Weak Monotone Comparative Statics

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Motivation

- **Comparative Statics:** how predicted behavior changes as environment changes.
- **Monotone Comparative Statics:** Topkis (1979, 1998) and Milgrom and Shannon (1994) provide a method that captures essential properties driving comparative statics.
  - Since predictions are often nonunique, set order matters.
  - Existing theory uses strong set order

Strong Set vs Weak Set Order

- Consider a partial order $(X, \geq)$, which induces set orders.

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- **Weak Set Order:** $X'' \geq_{ws} X'$ if
  - $X'' \geq_{uws} X' : \forall x' \in X'$, there exists $x'' \in X''$ with $x'' \geq x'$.
  - $X'' \geq_{lws} X' : \forall x'' \in X''$, there exists $x' \in X'$ with $x' \leq x''$.

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☐ The set $M(t) := \arg\max_{x \in X} u(x; t)$ increases in t in the strong set order if u satisfies **MS** conditions: *single crossing* in (x, t) and is *quasi-supermodular* in x.

☐ But beyond individual choices, MCS is difficult to achieve in the strong set order (e.g., social choice, games, and matching)

Illustration with Nash equilibria

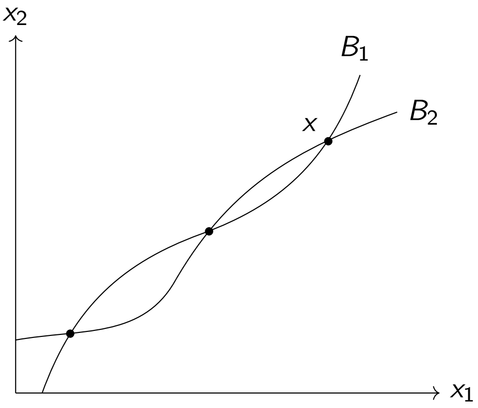

Figure: Failure of sMCS.

The MS conditions for payoffs guarantee monotonicity of best response.

Illustration with Nash equilibria

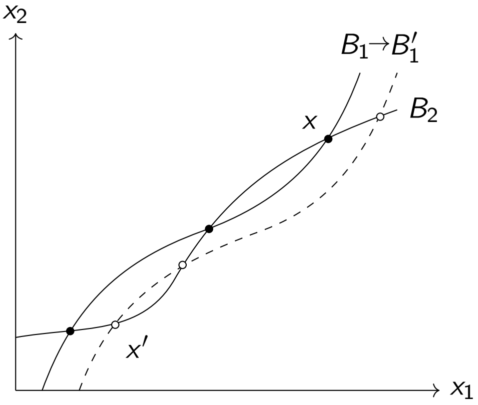

Figure: Failure of sMCS.

But equilibria do not shift in the strong set order. They do shift monotonically in the weak set order.

What We Do

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- Look for conditions for wMCS in the context of:
  - Individual choices
  - Pareto optimal choices
  - Games
  - Two-sided matching
- In the process, we make progress on
  - existence of fixed points and Nash equilibria in games
  - characterization and existence of stable matching in two-sided matching
- Expand applications of game theory and matching: to allow for indidviduals with incomplete preferences and multidivisional organizations.

Individual Choices

— characterizations along the lines of Milgrom and Shannon (1994) and Quah and Strulovici (2007)

— Omitted due to time constraint

Pareto Optimal Choices

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- I: finite set of individuals
- X: set of possible (social) choices; a *poset* with $\geq$
- $u_i : X \to \mathbb{R}$ payoff function for $i \in I$;

  $\mathbf{u} = (u_i)$ profile of payoff functions
- $P(\mathbf{u})$: set of Pareto optimal choices (POC) under $\mathbf{u}$.

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- *Does MS condition for individuals imply wMCS of POCs?*

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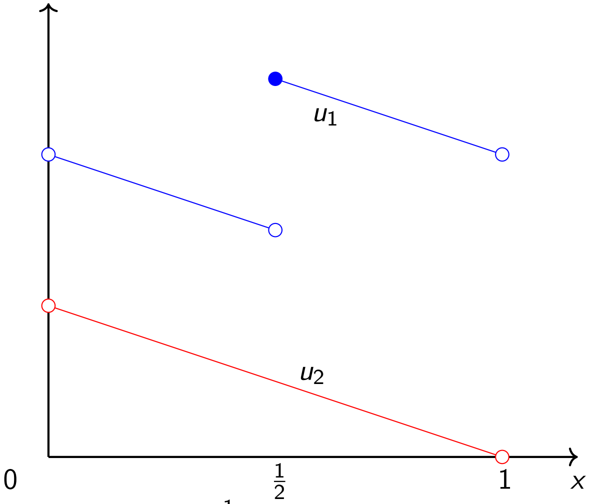
- *Does MS condition for individuals imply wMCS of POCs?* Not without additional condition.

