Nonlinear Pricing Schedule under Competition*

Yong Chao† Guofu Tan‡ Adam Chi Leung Wong§

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Abstract

Motivated by several recent antitrust cases, we study a strategic model of competition in intermediate-goods markets. Our model is a three-stage game with complete information in which a dominant firm offers a general tariff first and then a rival firm responds with a per-unit price, followed by a buyer making her decision to purchase from one or both firms. We characterize subgame perfect equilibria of the game and study the implications of the equilibrium outcome.

Our paper makes three main contributions. First, it provides a novel explanation for the prevalence of nonlinear pricing (a menu of offers conditional on volumes) under duopoly *in the absence of private information*: The dominant firm can use a menu of offers to constrain its rival's choices and extract surplus from the buyer. Second, it shows that when the capacity of the rival firm is constrained, as compared to linear pricing schemes, the nonlinear pricing tariff adopted by the dominant firm reduces the price, sales, and profits of the rival

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†College of Business, University of Louisville, yong.chao@louisville.edu.

‡Department of Economics, University of Southern California, guofutan@usc.edu.

§Department of Economics, Lingnan University, adamwong@ln.edu.hk.

firm as well as the buyer's surplus. In other words, nonlinear pricing may have antitrust implications in the sense that it can lead to partial foreclosure and harm consumer welfare. Third, we establish an equivalence between a subgame perfect equilibrium of the game and an optimal mechanism in a "virtual" principal-agent model with hidden action and hidden information. This involves treating the rival firm's (an agent's) price as its hidden action meanwhile letting the buyer (another agent) to report the rival firm's price as her private information to the dominant firm (the principal). As a result of such an equivalence, we can apply mechanism design techniques to solve for subgame perfect equilibria of the game.

Keywords: Nonlinear Pricing Schedule, Asymmetric Competition, Capacity Constraint, Complete Information, Subgame Perfect Equilibrium, Principal-agent Model, and Partial Foreclosure.

JEL Code: L13, L42, K21

1 Introduction

Nonlinear pricing is often observed in intermediate-goods markets. It takes the form of various rebates and discounts conditional on volumes (or share of the volumes among competitors) purchased by a buyer. An example is all-units discount pricing scheme that lowers a buyer's marginal price on every unit purchased when the buyer's purchase exceeds or is equal to a pre-specified volume threshold. The adoption of such conditional rebates and discounts by dominant firms has become a prominent antitrust issue. Indeed, in a number of recent antitrust cases in the U.S., E.U., Canada, and China, a plaintiff (a government antitrust agency or a rival firm) alleged that a dominant firm used pricing schemes such as conditional rebates/discounts to its downstream buyers to fully or partially exclude its rival firm(s) and that such an exclusion had harmed competition and consumer welfare. Those antitrust cases share some common features: First, there is a firm that is considered as "dominant" in market share, capacity, product lines, profits, and so on. Second, there is one or several smaller firms (or recent entrants) that have limited capacity, narrower product lines, or limited distribution channels. Third, the "dominant" firm typically offers more complex pricing schemes (e.g., rebates/discounts conditional on volumes) than its rival(s). What explains the observed practices of various nonlinear pricing schemes in intermediate-goods markets and what are the implications of those practices? The main objective of this paper is to provide an explanation for nonlinear pricing in the presence of asymmetric competition and in the absence of private information.

Motivated by recent antitrust cases, we study a stylized model of asymmetric competition. In the model, there are two firms, a dominant firm (Firm 1) and a rival firm (Firm 2). Both firms can produce a homogeneous product at constant marginal cost. However, the rival firm is capacity constrained. There is a representative downstream buyer who may purchase the product from one or both firms. We consider a three-stage game with complete information in which the dominant firm offers a general tariff first and then its rival firm responds with a per-unit price, followed by the buyer making her decision to purchase from one or both firms. We characterize subgame perfect equilibria of the game and study the implications of the equilibrium outcome.

Our model involves three kinds of asymmetries between the two firms. The first is concerned with pricing schemes: The dominant firm is able to make nonlinear tariff schedules, i.e., payments conditional on volumes, while the rival firm can only choose linear pricing schemes. This assumption appears to be consistent with the observations from the major antitrust cases, and is perhaps due to the fact that the dominant firm is more experienced in dealing with downstream buyers than new entrants to the market. The second asymmetry concerns the timing of the game: The dominant firm commits to offering tariffs before its rival. This might be related to the dominant firm's bargaining power and its willingness to commit its offers when dealing with the buyer. Another asymmetry is about capacity levels of the firms. That is, relative to the demand size the dominant firm has no capacity limit while its rival is capacity-constrained. Our analysis suggests that the asymmetry in capacity is not crucial for the equilibrium adoption of nonlinear pricing by the dominant firm, but is important for the results of partial foreclosure and harming the buyer welfare.

