Gender consistent resolving rules in marriage problems

Dinko Dimitrov, Laura Kasper, Yongjie Yang
Chair of Economic Theory, Saarland University, Germany

March 22, 2017

Abstract

The selection of blocking pairs to be matched plays an important role in the study of mechanisms converting arbitrary matchings into stable ones. We assume that a resolving rule guides the selection and show that two axioms (independence and top optimality) transform such a rule into a gender consistent one. That is, the rule is forced by the axioms to follow a linear order over acceptable pairs which is consistent with the preferences of either all men or all women. As shown by Abeledo and Rothblum (1995), stable matchings can be reached when starting from an arbitrary individually rational matching and iteratively satisfying the pair selected by a gender consistent resolving rule.

Keywords: gender consistency; paths to stability; stable marriage problem; stable matching; two-sided matching

1 Introduction

The selection of blocking pairs to be matched plays an important role in the study of mechanisms converting arbitrary matchings into stable ones as shown in the seminal works of Knuth (1976) and Roth and Vande Vate (1990). The latter work considers sequences of matchings, where each matching is obtained from the previous one by satisfying a single blocking pair, and introduces an algorithm which selects in a specific way the blocking pair to be satisfied at each iteration until a stable matching is reached.[1] The focus on matching a single blocking pair at each iteration

[1] As shown by Ma (1996), the mechanism suggested by Roth and Vande Vate (1990) does not always reach all stable matchings.

of an algorithm turned out to be fruitful also with respect to the original Gale-Shapley procedure (cf. Gale and Shapley, 1962) for showing the existence of a stable matching. For instance, McVitie and Wilson (1970) have modified Gale and Shapley's algorithm by letting at each iteration only one man propose to the woman he prefers most among those who have not yet rejected him.

The single-proposal variant of the Gale-Shapley algorithm and the Roth-Vande Vate algorithm were shown in Abeledo and Rothblum (1995) to share a common feature. More precisely, the common feature is shared by the corresponding *resolving rules*, that is, by the rules applied in both algorithms for the selection of a blocking pair to be matched from the set of all pairs blocking a given matching. These resolving rules turn out to follow a pre-specified *linear order* over the set of acceptable pairs which is *gender consistent*, that is, the linear order is consistent with the preferences of either all men or all women (see Sections 2 and 3 for the precise definitions).

In the current paper we provide an axiomatic foundation of gender consistent resolving rules. In doing so, we focus on male consistency, that is, the corresponding rule shall respect the preferences of all men. When defining the domain on which a resolving rule is supposed to work, we consider preference profiles differing only with respect to women's preferences and matchings that are individually rational (but not stable) at such profiles. Given such a domain, a resolving rule then assigns, to each preference profile and matching, a pair that blocks the matching at that preference profile.[2] It is worth mentioning that the above domain restrictions are in line with Abeledo and Rothblum (1995), where male consistency is defined with respect to a fixed profile of men's preferences and matchings are always supposed to be individually rational.

We formulate two requirements - an independence axiom and a top man optimality condition - and impose them on a resolving rule. The independence axiom works on different pairs consisting of a preference profile and a matching, and forces a resolving rule to select the same outcome, provided that the corresponding sets of blocking pairs do coincide. As for our second requirement, we slightly modify the notion of domination among blocking pairs introduced in Klijn and Massó (2003), and require in the top man

[2] At a very general level, (the algorithms based on) gender consistent resolving rules can be interpreted as describing different ways for assigning a stable matching to any (individually rational) initial matching. In that sense, our work can be seen in the same vain as the axiomatically driven papers of Kojima and Manea (2010) and Morrill (2013). The rules characterized in these papers are based on the deferred acceptance algorithm applied in the context of object allocation problems.

optimality condition any resolving rule outcome to be undominated.

The rest of the paper is organized as follows. Section 2 contains basic notation and definitions, as well as the precise formulations of the proposed axioms. We then show in Section 3 that a resolving rule satisfying these axioms must be male consistent. Finally, in Section 4, we embed two stable matching problems in our framework and discuss matching sequences that cycle when the resolving rule applied satisfies one of the axioms but not the other.