Example

- Suppose $X = (0, 1)$.



- Unique Pareto optimum $= \frac{1}{2}$.

Example: after a single crossing dominating shift

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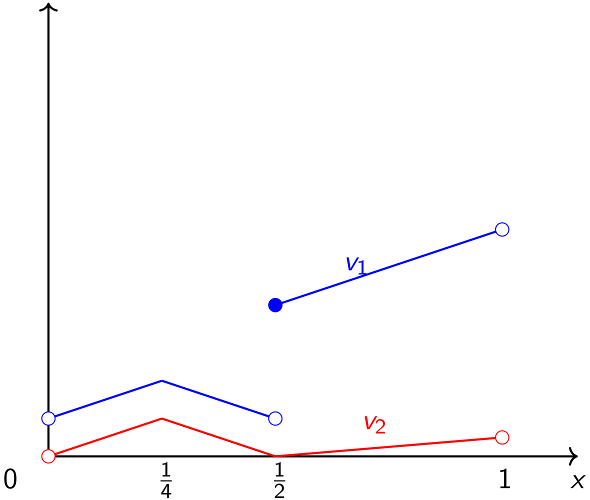

- Unique Pareto optimum $= \frac{1}{4}$ — Pareto optimum falls!!

wMCS of POC: one-dimensional X

If X is totally ordered, the condition is simple:

Theorem

Suppose

(i) X is compact and $\mathbf{u}$ and $\mathbf{v}$ are upper semicontinuous;

(ii) $\mathbf{v}$ single-crossing dominates $\mathbf{u}$.

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- In the example: If $X = [0, 1]$, then

$$P(\mathbf{u}) = \{0, \tfrac{1}{2}\} \leq_{ws} \{\tfrac{1}{4}, 1\} = P(\mathbf{v}).$$

Proof Sketch

- Any $x < \inf P(\mathbf{u})$ is Pareto dominated under $\mathbf{u}$
- In particular, it is Pareto dominated by some $x' \in P(\mathbf{u})$ (due to compactness), so $x' > x$;
- $\Leftrightarrow x$ Pareto dominated by x' under $\mathbf{u}$,.
- By SCP, x Pareto dominated (by x') under $\mathbf{v}$
- $\inf P(\mathbf{u}) \leq \inf P(\mathbf{v})$.

Similar argument shows $\sup P(\mathbf{u}) \leq \sup P(\mathbf{v})$. With a little more care, the result follows. $\square$

wMCS of POC: General X

Theorem

Suppose

(i) X is a convex, compact lattice

(ii) $\mathbf{u}$ and $\mathbf{v}$ are upper semicontinuous, concave, supermodular; and $\mathbf{v}$ increasing-difference dominates $\mathbf{u}$.

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- Upshot: Conditions guaranteeing sMCS for individual choices give wMCS for POCs, in a "well-behaved" environment.

Proof Skech

We utilize our new characterization of POC.

Theorem (Che, Kim, Kojima and Ryan, 2020)

Given our conditions, $x \in P(\mathbf{u})$ if and only if there exists a sequence $\{\phi^k\}_{k=1}^K$ of nonnegative welfare weights, ϕ^K strictly positive, such that $x \in X^k(\mathbf{u})$ for all $k = 1, ..., K$, where

$$X^0(\mathbf{u}) := X \text{ and } X^k(\mathbf{u}) := \arg \max_{x' \in X^{k-1}(\mathbf{u})} \sum_i \phi_i^k u_i(x'). \quad \Rightarrow$$

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- Fix any sequence $\{\phi^k\}$. Apply MS result inductively to get

$$P_{\{\phi^k\}}(\mathbf{u}) := X^K(\mathbf{u}) \leq_{ss} X^K(\mathbf{v}) =: P_{\{\phi^k\}}(\mathbf{v}).$$

