Lambda^{h(t)}$.

The situation in which $t'=t$ was already considered in Step 2. Therefore, we will assume that $h(t) \geq 3$ holds. Let $N=\{1,\ldots,n\}$ and consider the seeding rule $s \in \mathcal{S}^{(t,N)}$ assigning players 1, 2, and 3 to the leaves of t' as in the proof of Step 2. For any probability matrix p, denote by $\varphi_w(t',s,p)$, $w \in \{1,2,3\}$, the probability with which w wins the subcompetition t'. We will prove that there exists a probability matrix p^* such that $p^*_{23} > 0.5$ and $\varphi_3(t,s,p^*) > \varphi_2(t,s,p^*)$.

For this, take p^* to be such that $p^*_{12}=0.51$, $p^*_{13}=0.53$, $p^*_{23}=0.52$; notice that these are the same probability values also used in the proof of Step 2. Thus, we already know that $\varphi_3((t',\{1,2,3\}),s,p^*_{|\{1,2,3\}}) > \varphi_2((t',\{1,2,3\}),s,p^*_{|\{1,2,3\}})$ though $p^*_{23} > 0.5$. Moreover, we further assume that $p^*_{2i} > p^*_{3i}$ and $\frac{p^*_{2i}}{p^*_{3i}} \approx 1$ holds for each $i \in N \setminus \{1,2,3\}$.

Let $\mathbf{v}^{short}(t',t)$ be the set of all nodes of t which are on the shortest path from the root of t' to the root of t. Notice that for each $v(t) \in \mathbf{v}^{short}(t',t)$, $\ell(v(t)) \in \{0,1,\ldots,h(t)-2\}$ with $\ell(v(t))=0$ indicating that the corresponding node $v(t)$ is the root of t and $\ell(v(t))=h(t)-2$ indicating that the corresponding node $v(t)$ is the root of t'. Since t is a binary tree, each $v(t) \in \mathbf{v}^{short}(t',t)$ has exactly one successor node $v'(t)$ with $\ell(v'(t))=\ell(v(t))+1$ and $v'(t) \in V(t) \setminus \mathbf{v}^{short}(t',t)$. Clearly, $v'(t)$ could be either a terminal node or not.

We denote by $\Lambda_{v'(t)}$ the set of all leaves of t whose shortest path to the root of t contains $v'(t)$ and by $N_{v'(t)}$ the set of players seeded by s to some leaf from $\Lambda_{v'(t)}$. For $k \in N_{v'(t)}$, $\varphi_k^{v'(t)}(t,s,p^*)$ stands for the probability with which a player k seeded to a leaf from $\Lambda_{v'(t)}$ reaches the node $v'(t)$.

Suppose now that some player $w \in N$ has reached the node $\overline{v}(t) \in \mathbf{v}^{short}(t',t)$ and let $v(t) \in \mathbf{v}^{short}(t',t)$ be such that $\ell(\overline{v}(t))=\ell(v(t))+1$. Given the above additional notation, it is then clear that $\Sigma_{k \in N_{v'(t)}} p^*_{wk}\varphi_k^{v'(t)}(t,s,p^*)$ expresses the probability with which w reaches $v(t)$. Hence,

$$\varphi_2(t,s,p^*) = \varphi_2((t',\{1,2,3\}),s,p^*_{|\{1,2,3\}}) \cdot \prod_{v':\ell(v'(t))=1}^{h(t)-2} \sum_{k \in N_{v'(t)}} p^*_{2k}\varphi_k^{v'(t)}(t,s,p^*)$$

and

$$\varphi_3(t,s,p^*) = \varphi_3((t',\{1,2,3\}),s,p^*_{|\{1,2,3\}}) \cdot \prod_{v':\ell(v'(t))=1}^{h(t)-2} \sum_{k\in N_{v'(t)}} p^*_{3k}\varphi_k^{v'(t)}(t,s,p^*)$$

We already know that $\varphi_3((t',\{1,2,3\}),s,p^*_{|\{1,2,3\}}) > \varphi_2((t',\{1,2,3\}),s,p^*_{|\{1,2,3\}})$ and that $p^*_{2k} > p^*_{3k}$ holds for each $k \in N$ by construction. Moreover, since $\frac{p^*_{2i}}{p^*_{3i}} \approx 1$ holds for each $i \in N \setminus \{1,2,3\}$, we have

$$\prod_{v':\ell(v'(t))=1}^{h(t)-2} \sum_{k\in N_{v'(t)}} p^*_{2k}\varphi_k^{v'(t)}(t,s,p^*) \approx \prod_{v':\ell(v'(t))=1}^{h(t)-2} \sum_{k\in N_{v'(t)}} p^*_{3k}\varphi_k^{v'(t)}(t,s,p^*)$$

and thus, $\varphi_3(t,s,p^*) > \varphi_2(t,s,p^*)$ holds. Since $p^*_{23} > 0.5$, we conclude that (t,N) violates MS.

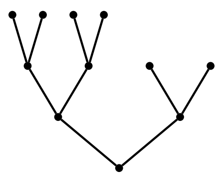
Case (ii) There exists a balanced subtree t'' of t with $h(t'') = 2$ and $\Lambda^{h(t'')} \subseteq \Lambda^{h(t)}$.

Consider the seeding rule $s \in \mathcal{S}^{(t,N)}$ assigning players 1, 2, 3, and 4 to the leaves of t'' in such a way that the initial matches are between players 1 and 2, and 3 and 4, respectively. Let p^* be such that $p^*_{|\{1,2,3,4\}} = p$ as defined in the proof of Step 1. As already shown in Step 1, $\varphi_3((t'',\{1,2,3,4\}),s,p^*_{|\{1,2,3,4\}}) > \varphi_2((t'',\{1,2,3,4\}),s,p^*_{|\{1,2,3,4\}})$. We can then further proceed as in the proof of Case (i) showing that $\varphi_3(t,s,p^*) > \varphi_2(t,s,p^*)$ holds though $p^*_{23} > 0.5$. Thus, (t,N) violates MS. ■

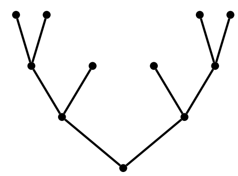
4.3 Antler-free competitions and weak monotonicity in strength

We start this section by introducing a special type of binary trees. An *antler* is a binary tree whose height is 3 and has six leaves: two leaves at level 2 and four leaves at level 3. There are two possible types of antlers: asymmetric antlers have the two leaves at level 2 in the same branch of the binary tree and symmetric antlers have one leaf at level 2 in each of the two branches of the binary tree. Formally defined, we say that a binary tree t with $h(t) = 3$ is

- an *antler*, if $|\Lambda(t)| = 6$ with $|\Lambda^3(t)| = 4$ and $|\Lambda^2(t)| = 2$;



(a)



(b)

Figure 6: The binary tree in (a) is an assymmetric antler, while the one in (b) is a asymmetric antler

- an *asymmetric antler*, if t is an antler with the leaves in $\Lambda^2(t)$ having a common immediate predecessor;
- a *symmetric antler*, if t is an antler with the leaves in $\Lambda^2(t)$ having distinct immediate predecessors.

Figure 6(a) displays an asymmetric antler and Figure 6(b) represent a symmetric one.

Antlers represent plausible competition systems that combine the characteristics of balanced elimination trees with the "byes" spirit of stepladders. However, antlers are difficult to be found in real competitions, one possible exception being, to the best of our knowledge, the Basque pelota in which an extended form of an antler is used.

Apart of their plausibility, antlers turn to be of a great theoretical interest. The reason is that, as we prove in this section, antlers constitute the minimal competition systems violatig WMS in the sense that removing any match from an antler gives as a result a system that satisfies WMS and any tree that contains (as a subgraph) an antler leads to the competition system violating WMS.

In order to provide a formal proof of the statement above, we need some additional notation and definitions.

We say that a binary tree is *antler-free* if it does not contain any (symmetric or asymmetric) antler.

A *root-to-leaf path* connects the root of t with a leaf of t. By $\gamma(t)$ we denote a root-to-leaf path of length $h(t)$ (i.e., $\gamma(t)$ is a *maximal root-to-leaf path* in t) and by $\mathbf{v}(\gamma(t))$ we denote the set of nodes of $\gamma(t)$. For $v \in \mathbf{v}(\gamma(t))$, t_v denotes the subtree of t with root v and $\Lambda_{-\gamma}(t_v)$ the set of leaves of t_v for which there is a shortest path to v *not including any other node* from $\mathbf{v}(\gamma(t))$.

Finally, we denote by $h_{-\gamma}(t_v)$ the maximal geodesic distance between v and the leaves in $\Lambda_{-\gamma}(t_v)$. We call then a binary tree t *an extended stepladder of degree* x, $x \in \{1, \ldots, h(t)\}$, if $\max_{v\in\mathbf{v}(\gamma(t))} h_{-\gamma}(t_v) = x$. That is, x represents the maximal distance that can be found between a node of $\gamma(t)$ and a leaf that is not in $\gamma(t)$.

According to the definition above a binary tree is in fact an extended stepladder of some degree. For example, balanced elimination-type competitions with four players and asymmetric antlers are extended stepladders of degree 2, balanced elimination-type competitions with eight players and symmetric antlers are extended stepladders of degree 3, while standard stepladders and elementary binary trees are extended stepladders of degree 1.

Clearly, by t being a binary tree, there are at least two maximal root-to-leaf paths in t. However, the next lemma shows that the degree of an extended stepladder is *robust* with respect to the selection of the maximal root-to-leaf path.

Lemma 3 *Let t be a binary tree with $\gamma'(t)$ and $\gamma''(t)$ being two different maximal root-to-leaf paths. If t is an extended stepladder of degree x' with respect to $\gamma'(t)$ and of degree x'' with respect to $\gamma''(t)$, then $x' = x''$.*

Proof. Assume that, w.l.o.g., $x'' > x'$ holds. Let v be the highest-level node in $\mathbf{v}(\gamma'(t)) \cap \mathbf{v}(\gamma''(t))$, $v' \in \mathbf{v}(\gamma'(t))$ be the lowest-level node such that $h_{-\gamma'}(t_{v'}) = x'$, and $v'' \in \mathbf{v}(\gamma''(t))$ be the lowest-level node such that $h_{-\gamma''}(t_{v''}) = x''$. Note that $\ell(v') \geq \ell(v)$ and $\ell(v'') \geq \ell(v)$ holds. Clearly, $\ell(v'') - \ell(v) + x''$ is the length of the shortest path from v to the corresponding leaf of t for which $h_{-\gamma''}(t_{v''}) = x''$ was calculated. Now, given that $v \in \mathbf{v}(\gamma'(t))$ and that the path from v to that leaf of t for which $h_{-\gamma''}(t_{v''}) = x''$ was calculated does not include any other node from $\mathbf{v}(\gamma'(t))$, it follows from the definition of an extended stepladder of degree x' that $\ell(v'') - \ell(v) + x'' \leq x'$ should hold. We have then $0 \leq \ell(v'') - \ell(v) \leq x' - x'' < 0$, a contradiction. ∎

The next lemma, which will be used in the proof of the main result of this section, shows that in any extended stepladder of degree 2 there are either two or four different maximal root-to-leaf paths. The situation with only two paths displays the fact that the competition among the players starts with a balanced competition between two players (the tree has only two leaves at level $h(t)$), while the situation with four paths implies the start to be given by a balanced four player competition (and the tree has four leaves at level $h(t)$).

Lemma 4 *Let t be an extended stepladder of degree 2 with $\gamma'(t)$ and $\gamma''(t)$ being two different maximal root-to-leaf paths. Then* $\max_{v\in\mathbf{v}(\gamma'(t))\cap\mathbf{v}(\gamma''(t))}\ell(v)\in\{h(t)-2,h(t)-1\}$.

Proof. Let v be the highest-level node in $\mathbf{v}(\gamma'(t))\cap\mathbf{v}(\gamma''(t))$. Given that $\gamma'(t)$ and $\gamma''(t)$ are different, $\max_{v\in\mathbf{v}(\gamma'(t))\cap\mathbf{v}(\gamma''(t))}\ell(v)\neq 0$. On the other hand, by t being an extended stepladder of degree 2, $h(t)-\ell(v)\leq 2$ should hold. The assertion then follows. ■

Next we denote by ES_x the set of extended stepladders of degree *at most* x (note that $ES_x\subseteq ES_{x'}$ for $x'\geq x$). Further, we use ES_2^* to denote the subclass of ES_2 defined as follows: An extended stepladder t of degree at most 2 belongs to ES_2^* only if there exists a maximal root-to-leaf path $\gamma(t)$ such that for all $v,v'\in\mathbf{v}(\gamma(t))$ with $|\ell(v)-\ell(v')|=1$, we have that $h_{-\gamma}(t_v)=2$ implies $h_{-\gamma}(t_{v'})=1$.

Remark 1. In view of Lemma 4, it is easy to see that an extended stepladder of degree 2 belongs to ES_2^* according to some maximal root-to-leaf path if and only if it belongs to ES_2^* according to any other such path. In other words, the belonging to ES_2^* is also robust with respect to the selection of the maximal root-to-leaf path.

Clearly, $ES_1\subseteq ES_2^*$ but not every extended stepladder of degree 2 belongs to ES_2^*. Figure 7 exemplifies two extended stepladders of degree 2 with only one of them (Figure 7(a)) belonging to ES_2^*. Notice further that asymmetric antlers do belong to ES_2 but not to ES_2^*, while symmetric antlers do not belong even to ES_2 as they are extended stepladders of degree 3.

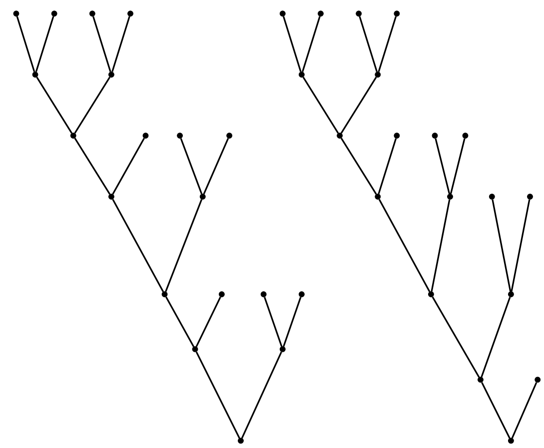
Lemma 5 establishes the relationship between antler-free binary trees and binary trees belonging to ES_2^*.

Lemma 5 *A binary tree belongs to ES_2^* if and only if it is antler-free.*

Proof. The proof consists of the following tree steps.

Step 1 If t is symmetric-antler-free, then $t\in ES_2$.

Proof. Let t be a symmetric-antler-free binary tree and suppose that $t\notin ES_2$. The latter implies that t is an extended stepladder of degree $x\geq 3$ with respect to a maximal root-to-leaf path $\gamma(t)$ as defined above. Therefore, there exists a node $v\in\gamma(t)$ and a leaf λ in $\Lambda_{-\gamma}(t_v)$ such that the distance between v and λ is x. Denote by π the path connecting v and λ. Let y be the immediate successor of v in π, y' the immediate successor of y in π and y'' the immediate successor of y' in π (notice that such nodes exist



(a) (b)

Figure 7: Extended stepladders of degree 2. Only the one displayed in (a) belongs to ES_2^*

because $x \geq 3$). Since t is a binary tree, y has another immediate successor $z' \neq y'$ and y' has another immediate successor $z' \neq y''$. On the other hand, given that $v \in \gamma(t)$ and that $\gamma(t)$ is a maximal root-to-leaf path, there are at least three consecutive successor nodes x, x' and x'' that belong to $\gamma(t)$ (otherwise there would be a longer root-to-leaf path connecting the root with λ). Again, since t is a binary tree, x has another immediate successor $w' \neq x'$ and x' has another immediate successor $w'' \neq y''$. Now, notice that the set of nodes $\{v, x, y, x', y', x'', y'', z', w', z'', w''\}$ and the corresponding edges form a symmetric antler, a contradiction.