2 Setup and axioms

We consider the standard two-sided one-to-one matching model introduced in Gale and Shapley (1962), in which there are two finite sets M and W of agents, called "men" and "women", respectively. These two sets will be kept fixed throughout the paper with each agent being endowed with a complete, transitive, and antisymmetric binary relation over the agents from the opposite sex and the possibility of remaining single. For instance, $w \succ_m w'$ expresses the fact that man $m \in M$ prefers women $w \in W$ to woman $w' \in W$; we write $w \succeq_m w'$ whenever either $w \succ_m w'$ or $w = w'$ holds. A preference profile is denoted by $\succeq = (\succeq_i)_{i \in M \cup W}$, and we say that a pair $(m, w) \in M \times W$ is *acceptable* at $\succeq$ if $w \succ_m m$ and $m \succ_w w$ holds. The set of all acceptable pairs at $\succeq$ is denoted by $A^{\succeq}$.

A *matching* μ is a function $\mu : M \cup W \to M \cup W$ such that $\mu(m) \in W \cup \{m\}$, $\mu(w) \in M \cup \{w\}$, and $\mu^2(i) = i$ hold for $m \in M$, $w \in W$, and $i \in M \cup W$. The interpretation of $\mu(i) = i$ for some $i \in M \cup W$ is that the corresponding agent is single under μ. If $\mu(i) = i$ holds for each $i \in M \cup W$, we say that the corresponding matching is *empty* and denote it throughout the paper by μ_0. A matching μ is called *stable* at $\succeq$ if (1) it consists of either singletons or acceptable pairs at $\succeq$ (*individual rationality*), and (2) there are no blocking pairs at $\succeq$ for it, i.e., there is no pair (m, w) such that $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. The set of blocking pairs for a matching μ at a profile $\succeq$ is denoted by $B^{\succeq}(\mu)$.

Let $\mathcal{M}$ stand for the set of all matchings, $(\succeq_m)_{m \in M}$ be a fixed profile of men's preferences, and $\mathcal{R}$ be the largest collection of preference profiles for which $\succeq' \in \mathcal{R}$ implies $\succeq'_m = \succeq_m$ for each $m \in M$. We collect in the set $\mathcal{D}$ all $(\succeq, \mu) \in \mathcal{R} \times \mathcal{M}$ such that μ is individually rational but not stable at $\succeq$. A *resolving rule* g is defined on $\mathcal{D}$ and assigns a pair $(m, w) \in B^{\succeq}(\mu)$ to each $(\succeq, \mu) \in \mathcal{D}$. Notice then that, for each $(\succeq, \mu) \in \mathcal{D}$, we have $B^{\succeq}(\mu) \subseteq A^{\succeq}$.

The first axiom we impose on a resolving rule is a standard independence requirement. In our context it captures the idea that, when comparing two different pairs of preference profiles and matchings, the outcome of a resolving rule remains the same if the corresponding sets of blocking pairs are related by set inclusion as specified below. As a consequence, the axiom allows us to conclude that it is the *set* of blocking pairs that matters when a rule satisfying the axiom selects a particular pair.

Independence of Irrelevant Alternatives (IIA): If $B^{\succeq}(\mu) \subseteq B^{\succeq'}(\mu')$ for some $(\succeq, \mu), (\succeq', \mu') \in \mathcal{D}$, then $g(\succeq', \mu') \in B^{\succeq}(\mu)$ implies $g(\succeq, \mu) = g(\succeq', \mu')$.

In order to introduce our second axiom, let us consider $(\succeq, \mu) \in \mathcal{D}$ with $(m, w), (m, w') \in B^{\succeq}(\mu)$. Moreover, let man m be the top man for both w and w' at $\succeq$; that is, $m \succeq_{w''} m''$ holds for $w'' \in \{w, w'\}$ and all $m'' \in M$. We deem then g *top man dominated* at $(\succeq, \mu)$ if $g(\succeq, \mu) = (m, w)$ and $w' \succ_m w$. A resolving rule g is *top man optimal* if it is not top man dominated at any $(\succeq, \mu) \in \mathcal{D}$.

Top Man Optimality (TMO): g is top man optimal.