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- The result then follows since

$$P(\mathbf{u}) = \bigcup_{\{\phi^k\}} P_{\{\phi^k\}}(\mathbf{u}) \leq_{ws} \bigcup_{\{\phi^k\}} P_{\{\phi^k\}}(\mathbf{v}) = P(\mathbf{v}).$$

(Strong set order is NOT closed under $\cup$, but weak set order is.) $\square$

Example

Let $X = [0,6]^2$, $I = \{1,2\}$ and

$$u_1(x,y) = -(x-1)^2 - (y-1)^2, \quad u_2(x,y) = -(x-4)^2 - (y-1)^2$$
$$v_1(x,y) = -(x-1)^2 - (y-4)^2, \quad v_2(x,y) = -(x-4)^2 - (y-2)^2.$$

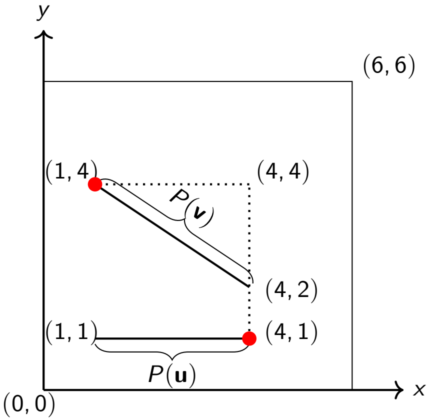

Figure: Failure of strong set monotonicity

Fixed Point Theorem and Applications

Tarski-Zhou Fixed Point Theorem

Theorem (Tarski-Zhou)

Suppose

- X: a complete lattice
- $F : X \rightrightarrows X$: non-empty, complete sublattice-valued, strong set monotonic

Then, the fixed point set is nonempty and a complete lattice.

New Fixed Point Theorem

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Theorem (Li-CKK)

Suppose

- X: partially ordered, and compact
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- regularity: $X_+(F)$ is non-empty.

Then, the fixed point set is nonempty and contains a maximal point.

- Note: analogous for "lower weak set monotonicity"

Comparison

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wMCS of Fixed Point Set

Let $\mathcal{E}(F)$ be the fixed point set of F.

Theorem (CKK)

Suppose X is compact, both F and G satisfy CKK conditions. If $G(x) \geq_{uws} F(x)$ for all x, then $\mathcal{E}(G) \geq_{uws} \mathcal{E}(F)$.

- analogous for "lower weak set monotonic."

Theorem

With order continuity (satisfied if X is finite), a fixed point can be found iterating F from a regular point (i.e., X_+ or X_-).

- But, can't guarantee obtaining a maximal or minimal fixed point this way. $\Rightarrow$

Application: Games with Weak Strategic Complementarities

- $\Gamma = (I, X, (B_i)_{i \in I})$ a game where
  - I: finite set of players
  - X: set of strategy profiles
  - B_i: best response correspondence

- Γ is a game with weak strategic complementarity if
  - for each i, B_i is. nonempty, compact valued and upper weak set monotonic
  - $B = (B_i)$ satisfies regularity.

wMCS of Nash equilibria

Theorem

1. A game Γ with weak strategic complementarities has a nonempty set of Nash equilibria.
2. Suppose that Γ' and Γ are both games with weak strategic complementarities, and $B_i'(s_{-i}) \geq_{uws} B_i(s_{-i})$ for every $i \in I$ and $s_{-i} \in S_{-i}$. Then, $\mathcal{NE}(\Gamma') \geq_{uws} \mathcal{NE}(\Gamma)$.

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- Requirement weaker than standard "(quasi)supermodular" games (Milgrom and Shannon (1994))
- Preferences don't need to be complete: B_i can simply be Pareto optimal choices (recall results before)

Application: General Model of Two Sided Matching with Contracts

- W: finite set of workers
- F: finite set of firms
- X: finite set of contracts; a contract $x \in X$ specifies a worker w and a firm f and a contract term (salary).
- **choice correspondence**: $C_a(X')$ are optimal choices by agent $a \in F \cup W$ from X':
- **stable allocation** suitably defined—*Individually Rational* and *No Blocking*.

Conditions on C_a

1. **Weak Substitutability**: the rejection correspondence $R_a(X') = \{Z : Z = X'_a \setminus Y \text{ for some } Y \in C_a(X')\}$ is weak set monotonic with "$\supset$" as order.