Step 2 If t is antler-free, then $t \in ES_2^*$.

Proof. Notice first that t being antler-free implies that t is both symmetric-antler free and asymmetric-antler free. Given that $ES_2^* \subset ES_2$ and in view of Step 1 it suffices to show that if $t \in ES_2$ does not contain an asymmetric antler, then $t \in ES_2^*$. Suppose not and let $\gamma(t)$ be a maximal root-to-leaf path in t. If $t \in ES_2 \setminus ES_2^*$ we can conclude in view of Remark 1 that there are two nodes $v, v' \in \mathbf{v}(\gamma(t))$ with $\ell(v') = \ell(v)+1$ and $h_{-\gamma}(t_v) = h_{-\gamma}(t_{v'}) = 2$.

Moreover, given that $\gamma(t)$ is a maximal root-to-leaf path, v' has at least two consecutive successors x and x' belonging to $\gamma(t)$. And given that t is a binary tree, x has another immediate successor $y \neq x'$. Consider then the set of nodes consisting of v, v', x, x', y, the immediate successors of v and v', as well as the leaves in $\Lambda_{-\gamma}(t_v) \cup \Lambda_{-\gamma}(t_{v'})$. Notice that this set of nodes together with the corresponding edges form an asymmetric antler, which is a contradiction.

Step 3 If $t \in ES_2^*$, then t is antler-free.

Proof. Notice first that if $t \in ES_1 \subseteq ES_2^*$, then it is antler-free. Suppose then that t is an extended stepladder of degree 2 belonging to ES_2^*. Clearly, t does not contain a symmetric antler t' since each symmetric antler is an extended stepladder of degree 3 and thus, t being an extension of t' implies that t should be an extended stepladder of degree at least 3, which is a contradiction. Let us show now that $t \in ES_2^*$ implies that t does not contain an asymmetric antler.

Suppose that, to the contrary, t contains an asymmetric antler t^A. Let $\gamma(t)$ and $\gamma'(t^A)$ be maximal root-to-leaf paths in t and t^A, respectively. There are two possibilities:

(i) $\mathbf{v}(\gamma'(t^A)) \cap \mathbf{v}(\gamma(t)) = \emptyset$. Consider the root v_0^A of t^A and the closest predecessor v of v_0^A such that $v \in \mathbf{v}(\gamma(t))$. Let d be the distance between v_0^A and v. Then we have that $h_{-\gamma}(t_v) > d + 3$ in contradiction to t being an extended stepladder of degree 2.

(ii) $\mathbf{v}(\gamma'(t^A)) \cap \mathbf{v}(\gamma(t)) \neq \emptyset$. Let $\mathbf{v}(\gamma'(t^A)) = \{v_0^A, v_1^A, v_2^A, v_3^A\}$ be such that, for all $i \in \{1, 2, 3\}$, v_i^A is the immediate successor of v_{i-1}^A and v_0^A is the root of t^A. Given that $\mathbf{v}(\gamma'(t^A)) \cap \mathbf{v}(\gamma(t)) \neq \emptyset$ there exists $v_i^A \in \mathbf{v}(\gamma'(t^A)) \cap \mathbf{v}(\gamma(t))$. Note that $v_i^A \in \mathbf{v}(\gamma(t))$ implies $v_j^A \in \mathbf{v}(\gamma(t))$ for all $j < i$. Therefore $v_0^A \in \mathbf{v}(\gamma(t))$. We distinguish then two cases: either $v_0^A \in \mathbf{v}(\gamma(t))$ and $v_1^A \notin \mathbf{v}(\gamma(t))$ or $v_0^A, v_1^A \in \mathbf{v}(\gamma(t))$. If $v_0^A \in \mathbf{v}(\gamma(t))$ and $v_1^A \notin \mathbf{v}(\gamma(t))$ then $h_{-\gamma}(t_{v_0^A}) \geq 3$, in contradiction to t being an extended stepladder of degree 2. If $v_0^A, v_1^A \in \mathbf{v}(\gamma(t))$ then, by the structure of an asymmetric antler, and given that t is an extended stepladder of degree 2, we know that $h_{-\gamma}(t_{v_0^A}) = h_{-\gamma}(t_{v_1^A}) = 2$, a contradiction to $t \in ES_2^*$. ■

The main result in this section (Theorem 3) characterizes the set of elimination-type competition systems that satisfy WMS as those displayed by an antler-free binary tree. Moreover, Theorem 3 also uniquely specifies the feasible seeding rule for which WMS is satisfied. This rule takes into account the following two characteristics of antler-free binary trees: (1) they allow for 4 players to be involved in a balanced elimination-type competition, and (2) any such tree might also incorporate byes at different levels having a crucial impact on players' probability of being the final winner.

For the formal definition of this rule which we call "increasingly balanced", let us first introduce the notion of a balanced seeding for balanced elimination-type competitions with four players. Such a seeding consists of matching the best player with the worst one at one branch of the tree, and the other two players being matched in the other branch. This kind of seeding is very often employed in knockout competitions because, presumably it does not give advantages to worse players with respect to better ones. Formally, given an admissible elimination-type competition system (t, N) such that $|N| = 4$ and a binary relation of strength R, we say that a seeding rule $s : \Lambda(t) \to N$ is *balanced* (and denote it by s_{b4}) if there are players $i, j \in N$ who are initially playing against each other under s such that iRk and kRj holds for each $k \in N \setminus \{i, j\}$. Thus, a seeding that matches 1 with 4 and 2 with 3 is always balanced. But a seeding that matches 1 with 3 and 2 with 4 would also be balanced if (and only if) $1I2$ or $3I4$. Similarly, a seeding that matches 1 with 2 and 3 with 4 would also be balanced if (and only if) $2I3I4$.

We define next the increasingly balanced seeding by making use of the notion of a balanced seeding. Let (t, N) be an elimination-type competition system with t being an extended stepladder of degree 2. Given a binary relation of strength R we say that a seeding rule $s : \Lambda(t) \to N$ is *increasingly balanced* (and denote it by s_{ib}) if the following three conditions hold:

(1) for all $\lambda, \lambda' \in \Lambda(t)$, $\lambda \in \Lambda^{\ell}(t)$ and $\lambda' \in \Lambda^{\ell'}(t)$ with $\ell > \ell'$ implies $s(\lambda')Rs(\lambda)$;

(2) for all $\ell \in \{1, \ldots, h(t) - 1\}$, $\Lambda^{\ell}(t) = \{\lambda, \lambda', \lambda''\}$ with λ' and λ'' having a common intermediate predecessor implies $s(\lambda)Rs(\lambda''')$ for each $\lambda''' \in \{\lambda', \lambda''\}$;

(3) $\left|\Lambda^{h(t)}(t)\right| = 4$ implies that: (a) iRj holds for each $i \in N$ with $\ell(s^{-1}(i)) < h(t)$ and $j \in N$ with $\ell(s^{-1}(j)) = h(t)$, and (b) $s(\lambda) = s_{b4}(\lambda)$ for each $\lambda \in \Lambda^{h(t)}(t)$.

In words, s_{ib} assigns players to leaves in such a way that weaker players are seeded to higher levels in the tree, and when more than one player is seeded at the same level, then the rule distinguishes between two possibilities: (a) if the level is not the maximal one, then the best player among those seeded at that level is seeded to the leaf that is closest to the root-to-leaf path $\gamma(t)$, and (b) if the level is the maximal one and there are four leaves at it, then among the weakest four players, the weakest one is matched with the fourth weakest, and the other two are matched together (see Figure 8).

Note also that there are two cases in which s_{ib} is silent. The first case is when there are three leaves of t at the same level and thus, the two weakest players among the three seeded at that level play their initial match. Clearly, in such a case, the two possible seedings of these players are equivalent. The second case is when there are only two leaves of t at level $h(t)$. In that case, the two seedings of the two weakest players are equivalent.

It should also be noted that, by the structure of the extended stepladder competition of degree 2, if there are three or four leaves at a certain level, then four players are playing a balanced elimination-type sub-competition. The key feature of s_{ib} is that it ensures the strongest of the newly seeded players at that level to play against the survivor of the previous elimination process who, by the construction of s_{ib}, is necessarily weaker than any of the newly seeded players. In other words, s_{ib} ensures that in any balanced elimination-type sub-competition played by four players the strongest one is matched with the weakest one.

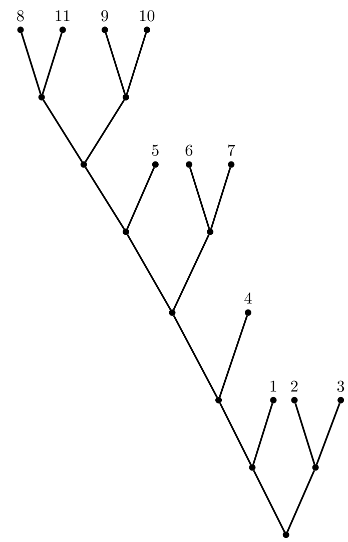


Figure 8: An increasingly balanced seeding in an extended stepladder of degree 2

We are now prepared to present three lemmas that will be used to prove the main characterization theorem of this section. Lemma 6 shows that the positions of two players, i and j, who are equally strong ($p_{ij} = 0.5$) can be exchanged without affecting the probabilities of winning of any player. In Lemma 7 we prove that a 4-player balanced elimination-type competition satisfies WMS only for the balanced seeding. Finally, in Lemma 8 we show that for a competition system to satisfy WMS, better players should not be seeded to leaves that are further away from the root of the tree.

For the statement of Lemma 6 we need to introduce the concept of *interchangeable* players and some additional notation.

Let (t, N) be an elimination-type competition system, $s \in \mathcal{S}^{(t,N)}$, and p a probability matrix. We say that players i and j are *interchangeable* with respect to p and s if:

(1) $\varphi_i(t, s, p) = \varphi_j(t, s', p)$ and $\varphi_j(t, s, p) = \varphi_i(t, s', p)$, and

(2) $\varphi_k(t, s, p) = \varphi_k(t, s', p)$ holds for each $k \in N \setminus \{i, j\}$,

where $s' \in \mathcal{S}^{(t,N)}$ is such that $s(s'^{-1}(i)) = j$ and $s'^{-1}(k) = s^{-1}(k)$ for each $k \in N \setminus \{i, j\}$.

Now, given $s \in \mathcal{S}^{(t,N)}$, for any $x \in N$, $\mathbf{v}^x$ denotes the set of nodes along the shortest path between $s(x)$ and the root of t. Notice that, for each $v^x \in \mathbf{v}^x$ with $v^x \neq s(x)$, v^x has a unique successor node v'^x with $v'^x \in V(t) \setminus \mathbf{v}^x$. We denote by $\Lambda_{v'^x}(t)$ the set of all leaves of t whose shortest path to the root of t contains v'^x and by $N_{v'^x}$ the set of players seeded by s to some leaf from $\Lambda_{v'^x}(t)$. Given a probability matrix p, a node v'^x, and a player $w \in N_{v'^x}$, $\varphi_w^{v'^x}(t,s,p)$ stands for the probability with which w reaches the node v'^x.

Lemma 6 *Let (t,N) be an elimination-type competition system and p a probability matrix with $p_{ij} = 0.5$ holding for some $i,j \in N$. Then i and j are interchangeable with respect to p and any $s \in \mathcal{S}^{(t,N)}$.*

Proof. Given the notation above, we have that $\varphi_i(t,s,p)$ can be expressed by:

$$\varphi_i(t,s,p) = \prod_{v'^i:\ell(v'^i)=1}^{\ell(s^{-1}(i))} \sum_{k \in N_{v'^i}} p_{ik}\varphi_k^{v'^i}(t,s,p)$$

and

$$\varphi_j(t,s,p) = \prod_{v'^j:\ell(v'^j)=1}^{\ell(s^{-1}(j))} \sum_{k \in N_{v'^j}} p_{ik}\varphi_k^{v'^j}(t,s,p).$$

Notice that $\varphi_i(t,s,p) = \varphi_j(t,s',p)$ holds due to the construction of s', $p_{ij} = 0.5$, and $p_{ik} = p_{jk}$ for each $k \in N \setminus \{i,j\}$, where the latter equality follows from $p_{ij} = 0.5$ and condition (2). Analogously $\varphi_j(t,s,p) = \varphi_i(t,s',p)$ holds as well. Moreover, again by the same argument and for each $k \in N \setminus \{i,j\}$, $\varphi_k(t,s,p) = \varphi_k(t,s',p)$ also holds, completing the proof. ■

Lemma 7 *A balanced elimination-type competition system (t,N) with $h(t) = 2$ satisfies WMS with respect to a seeding rule $s \in \mathcal{S}^{(t,N)}$ if and only if $s = s_{b4}$.*

Proof. Let (t,N) be as above with $N = \{1,2,3,4\}$ and recall that $(1R2R3R4)$. By Lemma 6 we can assume without loss of generality that s_{b4} is such that player 1 is matched with 4 and player 2 is matched with 3. In order to prove Lemma 7, let us consider the seeding rule $s = s_{b4}$ and fix *any* probability matrix $p \in p_R$; we have then to show that $\varphi_1(t,s_{b4},p) \geq \varphi_2(t,s_{b4},p) \geq \varphi_3(t,s_{b4},p) \geq \varphi_4(t,s_{b4},p)$ holds.

Note first that for the final winner's probabilities we have:

$\varphi_1(t,s_{b4},p) = (p_{14} \cdot p_{23} \cdot p_{12}) + (p_{14} \cdot p_{32} \cdot p_{13})$,
$\varphi_2(t,s_{b4},p) = (p_{23} \cdot p_{14} \cdot p_{21}) + (p_{23} \cdot p_{41} \cdot p_{24})$,
$\varphi_3(t,s_{b4},p) = (p_{32} \cdot p_{14} \cdot p_{31}) + (p_{32} \cdot p_{41} \cdot p_{34})$,
$\varphi_4(t,s_{b4},p) = (p_{41} \cdot p_{23} \cdot p_{42}) + (p_{41} \cdot p_{32} \cdot p_{43})$.

First, when comparing $\varphi_1(t,s_{b4},p)$ with $\varphi_2(t,s_{b4},p)$ we have from $p_{12} \geq 0.5$ and condition (1) that $p_{21} \leq p_{12}$ holds. Thus, $p_{14} \cdot p_{23} \cdot p_{12} \geq p_{23} \cdot p_{14} \cdot p_{21}$. On the other hand, $p_{14} \cdot p_{32} \cdot p_{13} \geq p_{23} \cdot p_{41} \cdot p_{24}$ holds by $p_{32} \geq p_{41}$ (due to

Lemma 2) and by $p_{14} \geq p_{24}$ and $p_{13} \geq p_{23}$ (following from $p_{12} \geq 0.5$ and condition (2)). Thus, $\varphi_1(t, s_{b4}, p) \geq \varphi_2(t, s_{b4}, p)$ holds.

As for players 2 and 3, $p_{23} \geq 0.5$ and (1) implies $p_{32} \leq p_{23}$. Moreover, $p_{21} \geq p_{31}$ and $p_{24} \geq p_{34}$ follow from condition (2) . Hence, $p_{23} \cdot p_{14} \cdot p_{21} \geq p_{32} \cdot p_{14} \cdot p_{31}$ and $p_{23} \cdot p_{41} \cdot p_{24} \geq p_{32} \cdot p_{41} \cdot p_{34}$, and therefore $\varphi_2(t, s_{b4}, p) > \varphi_3(t, s_{b4}, p)$.

