Global Games with Interim Information Acquisition *

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Abstract

We study global games with interim information acquisition where players acquire additional information after they observe private signals. In the first period, players receive private signals with unknown precision and then choose costly effort to investigate the precision of the signal. In the second period, players play a global game conditional on their signals and investigation results. We provide sufficient conditions under which the game has a unique equilibrium. The optimal information acquisition decision as a function of private signals is characterized. We analyze equilibrium behaviors of players with different private information. We show how players with different private information would react to changes of public information. It is also shown that information decisions may not always exhibit strategic complementarities even in a game with strategic complementarities in actions.

1 Introduction

Following the seminal contribution by Carlsson and van Damme (1993), global games of regime change have been extensively applied to explain economic phenomena such as currency attack, financial crisis, bank runs and political revolutions. A global game is a coordination game with incomplete information where players observe noisy signals about an unobservable state which influences the payoffs. This class of games usually achieves a unique equilibrium when private information becomes sufficiently precise. In this paper we study a global game with *interim* information acquisition. In a standard global game, the precision of private signals is assumed to be common knowledge. We relax this assumption and consider the case where players do not know the precision. In addition, we allow players

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to acquire costly information about the precision after they receive private signals. The aim of this paper is to analyze the role of this *interim* stage information acquisition in a global game. And since the information acquisition occurs after players receive private signals, information choice of players of different types can be studied. In particular, we would like to address the following questions through this work : How much information do players with different private signals acquire? How does public information influence the information choice of different players? Do information choices exhibit strategic complementarities when actions are strategic complements?

Considering *interim* information acquisition is not only of theoretical interest but also of practical importance. In practice, people receive information from various sources: social media, internet, friends, etc. But they may not know how precise the information is. For example, in the currency attack model following Morris and Shin (1998), the central bank pegs the exchange rate while a group of speculators decide whether to attack the regime by short selling the currency. Before taking the action, a speculator receives some information about the strength of the country's economic fundamentals, but he does not know the precision of the information. The information he receives could be rumor or fake news. But he could learn more about the precision of this piece of information before he uses it to update his belief through different ways. For instance, he could check whether the news has a discernible source, or search on the internet to see whether there is any supportive evidence.

To model the above situation, we establish a global game with *interim* information acquisition as follows. Consider a group of players who play a coordination game with incomplete information. The payoffs depend on an unknown state as well as players' actions. The game has two periods. In the first period, each player observes a private signal about the state. The precision of the signal is unknown and players have a common prior over it. After observing the signal, players choose how much effort to put on an investigation. If the investigation is successful, they know the precision while if it is unsuccessful they learn nothing. The more effort they put, the higher chance of a successful investigation. In the second period, players choose a binary action attack or not conditional on their private signals as well as the investigation result. Players' actions are strategic complements, namely, a player's payoff from choosing one action is increasing as more players choose the same action.

We firstly characterize the conditions under which this game has a unique equilibrium. Note that in the coordination game of the second period, there exist two groups of players, one of which knows the precision while the other one does not. It is shown that if we make the information decision nonstrategic and fixed, then under some conditions, this corresponding global game has a unique equilibrium. Similar as the standard global game, the equilibrium has a switching strategy structure in which players choose to attack if their signals are below a threshold and take the alternative action when their signals are above the threshold. Since we have two groups of players, there will be two thresholds in equilibrium. For those players who conduct successful investigation and know the precision, their threshold depends on the realization of the precision and for those who fail the investigation, the threshold does not depend on the precision of private signals. Then we go back to the first stage and figure out players' optimal information decisions as a function of private signals. It is shown that the optimal effort in period one achieves a maximum at the threshold point for players who fail the investigation. When the private signal is below the threshold, the optimal effort increases and it decreases if the private signal is above the threshold.

After that, we study how public information influences players' information decisions. It is shown that different players react differently to a change of public information. In particular, a weaker (stronger) public information will motivate players with weak (strong) private information to achieve less information and those with strong (weak) private information to acquire more information. The effect of the precision of public information is ambiguous. As a special case, we show that if the public signal is sufficiently strong, then players with weak private signals will acquire less information and those with strong private signals will acquire more information when public information becomes more precise. This paper also shows that information choices may not exhibit strategic complementarities when actions are strategic complements. In particular, when players with strong private signals acquire more information, players with relatively weak private information will acquire less information about the precision of private signals while others with relatively strong private information will put more effort on information acquisition. It is shown that players may regard information choices as strategic substitutes even though actions are strategic complements

Compared with previous literature, our work has two major differences. One is about the information acquisition technology. Suppose there is an unobservable state that influences the payoffs of a game and there is a machine which delivers information about the unknown state to players. Some previous literature including Nikitin and Smith (2008) and Zwart (2008) considers a model where players decide whether to buy a machine or not with a fixed cost. Some other paper focus on models where players could pay a cost to improve the machine. For example, Szkup and Trevino (2015) studies a model where players could purchase more precise signals about the fundamentals with a higher cost prior to a global game of regime change. Yang (2015) introduces flexible information acquisition to global games. In his work, any type of "machine" could be purchased by players with a cost proportional to the reduction of Shannon's entropy and they may coordinate with each other on the choice of "machines". However, in our model the machine itself is costless and fixed but there are some features of the machine that players do not observe ex ante. In particular, the unknown feature is the precision of private information in this context. Through costly effort, the aim of players is not to improve the machine but to discover the unknown feature of the machine.

The second difference is about the order of information acquisition. In our model, players acquire information after private signals are observed while most previous literature focus on *ex ante* information acquisition.[1] The order matters not only because such a model is able to deal with some cases in practice while models with *ex ante* information acquisition do not consider, but also provide us with tools to analyze the information acquisition decisions of players with different private signals in equilibrium. We contribute to the literature by finding out a few new features of equilibrium behaviors and information decisions of heterogeneous players in coordination games with incomplete information.

The remainder of the paper is organized as follows. The model is described in Section 2. Then we solve the game and provide conditions under which there exists a unique equilibrium in Section 3. We also characterize the information acquisition decision as a function of private information in this section. The effect of information acquisition and public information on equilibrium is studied in Section 4. Section 5 shows information decisions may not be strategic complements in this model. In particular, they can be strategic substitutes among players with different private information. We list some related literature in Section 6 and conclude in Section 7. All the proofs are relegated in the Appendix.

2 The Model

We build a two period game where players receive signals and acquire information in the first period and then play a coordination game in period two. There is a continuum of agents, indexed by $i \in [0, 1]$. The strength of the economic fundamentals of the economy is characterized by a random variable θ which is unobservable and follows an improper uniform prior over $\mathbb{R}$. In the beginning of period 1, players firstly receive a public signal μ where

$$\mu = \theta + \alpha^{-1/2}\eta$$

with $\alpha > 0$ and η is a standard normal random variable. Note that α is the precision of the public signal μ. This information structure is equivalent to assuming θ has a common prior given by a normal distribution $\theta \sim N(\mu, \alpha^{-1})$. The reason we incorporate public information is that such an interpretation is more straightforward when we analyze the effect of public information on equilibrium later.

[1] For example, Zwart (2008), Szkup and Trevino (2015), Yang (2015), etc.

Additionally, each player i receives a private signal s_i where

$$s_i = \theta + \tau^{-1/2}\epsilon_i$$

with $\tau > 0$ and ϵ_i is a standard normal random variable. We assume ϵ_i is independent and identically distributed across all players and is independent of η for all i.

We assume the value of α is common knowledge but τ is not. Instead we assume players do not know the value of τ but can choose effort to make investigation after they observe private signals. Players share a common belief that τ is distributed according to a probability density function g with support $[\underline{\tau}, \bar{\tau}]$ where $\underline{\tau}, \bar{\tau} \in \mathbb{R}^+$. The corresponding cumulative distribution function is denoted by G. τ is independent of θ and μ. We define information acquisition technology as follows. After receiving signals μ and s_i, player i chooses the probability that he conducts a successful investigation, denoted by $p_i \in [0, 1]$. The investigation is either successful or failed and the result is released by the end of period 1. If the investigation is successful, player i knows the value of τ while otherwise he learns nothing. An investigation decision p generates a personal cost $C(p)$. It is assumed that function $C(.)$ is smooth, strictly increasing, convex and satisfies the Inada condition $C'(0) = 0$ and $\lim_{p\to 1} C'(p) = \infty$. Throughout the context, we will interpret p as players' effort or information decision. When p_i is higher, we say player i acquires more information. Note that players' decisions in the first period can be represented by a function $P : \mathbb{R} \to [0, 1]$. The above information structure is common knowledge.

In the second period, players simultaneously choose a binary action $a \in \{0, 1\}$. The final payoff depends on the strength of the economic fundamentals as well as the proportion of players who choose action $a = 1$. The payoffs are given by:

$$u_i(1, \theta) = \begin{cases} 1 - t & \text{if } l > \theta \\ -t & \text{if } l \leq \theta \end{cases}$$

$$u_i(0, \theta) = 0$$

where $t \in (0, 1)$ and l is the proportion of players who choose $a = 1$. Though we provide a generic description, this game can be interpreted in different specific examples. In the currency crisis model following Morris and Shin (1998), action 1 represents attacking a fixed exchange rate regime by short selling the currency. In the pricing debt model following Morris and Shin (2004), action 1 represents rolling over the debt. In the bank run model following Goldstein and Pauzner (2005), action 1 represents early withdrawn from the bank. More details can be found in Morris and Shin (2003). In the following context, we will use attack and action 1 interchangeably and when the proportion of players who choose to attack is above θ, we say the regime change happens.

The timeline of the game is as follows. At the beginning of period one, the state variable θ is drawn by nature. Each player firstly receives a public signal and a private signal. After that, each player decides the effort to put on investigating the precision of the private signal. At the end of the first period, the investigation result, either successful or failed, is revealed. In the second period, each player i chooses a binary action $a_i \in \{0, 1\}$ based on the signals he receives as well as the investigation result. The final payoffs are realized by the end of the second period.

We will solve this model by steps. Firstly, we will solve the second stage coordination game for any fixed information decisions in the first period. It will be shown that under some assumptions, the coordination game has a unique "switching strategy" equilibrium. Then we go back to the first stage to figure out the optimal information decisions.