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   - Weaker than **WARP** $=$ Sen's α $+$ Sen's β.
   - Sen's β: $Y, Y' \in C_a(X'), Y \in C_a(X''), X' \subset X'' \Rightarrow Y' \in C_a(X'')$
   - Relaxing Sen's β accommodates incomplete preferences $\Rightarrow$

- cf. State of the art assumes a stronger version of 1 and WARP.

Fixed Point Characterization of Stability

Build a tâtonnement-like operator: $T(X', X'') = (T_1(X''), T_2(X'))$, for each $(X', X'') \in 2^X \times 2^X$, where

$$T_1(X'') = \{\tilde{X} \in 2^X : \tilde{X} = X \setminus \tilde{Y} \text{ for some } \tilde{Y} \in R_W(X'')\},$$
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Theorem

Suppose C_a satisfies Sen's α for all a. Then, Z is stable if and only if there exists a fixed point (X', X'') of T such that $Z \in C_F(X') \cap C_W(X'')$.

- cf. The state of art assumes WARP.

Existence of Stability

Theorem

Suppose choice correspondences satisfy Sen's α and weak substitutability. Then, a stable allocation exists.

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Proof Sketch.

- Define a partial order set $(2^X \times 2^X, \geq)$ with $\geq = (\supset, \subset)$.
- Weak Substitutability: T is weak set monotonic.
- Fixed Point Theorem: T has a fixed point

By our characterization, a stable allocation exists. □

Remark: Gale-Shapley is an iterative version of Tarski that works for a simple environment. We are generalizing it.

weak MCS

Theorem

Suppose that a firm's choice correspondence becomes more permissive (in set inclusion). Then, workers become better off and firms become worse off in the weak set order sense (under original preferences).

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Proof Sketch.

- Stable allocation = Fixed point of T
- Change in choice $\Rightarrow$ Change in T
- Use Comparative statics of fixed points

□

Applications:

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Choice correspondence satisfies weak substitutability and Sen's α while violating WARP

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- Corollaries: Existence of stable allocations, comparative statics: when the hiring constraint becomes more restrictive; all other firms benefit, workers are hurt.

Conclusion

- We propose weak monotone comparative statics (wMCS)
- Requirement is weaker, so wider applicability
- Analyzed: individual choices, Pareto optimal choices, games with weak strategic complementarity, matching theory
- Future Research:
  - Weaker sufficient conditions for wMCS of Pareto optimal choices
  - More applications

Thank You!

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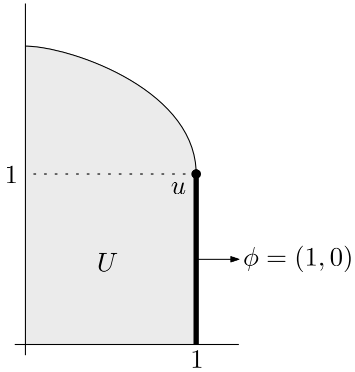
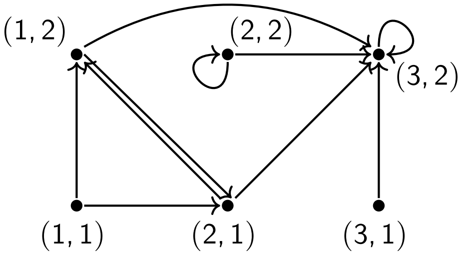
Illustration of Non-Exposed Pareto Optimum



- u is Pareto optimal but not exposed.

Return

Faiulure of any iteration to reach a minimal fixed point



- The minimal fixed point $(2, 2)$ cannot be reached from any iterative application of F starting from $(1, 1)$.

Return

Violation of Sen's β due to Preference Imcompleteness

- A firm f with two divisions, δ and δ', and three workers w, w', and w''.
- Workers are all acceptable to δ and δ' while $w'' \succ_{\delta'} w'$.
- Constrained to hire at most one worker across the divisions.
- No strict preferences over which division should hire a worker when both divisions have applicants.
- $C_f(\{(w,\delta),(w',\delta')\}) = \{\{(w,\delta)\},\{(w',\delta')\}\}$.
- But $C_f(\{(w,\delta),(w',\delta'),(w'',\delta')\}) = \{\{(w,\delta)\},\{(w'',\delta')\}\}$.

Return

Li (2014), Fleiner (2003), Che et al. (2020), Tarski (1955), Zhou (1994)