Our paper makes several major contributions. First, it provides a novel explanation for the prevalence of nonlinear pricing (a menu of offers conditional on volumes) under duopoly in the absence of private information: The dominant firm can use a menu of offers to constrain its rival's choices and extract surplus from the buyer. Second, it shows that when the capacity of the rival firm is relatively small, as compared to linear pricing schemes, the nonlinear pricing tariff adopted by the dominant firm reduces the price, sales, and profits of the rival firm as well as the buyer's surplus. In other words, nonlinear pricing in this context can lead to partial foreclosure and harm consumer welfare, which may have antitrust implications. Third, we establish an equivalence between a subgame perfect equilibrium of the game and an optimal mechanism in a "virtual" principal-agent model with hidden action and hidden information. This involves treating the rival firm's (an agent's) price as its hidden action meanwhile letting the buyer (another agent) to report the rival firm's price as her private information to the dominant firm (the principal). As a result of such an equivalence, we can apply mechanism design techniques to characterize subgame perfect equilibria of the game. Other properties of the equilibrium tariffs are also discussed in the paper.

[Relevant literature to be discussed.]

The remainder of the paper is organized as follows. In Section 2, we set up our model of asymmetric competition in intermediate-goods markets. Section 3 examines the buyer's problem in the last stage and points out a difficulty in applying standard backward induction procedure. Using simple price-quantity bundles, Section 4 demonstrates how an extra bundle could improve firm 1's profit, albeit it will not be chosen in equilibrium. Section 5 establishes an equivalence between a subgame perfect equilibrium of the game and an optimal mechanism in a "virtual" principal-agent model with hidden action and hidden information. Section 6 characterizes the equilibrium outcome of the game. Other properties and implications of the equilibrium are discussed in Section 7. Section 8 contains concluding remarks.

2 Model

There are three players in our model: two firms, producing a homogeneous product, and one buyer for the product. To capture a notion of dominance, we allow for a possible capacity asymmetry between the two firms. In particular, firm 1, as a dominant firm, can produce any quantity at a unit cost $c \geq 0$. Firm 2, as a possibly smaller firm, has a capacity $k \in (0, \infty]$, up to which it can produce any quantity at the same unit cost c. If the buyer chooses to buy $Q \geq 0$ units from firm 1 and $q \in [0, k]$ units from firm 2, his payoff is the gross utility given by $u(Q + q)$, less the payments to the two firms.

We consider a three-stage game as follows. First, firm 1 offers a nonlinear tariff $\tau(\cdot)$, which specifies the payment $\tau(Q) \in \mathbb{R} \cup \{\infty\}$ that the buyer has to make if she chooses to buy Q units from firm 1, with the restriction that $\tau(0) \leq 0$.[1] Second, after observing $\tau(\cdot)$, firm 2 offers a unit price p (up to k units). Third, after observing $\tau(\cdot)$ and p, the buyer chooses the quantities she buys from the two firms. As a necessary tie-breaking rule, when indifferent, the buyer purchases from firm 2. This is an

[1] $\tau(Q) = \infty$ means that purchasing Q units is not allowed.

extensive-form game with complete and perfect information. We use the equilibrium concept of pure strategy subgame-perfect equilibrium (SPE).

We say a nonlinear tariff is *regular* if the subgame after firm 1 offers such a tariff has some SPE. We do not allow firm 1 to offer irregular tariffs.[2] That is, the set of feasible tariffs firm 1 can choose from, denoted as $\mathcal{T}$, is the collection of $\tau : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ that is regular and satisfies $\tau(0) \leq 0$. Also, we do not allow firm 2 to offer prices below c, in which case it makes non-positive profit for sure. Therefore, the set of feasible unit prices firm 2 can choose from is

$$\mathcal{P} \equiv [c, \infty).$$

A SPE is composed of a firm 1's strategy $\tau^* \in \mathcal{T}$, a firm 2's strategy $p^* : \mathcal{T} \to \mathcal{P}$, and a buyer's strategy $q^* : \mathcal{T} \times \mathcal{P} \to \mathbb{R}_+ \times [0, k]$, such that

$$q^*(\tau, p) \in \underset{(Q,q) \in \mathbb{R}_+ \times [0,k]}{\operatorname{argmax}} \{u(Q + q) - pq - \tau(Q)\} \quad \forall (\tau, p) \in \mathcal{T} \times \mathcal{P}, \tag{1}$$

$$p^*(\tau) \in \underset{p \in \mathcal{P}}{\operatorname{argmax}} \{(p - c)q_2^*(\tau, p)\} \quad \forall \tau \in \mathcal{T}, \tag{2}$$

$$\tau^* \in \underset{\tau \in \mathcal{T}}{\operatorname{argmax}} \{\tau(q_1^*(\tau, p^*(\tau))) - cq_1^*(\tau, p^*(\tau))\}. \tag{3}$$

We make the following two regularity assumptions.