Finally, when comparing $\varphi_3(t, s_{b4}, p)$ with $\varphi_4(t, s_{b4}, p)$ we have that $p_{34} \geq 0.5$ and condition (2) implies $p_{32} \geq p_{42}$ and $p_{31} \geq p_{41}$. Moreover, it follows from Lemma 2 that $p_{14} \geq p_{23}$ holds and thus, $p_{32} \cdot p_{14} \cdot p_{31} \geq p_{41} \cdot p_{23} \cdot p_{42}$. On the other hand, $p_{34} \leq p_{43}$ holds by (1) and thus, $p_{32} \cdot p_{41} \cdot p_{34} \geq p_{41} \cdot p_{32} \cdot p_{43}$. We have then $\varphi_3(t, s_{b4}, p) > \varphi_4(t, s_{b4}, p)$, concluding that (t, N) satisfies WMS with respect to s_{b4}.

Let us now consider a seeding rule $s \in \mathcal{S}^{(t,N)}$ which differs from $s \neq s_{b4}$. Let $\varepsilon > 0$ be arbitrarily small and $p \in p_R$ be defined as follows:

$$p = \begin{pmatrix} 0.5 & 0.5+\varepsilon & 0.5+2\varepsilon & 1-\varepsilon \\ & 0.5 & 0.5+\varepsilon & 1-2\varepsilon \\ & & 0.5 & 1-3\varepsilon \\ & & & 0.5 \end{pmatrix}$$

There are two possible cases with respect to s.

Case 1 (s is such that the initial matches are between 1 and 2, and 3 and 4, respectively). We have in this case:

$$\varphi_3(t, s, p) = (p_{34} \cdot p_{12} \cdot p_{31}) + (p_{34} \cdot p_{21} \cdot p_{32}) \approx (1 \cdot 0.5 \cdot 0.5) + (1 \cdot 0.5 \cdot 0.5) \approx 0.5$$

and

$$\varphi_2(t, s, p) = (p_{21} \cdot p_{34} \cdot p_{23}) + (p_{21} \cdot p_{43} \cdot p_{24}) \approx (0.5 \cdot 1 \cdot 0.5) + (0.5 \cdot 0 \cdot 1) \approx 0.25,$$

in contradiction to $p_{23} > 0.5$ and (t, N) satisfying WMS.

Case 2 (s is such that the initial matches are between 1 and 3, and 2 and 4, respectively). Considering again the above probability matrix, we have

$$\varphi_1(t, s, p) = (p_{13} \cdot p_{24} \cdot p_{12}) + (p_{13} \cdot p_{42} \cdot p_{14}) \approx (0.5 \cdot 1 \cdot 0.5) + (0.5 \cdot 0 \cdot 1) \approx 0.25$$

and

$$\varphi_2(t, s, p) = (p_{24} \cdot p_{13} \cdot p_{21}) + (p_{24} \cdot p_{31} \cdot p_{23}) \approx (1 \cdot 0.5 \cdot 0.5) + (1 \cdot 0.5 \cdot 0.5) \approx 0.5,$$

in contradiction to $p_{12} > 0.5$ and (t, N) satisfying WMS. ■

Lemma 8 *Let R be a strength relation defined on N, (t, N) an elimination-type competition system, and $s \in \mathcal{S}^{(t,N)}$. If (t, N) satisfies WMS with respect* *to s, then $\ell(\lambda) > \ell(\lambda')$ for $\lambda, \lambda' \in \Lambda(t)$ implies $s(\lambda')Rs(\lambda)$.*

Proof. Suppose that the implication is false. That is, given R, let (t, N) satisfy WMS with respect to s such that $s(\lambda)Ps(\lambda')$ holds for some $\lambda, \lambda' \in \Lambda(t)$ with $\ell(\lambda) > \ell(\lambda')$. For (t, N) to satisfy WMS it is necessary that $\varphi_{s(\lambda)}(t, s, p') \geq \varphi_{s(\lambda')}(t, s, p')$ for all probability matrices $p \in p_R$ such that $p_{s(\lambda),s(\lambda')} > 0.5$. Let us consider a probability matrix $p \in p_R$ such that, for all $i, j \in N$, $p_{ij} \approx 0.5$ with $p_{s(\lambda),s(\lambda')} > 0.5$. Then $\varphi_{s(\lambda)}(t, s, p) \approx 0.5^{\ell(\lambda)}$ and $\varphi_{s(\lambda')}(t, s, p) \approx 0.5^{\ell(\lambda')}$. Since $\ell(\lambda) > \ell(\lambda')$, $\varphi_{s(\lambda)}(t, s, p) < \varphi_{s(\lambda')}(t, s, p)$. Taking into account that $p_{s(\lambda),s(\lambda')} > 0.5$, the latter inequality implies that (t, N) violates WMS with respect to s, a contradiction. ■

The main result in this section is given below and characterizes elimination-type competitions that satisfy WMS as those whose tree is antler-free. Moreover the theorem states that s_{ib} is the only seeding rule for which antler-free competitions satisfy WMS.

Theorem 3 *An elimination-type competition system (t, N) satisfies WMS with respect to $s \in \mathcal{S}^{(t,N)}$ if and only if t is antler-free and $s = s_{ib}$.*

The proof of Theorem 3 is relegated to the Appendix.

In many real situations there are 2^x players participating in balanced elimination-type competitions. It is then the balanced seeding which is profusely taken because it is broadly considered a fair solution.[8] A remarkable corollary of Theorem 3 is that no balanced competition satisfies WMS if $x \geq 3$, that is, if 8 or more players compete, even if the balanced seeding is used. The reason is that the binary trees that represent those competition systems do contain an antler. Broadly speaking, it is impossible to find a fair balanced playoff competition with more than 4 participants.

5 League-type competitions

In this section we explore the fulfillment of the four fairness requirements by league-type competitions. As already advanced in the Introduction, a league-type competition consists of a pair (G, N), where N is the player set and G is a forest of elementary binary trees. Recall that, as a consequence of the two assumptions on seeding rules, $s \in \mathcal{S}^{(G,N)}$ implies $2 \leq |N| \leq |\Lambda(G)|$.

[8]In the case of 8 players the balanced seeding matches 1 with 8 and 4 with 5 in one branch, and 2 with 7 and 3 with 6 in the other branch of the binary tree.

Before presenting the results in this section, the following additional notation and remarks will be needed. Recall that, given a competition (G, N), w_t denotes the player who reaches the root of the binary tree $t \in G$. Moreover, $i \in N$ is a winner of the competition (G, N) if $|\{t \in G : w_t = i\}| \geq |\{t \in G : w_t = j\}|$ holds for all $j \in N$. For every $t \in G$, λ_t and λ'_t denote the two leaves of t. Then, given $G = \{t_1, t_2, \ldots, t_{\#G}\}$ and $s \in \mathcal{S}^{(G,N)}$, $w = \left(w_{t_1}, w_{t_2}, \ldots, w_{t_{\#G}}\right)$ stands for the corresponding vector or *configuration* of tree winners, where $w_t \in \{s(\lambda_t), s(\lambda'_t)\}$ holds for each $t \in G$. Given $s \in \mathcal{S}^{(G,N)}$ and a probability matrix p, the probability $pr(w)$ for the occurrence of a configuration w is given by $pr(w) = \prod_{t \in G} \varphi_{w_t}(t, s, p)$. Then, for each $i \in N$, $\varphi_i(G, s, p)$ can be expressed by $\varphi_i(G, s, p) = \sum_{w \in \mathbf{w}_{[i]}} pr(w) = \sum_{w \in \mathbf{w}_{[i]}} \prod_{t \in G} \varphi_{w_t}(t, s, p)$, where $\mathbf{w}_{[i]}$ stands for the set of all configurations for which i is the final winner of the competition. Additionally, when the probability matrix p is such that $p_{ij} = 0.5$ holds for all $i, j \in N$, we have $pr(w) = (0.5)^{\#G}$ for each configuration of tree winners, and $\varphi_i(G, s, p) = (0.5)^{\#G} \cdot \left|\mathbf{w}_{[i]}\right|$ holding for each $i \in N$.

5.1 Equal participation and equal treatment

Like in the case of elimination-type competitions, the fulfillment of the two equal treatment axioms (ET and WET) by league-type competitions is closely related with the fact that players should play the same number of matches. The first result in this section shows that that a league satisfies ET if and only if either each player plays a unique match or there are only two players participating in all matches. As for the weak version, we present a result showing that every competition where each player participates the same number of times in at least $|N| - 1$ matches satisfies WET. This includes single round-robin and double round-robin competitions as particular cases.

Theorem 4 *A league-type competition system* (G, N) *satisfies ET if and only if either* $|\Lambda(G)| = |N|$ *or* $|\Lambda(G)| > |N| = 2$.

Proof. Let (G, N) be a league-type competition system with either $|\Lambda(G)| = |N|$ or $|\Lambda(G)| > |N| = 2$. We show first that (G, N) satisfies ET. Let $s \in \mathcal{S}^{(G,N)}$ be an arbitrary but fixed seeding rule and p be the probability matrix with $p_{ij} = 0.5$ for all $i, j \in N$. Notice then that each player from N participates either in exactly one match (when $|\Lambda(G)| = |N|$) or there are only two players that participate in all matches (when $|\Lambda(G)| > |N| = 2$). Let us consider the following cases separately.

Case 1 ($|\Lambda(G)| > |N| = 2$ and $\#G$ is odd). Let $N = \{1, 2\}$ and notice that in this case we have $\varphi_1(G, s, p) = (0.5)^{\#G} \cdot |\mathbf{w}_{[1]}|$ and $\varphi_2(G, s, p) = (0.5)^{\#G} \cdot |\mathbf{w}_{[2]}|$. By $\#G$ being odd, $w \in \mathbf{w}_{[1]}$ implies $w \notin \mathbf{w}_{[2]}$, and $w \in \mathbf{w}_{[2]}$ implies $w \notin \mathbf{w}_{[1]}$. Thus, $\mathbf{w}_{[1]} \cap \mathbf{w}_{[2]} = \emptyset$. Finally, by $N = \{1, 2\}$ and $\mathbf{w}_{[1]} \cap \mathbf{w}_{[2]} = \emptyset$, we can define a bijection $f : \mathbf{w}_{[1]} \to \mathbf{w}_{[2]}$ by just replacing 1 by 2 and 2 by 1 as tree winners at each $w \in \mathbf{w}_{[1]}$ as to get $f(w) \in \mathbf{w}_{[2]}$. Thus $|\mathbf{w}_{[1]}| = |\mathbf{w}_{[2]}|$ and $\varphi_1(G, s, p) = \varphi_2(G, s, p)$ holds. We conclude then that (G, N) satisfies ET.

Case 2 ($|\Lambda(G)| > |N| = 2$ and $\#G$ is even). Notice that in this case $\mathbf{w}_{[1]} \cap \mathbf{w}_{[2]} \neq \emptyset$. We have then that $w \in \mathbf{w}_{[1]} \setminus \mathbf{w}_{[2]}$ implies $w \notin \mathbf{w}_{[2]} \setminus \mathbf{w}_{[1]}$, and $w \in \mathbf{w}_{[2]} \setminus \mathbf{w}_{[1]}$ implies $w \notin \mathbf{w}_{[1]} \setminus \mathbf{w}_{[2]}$. Finally, we can define a bijection $f : \mathbf{w}_{[1]} \setminus \mathbf{w}_{[2]} \to \mathbf{w}_{[2]} \setminus \mathbf{w}_{[1]}$ in the same way as in Case 1 and conclude that $|\mathbf{w}_{[1]} \setminus \mathbf{w}_{[2]}| = |\mathbf{w}_{[2]} \setminus \mathbf{w}_{[1]}|$ should follow and thus, $|\mathbf{w}_{[1]}| = |\mathbf{w}_{[2]}|$ holds as well. We conclude that $\varphi_1(G, s, p) = \varphi_2(G, s, p)$ and thus, (G, N) satisfies ET also in this case.

Case 3 ($|\Lambda(G)| = |N|$). Since each player participates in exactly one match in this case, then for any configuration w of tree winners, a player is a final winner of the competition only if he is the winner of the match displayed by the unique tree he is seeded at by s. Thus, $|\mathbf{w}_{[i]}| = |\mathbf{w}_{[j]}|$ holds for all $i, j \in N$. By $\varphi_i(G, s, p) = (0.5)^{\#G} \cdot |\mathbf{w}_{[i]}|$ holding for each $i \in N$, (G, N) satisfies ET.

Suppose now that (G, N) is a league-type competition system satisfying ET. In order to show that either $|\Lambda(G)| = |N|$ or $|\Lambda(G)| > |N| = 2$ holds, let p be the probability matrix with $p_{ij} = 0.5$ for all $i, j \in N$. Since each binary tree of G is a balanced elementary tree, it suffices to show that, for some feasible seeding rule, $|\Lambda(G)| > |N| > 2$ leads to a contradiction (as $|N| \leq |\Lambda(G)|$ follows from (G, N) satisfying ET and the definition of a feasible seeding rule). Consider then the following possible cases.

Case 1 ($|N|$ is even and $|\Lambda(G)| = |N| + 2$). Take the seeding rule $s_1 : \Lambda(G) \to N$ defined in such a way that there are exactly $\frac{|\Lambda(G)| - |N|}{2} + 1 = 2$ matches between player 1 and player 2 (with the corresponding elementary binary trees being t' and t''), while the remaining $\frac{|N|-2}{2}$ matches are played only between players from $N \setminus \{1, 2\}$ with each of these players being assigned to exactly one leaf of G. Notice that $s_1 \in \mathcal{S}^{(G,N)}$.

Consider the competition system $(G\setminus\{t'\},N)$ and let $s_1':\Lambda(G\setminus\{t'\})\to N$ be such that $s_1'(\lambda)=s_1(\lambda)$ for each $\lambda\in\Lambda(G\setminus\{t'\})$. Notice that $s_1'\in\mathcal{S}^{(G\setminus\{t'\},N)}$. Denote by $\mathbf{w}_{[i]}^{-t'}$ the set of configurations of tree winners in the subcompetition $(G\setminus\{t'\},N)$ for $i\in N$ and observe that, by $|\Lambda(G\setminus\{t'\})|=|N|$ and as argued in Case 3 above, $\left|\mathbf{w}_{[1]}^{-t'}\right|=\left|\mathbf{w}_{[3]}^{-t'}\right|$. Take $w^*\in\mathbf{w}_{[1]}^{-t'}$ and note that this implies that player 1 wins the match t''. Thus, for each $w^*\in\mathbf{w}_{[1]}^{-t'}$ there are exactly two configurations w', $w''\in\mathbf{w}_{[1]}$ of tree winners for (G,N) where, everything else being the same, either player 1 wins at t' or player 2 wins at t'. On the other hand, for w^{**} to belong to $\mathbf{w}_{[3]}^{-t'}$ necessarily 3 wins his unique match. Thus, for each $w^{**}\in\mathbf{w}_{[3]}^{-t'}$ there is a unique configuration $w'''\in\mathbf{w}_{[3]}$ of tree winners for (G,N) where, everything else being the same, $w_{t'}\neq w_{t''}$. We conclude then that $\left|\mathbf{w}_{[1]}\right|>\left|\mathbf{w}_{[3]}\right|$ holds and thus, by $\varphi_1(G,s_1,p)=(0.5)^{\#G}\cdot\left|\mathbf{w}_{[1]}\right|>(0.5)^{\#G}\cdot\left|\mathbf{w}_{[3]}\right|=\varphi_3(G,s_1,p)$, we reach a contradiction to (G,N) satisfying ET.

Case 2 ($|N|$ is even and $|\Lambda(G)|>|N|+2$). Take the seeding rule s_1 as defined in Case 1, fix a configuration w, and note that, given the definition of s_1, player 3 is a final winner at w only if, for each $i\in N$, $w_t=i$ holds for at most one tree t of G. However, by $\frac{|\Lambda(G)|-|N|}{2}+1>2$, the latter condition is violated for either player 1 or for player 2, or for both players. We conclude then that $\mathbf{w}_{[3]}=\emptyset$ should hold. On the other hand, $\mathbf{w}_{[1]}\neq\emptyset$ since player 1 is a final winner of the competition if he wins all its matches. We have then $\varphi_1(G,s_1,p)=(0.5)^{\#G}\cdot\left|\mathbf{w}_{[1]}\right|>0=(0.5)^{\#G}\cdot|\emptyset|=(0.5)^{\#G}\cdot\left|\mathbf{w}_{[3]}\right|=\varphi_3(G,s_1,p)$ in contradiction to (G,N) satisfying ET.