3 Solving the Model

3.1 Second stage global game with fixed information decisions

In this section, we analyze the equilibrium behaviors in the second stage. We firstly assume the information decisions in the first stage are nonstrategic and fixed. Then the corresponding second stage game is a global game where players choose to attack or not conditional on their information. We solve this game and make appropriate assumptions so that the uniqueness as well as the structure of the equilibrium holds regardless of players' decisions in the first stage. The main result is presented in the beginning of the section and then we will prove it by steps. The proposition is as follows:

Proposition 1. *If $\sqrt{\tau}/(\alpha+\tau) > 2/\sqrt{2\pi}$ for every $\tau \in [\underline{\tau}, \bar{\tau}]$, then the global game with fixed information strategy P has a unique, dominance solvable equilibrium which is characterized by (x_P, y_P) and θ_P. x_P and θ_P are functions mapping from $[\underline{\tau}, \bar{\tau}]$ to $\mathbb{R}$ and y_P is a real number. Players who failed their investigation choose action 1 if and only if $s < y_P$ and players who conduct successful investigation and know that the value of precision is τ choose to take action 1 if and only if $s < x_P(\tau)$. When the precision of private signals is τ, the regime change occurs if and only if $\theta < \theta_P(\tau)$.*

Note that the assumption we make in the above proposition does not only impose restrictions on the range of τ but also on the value of α. Given α, $\sqrt{\tau}/(\alpha + \tau)$ achieves a maximum value of $1/(2\sqrt{\alpha})$ when $\tau = \alpha$. Then if $\alpha \geq \pi/8$, it is easy to find out that there does not exist any real number τ such that the assumption $\sqrt{\tau}/(\alpha+\tau) > 2/\sqrt{2\pi}$ could hold.

Similar as previous studies on global games, we start the analysis by assuming that all the players would choose switching strategies. For a switching strategy, there is a threshold below which the player will choose to take action 1 and above which he will take the alternative choice. Then we show it is not only unique but also the only equilibrium which survives the iterated elimination of strictly dominated strategies.

Before heading to the equilibrium analysis, let us introduce some notations first. Suppose a random variable z follows a normal distribution with mean a and precision b, then let us denote the corresponding density function as $f(z; a, b)$. In particular, if z is a standard normal random variable, i.e. $a = 0$ and $b = 1$, then denote the corresponding density function as $\phi(.)$ and distribution function as $\Phi(.)$.

According to the information structure given in the previous section, after observing a public signal μ, all the players update their beliefs about θ and the corresponding posterior is $\theta \sim N(\mu, \alpha^{-1})$. Then when a player receives a private signal s, he will update his belief again. Consider the case where the investigation is successful and the precision τ is known, the posterior of θ will follow a normal distribution, $\theta \sim N(\nu_s, \beta^{-1})$ where

$$\nu_s = \frac{\alpha\mu + \tau s}{\alpha + \tau}$$
$$\beta = \alpha + \tau$$

On the other hand, when the investigation is unsuccessful and the precision is unknown, the posterior of θ follows a density function as follows[2]

$$p(\theta|s) = \int f(\theta; \nu_s, \beta) dG(\tau)$$

Suppose all players take switching strategies. For a player whose investigation is unsuccessful, he gets no additional information about τ and therefore he will just use his prior knowledge. Since in this section it is assumed that the first period information decisions are fixed, to save notations we omit the subscript P. Denote the threshold for these players as y and note that y should not depend on τ. On the other hand, for those players who conduct successful investigations, they know the value of τ, so their cutoff point depends on τ. Denote the threshold as x_τ.

Given τ, we can define the corresponding critical mass (CM) condition and preference

[2] The joint distribution of (θ, τ):

$$p(\theta, \tau|s) \propto p(s|\theta, \tau)p(\theta)p(\tau) \Rightarrow p(\theta, \tau|s) \propto f(\theta; \nu_s, \beta)p(\tau)$$

So the posterior of θ follows

$$p(\theta|s) = \int p(\theta, \tau|s) dG(\tau) = \int f(\theta; \nu_s, \beta) dG(\tau)$$

indifference (PI) condition for players who know τ as follows:

$$\int_{-\infty}^{x_\tau} f(s;\theta_\tau,\tau)P(s)ds + \int_{-\infty}^{y} f(s;\theta_\tau,\tau)(1-P(s))ds = \theta_\tau \tag{CM}$$

$$\int_{-\infty}^{\theta_\tau} f(\theta;\nu_{x_\tau},\beta)d\theta = t \tag{PI_1}$$

And for those players who fail their investigations, the corresponding (PI) condition is given by

$$\int\int_{-\infty}^{\theta_\tau} f(\theta;\nu_y,\beta)d\theta dG(\tau) = t \tag{PI_2}$$

Note that PI_1 characterizes a point x_τ at which players who know the precision are indifferent between choosing 1 and 0 and PI_2 characterizes a point y at which players who fail the investigation are indifferent between the two actions. So given τ, (x_τ, y) are the cutoff points for players' actions. On the other hand, CM characterizes a critical point θ_τ for the regime change. Namely, the attack would be successful if and only if the economic fundamental variable θ is below θ_τ for given τ. By (PI_1),

$$\begin{aligned}
&\Phi(\frac{\theta_\tau - \frac{\alpha\mu+\tau x_\tau}{\alpha+\tau}}{\frac{1}{\sqrt{\alpha+\tau}}}) = t \\
\Rightarrow &\theta_\tau = \frac{\alpha\mu+\tau x_\tau}{\alpha+\tau} + \frac{1}{\sqrt{\alpha+\tau}}\Phi^{-1}(t) \\
\Rightarrow &x_\tau = (1+\frac{\alpha}{\tau})\theta_\tau - \frac{\alpha}{\tau}\mu - \frac{\sqrt{\alpha+\tau}}{\tau}\Phi^{-1}(t) = x(\theta_\tau)
\end{aligned}$$

We represent x_τ as a function of θ_τ. Substitute $x(\theta_\tau)$ to (CM), we get

$$\int_{-\infty}^{x(\theta_\tau)} f(s;\theta_\tau,\tau)P(s)ds + \int_{-\infty}^{y} f(s;\theta_\tau,\tau)(1-P(s))ds = \theta_\tau \tag{1}$$

Then we have the following lemma:

Lemma 1. *Given any τ and y, if $\sqrt{\tau}/(\tau_\theta+\tau) > 1/\sqrt{2\pi}$, then there exists a unique θ_τ such that equation (1) holds.*

By Lemma 1, given any τ and y, there exists a unique θ_τ satisfying equation (1). Namely, θ_τ can be expressed as a function of τ and y

$$\theta_\tau = \theta(\tau, y)$$

Then we substitute $\theta(\tau, y)$ into (PI_2), we have

$$\int\int_{-\infty}^{\theta(\tau,y)} f(\theta;\nu_y,\beta)d\theta dG(\tau) = t \tag{2}$$

We have the following lemma:

Lemma 2. *When $\sqrt{\tau}/(\tau_\theta + \tau) > 2/\sqrt{2\pi}$, there exists a unique y such that equation (2) holds.*

Lemma 1 and 2 together show that under appropriate assumptions, we have unique x_τ, y and θ_τ satisfying CM, PI_1 and PI_2 for all τ. In other words, a switching strategy exists and it is unique in this class of strategies. Then to show it is the unique equilibrium, we apply iterated elimination of strictly dominated strategies similar as that in Morris and Shin (2004) and prove the switching strategy is the only one which survives the process. The proof is left in the appendix.

3.2 Optimal information acquisition

We have shown that given any information decisions in period one, the second stage coordination game has a unique equilibrium under certain conditions. Now we go back to the first stage to see how players decide the optimal effort. We will show under some assumptions there exists a unique equilibrium. Since it is a function of private signal s, denote the optimal effort as $P^*(s)$. Denote the payoff of a player, who conducts successful investigation, in the unique equilibrium of the coordination game in period two for given s and τ as $\pi(s,\tau)$, one has

$$\pi(s,\tau) = \max\{\Phi(\frac{\theta_\tau - \nu_s}{\sqrt{\beta}^{-1}}) - t, 0\}$$

Or equivalently,

$$\pi(s,\tau) = \begin{cases} \Phi(\sqrt{\beta}(\theta_\tau - \nu_s)) - t & \text{if } s < x_\tau \\ 0 & \text{if } s \geq x_\tau \end{cases}$$

and denote the payoff of players who failed the investigation as $\pi(s)$, one has

$$\pi(s) = \max\{\int \Phi(\frac{\theta_\tau - \nu_s}{\sqrt{\beta}^{-1}})dG(\tau) - t, 0\}$$

Or equivalently,

$$\pi(s) = \begin{cases} \int \Phi(\sqrt{\beta}(\theta_\tau - \nu_s))dG(\tau) - t & \text{if } s < y \\ 0 & \text{if } s \geq y \end{cases}$$

Then in the first period, the optimization problem for a player with signal s is

$$\max_p \int \pi(s,\tau)dG(\tau)p + (1-p)\pi(s) - C(p)$$

Obviously, $\int \pi(s,\tau)dG(\tau) \geq \pi(s)$. The first order condition yields that

$$\begin{aligned} C'(P^*(s)) &= \int \pi(s,\tau)dG(\tau) - \pi(s) \\ \Rightarrow P^*(s) &= (C')^{-1}(\int \pi(s,\tau)dG(\tau) - \pi(s)) \end{aligned} \tag{3}$$

Since we assume C satisfies the Inada condition, $(C')^{-1}(\int \pi(s,\tau)dG(\tau) - \pi(s)) \in [0,1]$ and the solution is well defined. Note that $\int \pi(s,\tau)dG(\tau) - \pi(s)$ depends on a function $p(.)$ and by our assumptions C is smooth so in equilibrium P should be continuous. Therefore we have a mapping from the space of all continuous functions with range belonging to $[0,1]$ to itself characterized by equation (3). Let $\mathscr{C}$ represents the set of all continuous functions mapping from $\mathbb{R}$ to $[0,1]$. Equip the set with the sup norm and the space $\mathscr{C}$ is complete. Denote the mapping as T, one has

$$T \cdot P(s) = (C')^{-1}(\int \pi(s,\tau)dG(\tau) - \pi(s))$$

The next lemma shows T is a contraction from $\mathscr{C}$ to $\mathscr{C}$ under some assumptions. Then it follows that there exists a unique solution. Before presenting the result, let us make an additional assumption on the cost function C.