As to understand the power of TMO, suppose that $g(\succeq, \mu) = (m, w)$ holds for some $(\succeq, \mu) \in \mathcal{D}$. Notice then that g being top man optimal does not necessarily imply that m is a top man at $\succeq$ for w. It rather says that, *provided m is* the top man for w at $\succeq$, there is no other woman w' *with the same top man* at $\succeq$ such that $(m, w') \in B^{\succeq}(\mu)$ and $w' \succ_m w$.[3]

3 Characterization

In this section we provide our characterization result by first showing that the IIA axiom forces a resolving rule to become dependent on a linear order over acceptable pairs (Proposition 1). We then prove that adding TMO to IIA implies that the constructed linear order becomes male consistent (Proposition 2).

Let us start with the simple observation that the set $\mathcal{R}$ of preference profiles we consider contains a profile $\succeq^*$ at which $m \succ^*_w w$ holds for any

[3] Klijn and Massó (2003) introduce weak stability for the marriage model that allows for the existence of weak blocking pairs, and show that Zhou's (1994) bargaining set for their context does coincide with the set of weakly stable and weakly efficient matchings. More precisely, a pair $(m, w) \in B^{\succeq}(\mu)$ is called *weak* if there exists either a woman w' with $(m, w') \in B^{\succeq}(\mu)$ and $w' \succ_m w$, or a man m' with $(m', w) \in B^{\succeq}(\mu)$ and $m' \succ_w m$. Notice that our dominance notion is stronger and thus, the corresponding TMO condition allows for the selection of weak blocking pairs.

$m \in M$ and $w \in W$. Clearly then, $A^{\succeq} \subseteq A^{\succeq^*}$ holds for each $\succeq \in \mathcal{R}$. The following lemma will prove useful subsequently.

Lemma 1 *Let $A \subseteq A^{\succeq^*}$. Then there exists $(\succeq, \mu) \in \mathcal{D}$ such that $(m, w) \in B^{\succeq}(\mu)$ if and only if $(m, w) \in A$.*

Proof. Take $\mu = \mu_0$ and notice that μ_0 is individually rational at any preference profile. Recall that men's preferences are fixed and take $\succeq \in \mathcal{R}$ to be such that, for $m \in M$ and $w \in W$, $m \succeq_w w$ if and only if $(m, w) \in A$. By completeness of women's preferences, $(m', w) \in A^{\succeq^*} \setminus A$ for some $m' \in M$ implies $w \succ_w m'$. We conclude that $(m, w) \in B^{\succeq}(\mu_0)$ if and only if $(m, w) \in A$. ■

Our first result shows that rules satisfying IIA and those that are order dependent turn out to be equivalent. More precisely, a resolving rule g is *order dependent* if there exists a linear order $\trianglerighteq$ over $A^{\succeq^*}$ such that, for each $(\succeq, \mu) \in \mathcal{D}$, $g(\succeq, \mu) \trianglerighteq (m, w)$ holds for each $(m, w) \in B^{\succeq}(\mu)$. We write here $(m, w) \trianglerighteq (m', w')$ to express the fact that either $(m, w) \triangleright (m', w')$ or $(m, w) = (m', w')$ holds.

Proposition 1 *A resolving rule satisfies IIA if and only if it is order dependent.*

Proof. We first show that if a resolving rule g is order dependent with respect to some linear order $\trianglerighteq$ over $A^{\succeq^*}$, then it satisfies IIA. For this, take $(\succeq, \mu), (\succeq', \mu') \in \mathcal{D}$ and suppose that $B^{\succeq}(\mu) \subseteq B^{\succeq'}(\mu')$. By the order dependence of g, $g(\succeq', \mu') \trianglerighteq (m, w)$ holds for all $(m, w) \in B^{\succeq'}(\mu') \supseteq B^{\succeq}(\mu)$. Clearly then, $g(\succeq', \mu') \in B^{\succeq}(\mu)$ implies $g(\succeq, \mu) = g(\succeq', \mu')$ and thus, g satisfies IIA.