Case 3 ($|N|$ is odd). Let $N=\{1,\ldots,n\}$ and take in this case the seeding rule $s_2:\Lambda(G)\to N$ defined as follows. Exactly $\frac{|\Lambda(G)|-|N|+1}{2}$ matches are played between player 1 and player 2, exactly one match between player 1 and player n, and the remaining $\frac{|N|-3}{2}$ matches are played between players from $N\setminus\{1,n\}$ with each of these players being assigned to exactly one leaf of G. Notice that $s_2\in\mathcal{S}^{(G,N)}$. We proceed by considering the following three possible sub-cases.

Case 3.1 ($\frac{|\Lambda(G)|-|N|+1}{2}=1$). Denote by t' the unique tree at which players 1 and 2 are seeded, and let t'' be the unique tree at which players 1 and n are seeded. These are the four possible combinations of winners in $\{t',t''\}$: (i) 1 wins in both t' and t''; (ii) 1 wins in t' and n wins in t''; (iii) 2 wins in t' and 1 wins in t'' and (iv) 2 wins in t' and n wins in t''. Notice that for every configuration of winners in $G\setminus\{t',t''\}$ the maximal number of matches that a player from $N\setminus\{1,2,n\}$ wins is one. Therefore, for each of the configurations of winners in $G\setminus\{t',t''\}$, three out of the four possible combinations of winners in $\{t',t''\}$ give as a result a configuration $w\in\mathbf{w}_{[1]}$ and only two give as a result a configuration $w\in\mathbf{w}_{[n]}$. We conclude then that $\left|\mathbf{w}_{[1]}\right|>\left|\mathbf{w}_{[n]}\right|$ holds and thus, by $\varphi_1(G,s_2,p)=(0.5)^{\#G}\cdot\left|\mathbf{w}_{[1]}\right|>(0.5)^{\#G}\cdot\left|\mathbf{w}_{[n]}\right|=\varphi_n(G,s_2,p)$, we reach a contradiction to (G,N) satisfying ET.

Case 3.2 ($\frac{|\Lambda(G)|-|N|+1}{2}=2$). Denote by t' and t'' the two trees at which players 1 and 2 are seeded, and let t''' be the unique tree at which player n is seeded. Analogously to Case 3.1, it is easy to compute that, for each of the configurations of winners in $G\setminus\{t',t'',t'''\}$, three out of the eight possible combinations of winners in $\{t',t'',t'''\}$ give as a result a configuration $w\in\mathbf{w}_{[1]}$ and only two give as a result a configuration $w\in\mathbf{w}_{[n]}$. Therefore $\left|\mathbf{w}_{[1]}\right|>\left|\mathbf{w}_{[n]}\right|$ and thus, by $\varphi_1(G,s_2,p)=(0.5)^{\#G}\cdot\left|\mathbf{w}_{[1]}\right|>(0.5)^{\#G}\cdot\left|\mathbf{w}_{[n]}\right|=\varphi_n(G,s_2,p)$, we reach again a contradiction to (G,N) satisfying ET.

Case 3.3 ($\frac{|\Lambda(G)|-|N|+1}{2}>2$). Fix a configuration w and note that, given the definition of s_2, player n is a final winner at w only if, for any $i\in N$, $w_t=i$ holds for at most one tree t of G. However, by $\frac{|\Lambda(G)|-|N|+1}{2}>2$, the latter condition is violated for either player 1 or for player 2, or for both players. We conclude then that $\mathbf{w}_{[n]}=\emptyset$ should hold. On the other hand, $\mathbf{w}_{[1]}\neq\emptyset$ because player 1 is the winner of the entire competition if he wins all the matches that he plays. We have then $\varphi_1(G,s_2,p)=(0.5)^{\#G}\cdot\left|\mathbf{w}_{[1]}\right|>0=(0.5)^{\#G}\cdot|\emptyset|=(0.5)^{\#G}\cdot\left|\mathbf{w}_{[n]}\right|=\varphi_n(G,s_2,p)$ in contradiction to (G,N) satisfying ET. ■

It is sometimes suggested that leagues are fair competition systems. Strictly speaking this is not totally true because our definition of leagues is much broader than what is popularly understood as "leagues", which is usually identified with round-robin tournaments. In fact, the theorem above excludes leagues as fair structures, unless every player plays exactly one match or there are only two players. As shown next, weakening ET to WET extends the the class of league-type competitions that are fair to those where each player has the possibility to participate the same number of times in at least $|N|-1$ different matches. In particular, for $m\geq 1$ being an integer, we propose the seeding rule letting each player participate in exactly m matches against every other player. We call this class of competitions *m-round-robin tournaments*, which include as particular cases single round-robin tournaments ($m = 1$) and double round-robin tournaments ($m = 2$).

Theorem 5 *Any league-type competition* (G, N) *with* $|\Lambda(G)| = |N|$ *or* $|\Lambda(G)| = m \cdot (|N| - 1) \cdot |N|$ *for some integer* $m \geq 1$ *satisfies WET.*

Proof. Notice first that if $|\Lambda(G)| = |N|$ holds for some (G, N), then the assertion follows from Theorem 4 and the fact that WET is weaker than ET. Suppose now that (G, N) is such that $|\Lambda(G)| = m \cdot (|N| - 1) \cdot |N|$ for some integer $m \geq 1$. Let $s \in \mathcal{S}^{(G,N)}$ be the rule letting each player participate in m matches against every other player from N. Moreover, let p be the probability matrix with $p_{ij} = 0.5$ for all $i, j \in N$. We show that (G, N) satisfies WET with respect to s. Recalling that $\varphi_k(G, s, p) = (0.5)^{\#G} \cdot \left|\mathbf{w}_{[k]}\right|$ holds for each $k \in N$, it is then enough to prove that the number of configurations of tree winners at which a particular player is a final winner is the same across all players. We proceed as follows.

For $i, k \in N$, let $t_{ik} = (t^1_{ik}, t^2_{ik}, \ldots, t^m_{ik})$ stand for the vector of trees of G which display the corresponding m matches between player i and player k. Fix now two players i and j, and for each $w \in \mathbf{w}_{[j]}$ define the function $f(w)$ as follows:

(1) for each $x \in \{1, 2, \ldots, m\}$: set $f(w_{t^x_{ji}}) \neq w_{t^x_{ji}}$;

(2) for each $k \in N \setminus \{i, j\}$ and each $x \in \{1, 2, \ldots, m\}$: if $w_{t^x_{jk}} = j$ and $w_{t^x_{ik}} = k$, set $f(w_{t^x_{jk}}) = k$ and $f(w_{t^x_{ik}}) = i$;

(3) for each $k \in N \setminus \{i, j\}$ and each $x \in \{1, 2, \ldots, m\}$: if $w_{t^x_{jk}} = k$ and $w_{t^x_{ik}} = i$, set $f(w_{t^x_{jk}}) = j$ and $f(w_{t^x_{ik}}) = k$;

(4) for each $k \in N \setminus \{i, j\}$ and each $x \in \{1, 2, \ldots, m\}$: if $w_{t^x_{jk}} = w_{t^x_{ik}} = k$, set $f(w_{t^x_{jk}}) = f(w_{t^x_{ik}}) = k$;

(5) for each $k \in N \setminus \{i, j\}$ and each $x \in \{1, 2, \ldots, m\}$: if $w_{t^x_{jk}} = j$ and $w_{t^x_{ik}} = i$, set $f(w_{t^x_{jk}}) = j$ and $f(w_{t^x_{ik}}) = i$;

(6) for each $q \in N \setminus \{i, j\}$, each $k \in N \setminus \{q\}$, and each $x \in \{1, 2, \ldots, m\}$: set $f(w_{t^x_{qk}}) = w_{t^x_{qk}}$.

Notice that, by construction, the number of matches won by i at $f(w)$ is the same as the number of matches won by j at w and vice versa. Moreover, also by construction, for each $k \neq i, j$, the number of matches won by k is the same in both w and $f(w)$. Hence, $w \in \mathbf{w}_{[j]} \setminus \mathbf{w}_{[i]}$ implies $f(w) \in \mathbf{w}_{[i]} \setminus \mathbf{w}_{[j]}$, while $w \in \mathbf{w}_{[j]} \cap \mathbf{w}_{[i]}$ implies $f(w) \in \mathbf{w}_{[i]} \cap \mathbf{w}_{[j]}$ and thus, $f(w) \in \mathbf{w}_{[i]}$ holds.

Let us now show that f is a bijection, that is, that $w \neq w'$ for $w, w' \in \mathbf{w}_{[j]}$ implies $f(w) \neq f(w')$.

If $w_{t^x_{ij}} \neq w'_{t^x_{ij}}$ for some $x \in \{1, 2, \ldots, m\}$ or $w_{t^x_{qk}} \neq w'_{t^x_{qk}}$ for some $q \in N \setminus \{i, j\}$ and $k \in N \setminus \{q\}$, then $f(w_{t^x_{ij}}) \neq f(w'_{t^x_{ij}})$ and $f(w_{t^x_{qk}}) \neq f(w'_{t^x_{qk}})$ clearly holds due to parts (1) and (6), respectively, of the above construction.

If $j = w_{t^x_{jk}} \neq w'_{t^x_{jk}} = k$ for some $k \in N \setminus \{i, j\}$ and $x \in \{1, 2, \ldots, m\}$, then the following four cases are possible:

(a) $w_{t^x_{ik}} = w'_{t^x_{ik}} = i$. We have then $f(w_{t^x_{jk}}) = j$, $f(w_{t^x_{ik}}) = i$, $f(w'_{t^x_{jk}}) = j$, $f(w'_{t^x_{ik}}) = k$;

(b) $w_{t^x_{ik}} = w'_{t^x_{ik}} = k$. We have then $f(w_{t^x_{jk}}) = k$, $f(w_{t^x_{ik}}) = i$, $f(w'_{t^x_{jk}}) = k$, $f(w'_{t^x_{ik}}) = k$;

(c) $w_{t^x_{ik}} = i$ and $w'_{t^x_{ik}} = k$. We have then $f(w_{t^x_{jk}}) = j$, $f(w_{t^x_{ik}}) = i$, $f(w'_{t^x_{jk}}) = k$, $f(w'_{t^x_{ik}}) = k$;

(d) $w_{t^x_{ik}} = k$ and $w'_{t^x_{ik}} = i$. We have $f(w_{t^x_{jk}}) = k$, $f(w_{t^x_{ik}}) = i$, $f(w'_{t^x_{jk}}) = j$, $f(w'_{t^x_{ik}}) = k$.

Thus $f(w) \neq f(w')$ holds for each of the possible cases.

Finally, if $i = w_{t^x_{ik}} \neq w'_{t^x_{ik}} = k$ for some $k \in N \setminus \{i, j\}$ and $x \in \{1, 2, \ldots, m\}$ the situation is completely analogous to the previous one and thus, one can also in this last case conclude that $f(w) \neq f(w')$. We conclude that f is a bijection and therefore, $\left|\mathbf{w}_{[i]}\right| = \left|\mathbf{w}_{[j]}\right|$ holds. Hence, (G, N) satisfies WET with s. ■

It should be noted that Theorem 5 is not vacous in the sense that not every league-type competition satisfies WET. For example, it is easy to check that there is no way to seed 3 players in a two-matches competition so that WET is fulfilled.

5.2 Minimal leagues and monotonicity in strength

The next result shows that the monotonicity in strength requirement restricts the number of participants to two for league-type competitions, finding again an analogy between the results for elimination-type competitions and for league-type ones.

Theorem 6 *A league-type competition system* (G, N) *satisfies MS if and only if it is minimal.*

Proof. We first show that if $N = \{1, 2\}$, then (G, N) satisfies MS. Let p be an arbitrary but fixed probability matrix. Let $s \in \mathcal{S}^{(G,N)}$ and notice that, by the definition of a feasible seeding rule and $N = \{1, 2\}$, s assigns each player from N to exactly $\frac{|\Lambda(G)|}{2}$ leaves of G. Suppose now that $p_{12} > 0.5$. We have to show that $\varphi_1(G,s,p) - \varphi_2(G,s,p) \geq 0$ holds. If G contains only one tree, then the inequality directly follows from $\varphi_1(G,s,p) = p_{12} > p_{21} = \varphi_2(G,s,p)$. Suppose now that $\#G \geq 2$ and consider the following two cases.

Case 1 ($|\Lambda(G)| > |N| = 2$ and $\#G$ is odd). Note that in this case we have

$$\varphi_1(G,s,p) = p_{12}^{\#G} + p_{12}^{\#G-1} \cdot p_{21} + p_{12}^{\#G-2} \cdot p_{21}^2 + \ldots + p_{12}^{\frac{\#G+1}{2}} \cdot p_{21}^{\frac{\#G+1}{2}-1}$$

and

$$\varphi_2(G,s,p) = p_{21}^{\#G} + p_{21}^{\#G-1} \cdot p_{12} + p_{21}^{\#G-2} \cdot p_{12}^2 + \ldots + p_{21}^{\frac{\#G+1}{2}} \cdot p_{12}^{\frac{\#G+1}{2}-1}.$$

Thus, $\varphi_1(G,s,p) - \varphi_2(G,s,p) = \left(p_{12}^{\#G} - p_{21}^{\#G}\right) + p_{12} \cdot p_{21} \cdot \left(p_{12}^{\#G-2} - p_{21}^{\#G-2}\right) + \ldots + p_{12}^{\frac{\#G+1}{2}-1} \cdot p_{21}^{\frac{\#G+1}{2}-1} \cdot (p_{12} - p_{21}) > 0$, where the inequality follows from $p_{12} > 0.5$. We conclude that (G,N) satisfies MS.

Case 2 ($|\Lambda(G)| > |N| = 2$ and $\#G$ is even). When expressing $\varphi_1(G,s,p)$ and $\varphi_2(G,s,p)$ for this case there are two differences in comparison to the corresponding expressions in Case 1. First, $\frac{\#G+1}{2}$ should be replaced by $\frac{\#G}{2}+1$, and $\frac{\#G+1}{2}-1$ by $\frac{\#G}{2}-1$. Second, in both expressions the probabilities of the configurations where both players, 1 and 2, are winners should be considered, that is, the term $p_{12}^{\frac{\#G}{2}} + p_{21}^{\frac{\#G}{2}}$ should be added in both expressions. Since these configurations for player 1 and player 2 do coincide (due to $\#G$ being even), the added terms are the same for both players. Thus, they cancel out when taking the difference between $\varphi_1(G,s,p)$ and $\varphi_2(G,s,p)$. Thus, reproducing the same reasoning as in Case 1 we can conclude that (G,N) satisfies MS also in this case.

Suppose now that (G,N) satisfies MS. We show that $|N| > 2$ leads to a contradiction. Consider first the case where $|N| = 3$, $N = \{1,2,3\}$ and the following probability matrix:

$$p = \begin{pmatrix} 0.5 & 0.5+\varepsilon & 1-\varepsilon \\ & 0.5 & 1-2\varepsilon \\ & & 0.5 \end{pmatrix}$$

Take the seeding rule $s \in \mathcal{S}^{(G,N)}$ that assigns players to leaves in such a way that there is exactly one match between player 1 and player 2, and there are $\frac{|\Lambda(G)|-2}{2}$ matches between player 2 and player 3.