Assumption. *$C''(p) \geq \underline{c}$ for all $p \in [0,1]$ and for some $\underline{c} > 0$.*

Since we have assumed that C is convex, $C'' > 0$. The above assumption actually puts a lower bound for it. An example of a function satisfying such an assumption is the commonly used quadratic cost function where $C(p) = p^2$. Note that for quadratic cost, $C'' = 2$ for all p. Then we have the following lemma:

Lemma 3. *Suppose C has a positive lower bound in the second derivative. If $\sqrt{\tau}/(\alpha+\tau) \geq (1+(2+\Delta)\underline{c})/\sqrt{2\pi}\underline{c}$ for all $\tau \in [\underline{\tau}, \bar{\tau}]$ where $\Delta = \sqrt{\frac{\bar{\tau}}{\underline{\tau}}}$, then T is a contraction mapping from $\mathscr{C}$ to $\mathscr{C}$.*

The existence and uniqueness of a solution to the optimization problem in period one follows directly from the above result that T is a contraction. We summarize the result in the following proposition.

Proposition 2. *Suppose C has a positive lower bound in the second derivative. When $\sqrt{\tau}/(\alpha+\tau)$ is sufficiently large for all τ, this two period game has a unique equilibrium, characterized by $\{P^*, (x_{P^*}, y_{P^*}), \theta_{P^*}\}$. Equilibrium behaviors in the second stage characterized by (x_{P^*}, y_{P^*}) and θ_{P^*} are summarized in Proposition 1. The optimal information acquisition decision in the first stage P^* is the fixed point of the contraction mapping T.*

Proposition 2 states that under certain conditions, a unique equilibrium exists in the game. In period one, since player acquire information after private signals are observed, the optimal information acquisition decision is a function of private information. The following proposition presents some properties of this function:

Proposition 3. *For the optimal effort function $P^*(s)$, there exist $\underline{s}$ and $\bar{s}$ such that $P^*(s) = 0$ when $s \notin (\underline{s}, \bar{s})$. When $s \in (\underline{s}, \bar{s})$, $P^*(s)$ increases with s as $s < y$ and decreases with s as $s > y$. The maximum is achieved at $s = y$.*

Figure 1 gives a qualitative description of the function $p^*(s)$. The intuition is as follows.

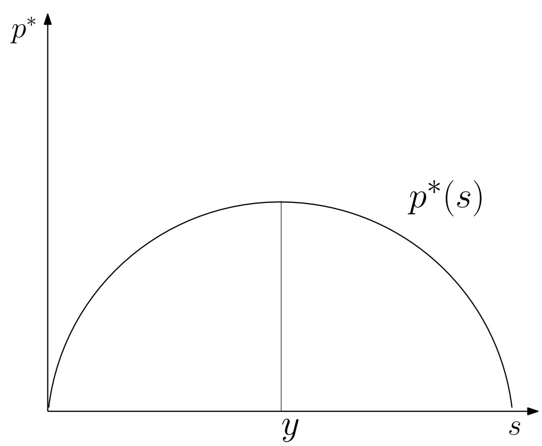

Figure 1: $P^*(s)$

When a player receives a sufficiently small private signal, even though he does not know how precise the private signal is, he believes the economic fundamental is weak enough and attack is a strictly dominant strategy regardless of the value of τ. Hence in this case he will have no incentives to acquire extra information about τ and the optimal information decision p^* should equal zero. Similar for the case when he receives a sufficiently large private signal.

On the other hand, if the private signal s is located in the medium range, the precision τ would influence the choice of actions. Consider the case where $s < y$. If the investigation is failed, players will choose to attack but if the investigation is successful players may choose to not attack if the realized τ satisfies $s > x_\tau$. So the benefit of acquiring additional information about τ is to avoid the risk of taking a bad attack decision. It is easy to observe that as s increases, the set of τ satisfying $s > x_\tau$ becomes larger, namely, the chance of taking a bad attack decision becomes higher. Therefore players have more incentives to acquire information about the precision τ and thereby p^* increases. Similarly, when $s > y$, players' default action is to not attack. In this case, the benefit of knowing the value of τ is to avoid taking the default action when he should have attacked the regime to get higher expected payoffs. Similarly, this happens when τ yields a threshold $x_\tau > s$. Now as s increases, the set of these τ becomes smaller. So the incentive to acquire extra information about τ becomes less and p^* decreases.

4 Equilibrium Analysis

In this section, we analyze the equilibrium in two aspects. Firstly, we examine the role of information acquisition by comparing the equilibrium in our model with that in a standard global game. Then we study the influence of public information on players' information acquisition decisions. It is shown that players with different private signals would react differently to a change of public signal.

4.1 Effect of information acquisition

As we introduced a global game with interim information acquisition, it is natural to ask what is the effect of this information acquisition technology on the equilibrium? We answer this question by comparing the equilibrium in our model with that in a standard global game. Consider a global game without the information acquisition stage. In particular, let us consider a coordination game with exactly the same structure as that in our model except that the precision of the private signal is now common knowledge. Previous literature established the well-known existence and uniqueness result of a switching strategy equilibrium. Denote the equilibrium in a standard global game when the precision of private signals is τ as $(\theta^s_\tau, x^s_\tau)$. Recall that in our model, θ_τ is the critical state for regime change in equilibrium and x_τ and y are the equilibrium switching points for players who construct successful investigation and unsuccessful investigation respectively, we have the following result

Proposition 4.
- *Suppose $x_\tau > y$, then $y < x_\tau < x^s_\tau$, $\theta_\tau < \theta^s_\tau$*
- *Suppose $x_\tau < y$, then $y > x_\tau > x^s_\tau$, $\theta_\tau > \theta^s_\tau$*

When x_τ is large, in particular when x_τ is larger than y, it is the case where players who know the precision are less willing to attack the regime. If we remove the information acquisition stage and let the precision of private signals to be common knowledge, we will have a larger threshold which indicates that if the information about precision becomes transparent, less players will choose to attack the regime and it is more difficult to have a regime change. This result is due to the existence of certain players who do not know the exact value of τ in our model. For these players, since they do know the exact value of the precision, they will apply certain average switching point. So even when the realized τ generates a higher threshold, this group of players will still use a switching strategy that has a lower cutoff point. Expecting this, the players who know the precision will decrease their threshold point since now there are more players willing to attack. Similar for the case where $x_\tau < y$, when players are more likely to attack the regime, the introduction of information acquisition makes a successful attack more difficult.

4.2 Effect of Public Information

Now we want to analyze the effect of public information. In our model, the public information is represented by μ which follows a normal distribution with mean θ and precision α. We are interested in the effect on equilibrium, especially on players' information acquisition decisions when μ and α change. Firstly, let us consider the case when μ changes. When μ increases, the public information indicates a stronger economic fundamental, which gives players less incentives to attack. The effect on players' information choices in the first period as μ changes is presented by the following proposition.

Proposition 5.

$$\frac{\partial P^*(s)}{\partial \mu} > 0 \quad \textit{for } s < y$$
$$\frac{\partial P^*(s)}{\partial \mu} < 0 \quad \textit{for } s > y$$

The above result shows that when public information indicates a weaker state, players who receive relatively weak private signals tend to increase their effort to acquire more information while those with relatively strong private signals are willing to decrease their effort. The intuition is as follows. When players receive a weaker public signal, their belief about the fundamental state θ decreases. The shifted belief makes it easier for players to coordinate on action attack, since the threshold in the global game decreases. Then for a player with a signal $s < y$, the risk for him to take a bad decision of attacking becomes larger. Hence he has more incentives to acquire more information. Similar intuition holds for those with signal $s > y$.

Then we analyze the effect on equilibrium when α changes. Note that in equilibrium, we have

$$x_\tau = (1+\frac{\alpha}{\tau})\theta_\tau - \frac{\alpha}{\tau}\mu - \frac{\sqrt{\alpha+\tau}}{\tau}\Phi^{-1}(t)$$

When α changes, x_τ can either increase or decrease depending on the value of μ, t and θ_τ. To get rid of the ambiguity, we make some assumptions on μ and give some partial results. The Proposition is as follows:

Proposition 6. *Assume in equilibrium, the following inequality holds for every* τ

$$\mu > \max\{0, \theta_\tau - \frac{1}{2}(\alpha+\tau)^{-1/2}\Phi^{-1}(t)\}$$

Then we have

$$\frac{\partial P^*(s)}{\partial \alpha} > 0 \quad \textit{for } s < y$$
$$\frac{\partial P^*(s)}{\partial \alpha} < 0 \quad \textit{for } s > y$$

The assumption we made guarantees when α increases, x_τ decreases for every τ. It holds if the public information is sufficiently strong. Then when the public information becomes more precise, players will have less incentives to attack so the threshold in equilibrium will decrease. In this case, for those players who receive relatively weak signals, the risk of taking a bad decision of attacking becomes larger and they will acquire more information to avoid the risk. On the other hand, for those players who receive strong signals, since a more precise public signal discourages players to attack, the chance that they should have chosen to attack while they do not becomes smaller. So they have less incentives to acquire additional information about the private signal's precision.