Suppose now that a resolving rule g satisfies IIA. As to show the order dependence of g, we shall construct a linear order $\trianglerighteq$ over $A^{\succeq^*}$ and then prove that g is order dependent with respect to $\trianglerighteq$.

Consider the set $A^{\succeq^*}$ and notice that by Lemma 1 there exists $(\succeq, \mu) \in \mathcal{D}$ such that $B^{\succeq}(\mu) = A^{\succeq^*}$. Set then $g(\succeq, \mu) \triangleright (m, w)$ to hold for all $(m, w) \in A^{\succeq^*} \setminus \{g(\succeq, \mu)\}$. Consider next the set $A' = A^{\succeq^*} \setminus \{g(\succeq, \mu)\}$ and apply again Lemma 1 to conclude the existence of $(\succeq', \mu') \in \mathcal{D}$ with $B^{\succeq'}(\mu') = A'$. Set then $g(\succeq', \mu') \triangleright (m', w')$ for all $(m', w') \in A' \setminus \{g(\succeq', \mu')\}$. Continue in this way until the set $A^{\succeq^*}$ is exhausted. Clearly, by its very construction, the binary relation $\trianglerighteq$ is complete, transitive, and antisymmetric and thus, a linear order over $A^{\succeq^*}$.

Let us now show that g is order dependent with respect to $\trianglerighteq$. Take $(\succeq, \mu) \in \mathcal{D}$ and let $(m, w) \trianglerighteq (m', w')$ hold for all $(m', w') \in B^{\succeq}(\mu)$. We show that $g(\succeq, \mu) = (m, w)$ should follow. For this, let

$$A'' = \left\{ (m'', w'') \in A^{\succeq^*} : (m, w) \trianglerighteq (m'', w'') \right\}.$$

By Lemma 1, there exists $(\succeq'', \mu'') \in \mathcal{D}$ such that $B^{\succeq''}(\mu'') = A''$. By the construction of $\trianglerighteq$ then, $g(\succeq'', \mu'') = (m, w)$ follows. Finally, from $(m, w) \in B^{\succeq}(\mu) \subseteq A''$ and IIA, we conclude that $g(\succeq, \mu) = (m, w)$ should hold. ∎

We are now ready for our main result that characterizes *gender consistent* resolving rules. These rules were introduced in Abeledo and Rothblum (1995) as a special class of order dependent resolving rules for which the corresponding linear order over acceptable pairs is either male consistent or female consistent. As already mentioned in the Introduction, we focus on male consistency.

For $\succeq \in \mathcal{R}$, we say that a linear order $\trianglerighteq$ over $A^{\succeq^*}$ is *male consistent* if for any $(m, w), (m, w') \in A^{\succeq^*}$, $(m, w) \trianglerighteq (m, w')$ if and only if $w \succeq_m w'$. Notice that, as men's preferences remain the same at each $\succeq \in \mathcal{R}$, this definition ensures male consistency of $\trianglerighteq$ with respect to any $\succeq \in \mathcal{R}$. We say finally that *a resolving rule g is male consistent* if it is order dependent with respect to some male consistent linear order over $A^{\succeq^*}$.

Our main result shows that any order dependent rule that is top man optimal must be male consistent.

Proposition 2 *A resolving rule satisfies IIA and TMO if and only if it is male consistent.*

Proof. Let g be order dependent with respect to some male consistent linear order $\trianglerighteq$ over $A^{\succeq^*}$. The fact that g satisfies IIA was shown in the proof of Proposition 1. On the other hand, g violating TMO would imply the existence of $(\succeq, \mu) \in \mathcal{D}$ and $(m, w') \in B^{\succeq}(\mu)$ such that $(m, w) = g(\succeq, \mu)$ and $w' \succ_m w$. By $\trianglerighteq$ being male consistent, $(m, w') \triangleright (m, w)$ should be the case. Thus, we reach a contradiction to $g(\succeq, \mu) = (m, w)$ and g being order dependent with respect to $\trianglerighteq$. We conclude that g satisfies TMO as well.

Suppose now that a resolving rule g satisfies IIA and TMO. Consider the linear order $\trianglerighteq$ constructed in the proof of Proposition 1 and recall that g is then order dependent with respect to $\trianglerighteq$. We show that TMO restricts $\trianglerighteq$ to be male consistent.