If $\#G = 2$, then $\varphi_1(G, s, p) \approx 0.5$ and $\varphi_2(G, s, p) \approx 1$ in contradiction to $p_{12} > 0.5$ and (G, N) satisfying MS.

If $\#G > 2$, given s and p, we have that player 2 wins with probability arbitrarily close to 1 all matches except one, and therefore he wins with probability close to one most of the matches. Thus, $\varphi_2(G, s, p) \approx 1$ and $\varphi_1(G, s, p) \approx 0$, reaching again a contradiction to $p_{12} > 0.5$ and (G, N) satisfying MS.

Suppose next that $|N| > 3$ holds and consider a probability matrix p such that $p_{23} > 0.5$ and $p_{3k} \approx 1$ for all $k \in N \setminus \{1, 2\}$. We distinguish the following two cases.

Case 1 ($|\Lambda(G)| = |N|$). In this case each feasible seeding rule assigns each player to exactly one leaf of G. Take $s_1 \in \mathcal{S}^{(G,N)}$ to be such that there are initial matches between players 1 and 2, and 3 and 4, respectively. Note that, for any configuration of tree winners, a player is a final winner in the entire competition only if he wins his unique match according to s_1. Thus, $\varphi_3(G, s_1, p) \approx 1$ and $\varphi_2(G, s_1, p) \leq 0.5$ in contradiction to $p_{23} > 0.5$ and (G, N) satisfying MS.

Case 2 ($|\Lambda(G)| > |N|$). Consider the seeding rule $s_2 \in \mathcal{S}^{(G,N)}$ defined as follows. Players are assigned to leaves in such a way that players 1 and 2 are seeded exactly once and they play against each other, while player 3 participates in each of the other $\frac{|\Lambda(G)|-2}{2}$ matches. Notice that, according to s_2 and the probability matrix p, player 3 wins with probability arbitrarily close to one all matches he is participating in, and therefore he wins with probability close to one every match of the competition except one (the one played between players 1 and 2). Hence, $\varphi_3(G, s_2, p) \approx 1$ and $\varphi_2(G, s_2, p) \approx 0$ in contradiction to $p_{23} > 0.5$ and (G, N) satisfying MS. ■

5.3 Leagues and weak monotonicity in strength

The main result in this section shows that any league-type competition satisfies WMS, provided that either each player participates in exactly one match or the total number of matches is at least $|N| - 1$. This in particular implies that any league competition covered by Theorem 5 satisfies the weak versions of both the equal treatment and monotonicity in strength properties.

Theorem 7 *Any league-type competition* (G, N) *with either* $|\Lambda(G)| = |N|$ *or* $|\Lambda(G)| \geq 2\,(|N| - 1)$ *satisfies WMS.*

Proof. Let us start with the case where $|\Lambda(G)| = |N|$. Let p be an arbitrary but fixed probability matrix and, recalling that $|N|$ is even, consider the seeding rule $s_1 : \Lambda(G) \to N$ defined by $s_1(\lambda(t)) + s_1(\lambda'(t)) = n + 1$ holding for each $t \in G$. That is, s_1 matches in the elementary binary trees of G the best player with the worst one, the second best with the second worst, and so on. Clearly, $s_1 \in \mathcal{S}^{(G,N)}$ with each player participating in exactly one match. Then, for any configuration of tree winners, a player is a final winner in the entire competition if and only if he is the winner of the match displayed by the unique tree he is seeded at by s_1. Thus, for $k \in N$, we have $\varphi_k(G, s_1, p) = p_{kk'}$ with $k' \in N$ being the player seeded by s_1 at the same tree as player k.

Suppose now that $p_{ij} > 0.5$ holds for some $i, j \in N$. We have to show that $\varphi_i(G, s_1, p) \geq \varphi_j(G, s_1, p)$ follows in such a case. For this, let $t^* \in G$ be the unique binary tree at which i is seeded by s_1 together with some other player i', and $t^{**} \in G$ be the unique binary tree at which j is seeded by s_1 together with some other player j'. If $i' = j$, then i and j are seeded by s_1 to the same tree and thus, $\varphi_i(G, s_1, p) = p_{ij} > p_{ji} = \varphi_j(G, s_1, p)$ follows. If $i' \neq j$, we have from $p_{ij} > 0.5$ that iRj holds, while $j'Ri'$ holds due to the construction of s_1. Thus, $p_{ii'} \geq p_{jj'}$ follows by Lemma 2. We have then $\varphi_i(G, s_1, p) = p_{ii'} \geq p_{jj'} = \varphi_j(G, s_1, p)$ as required for showing that (G, N) satisfies WMS with respect to s_1.

Let us now consider a league-type competition (G, N) such that $|\Lambda(G)| \geq 2\,(|N| - 1)$ holds. Fix a probability matrix p and consider the seeding rule s_2 defined as follows: player 1 is seeded to each tree of G, each player $k \in \{3, \ldots, n\}$ is seeded to exactly one tree of G, and player 2 is seeded to each of the remaining trees of G. Since there are at least $(|N| - 1)$ matches in the competition, $s_2 \in \mathcal{S}^{(G,N)}$ follows. We show that (G, N) satisfies WMS with respect to s_2 in tree steps. In what follows $\mathbf{t}^{(12)}$ stands for the set of matches played between 1 and 2 according to s_2.

Step 1: If $p_{12} > 0.5$, then $\varphi_1(G, s_2, p) \geq \varphi_2(G, s_2, p)$.

Proof. Recall that $\varphi_1(G, s_2, p) = \sum\limits_{w \in \mathbf{w}_{[1]}} pr(w)$ and $\varphi_2(G, s_2, p) = \sum\limits_{w \in \mathbf{w}_{[2]}} pr(w)$. Given a configuration w of winners let $\mathbf{t}^{(12)}_{w_t=1}$ be the set of trees t in $\mathbf{t}^{(12)}$ such that $w_t = 1$, and let $\mathbf{t}^{(12)}_{w_t=2}$ be the set of trees t in $\mathbf{t}^{(12)}$ such that $w_t = 2$.

Define the mapping $f : \mathbf{w}_{[2]} \to \mathbf{w}_{[1]}$ by just exchanging, for each $t \in \mathbf{t}^{(12)}$, the winner of the tree. That is, for all $t \in \mathbf{t}^{(12)}$, $w_t = 1$ if and only if $f(w)_t = 2$ and $w_t = 2$ if and only if $f(w)_t = 1$. Notice that for $w, w' \in \mathbf{w}_{[2]}$ with $w \neq w'$,

$f(w) \neq f(w')$ follows and thus, f is a bijection between $\mathbf{w}_{[2]}$ and a subset of $\mathbf{w}_{[1]}$.

Notice also that, for any configuration w of tree winners we have: $pr(w) = \prod_{t \in G \setminus \mathbf{t}^{(12)}} \varphi_{w_t}(t, s_2, p) \prod_{t \in \mathbf{t}^{(12)}_{w_t=1}} p_{12} \prod_{t \in \mathbf{t}^{(12)}_{w_t=2}} p_{21}$.

Finally, notice that, by $w \in \mathbf{w}_{[2]}$, player 2 wins at w at least as many matches against player 1 as player 1 against player 2, that is, $|\mathbf{t}^{(12)}_{w_t=2}| \geq |\mathbf{t}^{(12)}_{w_t=1}|$. Thus, $|\mathbf{t}^{(12)}_{f(w)_t=1}| \geq |\mathbf{t}^{(12)}_{f(w)_t=2}|$. We have then from $p_{12} > p_{21}$ that $\prod_{t \in \mathbf{t}^{(12)}_{w_t=1}} p_{12} \prod_{t \in \mathbf{t}^{(12)}_{w_t=2}} p_{21} \leq \prod_{t \in \mathbf{t}^{(12)}_{f(w)_t=1}} p_{12} \prod_{t \in \mathbf{t}^{(12)}_{(w)_t=2}} p_{21}$ holds. Therefore, $pr(w) \leq pr(f(w))$ holds for each $w \in \mathbf{w}_{[2]}$ which allows us to conclude that $\varphi_1(G, s_2, p) \geq \varphi_2(G, s_2, p)$.

Step 2: If $p_{ij} > 0.5$ for $i \in \{1, 2\}$ and $j \in N \setminus \{1, 2\}$, then $\varphi_i(G, s_2, p) \geq \varphi_j(G, s_2, p)$.

Proof. Notice first that if $|\mathbf{t}^{(12)}| > 2$ then $\mathbf{w}_{[j]} = \emptyset$ and $\varphi_i(G, s_2, p) \geq 0 = \varphi_j(G, s, p)$ immediately follows.

If $|\mathbf{t}^{(12)}| = 2$, then $w \in \mathbf{w}_{[j]}$ implies that every player wins exactly one match (including players 1 and 2) and all players are winners of the entire competition. Thus, $w \in \mathbf{w}_{[j]}$ implies $w \in \mathbf{w}_{[i]}$, that is, $\mathbf{w}_{[j]} \subseteq \mathbf{w}_{[i]}$ holds and hence, $\varphi_i(G, s_2, p) \geq \varphi_j(G, s_2, p)$ follows.

Suppose now that $|\mathbf{t}^{(12)}| = 1$. Recall that only configurations in $\mathbf{w}_{[i]} \setminus \mathbf{w}_{[j]}$ and in $\mathbf{w}_{[j]} \setminus \mathbf{w}_{[i]}$ do matter for the comparison of $\varphi_i(G, s_2, p)$ and $\varphi_j(G, s_2, p)$. If $i = 1$, then $\mathbf{w}_{[j]} \setminus \mathbf{w}_{[1]} = \{w\}$ with w consisting of player 1 losing all his matches and thus $pr(w) = p_{21} \cdot p_{31} \cdot \ldots \cdot p_{n1}$. On the other hand, the configuration w' of tree winners where player 1 wins all his matches does definitely belong to $\mathbf{w}_{[1]} \setminus \mathbf{w}_{[j]}$, with $pr(w') = p_{12} \cdot p_{13} \cdot \ldots \cdot p_{1n}$. Then, given that $p_{1k} \geq p_{k1}$ for each $k \in N \setminus \{1\}$ (with strict inequality holding for $k = j$), we have that $\varphi_1(G, s_2, p) > \varphi_j(G, s_2, p)$.

On the other hand, $i = 2$ implies $\mathbf{w}_{[j]} \setminus \mathbf{w}_{[2]} = \{w\}$ with w conisting of 1 only winning his match against 2 and $pr(w) = p_{12} \cdot p_{31} \cdot p_{41} \cdot \ldots \cdot p_{n1}$. Consider now the configuration w' of tree winners which differs from w only with respect to the fact that at w' player 2 wins his unique match against player 1 and player j loses his unique match against player 1. This involves that $w' \in \mathbf{w}_{[2]} \setminus \mathbf{w}_{[j]}$ with $pr(w') \geq pr(w)$ due to $p_{21} \geq p_{j1}$ following from $p_{2j} > 0.5$ and condition (2). We conclude then that $\varphi_2(G, s_2, p) \geq \varphi_j(G, s_2, p)$.

Step 3: If $p_{ij} > 0.5$ for some $i, j \in N \setminus \{1, 2\}$, then $\varphi_i(G, s_2, p) \geq \varphi_j(G, s_2, p)$.

Proof. Recall that in this situation each of the players i and j participates in exactly one match against player 1. Let us then consider the following three possible cases.

Case 1 ($|\mathbf{t}^{(12)}| = 1$). In this case there is a unique configuration $w \in \mathbf{w}_{[j]} \setminus \mathbf{w}_{[i]}$ consisting of player j winning his match against player 1 and player 1 winning only his match against player i. The probability of w is then $pr(w) = p_{j1} \cdot p_{1i} \cdot \prod_{k \neq i,j} p_{k1}$. Similarly, there is a unique configuration $w' \in \mathbf{w}_{[i]} \setminus \mathbf{w}_{[j]}$ with $pr(w') = p_{i1} \cdot p_{1j} \cdot \prod_{k \neq i,j} p_{k1}$. By $p_{ij} > 0.5$ and condition (2), $p_{1j} \cdot p_{i1} \geq p_{j1} \cdot p_{1i}$ follows. We have then $pr(w') \geq pr(w)$ which implies $\varphi_i(G, s_2, p) \geq \varphi_j(G, s_2, p)$.

Case 2 ($|\mathbf{t}^{(12)}| = 2$). In this case each of the two configurations of tree winners in $\mathbf{w}_{[j]}$ has the following structure: each of the players in $N \setminus \{1, 2\}$ (including i and j) wins his match against player 1; player 1 uniquely wins one of his matches against player 2, and player 2 wins the other match that players 1 and 2 are playing together. Clearly then, $\mathbf{w}_{[j]} = \mathbf{w}_{[i]}$ and thus, $\varphi_i(G, s_2, p) = \varphi_j(G, s_2, p)$ holds.

Case 3 ($|\mathbf{t}^{(12)}| > 2$). In this case, given that there are more than two matches between player 1 and 2, $\mathbf{w}_{[i]} = \mathbf{w}_{[j]} = \emptyset$. Thus $\varphi_i(G, s_2, p) = \varphi_j(G, s_2, p) = 0$.
■

It is not difficult to prove that the seeding used when $|N| = |\Lambda(G)|$ (where the best player is matched with the worst one, the second best with the second worst, etc.) is the only one for which (G, N) satisfies WMS. However, there are other seedings different from those used in the two cases where $|\Lambda(G)| \geq 2(|N| - 1)$ for which (G, N) satisfies WMS.

Finally, it should be noted that Theorem 7 is not vacous in the sense that there exist league-type competitions that do not satisfy WMS. For example, it can be proved that a league-type competition consisting of 3 matches and 5 players does not satisfy WMS.

6 Concluding remarks and further research

Our results enable the evaluation and comparison of different competition systems on the basis of two reasonable principles of fairness. The analysis has gone also into detail about the particular assignments assuring the fulfillment of these principles. In this way the model connects with the specific line of research in Management Mathematics devoted to the study of seeding procedures in elimination-type competitions with few players (cf. Horen and Riezmann 1985, Hwang 1982, Prince et al. 2013, and Schwenk 2000). Specially remarkable has turned out to be the the study of how weak monotonicity shapes the structure a competition. There, a singular structure, that we have called an *antler*, arises as a referential type of competition.

In general, our results show that there is a quite limited number of competition systems that are *fair* in the sense of *simultaneously* satisfying both types of fairness in their corresponding strong or weak forms.

Let us first consider elimination-type competitions. Due to Theorem 1 and Theorem 2, only balanced competitions with two players do satisfy MS together with either ET or WET. Clearly, this fact has the flavor of an impossibility result. Weakening MS to WMS (Theorem 3) does not heavily enlarge the set of admissible structures since the only balanced trees that are antler-free are either elementary or balanced of height 2. Thus, replacing MS by WMS adds only 4-players balanced competitions as "fair".

In the case of leagues, MS combined with either ET or WET produces again a degenerate competition consisting of a two-players league (Theorem 4). Weakening MS to WMS and imposing it together with ET slightly enlarges the class of admissible leagues to those where each participant plays a unique match. On the other hand, the combination of WMS and WET results in an expansion of the mentioned class of admissible leagues as to include leagues allowing for players to participate the same number of times in at least $|N| - 1$ different matches (Theorems 5 and 7).

Round-robin tournaments deserve a special attention. In Theorem 6 we prove that m-round-robin tournaments (which include single round-robin and double-round robin tournaments) satisfy WET. Presumably they satisfy WMS too, but the formal proof remains an open question.

In line with the closest literature, our analysis is based on a reasonable assumption with respect to the winning probabilities, namely that $p_{ij} \geq 0.5$ implies $p_{ik} \geq p_{jk}$ for each $k \in N \setminus \{i, j\}$. This is a sufficient but not a necessary condition for the associated binary relation of strength to be transitive. A natural extension of the model could consist of making weaker assumptions on p, for example, the minimal one to ensure transitivity of the strength relation, namely that $p_{ij} \geq 0.5$ and $p_{jk} \geq 0.5$ implies $p_{ik} \geq 0.5$. However, by such an assumption being weaker, the number of admissible probability matrices that satisfy it would be greater, and therefore the class of competition systems that would satisfy the different fairness axioms considered would be even smaller than the ones under our assumption on p.