Assumption 1. *$u : \mathbb{R}_+ \to \mathbb{R}$ is twice continuously differentiable, satisfies $u(0) = 0$, $u''(\cdot) < 0$, $u'(0) > c$, and there exists a unique $q^e > 0$ such that $u'(q^e) = c$.*

Denote the quantity demanded by the buyer at a per-unit price p as

$$D(p) \equiv \underset{q \geq 0}{\operatorname{argmax}} \{u(q) - pq\},$$

and the monopoly profit under linear pricing p as

$$\pi(p) \equiv (p - c)D(p).$$

[2]By definition, if we allow firm 1 to choose an irregular tariff, the whole game has no SPE.

Assumption 1 implies that $D(\cdot)$ and $\pi(\cdot)$ are continuously differentiable and $D(\cdot)$ is strictly decreasing on $[c, u'(0)]$.

Assumption 2. *The monopoly profit function $\pi(\cdot)$ is strictly concave on $[c, u'(0)]$.*

Assumption 2 implies that there is a unique optimal monopoly price $p^m \equiv \operatorname{argmax}_p \pi(p) \in (c, u'(0))$ given by $\pi'(p^m) = 0$.

3 Buyer's Problem

Since our game is a sequential-move complete information one and we try to determine its SPE outcome, standard backward induction method requires us to investigate the buyer's problem in the last stage first. However, as we shall see in this section, we immediately encounter a difficulty in characterizing the buyer's optimal purchase response in the last stage. As a result, the standard procedure of backward induction *cannot* be applied to our game.

In the last stage, given the two firms' offers $\tau \in \mathcal{T}$ and $p \in \mathcal{P}$, the buyer's optimal purchase is given by (1). However, (1) may not be a well behaved problem since firm 1's offer τ is an endogenous function, on which we have imposed only minimal restrictions. So solving the SPE outcome of our game seems unmanageable: without knowing what τ is, it is impossible to solve (1); nevertheless, without knowing the buyer's optimal purchase in the last stage, it is impossible for firms 2 and 1 to pin down their optimal pricing p_2 in (2) and τ in (3), respectively. Therefore, the standard backward induction loses its bite in our game, and we need a transformation of the whole game in order to solve its SPE.

As a first step of the transformation, we introduce *a buyer's conditional payoff* if she is endowed with Q units and can buy at most k more units at price p as

$$V(Q, p) \equiv \max_{q \in [0,k]} \{u(Q + q) - pq\}. \tag{4}$$

Accordingly, given the two firms' offers $\tau \in \mathcal{T}$ and $p \in \mathcal{P}$, we can decompose the buyer's maximization problem (1) into two sub-problems:

- in the first sub-problem, for any given Q, the buyer chooses q from firm 2 by solving (4);
- in the second sub-problem, the buyer chooses Q from firm 1, i.e.,

$$\max_{Q \geq 0} \left\{ V(Q,p) - \tau(Q) \right\}. \tag{5}$$

Even though we still cannot solve (5) without knowing τ, the buyer's conditional payoff $V(Q,p)$ in (4) has some nice properties which turn out to be important for us to transform the SPE and solve for the optimal general tariff τ.

Note that problem (4) has a unique maximizer[3]

$$\mathrm{Proj}_{[0,k]}(D(p) - Q) \equiv \max \left\{ \min \left\{ D(p) - Q, k \right\}, 0 \right\}. \tag{6}$$

By the Envelope Theorem, we have the following lemma.

Lemma 1. *For every* $(Q,p) \in \mathbb{R}_+ \times \mathcal{P}$,

$$V_p(Q,p) = -\mathrm{Proj}_{[0,k]}(D(p) - Q), \tag{7}$$

$$V_Q(Q,p) = u'(\mathrm{Proj}_{[Q,Q+k]}(D(p))) = \mathrm{Proj}_{[u'(Q+k),u'(Q)]}(p). \tag{8}$$

For every $(Q,p) \in \mathbb{R}_+ \times \mathcal{P}$ *such that* $Q \neq D(p)$ *and* $Q \neq D(p) - k$,

$$V_{Qp}(Q,p) = V_{pQ}(Q,p) = \begin{cases} 1 & \text{if } D(p) - k < Q < D(p) \\ 0 & \text{if } Q < D(p) - k \text{ or } Q > D(p) \end{cases}. \tag{9}$$

Note that, from (9), V satisfies weak increasing differences. On the region

$$\Phi \equiv \{(Q,p) \in \mathbb{R}_+ \times \mathcal{P} : D(p) - k \leq Q \leq D(p)\}, \tag{10}$$

the property of increasing differences is strict, which turns out to be important for our analysis.

[3]For any closed interval $X \subset \mathbb{R}$ and any point $x \in \mathbb{R}$, let $\mathrm{Proj}_X(x)$ denote the projection of x on X, that is, $\mathrm{argmin}_{y \in X} |y - x|$.

4 Why an Unchosen Bundle Helps

Before we characterize the optimal general tariff $\tau(\cdot)$ for firm 1, let us first look at some specific nonlinear tariffs: price-quantity point offers, and gain some insights on why an *unchosen* bundle could help improve firm 1's profits. In this section, we assume that firm 2 does not have capacity constraint, i.e., $k \geq q^e$ for expositional simplicity.