For the sake of contradiction, assume that $\trianglerighteq$ violates male consistency. Let (m, w) be a pair such that there is $w' \in W$ with $w' \succ_m w \succ_m m$ and $(m, w) \triangleright (m, w')$ (as $\trianglerighteq$ is assumed to violate male consistency, such a pair exists). Let $\succeq \in \mathcal{R}$ be a profile such that (m, w) and (m, w') are the only acceptable pairs and women's preferences be as follows:

- $m \succ_w w \succ_w m'$ and $m \succ_{w'} w' \succ_{w'} m'$ for all $m' \in M \setminus \{m\}$; and
- $w'' \succ_{w''} m'$ for all $w'' \in W \setminus \{w, w'\}$ and $m' \in M$.

Consider finally the matching μ_0 and notice that $B^{\succeq}(\mu_0) = \{(m, w), (m, w')\}$. By g being order dependent with respect to $\trianglerighteq$, it holds that $g(\succeq, \mu_0) = (m, w)$. However, this implies that g is top man dominated at $(\succeq, \mu_0)$, a contradiction. ■

Notice that the IIA axiom relates resolving rule outcomes at different pairs of preference profiles and matchings, while the TMO axiom is a condition imposed on the outcome of a resolving rule at a given profile-matching pair. Thus, it is natural to expect that these axioms are independent. Our next two examples do indeed show this fact. More precisely, we provide below two particular resolving rules each of which satisfying one of the axioms but not the other. In both examples, $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$.

We start by constructing a resolving rule g_1 fulfilling IIA but violating TMO. Let men's preferences be fixed at $w_1 \succ_{m_1} w_2 \succ_{m_1} m_1$ and $m_2 \succ_{m_2} w_1 \succ_{m_2} w_2$. Consider the preference profile $\succeq \in \mathcal{R}$ at which women's preferences are as follows: $m_1 \succ_{w_1} w_1 \succ_{w_1} m_2$ and $m_1 \succ_{w_2} w_2 \succ_{w_2} m_2$. Let g_1 be order dependent with respect to the following linear order $\trianglerighteq_1$ over the set $A^{\succeq^*} = A^{\succeq} = \{(m_1, w_1), (m_1, w_2)\}$: $(m_1, w_2) \triangleright_1 (m_1, w_1)$. By Proposition 1, g_1 satisfies IIA. Take μ_0 and notice that $B^{\succeq}(\mu_0) = A^{\succeq}$ and $g_1(\succeq, \mu_0) = (m_1, w_2)$ holds. Clearly then, g_1 is top man dominated at $(\succeq, \mu_0)$.

Let us now construct a resolving rule g_2 satisfying TMO but not IIA. Fix men's preferences as follows: $w_1 \succ_{m_1} m_1 \succ_{m_1} w_2$ and $w_1 \succ_{m_2} m_2 \succ_{m_2} w_2$. Consider the profiles $\succeq', \succeq'' \in \mathcal{R}$ at which w_1's preferences are $m_1 \succ'_{w_1} m_2 \succ'_{w_1} w_1$ and $m_2 \succ''_{w_1} m_1 \succ''_{w_1} w_1$, respectively. Let g_2 be any resolving rule satisfying $g_2(\succeq', \mu_0) = (m_1, w_1)$ and $g_2(\succeq'', \mu_0) = (m_2, w_1)$. Clearly then, g_2 vacuously satisfies TMO since only w_1 can form blocking pairs with m_1 and m_2. Notice however, that g_2 violates IIA since $B^{\succeq'}(\mu_0) = B^{\succeq''}(\mu_0)$ but $g_2(\succeq', \mu_0) \neq g_2(\succeq'', \mu_0)$. We conclude that IIA and TMO do not imply each other.