5 Strategic Complements or Substitutes

When players are involved in a game with strategic complementarities in actions, it is natural to ask whether it can be translated into strategic complementarities in information choices. In this section, this issue will be analyzed in our model. It is shown that when other players acquire more information, players with different signals would react differently. Before proceeding to the analysis, let us introduce some notations. Suppose in the first period, the information decision is P. Then by previous analysis, the second stage coordination game has a unique equilibrium denoted by (x_P, y) and θ_P. Now suppose all the other players choose P'. Denote the equilibrium in the second stage conditional on P' as $(x_{P'}, y_{P'})$ and $\theta_{P'}$. We have the following lemma

Lemma 4. *Suppose $P'(s) = P(s)$ for all $s \leq y$ and $P'(s) > P(s)$ for $s > y$, then $y_{P'} > y_P$, $x_{P'}(\tau) > x_P(\tau)$ and $\theta_{P'}(\tau) > \theta_P(\tau)$ for all τ. Similarly, $P'(s) = P(s)$ for all $s \geq y$ and $P'(s) > P(s)$ for $s < y$, then $y_{P'} < y_P$, $x_{P'}(\tau) < x_P(\tau)$ and $\theta_{P'}(\tau) < \theta_P(\tau)$ for all τ.*

The above lemma shows that if we only allow players with relatively high signals to increase their effort, then the threshold for players who failed the investigation will increase. Consider the result when we compare the equilibrium in our model with that in the standard global games in Proposition 3. If the precision of the private signals is common knowledge, the threshold increases if $x > y$. Here when we only allow players with stronger private information to increase their effort, there will be more players that know the value of τ. As a consequence, these players with successful investigation and discover that τ satisfies $x_P(\tau) > s$ will choose 1 instead of 0, which increases the threshold for the global game. Now we go back to analyze the effect on optimal information decisions. Suppose all the other players take a different information choice P', we want to know as a player who receives a signal s, whether he has more incentives to acquire more information. Denoted the optimal information decision when all other players choose P' as $P^{*\prime}$, we have the following result:

Proposition 7. *Suppose $P'(s) = P^*(s)$ for all $s \leq y_{P^*}$ and $P'(s) > P^*(s)$ for $s > y_{P^*}$, then there exists a $\hat{s} \in [y_{P^*}, y_{P'}]$, such that $P^{*\prime}(s) \geq P^*(s)$ for $s > \hat{s}$ and $P^{*\prime}(s) \leq P^*(s)$ for $s < \hat{s}$. Similarly, Suppose $P'(s) = P^*(s)$ for all $s > y$ and $P'(s) > P^*(s)$ for $s \leq y$, then there exists a $\hat{s} \in [y_{P'}, y_{P^*}]$, such that $P^{*\prime}(s) \leq P^*(s)$ for $s > \hat{s}$ and $P^{*\prime}(s) \geq P^*(s)$ for $s < \hat{s}$.*

We show that when part of the players increase their effort, the influence is different across players with different signals. In particular when players with stronger private signals acquire more information, those also with stronger private signals will acquire more information while those players with relatively weak signals acquire less information. The strategic complementarities in information decisions holds within players with similar signals or say types while information decisions become substitutes for players with different types. The result directly follows the lemma we present earlier in this section. When players with strong private signals increase their effort, it becomes easier for players to coordinate on action 1. Hence, the risk for player with weak signals to take a bad action 1 becomes smaller, so they are willing to acquire less information. Similar for those with strong signals.

6 Related Literature

Global games and applications. Our work is built upon the global game literature following Carlsson and van Damme (1993). In their seminal work, they study a 2×2 global game and show that a unique equilibrium exists and the equilibrium strategy is the risk dominant strategy in the coordination game. Later Frankel, Morris, and Pauzner (2003) extend the model to allow arbitrary number of players and actions and characterize the conditions under which the unique equilibrium exists. Morris, Shin, and Yildiz (2016) identify the driving force of this unique selection of equilibrium result as the approximate common certainty of uniform rank beliefs in global games.

As coordinating actions with others is ubiquitous, the global game model has been extensively applied to explain phenomena in crisis and regime change in macroeconomics, financial market and political systems. Morris and Shin (1998) introduce the framework to analyze currency attacks and relevant policies. Corsetti, Dasgupta, Morris, and Shin (2004) also builds a global game model to explain currency crisis. They focus on the impact of a large trader on the foreign exchange market in a currency crisis. Similar to our model, their equilibrium also includes two groups of players. The difference is that they assume a single large trader and a continuum of small traders exogenously while in our model the two groups of players are determined endogenously. Morris and Shin (2004) applies the model to study debt pricing. Goldstein and Pauzner (2005) incorporate global game information structure to the Diamond and Dybvig (1983) bank run model.

Information acquisition. Another related strand of literature also examines the role of endogenous information in coordination games. However, most of the previous studies make information acquisition as an ex-ante choice which means players make the decision before they observe their signals while in our work, information acquisition occurs after the observation of signals. Some of them apply information acquisition in a global game framework. Szkup and Trevino (2015) and Ahnert and Kakhbod (2017) study global games where players are able to purchase more precise signals. Zwart (2008) studies a liquidity run model where investor can choose to acquire a noisy signal about the underlying state with a fixed cost. Recently, Yang (2015) introduces the rational inattention framework following Sims (2003) to a coordination game with incomplete information. Prior to the play of the game, players simultaneously choose a flexible information structure subject to a cost proportional to reduction of Shannon's entropy. Hebert and Woodford (2017) and Morris and Yang (2016) extend this approach by considering more general cost functions. Besides global games, endogenous information in coordination games with linear-quadratic payoffs are studied by Hellwig and Veldkamp (2009), Myatt and Wallace (2012), Colombo, Femminis, and Pavan (2014), etc.

Note that all the literature mentioned above studies how players acquire information before they select actions. Some other paper assume the existence of a policy maker who can endogenously choose the information structure for the players in a global game and study the effect of policy interventions. The literature exploring this approach includes Angeletos, Hellwig, and Pavan (2006) and Angeletos and Pavan (2013).

7 Concluding Remarks

We study a global game with interim information acquisition. In our model, the information acquisition choice depends on the realization of private signals. Under certain conditions, unique equilibrium arises. In particular by analyzing the effect on equilibrium for players with different signals, our result contribute to the literature by showing equilibrium behaviors of heterogeneous players. We show a different piece of public information will make players with different types to take different information decisions. We also show that strategic complementarities in information decisions only hold within a group of players with similar types. For players with different information, information decision may become strategic substitutes.

Appendix

Proof of Lemma 1

Proof. Define the l.h.s. of equation (1) as a function of θ_τ, $h(\theta_\tau) = \int_{-\infty}^{x(\theta_\tau)} f(s;\theta_\tau,\tau)p(s)ds + \int_{-\infty}^{y} f(s;\theta_\tau,\tau)(1-p(s))ds$. It suffices to show that $h'(\theta_\tau) < 1$.

$$\begin{aligned}
h'(\theta_\tau) =& f(x(\theta_\tau);\theta_\tau,\tau)x'(\theta_\tau)P(x(\theta_\tau)) + \int_{-\infty}^{x(\theta_\tau)} \frac{\partial f(s;\theta_\tau,\tau)}{\partial\theta_\tau} P(s)ds \\
&+ \int_{-\infty}^{y} \frac{\partial f(s;\theta_\tau,\tau)}{\partial\theta_\tau}(1-P(s))ds \\
=& \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(x(\theta_\tau)-\theta_\tau)^2}{2})(1+\frac{\alpha}{\tau})P(x(\theta_\tau)) + \int_{-\infty}^{x(\theta_\tau)} \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(s-\theta_\tau)^2}{2})\tau(s-\theta_\tau)ds \\
&+ \int_{x(\theta_\tau)}^{y} \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(s-\theta_\tau)^2}{2})(1-P(s))\tau(s-\theta_\tau)ds
\end{aligned}$$

Since $P(x(\theta_\tau)) \leq 1$, one has

$$\begin{aligned}
h'(\theta_\tau) \leq& \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(x(\theta_\tau)-\theta_\tau)^2}{2})(1+\frac{\alpha}{\tau}) + \int_{-\infty}^{x(\theta_\tau)} \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(s-\theta_\tau)^2}{2})\tau(s-\theta_\tau)ds \\
&+ \int_{x(\theta_\tau)}^{y} \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(s-\theta_\tau)^2}{2})(1-p(s))\tau(s-\theta_\tau)ds
\end{aligned}$$

Note that $\int_{-\infty}^{x(\theta_\tau)} \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(s-\theta_\tau)^2}{2})\tau(s-\theta_\tau)ds = -\frac{\sqrt{\tau}}{\sqrt{2\pi}}\exp(-\frac{\tau(x(\theta_\tau)-\theta_\tau)^2}{2})$, one has

$$h'(\theta_\tau) = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(x(\theta_\tau)-\theta_\tau)^2}{2})\frac{\alpha}{\tau} + \int_{x(\theta_\tau)}^{y} \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(s-\theta_\tau)^2}{2})\tau(s-\theta_\tau)(1-p(s))ds$$

$$< \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(x(\theta_\tau)-\theta_\tau)^2}{2})\frac{\alpha}{\tau} + \int_0^\infty \frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(s-\theta_\tau)^2}{2})\tau(s-\theta_\tau)ds$$
$$< \frac{1}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})$$

If $\sqrt{\tau}/(\alpha+\tau) > 1/\sqrt{2\pi}$, $h'(\theta_\tau) < 1$. The conclusion follows. □

Proof of Lemma 2

Proof. Note that for each τ, as y goes to $-\infty$, l.h.s. of equation (2) goes to 1 and as y goes to ∞, l.h.s. of equation (2) goes to 0. For each τ, define

$$J(y) = \Phi(\frac{\theta_\tau - \frac{\alpha\mu+\tau y}{\alpha+\tau}}{\frac{1}{\sqrt{\alpha+\tau}}})$$

To prove the desired result, it suffices to show l.h.s. of equation (2) is strictly decreasing, i.e. $J'(y) < 0$.

$$J'(y) = \phi(z)\frac{\partial z}{\partial y}$$

where $z = \frac{\theta_\tau - \frac{\alpha\mu+\tau y}{\alpha+\tau}}{\frac{1}{\sqrt{\alpha+\tau}}}$. To show $J'(y) < 0$, it suffices to show that

$$\frac{\partial \theta(y,\tau)}{\partial y} < \frac{\tau}{\alpha+\tau}$$

Let $L(\theta_\tau, y) = h(\theta_\tau, y) - \theta_\tau$. By Lemma 1, $L(\theta_\tau, y) = 0$. Then by implicit function theorem,

$$\frac{\partial \theta_\tau}{\partial y} = -\frac{\partial L/\partial y}{\partial L/\partial \theta_\tau}$$
$$= \frac{\frac{\sqrt{\tau}}{\sqrt{2\pi}} \exp(-\frac{\tau(y-\theta_\tau)^2}{2})(1-P(y))}{1-h'(\theta_\tau)}$$

By Lemma 1, $h'(\theta_\tau) < \frac{1}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau}) < 1$, hence

$$\frac{\partial \theta_\tau}{\partial y} < \frac{\frac{\sqrt{\tau}}{\sqrt{2\pi}}}{1 - \frac{1}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})}$$

It suffices to show

$$\frac{\frac{\sqrt{\tau}}{\sqrt{2\pi}}}{1 - \frac{1}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})} < \frac{\tau}{\alpha+\tau}$$

through simple algebra, it is easy to figure out that the above relation is equivalent to

$$\frac{\sqrt{\tau}}{\alpha+\tau} > \frac{2}{\sqrt{2\pi}}$$

So when $\sqrt{\tau}/(\alpha+\tau) > 2/\sqrt{2\pi}$, $J'(y) < 0$. The conclusion follows. □

Proof of Proposition 1

Proof. The existence follows directly from Lemma 1 and 2. It also follows that the there is only one switching strategy equilibrium. It remains to show it is the only strategy which survives the iterated elimination of strictly dominated strategies.