Our starting point will be the simplest one, a single bundle offer (Q, T), which people might think as optimal for firm 1. After all, there is only one buyer and there is no demand uncertainty, so there can be only one quantity that the buyer will purchase from firm 1 in equilibrium, regardless how many quantities offered by firm 1. Nevertheless, as we shall see, firm 1 can strictly improve its profit over its optimal profit level in the "one-bundle equilibrium," by offering an extra bundle which will not be chosen in equilibrium!

4.1 Optimal One-Bundle Offer

In this section, we show how the optimal one-bundle offer is determined. Let Q^* and T^* denote the optimal bundle quantity and bundle price offered by firm 1, respectively, in "one-bundle equilibrium."

First, given firm 1's one bundle (Q^*, T^*), the buyer may accept or reject it. By accepting (Q^*, T^*), the buyer's surplus is $V(Q^*, p) - T^*$; otherwise it is $V(0, p)$. From Lemma 1, we have $V_{Qp}(Q, p) > 0$ for $0 \leq Q < D(p)$. It implies the two surplus curves must cross once and only once at x^* as shown in Figure 1a, i.e.,

$$V(0, x^*) = V(Q^*, x^*) - T^*. \tag{11}$$

It follows that the buyer will accept (Q^*, T^*) from firm 1 and thus buy $D(p) - Q^*$ from firm 2 if and only if $p > x^*$; otherwise buy $D(p)$ solely from firm 2.

Accordingly, firm 2's profit function, as shown in Figure 1b, will consist of two pieces as

$$\Pi_2(p) = \begin{cases} (p-c)D(p) & \text{if } p \leq x^* \\ (p-c)(D(p)-Q^*) & \text{if } p > x^* \end{cases}.$$

Since firm 1 has positive sales if and only if it can induce firm 2 to set $p > x^*$, firm 1 must ensure

$$\max_{p>x^*} \{(p-c)(D(p)-Q^*)\} \geq \max_{p\leq x^*}(p-c)D(p). \tag{12}$$

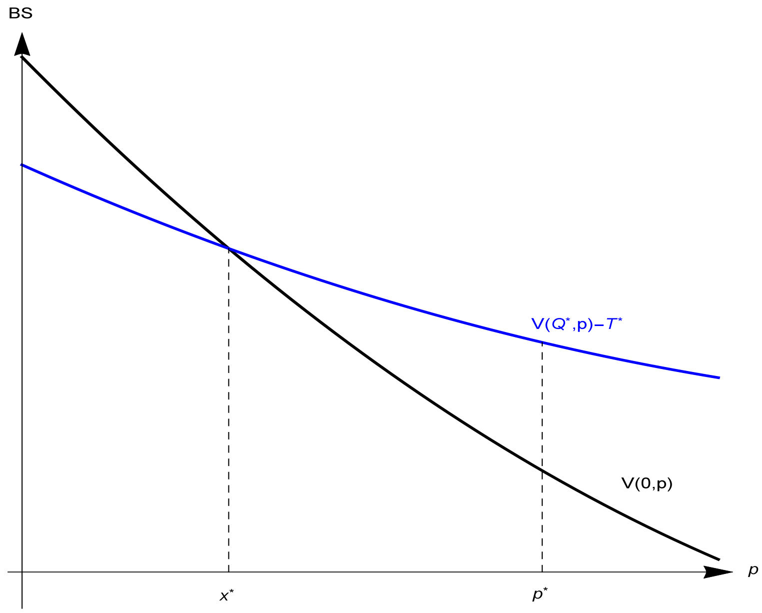
The optimal (Q^*, T^*) in "one-bundle equilibrium" must solve the following firm 1's optimization problem

$$\max_{(Q,T)} \{T - c\cdot Q \text{ s.t.(11) and (12)}\}. \tag{13}$$

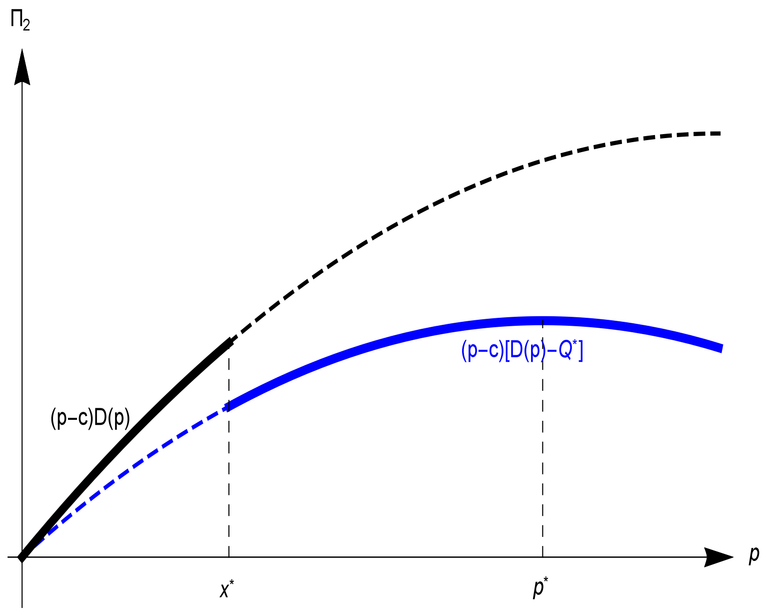
Note that

$$\begin{aligned} \Pi_1^* &= T^* - c\cdot Q^* \\ &= V(Q^*, x^*) - V(0, x^*) - c\cdot Q^*. \text{ (by (11))} \end{aligned}$$