4 Discussion

A natural question that arises is whether it is always possible to convert an initial matching into a stable one when the resolving rule applied at each iteration fulfills one of the proposed axioms but not the other. We answer this question into the negative by embedding two stable matching problems in our framework. The first one is based on an example due to Knuth

(1976) and the corresponding resolving rule applied is order dependent but not top man optimal. For the second example we utilize the stable matching problem discussed by Abeledo and Rothblum (1995) in their Example 2, for which the resolving rule we construct and apply is top man optimal but it violates the independence of irrelevant alternatives axiom. In each of the corresponding sequence of matchings, whenever a pair (m, w) is blocking a matching μ and selected by the resolving rule, we say that the next matching μ' in the sequence is obtained from μ by satisfying that blocking pair if m and w are married under μ', their partners under μ (if any) are unmatched at μ', and all other agents are matched to the same mates under μ' as they were under μ.

Example 1 The set M of men and the set W of women consist of three agents each with the corresponding preference profile $\succeq$ being as follows:

$$\begin{array}{ll} w_2 \succ_{m_1} w_1 \succ_{m_1} w_3 \succ_{m_1} m_1 & m_1 \succ_{w_1} m_3 \succ_{w_1} m_2 \succ_{w_1} w_1 \\ w_1 \succ_{m_2} w_3 \succ_{m_2} w_2 \succ_{m_2} m_2 & m_3 \succ_{w_2} m_1 \succ_{w_2} m_2 \succ_{w_2} w_2 \\ w_1 \succ_{m_3} w_2 \succ_{m_3} w_3 \succ_{m_2} m_3 & m_1 \succ_{w_3} m_3 \succ_{w_3} m_2 \succ_{w_3} w_3 \end{array}$$

Notice that $A^{\succeq^*} = A^{\succeq} = M \times W$ and let $\trianglerighteq$ be a linear order over $A^{\succeq^*}$ for which the following holds: $(m_2, w_2) \triangleright (m_2, w_1) \triangleright (m_3, w_3) \triangleright (m_1, w_3) \triangleright (m_1, w_2) \triangleright (m_3, w_2) \triangleright (m_3, w_1) \triangleright (m_1, w_1) \triangleright (m_1, w_2) \triangleright \ldots$. Suppose that the resolving rule g is order dependent with respect to $\trianglerighteq$ and hence, by Proposition 1, g satisfies IIA on the domain $\mathcal{D}$ generated by the above profile of men's preferences. Take the matching $\mu_1 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$ as an initial one, and let us apply g when it comes to the selection of a blocking pair. In the following table, agents not included in a matching are single at it and each next matching is obtained from the previous one by satisfying the first pair from the the correspondingly listed blocking pairs.

Matchings	Blocking pairs
$\mu_1 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$	$(m_1, w_2), (m_3, w_2)$
$\mu_2 = \{(m_1, w_2), (m_3, w_3)\}$	$(m_2, w_1), (m_3, w_2), (m_3, w_1)$
$\mu_3 = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$	$(m_3, w_2), (m_3, w_1)$
$\mu_4 = \{(m_2, w_1), (m_3, w_2)\}$	$(m_1, w_3), (m_1, w_1), (m_3, w_1)$
$\mu_5 = \{(m_1, w_3), (m_2, w_1), (m_3, w_2)\}$	$(m_3, w_1), (m_1, w_1)$
$\mu_6 = \{(m_1, w_3), (m_3, w_1)\}$	$(m_2, w_2), (m_1, w_1), (m_1, w_2)$
$\mu_7 = \{(m_1, w_3), (m_2, w_2), (m_3, w_1)\}$	$(m_1, w_1), (m_1, w_2)$
$\mu_8 = \{(m_1, w_1), (m_2, w_2)\}$	$(m_3, w_3), (m_1, w_2), (m_3, w_2)$

As the reader can easily see, μ_1 is obtained from μ_8 by satisfying $g(\succeq, \mu_8) = (m_3, w_3)$ and thus, the sequence cycles and does not deliver a stable matching. Notice that we have $g(\succeq, \mu_4) = (m_1, w_3)$, $(m_1, w_1) \in B^{\succeq}(\mu_4)$, m_1 is the top man of both w_1 and w_3, and $w_1 \succ_{m_1} w_3$. Thus, g is not top man optimal as it is top man dominated by (m_1, w_1) at $(\succeq, \mu_4) \in \mathcal{D}$.