Consider when all the other players follow switching strategies with thresholds denoted by $\hat{x_\tau}$, $\hat{y}$, the expected payoff to choosing action 1 for a player who observes a signal x and knows the value of τ :

$$u(x_\tau, \hat{x_\tau}, \hat{y}) = \int_{-\infty}^{\theta(\hat{x_\tau},\hat{y})} f(\theta; \nu_{x_\tau}, \beta) d\theta - t$$

and for a player who observes a signal y and does not know the value of τ:

$$u(y, \hat{x_\tau}, \hat{y}) = \int \int_{-\infty}^{\theta(\hat{x_\tau},\hat{y})} f(\theta; \nu_y, \beta) d\theta dG(\tau) - t$$

where $\theta(\hat{x_\tau}, \hat{y})$ is determined by

$$\int_{-\infty}^{x_\tau} f(s; \theta_\tau, \tau) P(s) ds + \int_{-\infty}^{y} f(s; \theta_\tau, \tau)(1 - P(s)) ds = \theta_\tau$$

Note that both $u(x_\tau, \hat{x_\tau}, \hat{y})$ and $u(y, \hat{x_\tau}, \hat{y})$ are decreasing with respect to the first argument and increasing in the second and third argument.[3]

When a signal is sufficiently small, attack is the strictly dominant strategy. Suppose all the other players choose attack, this is the case where $\bar{x_\tau}^0 = \infty$ for all τ and $\bar{y}^0 = \infty$, we solve $\bar{x_\tau}^1$ and $\bar{y}^1$ satisfying:

$$u(\bar{x_\tau}^1, \bar{x_\tau}^0, \bar{y}^0) = t$$
$$u(\bar{y}^1, \bar{x_\tau}^0, \bar{y}^0) = t$$

It is easy to see that $\bar{x_\tau}^1 < \bar{x_\tau}^0$ for all τ and $\bar{y}^1 < \bar{y}^0$. Note that even if all the other players choose to attack, a players who observes a signal above $\bar{x_\tau}^1$ and conducts a successful investigation will not take attack as a rational choice. Similarly action attack cannot be rational

[3]More precisely, for $u(y, \hat{x_\tau}, \hat{y})$, we should say when $\hat{x_\tau}$ increases for all τ, $u(y, \hat{x_\tau}, \hat{y})$ decreases.

for a player who observes a signal above $\bar{y}^1$ and conducts an unsuccessful investigation. After eliminating all the strictly dominated strategies, let us repeat the process by taking $\bar{x_\tau}^1$ and $\bar{y}^1$ to the above equation systems. In other words, suppose all the players follow switching strategies with thresholds denoted by $\bar{x_\tau}^1$ and $\bar{y}^1$. Then we will get $\bar{x_\tau}^2$ and $\bar{y}^2$ and $\bar{x_\tau}^2 < x_\tau^1$ for all τ and $\bar{y}^2 < \bar{y}^1$. Proceeding in this way, we will get two decreasing sequences $(\bar{x_\tau}^k)$ for each τ, and $\bar{y}^k$ satisfying

$$\infty = \bar{x_\tau}^0 > \bar{x_\tau}^1 > \bar{x_\tau}^2 > ... > \bar{x_\tau}^k > ... \quad \forall \tau$$
$$\infty = \bar{y}^0 > \bar{y}^1 > \bar{y}^2 > ... > \bar{y}^k > ...$$

Note that for choosing to attack for a player who observes a signal above $\bar{x_k}$ and knows the value of τ will not survive k rounds of elimination of strictly dominated strategies. The limits of the sequences denoted by $\bar{x_\tau}$ and $\bar{y}$ should satisfy

$$u(\bar{x_\tau}, \bar{x_\tau}, \bar{y}) = t$$
$$u(\bar{y}, \bar{x_\tau}, \bar{y}) = t$$

On the other hand, by similar logic, we can start our analysis by assuming all the other players choose to not attack and eliminate the irrational strategies of not attacking for players who observe signals below certain thresholds. We start from taking $\underline{x_\tau}^0$ and $\underline{y}^0$ from $-\infty$. Then we will get two increasing sequences $\underline{x_\tau}^k$ and $\underline{y}^k$. The corresponding limits should $\underline{x_\tau}$ and $\underline{y}$ satisfy:

$$u(\underline{x_\tau}, \underline{x_\tau}, \underline{y}) = t$$
$$u(\underline{y}, \underline{x_\tau}, \underline{y}) = t$$

Note that $u(x_\tau, x_\tau, y) = t$ and $u(y, x_\tau, y) = t$ have unique solutions by Lemma 1 and Lemma 2. Hence $\underline{x_\tau} = \bar{x_\tau} = x_\tau$ and $\underline{y} = \bar{y} = y$. And this unique strategy which survives the iterated elimination of strictly dominated strategies is exactly the switching strategy that we present in Proposition 1. The conclusion follows. □

Proof of Lemma 3

Proof. Consider $P, P' \in \mathscr{C}$. Let $\pi'(s,\tau)$ and $\pi'(s)$ represent the payoffs conditional on P'.

$$|T_P(s) - T_{P'}(s)| = |(C')^{-1}(\int \pi(s,\tau)dG(\tau) - \pi(s)) - (C')^{-1}(\int \pi'(s,\tau)dG(\tau) - \pi'(s))|$$

Since C is smooth, there exists $a_1, a_2 \in \mathbb{R}$ such that

$$\begin{aligned}|T_P(s) - T_{P'}(s)| &= |[(C')^{-1}]'(a_1)|| \int \pi(s,\tau)dG(\tau) - \pi(s) - \int \pi'(s,\tau)dG(\tau) + \pi'(s)| \\ &\leq \frac{1}{C''(a_2)}[\int |\pi(s,\tau) - \pi'(s,\tau)|dG(\tau) + |\pi(s) - \pi'(s)|]\end{aligned}$$

Let $\int |\pi(s,\tau) - \pi'(s,\tau)|dG(\tau) + |\pi(s) - \pi'(s)| = D$, since we have assumed that $C'' \geq c$, one has

$$|T_P(s) - T_{P'}(s)| \leq \frac{1}{\underline{c}} D$$

And for D, we have

$$\begin{aligned}D &= \int |\pi(s,\tau) - \pi'(s,\tau)|dG(\tau) + |\pi(s) - \pi'(s)| \\ &\leq 2 \int |\Phi(\sqrt{\beta}(\theta_\tau - \nu_s)) - \Phi(\sqrt{\beta}(\theta'_\tau - \nu_s))|dG(\tau)\end{aligned}$$

For each τ,

$$\begin{aligned}|\Phi(\sqrt{\beta}(\theta_\tau - \nu_s)) - \Phi(\sqrt{\beta}(\theta'_\tau - \nu_s))| &\leq \frac{\sqrt{\alpha+\tau}}{\sqrt{2\pi}}|\theta_\tau - \theta'_\tau| \\ &\leq \sup_\tau \frac{\sqrt{\alpha+\tau}}{\sqrt{2\pi}} \sup_\tau |\theta_\tau - \theta'_\tau| \\ &= \sup_\tau \frac{\tau+\alpha}{\sqrt{2\pi}\sqrt{\tau}} \frac{\sqrt{\tau}}{\sqrt{\tau+\alpha}} \sup_\tau |\theta_\tau - \theta'_\tau| \\ &< \sup_\tau \frac{\tau+\alpha}{\sqrt{2\pi}\sqrt{\tau}} \sup_\tau |\theta_\tau - \theta'_\tau|\end{aligned}$$

By equation (1),

$$\begin{aligned}\theta_\tau &= \int_{-\infty}^{x_\tau} f(s;\theta_\tau,\tau)ds + \int_{x_\tau}^{y} f(s;\theta_\tau,\tau)(1-P(s))ds \\ \Rightarrow |\theta_\tau - \theta'_\tau| &\leq |\int_{-\infty}^{x_\tau} f(s;\theta_\tau,\tau)ds - \int_{-\infty}^{x'_\tau} f(s;\theta'_\tau,\tau)ds| \\ &+ |\int_{x_\tau}^{y} f(s;\theta_\tau,\tau)(1-P(s))ds - \int_{x'_\tau}^{y'} f(s;\theta'_\tau,\tau)(1-P'(s))ds|\end{aligned}$$

For the first term on the r.h.s. of the above inequality, one has

$$|\int_{-\infty}^{x_\tau} f(s;\theta_\tau,\tau)ds - \int_{-\infty}^{x'_\tau} f(s;\theta'_\tau,\tau)ds| \leq \frac{\sqrt{\tau}}{\sqrt{2\pi}}|(x_\tau - \theta_\tau) - (x'_\tau - \theta'_\tau)| \tag{4}$$

Note that

$$x_\tau = (1+\frac{\alpha}{\tau})\theta_\tau - \frac{\alpha}{\tau}\mu - \frac{\sqrt{\alpha+\tau}}{\tau}\Phi^{-1}(t)$$
$$\Rightarrow x_\tau - \theta_\tau = \frac{\alpha}{\tau}\theta_\tau - \frac{\alpha}{\tau}\mu - \frac{\sqrt{\alpha+\tau}}{\tau}\Phi^{-1}(t)$$

Substitute the relation into inequality (4), we get

$$\begin{aligned}|\int_{-\infty}^{x_\tau} f(s;\theta_\tau,\tau)ds - \int_{-\infty}^{x'_\tau} f(s;\theta_\tau,\tau)ds| &\leq \frac{\sqrt{\tau}}{\sqrt{2\pi}}\frac{\alpha}{\tau}|\theta_\tau - \theta'_\tau| \\ &= \frac{\alpha}{\sqrt{2\pi}\sqrt{\tau}}|\theta_\tau - \theta'_\tau|\end{aligned}$$