We have excluded from the analysis the possibility of random seedings and reseedings as well as the study of double elimination competitions and the typical two-stage competitions consisting of qualification parallel round-robin tournaments followed by a knockout competition. However, our model sets the fundamentals for approaching such problems.

Another intriguing extension of our setup concerns the analysis of competition systems representable by forests of non-elementary binary trees. For example, one could imagine a variant of a round-robin competition where, at each round, players do not play a single match but are grouped to play 4-player (or larger) knockout or stepladder competitions with the final winner being the player who win the most sub-competitions. David (1959) considers "repeated knockout tournaments" as interesting systems to be studied but, to the best of our knowledge, these kinds of competitions have neither been applied in sports nor theoretically analyzed.

7 Appendix

This appendix contains the proof of Theorem 3 from Section 4.

Before presenting the proof of Theorem 3, we need to introduce the following additional concept. We say that a binary tree t with $h(t) = 3$ is an *one-bye antler*, if $|\Lambda(t)| = 7$ with $|\Lambda^3(t)| = 6$ and $|\Lambda^2(t)| = 1$. Clearly, any one-bye antler is an extended stepladder of degree 3. Further, for t and t' being binary trees, we say that:

- t' is an *extension from the leaves* of t if t' and t have the same root and $\Lambda(t) \subseteq \Lambda(t')$;
- t' is an *extension from the root* of t if t is a subtree of t';
- t' is a *limited extension from the root* of t, if t is a subtree of t' and $\Lambda^{h(t)}(t) \subseteq \Lambda^{h(t')}(t')$.

Thus, a limited extension from the root of a tree t never has leaves at a height that is greater than the height of any of the leaves of t.

We are now ready to present the proof of Theorem 3.

Proof (Sufficiency). Given a strength relation R defined on the player set N, we have to prove that an elimination-type competition system (t, N) with t being antler-free satisfies WMS with respect to s_{ib}.

Let (t, N) be such that t is antler-free and $s = s_{ib}$. By Lemma 5, $t \in ES_2^*$. Take a maximal root-to-leaf path $\gamma(t)$ and notice that, in view of Remark 1, $t \in ES_2^*$ irrespective of the choice of $\gamma(t)$. For $s \in \mathcal{S}^{(t,N)}$, $v \in \mathbf{v}(\gamma(t))$, and any probability matrix p, we denote by $p_i^v(s)$ the probability with which player $i \in N$ reaches v under a given seeding s and by v_h the unique leaf in $\mathbf{v}(\gamma(t))$. Moreover, we collect in the set $S_v^1(s)$ all players whose first match in the competition is against a player who has already reached some $v' \in \mathbf{v}(\gamma(t))$ with $\ell(v') > \ell(v)$; correspondingly, $S_v^2(s)$ stands for the set of all players who had to play an initial match before having the possibility to meet a player who has already reached some node from $\mathbf{v}(\gamma(t))$ at a higher level than v. Notice that for each $i \in N$ we have due to $t \in ES_2^*$ that either $i = s(v_h)$ or $i \in (S_v^1(s) \cup S_v^2(s))$ holds for some $v \in \mathbf{v}(\gamma(t))$.

We denote by v^x the closest predecessor belonging to $\mathbf{v}(\gamma(t))$ of $x = s(\lambda)$ for some $\lambda \in \Lambda(t)$. Note that, for each $v \in \mathbf{v}(\gamma(t))$, any probability matrix p, and any two players $k, j \in N$ with $p_{kj} > 0.5$ and $s^{-1}(k), s^{-1}(j) \in \Lambda(t_v)$, we have that $p_k^v(s) > p_j^v(s)$ implies $\varphi_k(t, s, p) > \varphi_j(t, s, p)$. The reason is that for each $i \in N$ with $s^{-1}(i) \in \Lambda(t_v)$ we have

$$\varphi_i(t, s, p) = p_i^v(s) \times \prod_{x \in S_{v'}^1(s): \ell(v') < \ell(v)} p_{ix} \times \prod_{y,z \in S_{v'}^2(s): \ell(v') < \ell(v)} (p_{iy}p_{yz} + p_{iz}p_{zy}).$$

Hence, $\varphi_k(t, s, p) > \varphi_j(t, s, p)$ is just implied by $p_k^v(s) > p_j^v(s)$, $p_{kx} \geq p_{jx}$ for each $x \in N$ following from condition (2), and p_{yz} (p_{zy}) being independent of any other parameter in the respective formulae for k and j.

Thus, in order to prove the sufficiency part of Theorem 3, let us now consider the increasingly balanced rule s_{ib}. We have to show that (t, N) satisfies WMS with respect to s_{ib}. In view of the argument just explained, assuming that $p_{kj} > 0.5$, it is enough to find a node $v \in \mathbf{v}(\gamma(t))$ with $s_{ib}^{-1}(k), s_{ib}^{-1}(j) \in \Lambda(t)$ and $p_k^v(s_{ib}) > p_j^v(s_{ib})$. We distinguish the following three possible cases:

(i) $\ell(s_{ib}^{-1}(k)) < \ell(s_{ib}^{-1}(j))$ and there is no $m \in N$ with $\ell(s_{ib}^{-1}(m)) = \ell(s_{ib}^{-1}(k))$. Clearly then, player k does not need to win any match to reach $v^k \in \mathbf{v}(\gamma(t))$. Therefore, given that s_{ib} seeds worse players to higher levels, $p_k^{v^k}(s_{ib}) > 0.5$ and, since j has to defeat k in order to reach v^k, $p_j^{v^k}(s_{ib}) < 0.5$.

(ii) $\ell(s_{ib}^{-1}(k)) < \ell(s_{ib}^{-1}(j))$ and there exists $m \in N$ with $\ell(s_{ib}^{-1}(m)) = \ell(s_{ib}^{-1}(k))$. In this case player k is involved in a balanced sub-competition of four players. Let v^* be the root of the sub-competition (note that $v^* \in \mathbf{v}(\gamma(t))$) with $p_k^{v^*}(s_{ib})$ being the probability for player k to win the sub-competition. As for player j, $p_j^{v^*}(s_{ib})$ is the product of two probabilities:

the probability to reach the sub-competition, that is, to reach the node $v \in \mathbf{v}(\gamma(t))$ such that $\ell(v) = \ell(s_{ib}^{-1}(k))$, and the probability to win the sub-competition. Given that the sub-competition is played under a balanced seeding, we know by Lemma 7 that for any probability matrix with $p_{kj} > 0.5$, the probability for k to win the sub-competition is weakly greater than the one for j. We conclude then that $p_k^{v^*}(s_{ib}) > p_j^{v^*}(s_{ib})$ should hold.

(iii) $\ell(s_{ib}^{-1}(k)) = \ell(s_{ib}^{-1}(j))$. Also in this case players k and j are involved in a balanced sub-competition of four players. Following the same reasoning as in (ii), we obtain $p_k^{v^*}(s_{ib}) > p_j^{v^*}(s_{ib})$.

Proof of Theorem 3 (Necessity). We have to prove that if (t, N) satisfies WMS with respect to some seeding rule s, then t is antler-free and $s = s_{ib}$. In order to prove that t is antler-free in such a case, we will show that if t contains an antler, then the competition system (t, N) violates WMS. More precisely, in Steps 1 to 9 of the proof we show progressively and in an exhaustive way that all the different types of structures that can contain an antler do violate WMS. In Step 10 we finally prove that the seeding rule s with respect to which (t, N) satisfies WMS is necessarily $s = s_{ib}$.

Step 1 Let (t, N) be an elimination-type competition system with t being a symmetric antler. Then (t, N) violates WMS.

Proof. Notice that $N = \{1, \ldots, 6\}$ holds in this case. Let λ_ℓ^2 and λ_r^2 be the two leaves of t that are at level 2 of its left and right branch, respectively. Similarly, let λ_ℓ^{3a} and λ_ℓ^{3b} be the two leaves at level 3 of t's left branch, while λ_r^{3a} and λ_r^{3b} be the two leaves at level 3 of t's right branch. We proceed by reduction to the absurd, that is, we assume that (t, N) satisfies WMS and then prove that we reach a contradiction. By Lemma 8, any $s \in \mathcal{S}^{(t,N)}$ with respect to which (t, N) satisfies WMS should be such that the two strongest players are seeded to λ_ℓ^2 and λ_r^2. In view of Lemma 6 we assume without loss of generality that these players are 1 and 2, and that $s(\lambda_\ell^2) = 1$ and $s(\lambda_r^2) = 2$. There are then six possible non-equivalent seedings for the remaining players:

(i) $s(\lambda_\ell^{3a}) = 3$, $s(\lambda_\ell^{3b}) = 4$, $s(\lambda_r^{3a}) = 5$, $s(\lambda_r^{3b}) = 6$.
(ii) $s(\lambda_\ell^{3a}) = 3$, $s(\lambda_\ell^{3b}) = 5$, $s(\lambda_r^{3a}) = 4$, $s(\lambda_r^{3b}) = 6$.
(iii) $s(\lambda_\ell^{3a}) = 3$, $s(\lambda_\ell^{3b}) = 6$, $s(\lambda_r^{3a}) = 4$, $s(\lambda_r^{3b}) = 5$.
(iv) $s(\lambda_\ell^{3a}) = 4$, $s(\lambda_\ell^{3b}) = 5$, $s(\lambda_r^{3a}) = 3$, $s(\lambda_r^{3b}) = 6$.
(v) $s(\lambda_\ell^{3a}) = 4$, $s(\lambda_\ell^{3b}) = 6$, $s(\lambda_r^{3a}) = 3$, $s(\lambda_r^{3b}) = 5$.
(vi) $s(\lambda_\ell^{3a}) = 5$, $s(\lambda_\ell^{3b}) = 6$, $s(\lambda_r^{3a}) = 3$, $s(\lambda_r^{3b}) = 4$.

In order to prove that (t, N) violates WMS we next show that for each of the six possible seedings we can find a probability matrix $p \in p_R$ defined on N such that there exists $i \in N$ with $p_{i-1,i} > 0.5$ (and therefore $(i-1)Pi$) and $\varphi_i(t,s,p) > \varphi_{i-1}(t,s,p)$.

(i) Take p as follows: $p_{jk} > 0.5$ if $j < k$; $p_{j6} \approx 1$ for all $j < 6$, and $p_{jk} \approx 0.5$ for all $j,k < 6$. We have then $\varphi_5(t,s,p) \approx 0.25 > 0.125 \approx \varphi_4(t,s,p)$ while $p_{45} > 0.5$.

(ii) Consider the same probability matrix p as in case (i). Then $\varphi_4(t,s,p) \approx 0.25 > 0.125 \approx \varphi_3(t,s,p)$ while $p_{34} > 0.5$.

(iii) Let p be such that $p_{jk} > 0.5$ if $j < k$; $p_{jk} \approx 1$ if $j \in \{1,2\}$ and $k \in \{4,5,6\}$, and $p_{jk} \approx 0.5$, otherwise. Then $\varphi_2(t,s,p) \approx 0.5 > 0.375 \approx \varphi_1(t,s,p)$ while $p_{12} > 0.5$.

(iv) Take p as follows: $p_{jk} > 0.5$ if $j < k$; $p_{jk} \approx 1$ if $j \in \{1,2\}$ and $k = 6$, and $p_{jk} \approx 0.5$, otherwise. Then $\varphi_2(t,s,p) \approx 0.375 > 0.25 \approx \varphi_1(t,s,p)$ while $p_{12} > 0.5$.

(v) Let p be as follows: $p_{jk} > 0.5$ if $j < k$; $1 \approx p_{15} \approx p_{16} \approx p_{25} \approx p_{26} \approx p_{36} \approx p_{46}$, and $p_{jk} \approx 0.5$, otherwise. Then $\varphi_2(t,s,p) \approx 0.375 > 0.25 \approx \varphi_1(t,s,p)$ while $p_{12} > 0.5$.

(vi) Consider the same probability matrix p as in cases (i) and (ii). Then $\varphi_5(t,s,p) \approx 0.25 > 0.125 \approx \varphi_4(t,s,p)$ while $p_{45} > 0.5$.

For later steps in the proof it is important to remark that, according to the probability matrices shown above and the one shown in the proof of Lemma 8, whatever seeding $s \in \mathcal{S}^{(t,N)}$ we consider in a symmetric antler t, not only there exist a probability matrix p and $i \in N$ such that $p_{i-1,i} > 0.5$ and $\varphi_i(t,s,p) > \varphi_{i-1}(t,s,p)$. It also holds that it is possible to find such a matrix p where $p_{i-1,i} \approx 0.5$ and $p_{ik} \approx p_{i-1,k}$ for all $k \in N \setminus \{i-1,i\}$.

Step 2 Let (t,N) be an elimination-type competition system with t being an asymmetric antler. Then (t,N) violates WMS.

Proof. Clearly $N = \{1,\dots,6\}$ holds also in this case. Assume without loss of generality that t's left branch has four leaves at level $h(t) = 3$, and denote them (from left to right) by λ_ℓ^{3a}, λ_ℓ^{3b}, λ_ℓ^{3c} and λ_ℓ^{3d}. Clearly then, t's right branch has two leaves (λ_r^{2a} and λ_r^{2b}) at level 2. Let v_1 be the node in the left branch of t which is an immediate successor of the root of t. Note that $\{\lambda_\ell^{3a}, \lambda_\ell^{3b}, \lambda_\ell^{3c}, \lambda_\ell^{3d}\}$ are the leaves of the balanced subtree t_1 of t whose root is v_1. We proceed again by reduction to the absurd. Assume that (t,N) satisfies WMS. By Lemma 8, the two strongest players should be seeded to the two leaves at level 2. In view of Lemma 6, we assume without loss of generality that these players are 1 and 2 and that $s(\lambda_\ell^2) = 1$ and $s(\lambda_r^2) = 2$. Fix then a seeding rule $s' : \Lambda(t_1) \to \{3,4,5,6\}$, note that $s' \in \mathcal{S}^{(t_1, N\setminus\{1,2\})}$, and consider the following two possibilities for it.

Case 1 ($s' \neq s'_{b4}$). Consider the matrix used in the proof of Lemma 7 and apply it to the set of players $\{3,4,5,6\}$. According to the mentioned proof, if $s \neq s_{b4}$ we can always find a pair of players $i, i-1 \in \{3,4,5,6\}$ and a probability matrix $p' \in p_{R_{|\{3,4,5,6\}}}$ such that $p'_{i-1,i} > 0.5$ and $p_i'^{v_1} > p_{i-1}'^{v_1}$. Moreover $p'_{i-1,i} \approx 0.5$ and $p'_{i-1,k} \approx p'_{i,k}$ holds for all $k \in \{3,4,5,6\}$ as well. Let then p be defined on N such that $p_{jk} = p'_{jk}$ for all $j,k > 2$; $p_{13} = p_{23} = p_{12} = 0.5$, and therefore $p_{jk} = p_{3k}$ for all $j < 3$ and all $k \in N$. By $p_{i3} \approx p_{i-1,3}$ and by p satisfying condition (2) we have $p_{i2} \approx p_{i-1,2}$ and $p_{i1} \approx p_{i-1,1}$. Thus, $p_{i-1,k} \approx p_{i,k}$ for all $k \in N$.