Example 2 The set M of men and the set W of women consist of four agents each, where the corresponding preference profile $\succeq$ is as follows:

$$\begin{array}{ll}
w_2 \succ_{m_1} w_3 \succ_{m_1} m_1 \succ_{m_1} w_1 \succ_{m_1} w_4 & m_3 \succ_{w_1} m_4 \succ_{w_1} w_1 \succ_{w_1} m_1 \succ_{w_1} m_2 \\
w_3 \succ_{m_2} w_4 \succ_{m_2} m_2 \succ_{m_2} w_1 \succ_{m_2} w_2 & m_4 \succ_{w_2} m_1 \succ_{w_2} w_2 \succ_{w_2} m_2 \succ_{w_2} m_3 \\
w_4 \succ_{m_3} w_1 \succ_{m_3} m_3 \succ_{m_3} w_2 \succ_{m_3} w_3 & m_1 \succ_{w_3} m_2 \succ_{w_3} w_3 \succ_{w_3} m_3 \succ_{w_3} m_4 \\
w_1 \succ_{m_4} w_2 \succ_{m_4} m_4 \succ_{m_4} w_3 \succ_{m_4} w_4 & m_2 \succ_{w_4} m_3 \succ_{w_4} w_4 \succ_{w_4} m_1 \succ_{w_4} m_4
\end{array}$$

Consider the domain $\mathcal{D}$ generated by the above profile of men's preferences and let $\trianglerighteq$ be a male consistent linear order over the corresponding set $A^{\succeq^*}$. Let g be order dependent with respect to $\trianglerighteq$ except at profile-matching pairs from $\mathcal{D}$, for which the profile is $\succeq$ and the matching is one from the eight matchings listed in the table below. For these cases, the selected blocking pair to be satisfied in the next matching is always the first from the listed ones.

Matchings	Blocking pairs
$\mu_1 = \{(m_1, w_1), (m_2, w_4), (m_3, w_1)\}$	$(m_4, w_2), (m_2, w_3)$
$\mu_2 = \{(m_2, w_4), (m_3, w_1), (m_4, w_2)\}$	$(m_2, w_3), (m_1, w_3)$
$\mu_3 = \{(m_2, w_3), (m_3, w_1), (m_4, w_2)\}$	$(m_1, w_3), (m_3, w_4)$
$\mu_4 = \{(m_1, w_3), (m_3, w_1), (m_4, w_2)\}$	$(m_3, w_4), (m_2, w_4)$
$\mu_5 = \{(m_1, w_3), (m_3, w_4), (m_4, w_2)\}$	$(m_2, w_4), (m_4, w_1)$
$\mu_6 = \{(m_1, w_3), (m_2, w_4), (m_4, w_2)\}$	$(m_4, w_1), (m_3, w_1)$
$\mu_7 = \{(m_1, w_3), (m_2, w_4), (m_4, w_1)\}$	$(m_3, w_1), (m_1, w_2)$
$\mu_8 = \{(m_1, w_3), (m_2, w_4), (m_3, w_1)\}$	$(m_1, w_2), (m_4, w_2)$

Notice that μ_1 is obtained from μ_8 by satisfying $g(\succeq, \mu_8) = (m_1, w_2)$ and thus, the sequence cycles and does not deliver a stable matching. Additionally, for $k = 1, 2, \ldots, 8$, there are no blocking pairs in $B^{\succeq}(\mu_k)$ which contain the same man and thus, g is not top man dominated at $(\succeq, \mu_k)$. Since on the rest of the domain $\mathcal{D}$ the resolving rule g follows a male consistent linear order, we conclude that g is top man optimal as there is no $(\succeq', \mu) \in \mathcal{D}$ at which g is top man dominated. However, g is not order dependent (and thus, by Proposition 1, it violates IIA). In order to see it, notice that a possible order dependence with respect to a linear order $\trianglerighteq$ over $A^{\succeq^*}$ would require $g(\succeq, \mu_1) = (m_4, w_2) \triangleright (m_2, w_3) \triangleright \ldots \triangleright (m_1, w_2) \triangleright (m_4, w_2) = g(\succeq, \mu_1)$ in contradiction to the transitivity of $\trianglerighteq$.

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