For $|\int_{x_\tau}^{y} f(s;\theta_\tau,\tau)(1-P(s))ds - \int_{x'_\tau}^{y'} f(s;\theta'_\tau,\tau)(1-P'(s))| = \xi$, we have

$$\begin{aligned}\xi \leq& |\int_{x_\tau}^{x'_\tau} f(s;\theta_\tau,\tau)(1-P(s))ds| + |\int_{y}^{y'} f(s;\theta_\tau,\tau)(1-P(s))ds| \\ &+ |\int_{x'_\tau}^{y'} [f(s;\theta_\tau,\tau)(1-P(s)) - f(s;\theta'_\tau,\tau)(1-P'(s))]ds| \\ \leq& \frac{\sqrt{\tau}}{\sqrt{2\pi}}(|x_\tau - x'_\tau| + |y-y'|) + |\int_{x'_\tau}^{y'} [f(s;\theta_\tau,\tau)(1-P(s)) - f(s;\theta'_\tau,\tau)(1-P'(s))]ds| \\ =& \frac{\sqrt{\tau}}{\sqrt{2\pi}}(1+\frac{\alpha}{\tau})|\theta_\tau - \theta'_\tau| + \frac{\sqrt{\tau}}{\sqrt{2\pi}}|y-y'| + |\int_{x'_\tau}^{y'} [f(s;\theta_\tau,\tau)(1-P(s)) - f(s;\theta'_\tau,\tau)(1-P'(s))]ds|\end{aligned}$$

Claim: $|y-y'| \leq \sup_\tau |x_\tau - x'_\tau|$
Suppose not, $|y-y'| > \sup_\tau |x_\tau - x'_\tau|$, w.l.o.g., suppose $y > y'$, then

$$\begin{aligned}&(\frac{\alpha\mu+\tau y}{\alpha+\tau} - \theta_\tau) - (\frac{\alpha\mu+\tau y'}{\alpha+\tau} - \theta'_\tau) \\ =& \frac{\tau}{\alpha+\tau}(y - y' - \frac{\alpha+\tau}{\tau}(\theta_\tau - \theta'_\tau)) \\ >& \frac{\tau}{\alpha+\tau}(x_\tau - x'_\tau - \frac{\alpha+\tau}{\tau}(\theta_\tau - \theta'_\tau)) \\ =& 0\end{aligned}$$

It implies that

$$\Phi(\sqrt{\beta}(\theta_\tau - \nu_y)) - \Phi(\sqrt{\beta}(\theta'_\tau - \nu_y)) < 0$$

Since it holds for all τ, one has

$$\int [\Phi(\sqrt{\beta}(\theta_\tau - \nu_y)) - \Phi(\sqrt{\beta}(\theta'_\tau - \nu_y))]dG(\tau) < 0$$
$$\Rightarrow \int \Phi(\sqrt{\beta}(\theta_\tau - \nu_y))dG(\tau) < \int \Phi(\sqrt{\beta}(\theta'_\tau - \nu_y))dG(\tau)$$

Then we have a contradiction since

$$\int \Phi(\sqrt{\beta}(\theta_\tau - \nu_y))dG(\tau) = \int \Phi(\sqrt{\beta}(\theta'_\tau - \nu_y))dG(\tau) = t$$

Similarly for the case where $y < y'$. So we conclude that $|y - y'| \leq \sup_\tau |x_\tau - x'_\tau|$.
Then we provide an upper bound for $|\int_{x'_\tau}^{y'} [f(s;\theta_\tau,\tau)(1-P(s)) - f(s;\theta'_\tau,\tau)(1-P'(s))]ds| = \psi$

$$\begin{aligned}
\psi \leq& |\int_{x'_\tau}^{y'} [f(s;\theta_\tau,\tau)(1-P(s)) - f(s;\theta_\tau,\tau)(1-P'(s))]ds| \\
&+ |\int_{x'_\tau}^{y'} [f(s;\theta_\tau,\tau)(1-P'(s)) - f(s;\theta'_\tau,\tau)(1-P'(s))]ds| \\
\leq& |\int_{x'_\tau}^{y'} f(s;\theta_\tau,\tau)|P(s) - P'(s)|ds| + \frac{\sqrt{\tau}}{\sqrt{2\pi}}|\theta_\tau - \theta'_\tau| \\
\leq& \sup_s |P(s) - P'(s)| + \frac{\sqrt{\tau}}{\sqrt{2\pi}}|\theta_\tau - \theta'_\tau|
\end{aligned}$$

Overall, we have

$$\begin{aligned}
&|\theta_\tau - \theta'_\tau| \leq \frac{\alpha}{\sqrt{2\pi}\sqrt{\tau}}|\theta_\tau - \theta'_\tau| + \frac{1}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})|\theta_\tau - \theta'_\tau| \\
&+ \frac{\sqrt{\tau}}{\sqrt{2\pi}} \sup_\tau (1 + \frac{\alpha}{\tau}) \sup_\tau |\theta_\tau - \theta'_\tau| + \frac{\sqrt{\tau}}{\sqrt{2\pi}}|\theta_\tau - \theta'_\tau| + \sup_s |P(s) - P'(s)| \\
&\Rightarrow [1 - \frac{2}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})]|\theta_\tau - \theta'_\tau| \leq \frac{\sqrt{\tau}}{\sqrt{2\pi}} \sup_\tau \frac{1}{\sqrt{\tau}} \sup_\tau (\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})|\theta_\tau - \theta'_\tau| + ||P - P'||_\infty \\
&\leq \frac{1}{\sqrt{2\pi}} \frac{\sup_\tau \sqrt{\tau}}{\inf_\tau \sqrt{\tau}} \sup_\tau (\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau}) \sup_\tau |\theta_\tau - \theta'_\tau| + ||p - p'||_\infty \\
&\Rightarrow |\theta_\tau - \theta'_\tau| \leq \frac{1}{\sqrt{2\pi}} \Delta \frac{\sup_\tau (\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})}{1 - \frac{2}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})} \sup_\tau |\theta_\tau - \theta'_\tau| + \frac{1}{1 - \frac{2}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})}||P - P'||_\infty \\
&\Rightarrow \sup_\tau |\theta_\tau - \theta'_\tau| \leq \Delta \sup_\tau \frac{\sup_\tau \frac{1}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})}{1 - \frac{2}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})} \sup_\tau |\theta_\tau - \theta'_\tau| + \sup_\tau \frac{1}{1 - \frac{2}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})}||P - P'||_\infty
\end{aligned}$$

Let $\lambda = \sup_\tau \frac{1}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau})$, then we have

$$(1 - \frac{\Delta\lambda}{1-2\lambda}) \sup_\tau |\theta_\tau - \theta'_\tau| \leq \frac{1}{1-2\lambda}||p - p'||_\infty$$
$$\Rightarrow \sup_\tau |\theta_\tau - \theta'_\tau| \leq \frac{\frac{1}{1-2\lambda}}{1 - \Delta\frac{\lambda}{1-2\lambda}}||p - p'||_\infty$$

So for D, we have

$$D < \sup_\tau \frac{1}{\sqrt{2\pi}}(\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau}) \sup_\tau |\theta_\tau - \theta'_\tau|$$
$$\leq \frac{\frac{\lambda}{1-2\lambda}}{1 - \Delta\frac{\lambda}{1-2\lambda}}||P - P'||_\infty$$

To show T is a contraction mapping, it suffices to show $\frac{\frac{\lambda}{1-2\lambda}}{1-\Delta\frac{\lambda}{1-2\lambda}} \leq \underline{c}$. Note that

$$\frac{\frac{\lambda}{1-2\lambda}}{1 - \Delta\frac{\lambda}{1-2\lambda}} \leq \underline{c}$$

which is equivalent to

$$\frac{\alpha}{\sqrt{\tau}} + \sqrt{\tau} \leq \frac{\sqrt{2\pi}\underline{c}}{1 + (2 + \Delta)\underline{c}}$$

The conclusion follows. □

Proof of Proposition 2

Proof. It follows directly from Proposition 1 and Lemma 3. □

Proof of Proposition 3

Proof. The optimal effort P^* is given by

$$P^*(s) = (C')^{-1}(\int \pi(s,\tau)dG(\tau) - \pi(s))$$

Recall that

$$\pi(s,\tau) = \max\{\Phi((\theta_\tau - \nu_s)\sqrt{\beta}) - t, 0\}$$
$$\pi(s) = \max\{\int \Phi((\theta_\tau - \nu_s)\sqrt{\beta})dG(\tau) - t, 0\}$$

Consider the case where $s < y$, for each τ, as s goes to $-\infty$, $\Phi((\theta_\tau - \nu_s)\sqrt{\beta})$ goes to 1. So when s is sufficiently small, we have

$$\Phi((\theta_\tau - \nu_s)\sqrt{\beta}) - t > 0$$

for all τ. Then $\int \pi(s,\tau)dG(\tau) = \pi(s) = \int \Phi((\theta_\tau - \nu_s)\sqrt{\beta})dG(\tau) - t$ which implies that $p^*(s) = 0$. Consider the case where there exist some τ such that $\pi(s,\tau) < 0$. It is the case where we have $p^*(s) > 0$. For these s, we have

$$\int \pi(s,\tau)dG(\tau) - \pi(s) = \int_{T_1} [t - \Phi((\theta_\tau - \nu_s)\sqrt{\beta})]dG(\tau)$$

where $T_1 = \{\tau \in [\underline{\tau}, \bar{\tau}] | x(\theta_\tau) < s\}$. As s increases, the set T_1 becomes larger. In addition, for each $\tau \in T_1$, $t - \Phi((\theta_\tau - \nu_s)\sqrt{\beta})$ increases as s increases. It follows that $p^*(s)$ increases. Similar for the case where $s > y$. And since p^* is continuous, the existence of $\underline{s}$ and $\bar{s}$ is immediate. The conclusion follows. □

Proof of Proposition 4

Proof. Note that the equilibrium in the global game are characterized by the following two equations

$$\int_{-\infty}^{x(\theta_\tau)} f(s;\theta_\tau,\tau)P(s)ds + \int_{-\infty}^{y} f(s;\theta_\tau,\tau)(1-P(s))ds = \theta_\tau \tag{1}$$

$$\int\int_{-\infty}^{\theta_\tau} f(\theta;\nu_y,\beta)d\theta dG(\tau) = t \tag{2}$$

When $x_\tau > y$, by equation (1) we have

$$\int_{-\infty}^{x_\tau} f(s;\theta_\tau,\tau)ds - \int_{y}^{x_\tau} f(s;\theta_\tau,\tau)(1-P(s)) = \theta_\tau$$
$$\Rightarrow \int_{-\infty}^{x_\tau} f(s;\theta_\tau,\tau)ds > \theta_\tau$$