To maximize $V(Q^*, x^*) - V(0, x^*)$ and hence its profit, firm 1 would like x^* to be as large as possible, because $V_{Qp}(Q, p) > 0$ for $0 \leq Q < D(p)$ from Lemma 1. However, firm 1 faces competitive constraint (12) from firm 2, whose right-hand side (monopoly profit) increases faster than the left-hand side (residual demand profit) as x^* increases. Consequently, at the optimal one bundle (Q^*, T^*), (12) must be binding, i.e., $\Pi_2(p^*) = (x^* - c)D(x^*)$, where $p^* \in \text{argmax}_{p>x^*}(p-c)(D(p)-Q^*)$ is firm 2's equilibrium price. Figure 1b illustrates it.



(a) Buyer's Surplus under One Bundle



(b) Firm 2's Profit under One Bundle

Figure 1: One Bundle

4.2 Adding an Unchosen Bundle to Improve

We now, given the optimal one-bundle offer (Q^*, T^*), construct *one extra bundle to relax the originally binding competitive constraints from firm 2.* As a result, firm 1 can strictly improve its profit over Π_1^* with the unchosen bundle.

We demonstrate the profitable improvement through the following two steps.

Step 1: Given (Q^*, T^*) , add an extra unchosen bundle $(Q_1, T_1(\epsilon))$ to relax originally binding competitive constraints.

Pick any $Q_1 \in (0, Q^*)$, and let $T_1(\epsilon) = V(Q_1, x^*) - V(0, x^*) - \epsilon$ $(\epsilon > 0)$. Recall from Lemma 1 that $V_{Qp}(Q, p) > 0$ for $0 \leq Q < D(p)$. It follows that $V(Q_1, p) - T_1(\epsilon)$ (the solid red curve in Figure 2a) must uniquely crosses $V(0, p)$ and $V(Q^*, p) - T^*$ at $x_0(\epsilon)$ and $x_1(\epsilon)$, respectively. Here $x_0(\epsilon)$ and $x_1(\epsilon)$ are given by

$$V(Q_1, x_0(\epsilon)) - T_1(\epsilon) = V(0, x_0(\epsilon)) \tag{14}$$

$$V(Q_1, x_1(\epsilon)) - T_1(\epsilon) = V(Q^*, x_1(\epsilon)) - T^*. \tag{15}$$

Note that (11) implies $x_0(0) = x_1(0) = x^*$, as indicated by the dashed red curve in Figure 2a. For small $\epsilon > 0$, we have $0 < x_0(\epsilon) < x^* < x_1(\epsilon) < p^*$ so that, if firm 1 offers two bundles characterized by $(Q_1, T_1(\epsilon))$ and (Q^*, T^*), the buyer would pick the large bundle (Q^*, T^*) when firm 2's price is above $x_1(\epsilon)$, and would pick the small bundle $(Q_1, T_1(\epsilon))$ when firm 2's price is between $x_0(\epsilon)$ and $x_1(\epsilon)$, and would not pick any bundle from firm 1 when firm 2's price is below $x_0(\epsilon)$.

Accordingly, firm 2's profit function, as shown in Figure 2b, will consist of three pieces as

$$\Pi_2(p) = \begin{cases} (p-c)D(p) & \text{if } p \leq x_0(\epsilon) \\ (p-c)(D(p) - Q_1) & \text{if } x_0(\epsilon) < p \leq x_1(\epsilon) \\ (p-c)(D(p) - Q^*) & \text{if } p > x_1(\epsilon) \end{cases}.$$

Again, to ensure the buyer still choose the large bundle (Q^*, T^*), firm 1 must ensure

$$\max_{p > x_1(\epsilon)} (p-c)(D(p) - Q^*) \geq \max_{p \leq x_0(\epsilon)} (p-c)D(p) \tag{16}$$

and

$$\max_{p>x_1(\epsilon)}(p-c)(D(p)-Q^*) \geq \max_{x(\epsilon)<p\leq x_1(\epsilon)}(p-c)(D(p)-Q_1). \tag{17}$$

Thus, by offering an extra bundle, firm 1 breaks firm 2's profit function from two pieces to three pieces, and hence replace constraints (11) and (12) with (14), (15), (16), and (17).[4] With the two bundles, firm 1's optimization problem now becomes