Notice then that $\varphi_i(t,s,p) = p_i'^{v_1} \cdot (p_{12}p_{i1} + p_{21}p_{i2})$ and $\varphi_{i-1}(t,s,p) = p_{i-1}'^{v_1} \cdot (p_{12}p_{i-1,1} + p_{21}p_{i-1,2})$. By $p_i'^{v_1} > p_{i-1}'^{v_1}$, $p_{i1} \approx p_{i-1,1}$ and $p_{i2} \approx p_{i-1,2}$, we have $\varphi_i(t,s,p) > \varphi_{i-1}(t,s,p)$ in contradiction to (t,N) satisfying WMS with respect to s.

Case 2 ($s' = s'_{b4}$). Consider the following probability matrix $p \in p_R$:

$$p = \begin{pmatrix} 0.5 & 0.5 & 0.5+\varepsilon & 1-2\varepsilon & 1-\varepsilon & 1-\varepsilon \\ & 0.5 & 0.5+\varepsilon & 1-2\varepsilon & 1-\varepsilon & 1-\varepsilon \\ & & 0.5 & 1-3\varepsilon & 1-2\varepsilon & 1-2\varepsilon \\ & & & 0.5 & 1-3\varepsilon & 1-3\varepsilon \\ & & & & 0.5 & 0.5 \\ & & & & & 0.5 \end{pmatrix}$$

According to p, s'_{b4} matches either 3 with 6 and 4 with 5, or it matches 3 with 5 and 4 with 6. In any case we have $p_{23} > 0.5$ and, after making the necessary computations, $\varphi_3(t,s,p) \approx 0.5 > 0.25 \approx \varphi_2(t,s,p)$. Thus, (t,N) violates WMS with respect to s.

Like in Step 1, it is important to remark that, according to the probability matrix shown above and the ones shown in the proofs of Lemma 7 and Lemma 8, whatever seeding $s \in \mathcal{S}^{(t,N)}$ we consider in an asymmetric antler t, not only there exist a probability matrix p and $i \in N$ such that $p_{i-1,i} > 0.5$ and $\varphi_i(t,s,p) > \varphi_{i-1}(t,s,p)$. It also holds that it is possible to find such a matrix p where $p_{i-1,i} \approx 0.5$ and $p_{ik} \approx p_{i-1,k}$ for all $k \in N \setminus \{i-1, i\}$.

Step 3 Let (t,N) be an elimination-type competition system with t being a one-bye antler. Then (t,N) violates WMS.

Proof. We proceed again by reduction to the absurd assuming that (t,N) violates WMS. Notice that $N = \{1,\ldots,7\}$ holds in this case. Assume without loss of generality that t's left branch has four leaves at level $h(t) = 3$, and denote them (from left to right) by λ_ℓ^{3a}, λ_ℓ^{3b}, λ_ℓ^{3c} and λ_ℓ^{3d}. Clearly then t's right branch has two leaves (λ_r^{3a} and λ_r^{3b}) at that same level and one leaf (λ_r^2) at level 2. By Lemma 8 the best player should be seeded to $s(\lambda_r^2)$. In view of Lemma 6 assume without loss of generality that $s(\lambda_r^2) = 1$. We distinguish now two possibilities depending on the leaf player 7 has been seeded at.

Case 1 ($s(\lambda) = 7$ for some λ of t's right branch). There are two subcases: (i) $s(\lambda_r^{4a}) = 6$ and $s(\lambda_r^{4b}) = 7$ (or vice versa without loss of generality) and (ii): $s(\lambda_r^{4a}) = x$ and $s(\lambda_r^{4b}) = 7$ (or vice versa) with $x < 6$.

(i) If $s(\lambda_r^{4a}) = 6$ and $s(\lambda_r^{4b}) = 7$, then let $p \in p_R$ be a probability matrix such that $p_{jk} > 0.5$ for all $j, k \in N$ with $j < k$, $p_{j7} \approx 1$ for all $j \in N \setminus \{7\}$, and $p_{jk} \approx 0.5$ for all $j, k \in N \setminus \{7\}$. By making the necessary calculations, we obtain $\varphi_6(t, s, p) \approx 0.25 > 0.125 \approx \varphi_5(t, s, p)$ while $p_{67} > 0.5$, in contradiction to (t, N) satisfying WMS.

(ii) If $s(\lambda_r^{4a}) = x$ and $s(\lambda_r^{4b}) = 7$, then let $p \in p_R$ be a probability matrix such that $p_{jk} > 0.5$ for all $j, k \in N$ with $j < k$, $p_{jk} \approx 0.5$ for all $j, k \in N \setminus \{1\}$, and $p_{1k} = 0.7$ for all $k \in N \setminus \{1\}$. By making the necessary calculations, we obtain $\varphi_6(t, s, p) \approx 0.09 > 0.075 \approx \varphi_x(t, s, p)$ while $p_{x6} > 0.5$, reaching again a contradiction.

Case 2 ($s(\lambda) = 7$ for some λ of t's left branch). Let x be the player whose initial match is against player 7 and suppose w.l.o.g., that $s(\lambda_\ell^{4c}) = 7$ and $s(\lambda_\ell^{4d}) = x$. Then remove from t the nodes λ_ℓ^{4c} and λ_ℓ^{4d} as well as the corresponding edges to their immediate predecessor v_λ. Notice that the remaining subgraph t^A of t is a symmetric antler with v_λ being now a leaf of t^A. Consider the seeding rule $s' : \Lambda(t^A) \to \{1, \ldots, 6\}$ defined as follows: $s'(v_\lambda) = x$ and $s'(\lambda) = s(\lambda)$ for each $\lambda \in \Lambda(t^A) \setminus \{v_\lambda\}$, and notice that $s' \in \mathcal{S}^{(t^A, N\setminus\{7\})}$. In other words, s' can be interpreted as a situation in which x wins his match against 7 and the remaining matches are not played yet.

By Step 1, the competition system $(t^A, N \setminus \{7\})$ violates WMS. That is, there exists a probability matrix $p' \in p_{R_{|N\setminus\{7\}}}$ such that, for some $i \in N \setminus \{7\}$, $p'_{i-1,i} > 0.5$ and $\varphi'_i(t^A, s', p') > \varphi'_{i-1}(t^A, s', p')$. Moreover, we know that p' can be constructed in such a way that $p'_{i-1,i} \approx 0.5$ and $p'_{ik} \approx p'_{i-1,k}$ holds for each $k \in N \setminus \{i-1, i, 7\}$.

Consider now the probability matrix $p \in p_R$ such that $p_{jk} = p'_{jk}$ for all $j, k \in N \setminus \{7\}$, and $p_{k7} \approx 1$ for all $k \in N \setminus \{7\}$. For the final winning probabilities of each $k < 7$ we have by construction that $\varphi_k(t, s, p) \approx \varphi'_k(t^A, s', p')$. By hypothesis, $p'_{i-1,i} > 0.5$ and $\varphi'_i(t^A, s', p') > \varphi'_{i-1}(t^A, s', p')$ holds and thus,

$p_{i-1,i} > 0.5$ and $\varphi_i(t,s,p) > \varphi_{i-1}(t,s,p)$ holds as well. Hence, (t,N) violates WMS in this case too. Moreover, like in Step 1, it is interesting to remark for the later steps in the proof that, for any seeding in a one-bye antler t we can always find a probability matrix $p \in p_R$ that makes the competition system (t,N) violate WMS and such that $p_{i-1,i} \approx 0.5$ and $p_{ik} \approx p_{i-1,k}$ holding for some $i \in N$ and all $k \in N \setminus \{i-1, i\}$.

Step 4 Let (t,N) be an elimination-type competition system with $h(t) = 3$ and t being balanced. Then (t,N) violates WMS.

Proof. Notice that $N = \{1,\ldots,8\}$ holds in this case. Let x be the player whose initial match is against player 8, and remove from t the nodes $s^{-1}(8)$ and $s^{-1}(x)$ together with the corresponding edges to their immediate predecessor, v_λ. Notice that the remaining subgraph t^A of t is a one-bye antler with v_λ being now a leaf of t^A. Consider then the seeding rule $s' : \Lambda(t^A) \to \{1,\ldots,7\}$ defined as follows: $s'(v_\lambda) = x$ and $s'(\lambda) = s(\lambda)$ for each $\lambda \in \Lambda(t_A) \setminus \{v_\lambda\}$, and notice that $s' \in \mathcal{S}^{(t^A, N\setminus\{8\})}$.

By Step 3, $(t^A, N \setminus \{8\})$ violates WMS. That is, there exists a probability matrix $p' \in p_{R_{|N\setminus\{8\}}}$ such that $p'_{ij} > 0.5$ and $\varphi'_j(t^A,s',p') > \varphi'_i(t^A,s',p')$ for the corresponding final winning probabilities of some $i,j \in N \setminus \{8\}$. Moreover, we know that p' can be constructed in such a way that $p'_{ij} \approx 0.5$ and $p'_{ik} \approx p'_{jk}$ holds for each $k \in N \setminus \{8\}$.

Consider now the probability matrix p defined on N such that $p_{jk} \approx p'_{jk}$ holds for all $j,k < 8$ and $p_{k8} \approx 1$ holds for all $k < 8$. For the final winning probabilities of each $k < 8$ we have by construction that $\varphi_k(t,s,p) \approx \varphi'_k(t^A,s',p')$. Therefore, $p_{ij} > 0.5$ and $\varphi_j(t,s,p) > \varphi_i(t,s,p)$ as it is required to prove that the competition system (t,N) violates WMS. Moreover, by construction, p is such that $p_{ij} \approx 0.5$ and $p_{ik} \approx p_{jk}$ for each $k \in N$.

Step 5 Let (t,N) be an elimination-type competition system with t being a limited extension from the root of an antler. Then (t,N) violates WMS.

Proof. Let t^A be the (symmetric or asymmetric) antler contained in t and fix *any* $s \in \mathcal{S}^{(t,N)}$. Since s is arbitrary, in order to show that (t,N) violates WMS it suffices to show that the violation holds with respect to s. Let $N'_{t_A}(s)$ be the set of players seeded by s to a leaf of t^A. For notational convenience, when $i \in N'_{t^A}(s)$ we will label him as i'.

By Lemma 8, Step 1 in the case of symmetric antlers, and Step 2 in the case of asymmetric antlers, we know that for any seeding in t^A we can find a probability matrix $p \in p_{R_{|N'_{t^A}(s)}}$ that makes $\left(t^A, N\right)$ violate WMS. In particular, for $s' = s_{|\Lambda(t^A)}$, there are such a matrix p' and players $i', h' \in N'_{t^A}(s)$ with $p'_{h',i'} > 0.5$ and $\varphi_{i'}(t^A, s', p') > \varphi_{h'}(t^A, s', p')$. Moreover, we know that p' can be constructed in such a way that $p'_{h',i'} \approx 0.5$ and $p'_{h',k'} \approx p'_{i',k'}$ holding for each $k' \in N'_{t^A}(s)$.

Now, for all $k \in N \setminus N'_{t^A}(s)$ let $sup(k) = min\{x' \in N'_{t^A}(s)$ such that $x' > k\}$ and $inf(k) = max\{x' \in N'_{t^A}(s)$ such that $x' < k\}$.

Let us now define a probability matrix p on N such that:

- for all $x', y' \in N'_{t^A}(s)$, $p_{x'y'} = p'_{x'y'}$
- for all $k \in N \setminus N'_{t^A}(s)$ such that $sup(k)$ exists, $p_{k,sup(k)} = 0.5$ (and $p_{kw} = p_{sup(k),w}$ $\forall w \in N$).
- for all $k \in N'_{t^A}(s)$ such that $sup(k)$ does not exist, $p_{k,inf(k)} = 0.5$ (and $p_{kw} = p_{inf(k),w}$ $\forall w \in N$).

That is, p restricted to the elements of $N'_{t^A}(s)$ is equal to p', and all the players that are not seeded to t^A are *assimilated* as equally strong as his immediately weaker player in $N'_{t^A}(s)$. Moreover, if for some element k not seeded to t^A there is no weaker player in $N'_{t^A}(s)$, then k is considered as equally strong as his immediately stronger player in $N'_{t^A}(s)$. Thus, by construction, $p \in p_R$.

Notice that, by $p'_{h'w'} \approx p'_{i'w'}$ for each $w' \in N'_{t^A}(s)$, we have by construction that $p_{h'w} \approx p_{i'w}$ holds for each $w \in N$.

Now, let γ be the shortest path between the root v_0 of t and the root v_0^A of t^A with $\mathbf{v}(\gamma)$ being its set of nodes. Notice that, due to t being a binary tree, for each $v \in \mathbf{v}(\gamma)$ with $\ell(v) \geq 1$ there always exists a unique node $v' \notin \mathbf{v}(\gamma)$ with $\ell(v') = \ell(v)$ at distance 2 from v. That is, the two players having reached these two nodes play against each other in order to arrive at their common immediate predecessor $v'' \in \mathbf{v}(\gamma)$ with $\ell(v'') = \ell(v) - 1$. Letting $N_{v'}$ to be the set of all players seeded by s to some leaf of the subtree of t rooted at v', we get $\varphi_i(t, s, p) = \varphi_{i'}(t^A, s', p') \times \prod_{v \in \mathbf{v}(\gamma), \ell(v) \geq 1} \sum_{k \in N_{v'}} p_{i'k} p_k^{v'}$ and $\varphi_h(t, s, p) = \varphi_{h'}(t^A, s', p') \times \prod_{v \in \mathbf{v}(\gamma), \ell(v) \geq 1} \sum_{k \in N_{v'}} p_{h'k} p_k^{v'}$.

Recall that $p_{i',w} \approx p_{h',w}$ holds for each $w \in N$. Moreover, $p_k^{v'}$ is independent of whether i' or h' have reached node $v \in \mathbf{v}(\gamma)$. Therefore, $\varphi_{i'}(t^A, s', p') > \varphi_{h'}(t^A, s', p')$ implies $\varphi_i(t, s, p) > \varphi_h(t, s, p)$. Thus, the competition system (t, N) violates WMS.

Step 6 Let (t, N) be an elimination-type competition system with t being a limited extension from the root of a one-bye antler. Then (t, N) violates WMS.

The proof is analogous to the one of Step 5.

Step 7 Let (t, N) be an elimination-type competition system with t being a limited extension from the root of a balanced tree of height 3. Then (t, N) violates WMS.

Again, the proof is analogous to the one of Step 5.

Step 8 Let t^* be a limited extension from the root of an antler t^A and (t, N) be an elimination-type competition system with t being an extension from the leaves of t^*. Then (t, N) violates WMS.

Proof. For the proof of the statement of Step 8, we will need the following additional notation.

Let $d(v_0, v_0^A)$ stand for the geodesic distance between the root v_0 of t and the root v_0^A of t^A. For $x \in \{0, \ldots, h(t) - d(v_0, v_0^A)\}$ we denote by t_0^x the subgraph of t consisting of all nodes $v \in \mathbf{v}(t)$ with $\ell(v) \leq d(v_0, v_0^A) + x$ and the corresponding edges of t connecting them. That is, t_0^x is just the tree t when being truncated at level $d(v_0, v_0^A) + x$. Clearly, $x = h(t^A) = 3$ implies $t_0^x = t^*$.

We denote by $\mathbf{m}^x$ the set of matches at level $d(v_0, v_0^A) + x$ of t_0^x (with m^x being a typical element of $\mathbf{m}^x$), and by $\mathbf{t}_k^x$ the set of subgraphs of t_0^x that can be obtained from t_0^x by removing a number k of matches at level $d(v_0, v_0^A)+x$ (with t_k^x being a typical element of $\mathbf{t}_k^x$). Clearly, $\mathbf{t}_{|\mathbf{m}^x|}^x = t_{|\mathbf{m}^x|}^x = t_0^{x-1}$.

On the other hand, for any tree $t_k^x \in t$ we consider a set of players $N_k^x = \{1, \ldots, n_k^x\}$ that makes competition (t_k^x, N_k^x) feasible, that is, a set of players whose cardinality is $n_k^x = |\Lambda(t_k^x)|$.