In a standard global game, $\int_{-\infty}^{x_\tau^s} f(s;\theta_\tau^s,\tau)ds = \theta_\tau^s$. And for $(x,\theta) \in \{(x_\tau,\theta_\tau),(x_\tau^s,\theta_\tau^s)\}$, we have

$$x = (1+\frac{\alpha}{\tau})\theta - \frac{\alpha}{\tau}\mu - \frac{\sqrt{\alpha+\tau}}{\tau}\Phi^{-1}(t)$$

And $I(\theta) = \theta - \int_{-\infty}^{x(\theta)} f(s;\theta,\tau)$ satisfies $I'(\theta) > 0$. So $\theta_\tau^s > \theta_\tau$, $x_\tau^s > x_\tau$. Similar for the case where $x_\tau < y$. The conclusion follows. □

Proof of Proposition 5

Proof. Note that the equilibrium in the global game are characterized by the following two equations

$$\int_{-\infty}^{x(\theta_\tau)} f(s;\theta_\tau,\tau)P(s)ds + \int_{-\infty}^{y} f(s;\theta_\tau,\tau)(1-P(s))ds = \theta_\tau \tag{1}$$

$$\int\int_{-\infty}^{\theta_\tau} f(\theta;\nu_y,\beta)d\theta dG(\tau) = t \tag{2}$$

where $x(\theta_\tau) = (1+\frac{\alpha}{\tau})\theta_\tau - \frac{\alpha}{\tau}\mu - \frac{\sqrt{\alpha+\tau}}{\tau}\Phi^{-1}(t)$. We have shown that from (1), θ_τ can be expressed as a function of y and τ, denoted by $\theta(y,\tau)$ with parameters α and μ. Substitute $\theta(y,\tau)$ into (2), we have

$$\int\int_{-\infty}^{\theta(y,\tau)} f(\theta;\nu_y,\beta)d\theta dG(\tau) = t$$

This is the equation determining y. Let $H(y;\alpha,\mu) = \int\int_{-\infty}^{\theta(y,\tau)} f(\theta;\nu_y,\beta)d\theta dG(\tau) - t$, then by implicit function theorem

$$\frac{dy}{d\mu} = -\frac{\partial H/\partial\mu}{\partial H/\partial y}$$

We have shown that $\partial H/\partial y < 0$ in Lemma 2. It is in our interest to analyze the sign of $\partial H/\partial\mu$.

$$\frac{\partial H}{\partial\mu} = \int[\frac{\partial\theta(y,\tau)}{\partial\mu}\frac{\sqrt{\beta}}{\sqrt{2\pi}}\exp(-\frac{\beta(\theta-\nu_y)^2}{2}) + \int_{-\infty}^{\theta(y,\tau)}\frac{\sqrt{\beta}}{\sqrt{2\pi}}\exp(-\frac{\beta(\theta-\nu_y)^2}{2})\beta(\theta-\nu_y)\frac{\alpha}{\tau+\alpha}d\theta]dG(\tau)$$

Note that $\theta(y,\tau)$ is achieved by equation (1), let $h(\theta_\tau) = \int_{-\infty}^{x(\theta_\tau)} f(s;\theta_\tau,\tau)p(s)ds + \int_{-\infty}^{y} f(s;\theta_\tau,\tau)(1-p(s))ds$, we have

$$\frac{\partial\theta(y,\tau)}{\partial\mu} = -\frac{\partial h/\partial\mu}{h'(\theta_\tau)-1}$$

From Lemma 1, we have shown that $h'(\theta_\tau) < 1$, and since $x(\theta_\tau) = (1+\frac{\alpha}{\tau})\theta_\tau - \frac{\alpha}{\tau}\mu - \frac{\sqrt{\alpha+\tau}}{\tau}\Phi^{-1}(t)$, $\partial x/\partial\mu < 0$, it implies that $\partial h/\partial\mu < 0$, so $\frac{\partial\theta(y,\tau)}{\partial\mu} < 0$. And

$$\int_{-\infty}^{\infty}\frac{\sqrt{\beta}}{\sqrt{2\pi}}\exp(-\frac{\beta(\theta-\nu_y)^2}{2})\beta(\theta-\nu_y)d\theta = 0$$

$$\Rightarrow \int_{-\infty}^{\theta(y,\tau)}\frac{\sqrt{\beta}}{\sqrt{2\pi}}\exp(-\frac{\beta(\theta-\nu_y)^2}{2})\beta(\theta-\nu_y)d\theta < 0$$

Then we have

$$\frac{\partial H}{\partial \mu} < 0 \Rightarrow \frac{dy}{d\mu} < 0$$

Hence $\frac{d\theta(y,\tau)}{d\mu} = \frac{\partial \theta}{\partial \mu} + \frac{\partial \theta}{\partial y}\frac{dy}{d\mu} < 0$. It implies that $\frac{dx}{d\mu} < 0$ since $x(\theta_\tau) = (1+\frac{\alpha}{\tau})\theta_\tau - \frac{\alpha}{\tau}\mu - \frac{\sqrt{\alpha+\tau}}{\tau}\Phi^{-1}(t)$. Then we analyze the effect on optimal effort to acquire information. Note that

$$\pi(s,\tau) = \max\{\Phi(\frac{\theta_\tau - \nu_s}{\sqrt{\beta}^{-1}}) - t, 0\}$$
$$\pi(s) = \max\{\int \Phi(\frac{\theta_\tau - \nu_s}{\sqrt{\beta}^{-1}})dG(\tau) - t, 0\}$$
$$p^*(s) = (C')^{-1}(\int \pi(s,\tau)dG(\tau) - \pi(s))$$

By Proposition 4, we have

$$\int \pi(s,\tau)dG(\tau) - \pi(s) = \int_{T_1} [t - \Phi((\theta_\tau - \nu_s)\sqrt{\beta})]dG(\tau)$$

$\int \pi(s,\tau)dG(\tau)$ is differentiable w.r.t. θ_τ, therefore differentiable w.r.t. μ. But for $\pi(s)$ it is not the case. $\pi(s)$ is differentiable w.r.t. μ everywhere but $s = y$. When $s < y$, since $d\theta/d\mu < 0$ and $dx/d\mu < 0$, for every $\tau \in T_1$

$$\frac{d[t - \Phi((\theta_\tau - \nu_s)\sqrt{\beta})]}{d\mu} > 0$$

Furthermore since when x_τ decreases for all τ, the set T_1 becomes larger. Hence we conclude that $\frac{\partial p^*(s)}{\partial \mu} \geq 0$ when $s < y$. And similarly when $s > y$, $\frac{\partial p^*(s)}{\partial \mu} \leq 0$. The conclusion follows. □

Proof of Proposition 6

Proof. Note that in equilibrium we have

$$x_\tau = (1 + \frac{\alpha}{\tau})\theta_\tau - \frac{\alpha}{\tau}\mu - \frac{\sqrt{\alpha+\tau}}{\tau}\Phi^{-1}(t)$$

which implies that

$$\frac{\partial x_\tau}{\partial \alpha} = \frac{1}{\tau}(\theta_\tau - \mu - \frac{1}{2}(\alpha+\tau)^{-1/2}\Phi^{-1}(t))$$

which is negative under our assumptions. In the proof of Proposition 5, we define $H(y;\alpha,\mu) = \int\int_{-\infty}^{\theta(y,\tau)} f(\theta;\nu_y,\beta)d\theta dG(\tau) - t$, then by implicit function theorem, we have

$$\frac{dy}{d\alpha} = -\frac{\partial H/\partial\alpha}{\partial H/\partial\alpha}$$

And

$$\frac{\partial H}{\partial\alpha} = \int[\frac{\partial\theta(y,\tau)}{\partial\alpha}\frac{\sqrt{\beta}}{\sqrt{2\pi}}\exp(-\frac{\beta(\theta-\nu_y)^2}{2}) + \int_{-\infty}^{\theta(y,\tau)}\frac{\sqrt{\beta}}{\sqrt{2\pi}}\exp(-\frac{\beta(\theta-\nu_y)^2}{2})\beta(\theta-\nu_y)\frac{\tau\mu}{(\tau+\alpha)^2}d\theta]dG(\tau)$$

Similar as that in the proof of Proposition 5, we define $h(\theta_\tau) = \int_{-\infty}^{x(\theta_\tau)} f(s;\theta_\tau,\tau)p(s)ds + \int_{-\infty}^{y} f(s;\theta_\tau,\tau)(1-p(s))ds$, we have

$$\frac{\partial\theta(y,\tau)}{\partial\alpha} = -\frac{\partial h/\partial\alpha}{h'(\theta_\tau)-1}$$

From Lemma 1, we have shown that $h'(\theta_\tau) < 1$, and since by our assumptions $\partial x/\partial\alpha < 0$, it implies that $\partial h/\partial\alpha < 0$, so $\frac{\partial\theta(y,\tau)}{\partial\alpha} < 0$. And So we have

$$\frac{\partial H}{\partial\alpha} < 0 \Rightarrow \frac{dy}{d\alpha} < 0$$

Hence $\frac{d\theta(y,\tau)}{d\alpha} = \frac{\partial\theta}{\partial\alpha} + \frac{\partial\theta}{\partial y}\frac{dy}{d\alpha} < 0$. It also implies that $\frac{dx}{d\alpha} < 0$. The remaining part is the same as the proof in the last section. □

Proof of Lemma 4

Proof. We show the result by contradiction. Suppose when $P'(s) = P(s)$ for all $s \leq y_P$ and $P'(s) > P(s)$ for all $s > y_P$, we have $y_{P'} < y_P$. The equations characterizing the equilibrium conditional on P are as follows:

$$\int_{-\infty}^{x(\theta_\tau)} f(s;\theta_\tau,\tau)P(s)ds + \int_{-\infty}^{y} f(s;\theta_\tau,\tau)(1-P(s))ds = \theta_\tau \quad (1)$$

$$\int\int_{-\infty}^{\theta_\tau} f(\theta;\nu_y,\beta)d\theta dG(\tau) = t \quad (2)$$

By Lemma 1, equation 1 allows us to represent θ as a function of y, denoted by $\theta_P(y,\tau)$. Then we substitute $\theta_P(y,\tau)$ into equation 2, one has

$$\int\int_{-\infty}^{\theta_P(y,\tau)} f(\theta_P(y,\tau);\nu_y,\beta)d\theta dG(\tau) = t$$