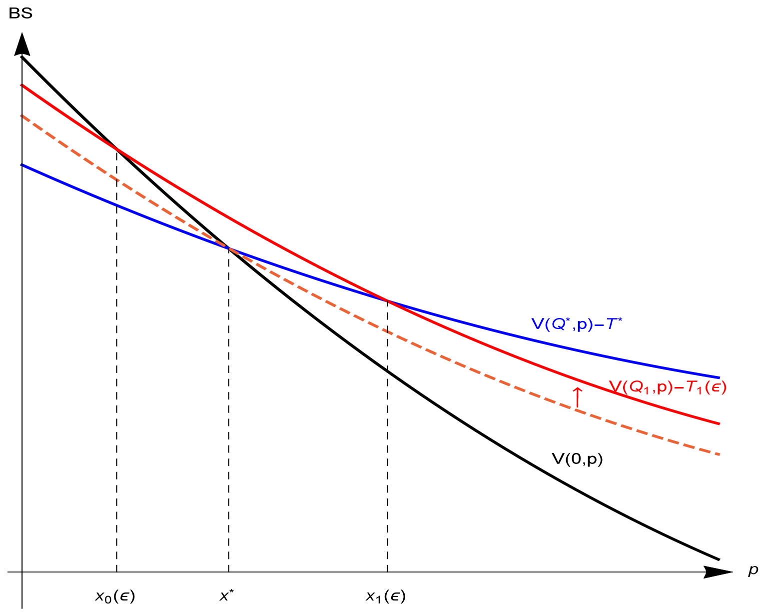
$$\max_{(Q^*,T^*),(Q_1,T_1)} \left\{T^* - c\cdot Q^* \text{ s.t.}(14), (15), (16), \text{ and } (17)\right\}, \tag{18}$$

Interestingly, firm 1 *relaxes* originally binding competitive constraint (12), by *adding* one extra unchosen bundle. This can be seen from Figure 2b. When $\epsilon > 0$ is small, $0 < x_0(\epsilon) < x^* < x_1(\epsilon) < p^*$. $\Pi_2(p^*) = \pi(x^*) > \pi(x_0(\epsilon))$ follows from binding (12) and $x_0(\epsilon) < x^*$. So (16) is *not* binding. In addition, because $(p-c)(D(p)-Q_1) < \pi(p)$ for any p, we have $(x^*-c)(D(x^*)-Q_1) < \pi(x^*)$. Note that for $\epsilon > 0$ small enough, we must have $x_1(\epsilon)$ is close enough to x^*, so that $(x_1(\epsilon)-c)(D(x_1(\epsilon))-Q_1) < \pi(x^*)$. It follows that $\Pi_2(p^*) = \pi(x^*) > (x_1(\epsilon)-c)(D(x_1(\epsilon))-Q_1)$. So (17) is *not* binding, neither.

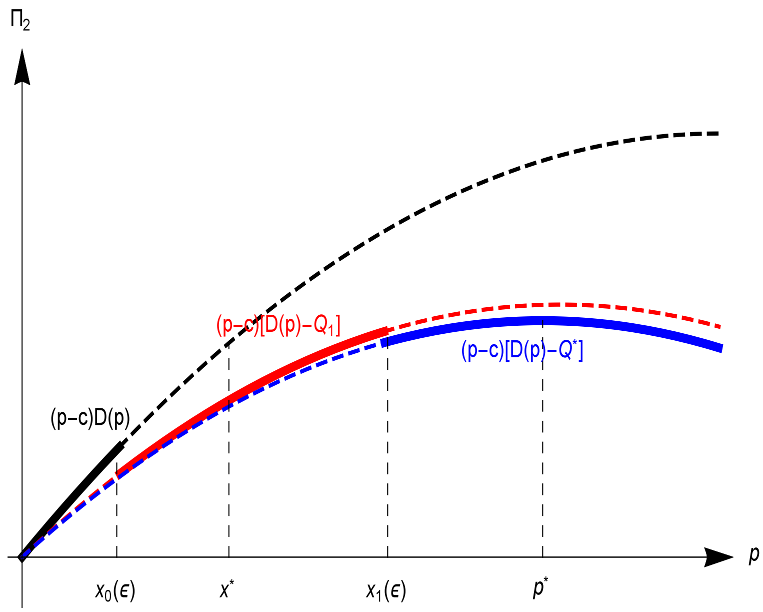
Step 2: Improve profit by increasing T^* to $T^*(\epsilon) = T^* + \epsilon$.

From Step 1, the newly added unchosen bundle $(Q_1, T_1(\epsilon))$ can make constraints (16) and (17) non-binding. As long as these constraints are non-binding, firm 1 will be able to strictly increase its profit by increasing T^* to $T^*(\epsilon) = T^* + \epsilon$ with $(Q^*, T^*(\epsilon))$ being chosen.

[4] Note when $\epsilon = 0$, $x_0(0) = x_1(0) = x^*$, (14) and (15) are reduced to (11), (16) and (17) are reduced to (12).