Consider now, for any $k \leq |\mathbf{m}^4|$, any tree $t_k^4 \in \mathbf{t}_k^4$ and the corresponding set of players N_k^4 that makes (t_k^4, N_k^4) feasible. Let R be the ordering of strength defined on N_k^4. Assume that (t_k^4, N_k^4) satisfies WMS. By Lemma 8, and without loss of generality according to Lemma 6, we know that, for t_k^4 to satisfy WMS with respect to some seeding rule $s \in \mathcal{S}^{\left(t_k^4, N_k^4\right)}$, the worst player n_k^4 according to R should be seeded to some leaf of t_k^4 that belongs to some match in $\mathbf{m}^4$. Let us denote by m'^4 the match to which n_k^4 is seeded, by $(\lambda_a'^4)$ and $(\lambda_b'^4)$ its two leaves, and by $(\overline{n}_k^4) \neq n_k^4$ the second player seeded to m'^4, that is, the opponent of n_k^4. Now, let $(t_{k+1}'^4, N_{k+1}^4)$ be the competition system in which $t_{k+1}'^4$ has been obtained from t_k^4 by removing the match m'^4 and

N^4_{k+1} is a set of $n_k - 1$ players. Clearly, the common immediate predecessor w of λ'^4_a and λ'^4_b becomes now a leaf of t'^4_{k+1} to be denoted by λ_w. Hence, $\Lambda(t'^4_{k+1}) = \Lambda(t^4_k) \cup \{\lambda_w\} \setminus \{\lambda'^4_a, \lambda'^4_b\}$.

The inductive reasoning starts by proving the following claim, which roughly speaking says that if the competition $((t^4_k, N^4_k))$ satisfies WMS and the match where the worst player is seeded is removed, then the remaining structure satisfies WMS too.

Claim Let (t^4_k, N^4_k) and (t'^4_{k+1}, N^4_{k+1}) be as above. If (t^4_k, N^4_k) satisfies WMS, then (t'^4_{k+1}, N^4_{k+1}) also satisfies WMS.

Proof of the Claim. Assume that (t^4_k, N^4_k) satisfies WMS but (t'^4_{k+1}, N^4_{k+1}) does not. Let R' be defined on N^4_{k+1} such that $R' = R_{|N^4_{k+1} \setminus \{n^4_k\}}$. Consider the seeding rule $s' : \Lambda(t'^4_{k+1}) \to N^4_{k+1}$ defined as follows: for each $\lambda \in \Lambda(t'^4_{k+1}) \setminus \{\lambda_w\}$, $s'(\lambda) = s(\lambda)$ and $s(\lambda_w) = \overline{n}^4_k$ (note that $N^4_{k+1} = N^4_k \setminus \{n^4_k\}$ and that $\overline{n}^4_k \in N^4_{k+1}$). That is, s' can be interpreted as a situation in which $\overline{n}^4_k$ wins his match against n^4_k and the remaining matches are not played yet. By hypothesis (t'^4_{k+1}, N^4_{k+1}) violates WMS. This implies that for the seeding rule s' there exists some probability matrix $p' \in p_{R'}$ defined on N^4_{k+1} such that $p'_{ij} > 0.5$ and $\varphi_j(t'^4_{k+1}, s', p') > \varphi_i(t'^4_{k+1}, s', p')$ holds for some $i, j \in N^4_{k+1}$.

Let p be a probability matrix on N^4_k defined as follows: $p_{ij} = p'_{ij}$ for all $i, j \in N^4_k \setminus \{n^4_k\}$, and $p_{i,n^4_k} \approx 1$ for each $i \in N^4_k \setminus \{n^4_k\}$. Notice that $p \in p_{R_{|N^4_k}}$ by construction. Also by construction, $\varphi_i(t'^4_{k+1}, s', p') \approx \varphi_i(t^4_k, s, p)$ holds for each $i \in N^4_k \setminus \{n^4_k\}$. Therefore, $p_{ij} > 0.5$ and $\varphi_j(t^4_k, s, p) > \varphi_i(t^4_k, s, p)$ holds for some $i, j \in N^4_k$. Hence, we have a contradiction to the hypothesis that (t^4_k, N^4_k) satisfies WMS, completing the proof of the claim.

Notice that the above claim holds also for $k + 1 = |\mathbf{m}^4|$. In this particular case, $t_{k+1} = t^4_{|\mathbf{m}^4|} = t^3_0 = t^*$ which leaves us with the following three possibilities:

(i) There are two leaves at distance 2 from v^A_0, that is, there is no extension from any leaf at distance 2 from v^A_0 and therefore t^* is a limited extension from the root of a (symmetric or asymmetric) antler.

(ii) There is a unique leaf at distance 2 from v^A_0. In that case t^* is a limited extension from the root of a one-bye antler.

(iii) There are no leaves at distance 2 from v^A_0. In that case t^* is a limited extension from the root of a balanced tree of height 3.

For each of these three possible cases we have proved in the previous steps that no competition system whose graph is $t^* = t^4_{|\mathbf{m}^4|}$ does satisfy WMS.

Now, for any $k \in \{0, \dots, |\mathbf{m}^4| - 1\}$, take any competition system (t^4_k, N^4_k) with $t^4_k \in \mathbf{t}^4_k$. Notice that from (t^4_k, N^4_k) it is always possible to define a sequence (t^4_k, N^4_k), $(t^4_{k+1}, N^4_{k+1}), \dots, (t^4_{|\mathbf{m}^4|}, N^4_{|\mathbf{m}^4|})$ by removing the match at level $d(v_0, v^A_0) + 4$ where the corresponding worst player n^4_k, n^4_{k+1}, ..., $n^4_{|\mathbf{m}^4|-1}$ has been seeded. Given that $(t^4_{|\mathbf{m}^4|}, N^4_{|\mathbf{m}^4|})$ violates WMS, and considering the Claim, and inductive argument allows to prove that (t^4_k, N^4_k) violates WMS too. Therefore, in particular, (t^4_0, N^4_0) violates WMS. Recalling that $t^5_{|\mathbf{m}^5|} = t^4_0$, we can recursively replicate the inductive argument at level $d(v_0, v^A_0) + 5$ as to conclude that for all $k \in \{0, 1, \dots, |\mathbf{m}^5| - 1\}$, every competition system (t^5_k, N^5_k) with $t^5_k \in \mathbf{t}^5_k$ violates WMS. The reasoning can be successively applied when t has been truncated at higher levels until we reach the tree $t = t_0^{h(t)-d(v_0,v^A_0)}$, proving that the competition system (t, N) violates WMS.

Step 9 Let (t, N) be an elimination-type competition with t containing an antler. Then (t, N) violates WMS.

Proof. The statement follows from the fact that if a tree t contains an antler then, clearly, it is some form of extension from the leaves of a limited extension from the root of an antler. By Step 8 this implies that, if (t, N) is an elimination-type competition with t containing an antler, then it does not satisfy WMS.

Step 10 Let (t, N) be an elimination-type competition with t being antler-free. Then (t, N) satisfies WMS with respect to $s \in \mathcal{S}^{(t,N)}$ only if $s = s_{ib}$.

Proof. We proceed by reduction to the absurd: We assume that (t, N) satisfies WMS, $s \neq s_{ib}$, and show that this leads to a contradiction, in other words, we show that given a strength ordering R defined on N, it is possible to find a probability matrix $p \in p_R$ such that if $s \neq s_{ib}$ then there exist $i, j \in N$ such that $p_{ij} > 0.5$ but $\varphi_i(t, s, p) < \varphi_j(t, s, p)$.

We know from Lemma 8 that for (t, N) to satisfy WMS it should be the case that for any probability matrix p, and any $\lambda, \lambda' \in \Lambda(t)$, $\ell(\lambda) < \ell(\lambda')$ implies $p_{s(\lambda),s(\lambda')} \geq 0.5$. On the other hand, by Lemma 5, $t \in ES^*_2$. Take now a maximal root-to-leaf path $\gamma(t)$ and notice that, for any probability matrix p, $s \neq s_{ib}$ implies either

(i) there exist leaves $\lambda_a, \lambda_b \in \Lambda(t)$ with $\ell(\lambda_a) = \ell(\lambda_b) < h(t)$ such that: (1) only λ_a has an immediate predecessor belonging to $\mathbf{v}(\gamma(t))$ and (2) $p_{s(\lambda_b),s(\lambda_a)} > 0.5$, or

(ii) $\left|\Lambda^{h(t)}(t)\right| = 4$ with the players in $\left\{s(\lambda) : \lambda \in \Lambda^{h(t)}(t)\right\}$ not being seeded in a balanced way.

We proceed by showing that in each of these cases we reach a contradiction.

Case (i) Let k be the number of players seeded by s to leaves at higher level than $\ell(\lambda_a) = \ell(\lambda_b)$. We construct the desired p in three steps.

First, we set $p_{n-k,n-k+1} > 0.5$ and $p_{n-k+1,z} \approx 1$ to hold for each $z > n-k+1$. By Lemma 8, the set of players seeded to the leaves at higher level than $\ell(\lambda_a)$ is then $\{n, n-1, \ldots, n-k+1\}$. Thus, the probability $p_{n-k+1}^{v^{s(\lambda_a)}}$ with which player $n-k+1$ reaches node $v^{s(\lambda_a)} \in \mathbf{v}(\gamma(t))$ is arbitrarily close to 1.

Second, let x_1, x_2, and x_3 be the three players seeded to the three leaves at level $\ell(\lambda_a)$ and set $p_{x_1x_2} \geq 0.5$ and $p_{x_2x_3} \geq 0.5$. By construction, $p_{x_3,(n-k+1)} > 0.5$. Note also that, with a probability arbitrary close to 1, the players $(n-k+1)$, x_1, x_2, and x_3 play a balanced sub-competition at level $\ell(\lambda_a)$ with the root of the sub-competition being $v \in \mathbf{v}(\gamma(t))$ with $\ell(v) = \ell(\lambda_a) - 2$. Moreover, by hypothesis, $n-k+1$ plays a match against some player x_i ($i \in \{2,3\}$) such that $p_{x_1x_i} > 0.5$. Therefore, by Lemma 7, it is possible to define a probability matrix p' on the player set $\{n-k+1, x_1, x_2, x_3\}$ such that there are players $i, j \in \{n-k+1, x_1, x_2, x_3\}$ with $p'_{ij} > 0.5$ and $p_i'^v < p_j'^v$. Moreover, we know by the proof of Lemma 7 that p' can always be constructed in such a way that $p'_{iw} \approx p'_{jw}$ holds for each $w \in \{n-k+1, x_1, x_2, x_3\}$. We take then $p_{xy} = p'_{xy}$ to hold for all $x, y \in \{n-k+1, x_1, x_2, x_3\}$. This implies that, according to p, $p_i^v < p_j^v$. It also implies $p_{iw} \approx p_{jw}$ for each $w \in \{n-k+1, x_1, x_2, x_3\}$.

Third, we take $p_{zx_1} = 0.5$ to hold for each $z \in N$ who is seeded at a lower level than $\ell(\lambda_a)$. That is, every player who is seeded at a lower level than $\ell(\lambda_a)$ is considered as bequally strong as the strongest player at level $\ell(\lambda_a)$. Notice that the latter fact together with $p_{n-k+1,z} \approx 1$ for each $z > n-k+1$ implies $p_{iw} \approx p_{jw}$ for each $w \in N$.

Thus, the constructed probability matrix p is as follows: the restriction of p on the player set $\{n-k+1, x_1, x_2, x_3\}$ is p'; x_1 is equally strong as every player who is seeded at lower levels; and each player in $\{n-k+1, x_1, x_2, x_3\}$ wins with a probability arbitrarily close to 1 the match against any player being seeded at higher levels and is different from $n-k+1$. Now, by using the notation of Step 1 and recalling that $\ell(v) = \ell(\lambda_a) - 2$, we have

$$\varphi_i(t,s,p) = p_i^v(s) \times \prod_{x \in S^1_{v'}(s), \ell(v') < \ell(v)} p_{ix} \times \prod_{y,z \in S^2_{v'}(s), \ell(v') < \ell(v)} (p_{iy}p_{yz} + p_{iz}p_{zy})$$

and

$$\varphi_j(t,s,p) = p_j^v(s) \times \prod_{x \in S^1_{v'}(s), \ell(v') < \ell(v)} p_{jx} \times \prod_{y,z \in S^2_{v'}(s), \ell(v') < \ell(v)} (p_{jy}p_{yz} + p_{jz}p_{zy}).$$

Then we have that $p_j^v > p_i^v$, $p_{ix} \approx p_{jx}$ for each $x \in N$, and since p_{yz} (p_{zy}) is independent of any other parameter in the respective formulae for i and j we conclude that $\varphi_j(t,s,p) > \varphi_i(t,s,p)$ should also hold, in contradiction to (t,N) satisfying WMS with respect to s in this case.

Case (ii) Notice that in this case $\left|\Lambda^{h(t)}\right| = 4$ implies that the node $v \in \mathbf{v}(\gamma(t))$ with $\ell(v) = h(t) - 2$ is the root of a balanced subtree of t. We construct the desired p in two steps.

First, let $\{a,b,c,d\} \subseteq N$ be the set of players seeded to the leaves in $\Lambda^{h(t)}$. It follows then from Lemma 7 that, for each of the two possible non-balanced seedings of the players in $\{a,b,c,d\}$ to the leaves in $\Lambda^{h(t)}$, there exists a probability matrix p' on $\{a,b,c,d\}$ such that $p'_{ij} > 0.5$ and $p'^v_j > p'^v_i$ holds for some $i,j \in \{a,b,c,d\}$. Moreover, it follows from the proof of Lemma 7 that p' can be constructed in such a way that $p'_{ij} \approx 0.5$ and $p'_{iw} \approx p'_{jw}$ for each $w \in \{a,b,c,d\}$. Thus, we take p to be such that $p_{xy} = p'_{xy}$ for all $x,y \in \{a,b,c,d\}$. Let a be a strongest player and d a weakest player among those in $\{a,b,c,d\}$. By Lemma 8, for all players $x \in N \setminus \{a,b,c,d\}$ and $i \in \{a,b,c,d\}$, $p_{xi} \geq 0.5$.

Second, we set $p_{zw} = 0.5$ to hold for all $z,w \in N \setminus \{b,c,d\}$.

Thus, the constructed probability matrix p is as follows: the restriction of p on the player set $\{a,b,c,d\}$ is p', while each of the remaining players (who are seeded to leaves at lower levels that $h(t)$ in the tree) is considered as equally strong as the strongest player in $\{a,b,c,d\}$. Moreover, $p'_{iw} \approx p'_{jw}$ holding for each $w \in \{a,b,c,d\}$ implies by construction that $p_{iw} \approx p_{jw}$ is also true for each $w \in N$.

By using the notation of Step 1 and recalling that $\ell(v) = h(t) - 2$,, we have

$$\varphi_i(t,s,p) = p_i^v(s) \times \prod_{x \in S^1_{v'}(s) : \ell(v') < \ell(v)} p_{ix} \times \prod_{y,z \in S^2_{v'}(s) : \ell(v') < \ell(v)} (p_{iy}p_{yz} + p_{iz}p_{zy})$$

and

$$\varphi_j(t,s,p) = p_j^v(s) \times \prod_{x \in S^1_{v'}(s) : \ell(v') < \ell(v)} p_{jx} \times \prod_{y,z \in S^2_{v'}(s) : \ell(v') < \ell(v)} (p_{jy}p_{yz} + p_{jz}p_{zy}).$$

Again we have that $p_j^v > p_i^v$, $p_{ix} \approx p_{jx}$ for each $x \in N$, and since p_{yz} (p_{zy}) is independent of any other parameter in the respective formulae for i and j we conclude that $\varphi_j(t,s,p) > \varphi_i(t,s,p)$ holds. Thus, we have again a contradiction to (t,N) satisfying WMS with respect to s. This completes the proof of the Theorem. ■

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