Lemma 2 shows the above equation has a unique solution y which is the equilibrium threshold y_P. y_P satisfies

$$\int\int_{-\infty}^{\theta_P(y_P,\tau)} f(\theta_P(y_P,\tau);\nu_{y_P},\beta)d\theta dG(\tau)=t$$

Since we have assumed that $y_{P'}<y_P$, and by lemma 2 $\int_{-\infty}^{\theta_P(y,\tau)} f(\theta_P(y,\tau);\nu_y,\beta)d\theta$ decreases w.r.t. y. So we have

$$\int_{-\infty}^{\theta_P(y_{P'},\tau)} f(\theta_P(y_{P'},\tau);\nu_{y_{P'}},\beta)d\theta > \int_{-\infty}^{\theta_P(y_P,\tau)} f(\theta_P(y_P);\nu_y,\beta)d\theta \quad \forall\tau$$
$$\Rightarrow \int\int_{-\infty}^{\theta(y_{P'},\tau)} f(\theta_P(y_{P'},\tau);\nu_{y_{P'}},\beta)d\theta dG(\tau) > \int\int_{-\infty}^{\theta(y_P,\tau)} f(\theta_P(y_P,\tau);\nu_y,\beta)d\theta dG(\tau)=t$$

Similarly, the equations characterizing the equilibrium conditional on P' are

$$\int_{-\infty}^{x(\theta_\tau)} f(s;\theta_\tau,\tau)P'(s)ds+\int_{-\infty}^{y} f(s;\theta_\tau,\tau)(1-P'(s))ds=\theta_\tau \tag{1'}$$
$$\int\int_{-\infty}^{\theta_\tau} f(\theta;\nu_y,\beta)d\theta dG(\tau)=t \tag{2'}$$

Now for equation (1′), we can represent θ as a function of y and τ, denoted by $\theta_{P'}(y,\tau)$. In equilibrium we have

$$\int_{-\infty}^{x(\theta_{P'}(y_{P'},\tau))} f(s;\theta_{P'}(y_{P'},\tau),\tau)P'(s)ds+\int_{-\infty}^{y_{P'}} f(s;\theta_{P'}(y_{P'},\tau),\tau)(1-P'(s))ds=\theta_{P'}(y_{P'},\tau)$$
$$\Rightarrow \int_{-\infty}^{y_{P'}} f(s;\theta_{P'}(y_{P'},\tau),\tau)ds+\int_{y_{P'}}^{x(\theta_{P'}(y_{P'},\tau))} f(s;\theta_{P'}(y_{P'},\tau),\tau)P'(s)ds=\theta_{P'}(y_{P'},\tau)$$

If we fix $y_{P'}$ and substitute P' by P, one has

$$\int_{-\infty}^{x(\theta_P(y_{P'},\tau))} f(s;\theta_P(y_{P'},\tau),\tau)P(s)ds+\int_{-\infty}^{y_{P'}} f(s;\theta_P(y_{P'},\tau),\tau)(1-P(s))ds=\theta_P(y_{P'},\tau)$$
$$\Rightarrow \int_{-\infty}^{y_{P'}} f(s;\theta_P(y_{P'},\tau),\tau)ds+\int_{y_{P'}}^{x(\theta_P(y_{P'},\tau))} f(s;\theta_P(y_{P'},\tau),\tau)P(s)ds=\theta_P(y_{P'},\tau)$$

Since $P'(s)>P(s)$ for $s>y_P>y_{P'}$, we have $\theta_P(y_P,\tau)\le\theta_{P'}(y_{P'},\tau)$ and the strict inequality holds for τ such that $x_P(\tau)>y_P$. Then

$$\int_{-\infty}^{\theta_{P'}(y_{P'},\tau)} f(\theta_{P'}(y_{P'},\tau);\nu_{y_{P'}},\beta)d\theta > \int_{-\infty}^{\theta_P(y_{P'},\tau)} f(\theta_P(y_{P'});\nu_{y_{P'}},\beta)d\theta \quad \forall\tau$$

$$\Rightarrow \int \int_{-\infty}^{\theta_{P'}(y_{P'},\tau)} f(\theta_{P'}(y_{P'},\tau);\nu_{y_{P'}},\beta) d\theta dG(\tau) > \int \int_{-\infty}^{\theta_{P}(y_{P'},\tau)} f(\theta_{P}(y_{P'},\tau);\nu_{y_{P'}},\beta) d\theta dG(\tau) > t$$

which shows that it cannot be an equilibrium. So $y_{P'} > y_P$. For the (CM) condition,

$$\int_{-\infty}^{x_\tau} f(s;\theta_\tau,\tau)P(s)ds + \int_{-\infty}^{y} f(s;\theta_\tau,\tau)(1-P(s))ds = \theta_\tau \qquad (CM)$$

When y increases, the l.h.s. of the above equation increases. And if P increases for $s > y_P$, l.h.s. also increases. So since we have $P' > p$ for $s > y_P$ and $y_{P'} > y_P$, it follows that $\theta_{P'}(\tau) > \theta_P(\tau)$. In addition, in equilibrium we have $x(\theta_\tau) = (1+\frac{\alpha}{\tau})\theta_\tau - \frac{\alpha}{\tau}\mu - \frac{\sqrt{\alpha+\tau}}{\tau}\Phi^{-1}(t)$. So $x_{P'}(\tau) > x_P(\tau)$. Similar logic applies when $P' > P$ for $s < y_P$. The conclusion follows. □

Proof of Proposition 7

Proof. The optimal effort P^* is given by

$$P^*(s) = (C')^{-1}(\int \pi(s,\tau)dG(\tau) - \pi(s))$$

If $s < y$, recall that

$$\int \pi(s,\tau)dG(\tau) - \pi(s) = \int_{T_1} [t - \Phi((\theta_\tau - \nu_s)\sqrt{\beta})]dG(\tau)$$

where $T_1 = \{\tau \in [\underline{\tau},\bar{\tau}] | x(\theta_\tau) < s\}$. Suppose $P'(s) = P(s)$ for all $s \le y$ and $P'(s) > P(s)$ for $s > y$. By Lemma 4, we have $x'_\tau > x_\tau$ and $\theta'_\tau > \theta_\tau$ for all τ. Hence $T'_1 \subset T_1$. In addition, for each $\tau \in T'_1$, $t - \Phi((\theta'_\tau - \nu_s)\sqrt{\beta}) < t - \Phi((\theta_\tau - \nu_s)\sqrt{\beta})$. So it follows that $P^{*\prime}(s) > P^*(s)$ for $s < y_P$ when $T'_1 \neq \emptyset$. Similarly, for $s > y_{P'}$, we get $P^{*\prime}(s) \le P^*(s)$. Note that the property of P^* proved by Proposition 3 also applies to $P^{*\prime}$. Since $P^{*\prime}$ is continuous, there must exists a $\hat{s} \in [y_P, y_{P'}]$ such that $P^{*\prime}(s) \ge P^*(s)$ for $s > \hat{s}$ and $P^{*\prime}(s) \le P^*(s)$ for $s < \hat{s}$. The conclusion follows. □

References

Ahnert, T., and A. Kakhbod (2017): "Information Choice and Amplification of Financial Crises," *The Review of Financial Studies*, 30(6), 2130–2178.

Angeletos, G., C. Hellwig, and A. Pavan (2006): "Signaling in a Global Game: Coordination and Policy Traps," *Journal of Political Economy*, 114(3), 452–484.

Angeletos, G., and A. Pavan (2013): "Selection-free predictions in global games with endogenous information and multiple equilibria," *Theoretical Economics*, 8(3), 883–938.

Carlsson, H., and E. van Damme (1993): "Global Games and Equilibrium Selection," *Econometrica*, 61(5), 989–1018.

Colombo, L., G. Femminis, and A. Pavan (2014): "Information Acquisition and Welfare," *Review of Economic Studies*, 81, 1438–1483.

Corsetti, G., A. Dasgupta, S. Morris, and H. S. Shin (2004): "Does One Soros Make a Difference? A Theory of Currency Crises with Large and Small Traders," *Review of Economic Studies*, 71(1), 87–113.

Diamond, D. W., and P. H. Dybvig (1983): "Bank runs, deposit insurance, and liquidity," *The Journal of Political Economy*, 91(3), 401–419.

Frankel, D. M., S. Morris, and A. Pauzner (2003): "Equilibrium Selection in Global Games with Strategic Complementarities," *Journal of Economic Theory*, 108, 1–44.

Goldstein, I., and A. Pauzner (2005): "Coordination with Flexible Information Acquisition," *Journal of Finace*, 60(3), 1293–1327.

Hebert, B., and M. Woodford (2017): "Rational inattention with sequential information sampling," Working paper, Stanford University and Columbia University.

Hellwig, C., and L. Veldkamp (2009): "Knowing What Others Know: Coordination Motives in Information Acquisition," *Review of Economic Studies*, 76, 223–251.

Morris, S., and H. S. Shin (1998): "Unique Equilibrium in a Model of Self-fulfilling Currency Attacks," *American Economic Review*, 88(3), 587–597.

——— (2003): "Global Games: Theory and Applications," in *Advances in Economics and Econometrics*, ed. by M. Dewatripont, L. P. Hansen, and S. J. Turnovsky, chap. 3, pp. 56–114. Cambridge University Press.

——— (2004): "Coordination Risk and the Price of Debt," *European Economic Review*, 48, 133–153.

Morris, S., H. S. Shin, and M. Yildiz (2016): "Common belief foundations of global games," *Journal of Economic Theory*, 163, 826–848.

Morris, S., and M. Yang (2016): "Coordination and Continuous Choice," Working paper, Princeton University and Duke University.

Myatt, D. P., and C. Wallace (2012): "Endogeneous Information Acquisition in Coordination Games," *Review of Economic Studies*, 79, 340–374.

Nikitin, M., and R. T. Smith (2008): "Information acquisition, coordination, and fundamentals in a financial crisis," *Journal of Banking & Finance*, 32(6), 907–914.

Sims, C. A. (2003): "Implications of rational inattention," *Journal of Monetary Economics*, 50(3), 665–690.

Szkup, M., and I. Trevino (2015): "Information Acquisition in Global Games of Regime Change," *Journal of Economic Theory*, 160, 387–428.

Yang, M. (2015): "Coordination with Flexible Information Acquisition," *Journal of Economic Theory*, 158, 721–738.

Zwart, S. (2008): "Liquidity runs with endogenous information acquisition," *Economics Letters*, 100(1), 64–67.