(a) Buyer's Surplus under Two Bundles



(b) Firm 2's Profit under Two Bundles

Figure 2: Two Bundles

4.3 Summary

From the above one-bundle and two-bundle cases, we can see that, by adding extra bundles, firm 1 provides the buyer extra options. *Such extra unchosen options, as a latent strategy, reduce the profitability for firm 2 to deviate from the equilibrium.* This can be seen from Figures 1b and 2b. Previously, if firm 2 cuts to the original threat price x^*, the buyer buys nothing from firm 1. However, with the extra bundle, if firm 2 cuts to the original threat price x^*, the buyer still buys something, rather than nothing, from firm 1. So now the increases in firm 2's sales and profits will be limited. If firm 2 wants to induce single-sourcing with the extra bundle, a deeper price cut is needed so that the increase in firm 2's profit is limited too.

In this simple price-quantity point offers example, we have seen having two-bundle offers can improve firm 1's profit over the optimal one-bundle offer. As we add more extra bundles, firm 1's profit can be increased even further. Indeed, we shall see in the following sections that firm 1's profit maximizing tariff involves a continuum of quantities for the buyer to choose from, although firm 1 understands that *all but one* would never be chosen by the buyer in equilibrium.

5 Equivalence between SPE and a Mechanism Design Problem

As we argued in Section 3, the standard backward induction cannot be applied to solve the SPE. Therefore, we shall transform our original problem as follows.

Let $\pi(Q, \cdot)$ denote firm 2's profit function conditional on the buyer's purchase from firm 1 being Q, i.e.,

$$\pi(Q, p) \equiv (p - c)\,\text{Proj}_{[0,k]}(D(p) - Q). \tag{19}$$

Now we are ready to formulate a mechanism design problem that allows us to determine SPE outcomes. Observe that, every tariff $\tau \in \mathcal{T}$ firm 1 might offer induces a continuation subgame in which firm 2 and the buyer sequentially choose their actions. When choosing τ, firm 1 understands that firm 2 and the buyer would play a SPE of the continuation subgame. Given τ, the buyer would optimally choose some purchase $Q(p) \geq 0$ from firm 1, contingent on any possible price $p \in \mathcal{P}$ chosen by firm 2. The payment for this purchase is thus $\tau(Q(p)) \equiv T(p)$. Given that the buyer's optimal purchase from firm 1 is $Q(p)$, and hence the optimal purchase from firm 2 is $\text{Proj}_{[0,k]}(D(p) - Q(p))$, firm 2 would optimally choose some price $\bar{p} \in \mathcal{P}$.

Virtually, we have a one-principal-two-agent model, in which firm 1 (the principal) offers a direct revelation mechanism $Q : \mathcal{P} \to \mathbb{R}_+$ and $T : \mathcal{P} \to \mathbb{R}$ to the buyer (Agent 1), and recommends a price $\bar{p} \in \mathcal{P}$ for firm 2 (Agent 2). In the spirit of Revelation Principle (*imagining firm 1 asks the buyer to report firm 2's price*), solving SPE for the whole game is equivalent to solving the following constrained optimization problem (OP1):

$$\underset{Q(\cdot),T(\cdot),\bar{p}}{\text{Maximize}}\, T(\bar{p}) - c \cdot Q(\bar{p}) \tag{OP1}$$

subject to

$$V(Q(p), p) - T(p) \geq V(Q(\tilde{p}), p) - T(\tilde{p}) \ \ \forall p, \tilde{p} \in \mathcal{P} \tag{B-IC}$$

$$V(Q(p), p) - T(p) \geq V(0, p) \ \ \forall p \in \mathcal{P} \tag{B-IR}$$

$$\pi(Q(\bar{p}), \bar{p}) \geq \pi(Q(p), p) \ \ \forall p \in \mathcal{P}. \tag{F2-IC}$$

Constraint (B-IC) is the incentive compatibility constraint for the buyer, i.e., the buyer has incentive to report firm 2's price truthfully. Constraint (B-IR) is the individual rationality constraint for the buyer, i.e., the buyer is willing to participate in the mechanism rather than obtaining nothing from and paying nothing to firm 1 (and single-sourcing from firm 2). Constraint (F2-IC) is the incentive compatibility constraint for firm 2, i.e., firm 2 has incentive to charge the recommended price $\bar{p}$, understanding that the buyer will always report its price truthfully. Finally, the objective function of (OP1) is firm 1's profit provided firm 2 follows the recommendation $\bar{p}$ and the buyer reports truthfully.

The equivalence between SPE and the optimization problem (OP1) is formalized in the following theorem.

Theorem 1. (Equivalence) *Take any $Q^* : \mathcal{P} \to \mathbb{R}_+$, $T^* : \mathcal{P} \to \mathbb{R}$, and $\bar{p}^* \in \mathcal{P}$. $(Q^*(\cdot), T^*(\cdot), \bar{p}^*)$ is a solution of (OP1) if and only if there is a SPE (τ^*, p^*, q^*) such that*

$$Q^*(p) = q_1^*(\tau^*, p) \ \ \forall p \in \mathcal{P}, \tag{20}$$

$$\text{Proj}_{[0,k]}(D(p) - Q^*(p)) = q_2^*(\tau^*, p) \ \ \forall p \in \mathcal{P}, \tag{21}$$

$$T^*(p) = \tau^*(Q^*(p)) \ \ \forall p \in \mathcal{P}, \tag{22}$$

$$\bar{p}^* = p^*(\tau^*). \tag{23}$$

By virtue of Theorem 1, we reduce our task of finding SPE to determining the solution to (OP1'). From now on we also call any solution of (OP1) an equilibrium.

6 Equilibrium Characterization

Following Theorem 1, in this section we solve the optimization problem (OP1).

6.1 Constraints for Buyer

The following lemma characterizes the incentive constraint (B-IC) and individual rationality constraint (B-IR) for the buyer.

Lemma 2. (Constraints for Buyer) *Any $Q : \mathcal{P} \to \mathbb{R}_+$ and $T : \mathcal{P} \to \mathbb{R}$ satisfy (B-IC) and (B-IR) if and only if the following conditions hold:*

$$\begin{aligned} &\forall p_1, p_2 \in \mathcal{P} \text{ with } p_1 \le p_2, \text{either } Q(p_1) \le Q(p_2) \\ &\qquad\qquad \text{or } D(p_1) \le Q(p_2) \text{ or } Q(p_1) \le D(p_2) - k \end{aligned} \tag{24}$$

$$\forall p \in \mathcal{P},\ T(p) - T(c) = V(Q(p), p) - V(Q(c), c) - \int_c^p V_p(Q(t), t)dt \tag{25}$$

$$V(Q(c), c) - T(c) \ge V(0, c) \tag{26}$$

Condition (24) is a weakened version of the standard monotonicity condition for mechanism design problems; it is weakened because the increasing differences property (9) of V is strict only on Φ. If $Q(\cdot)$ is non-decreasing, condition (24) automatically holds. The converse holds only partially: if condition (24) holds, $Q(\cdot)$ is non-decreasing for all $p \in \mathcal{P}$ with $(Q(p), p) \in \Phi$, but may be decreasing when $(Q(p), p) \notin \Phi$. Condition (24) says that $Q(\cdot)$ may be decreasing only in a particular way: whenever $p_1 < p_2$ and $Q(p_1) > Q(p_2)$, the rectangle $[Q(p_2), Q(p_1)] \times [p_1, p_2]$ must not intersect the region Φ. Such weakened monotonicity implies the following result.

Corollary 1. *(24) implies* $\mathrm{Proj}_{[0,k]}(D(p) - Q(p))$ *is non-increasing in* p *on* $\mathcal{P}$.

Condition (25) is the envelope formula for payment in standard mechanism design problems. Condition (24) and condition (25) together are necessary and sufficient conditions for (B-IC). Condition (26) is a necessary and sufficient condition for (B-IR), since $V(Q(p), p) - T(p) - V(0, p)$ is non-decreasing in p, which is a result of (25).

Once the constraints (B-IC) and (B-IR) are replaced with (24), (25), and (26), we see that (26) must be binding, for otherwise firm 1 can increase its profit $T(\bar{p}) - c \cdot Q(\bar{p})$ by increasing $T(p)$ for every $p \in \mathcal{P}$ by a constant, after which all other constraints ((24), (25), and (F2-IC)) are intact. Therefore,

$$T(c) = V(Q(c), c) - V(0, c). \tag{27}$$

6.2 Constraints for Firm 2

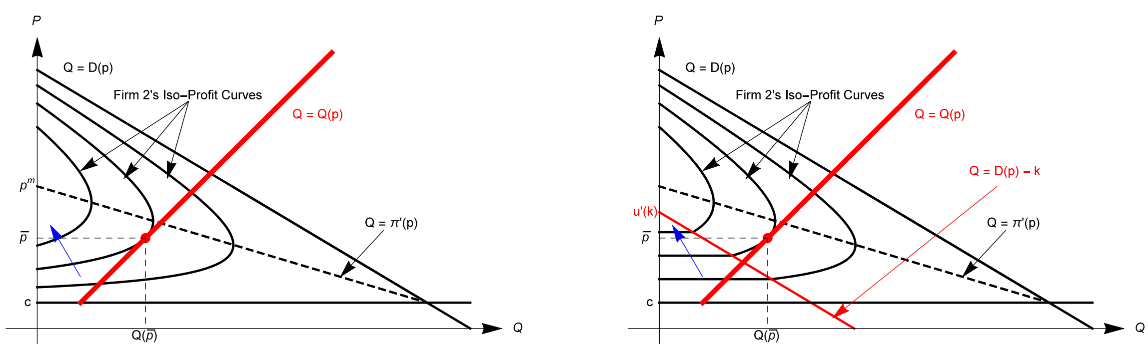
We now take a closer look at the incentive constraint (F2-IC) for firm 2. Given that firm 1 offers the buyer $Q(\cdot)$ and recommends firm 2 to charge $\bar{p}$, firm 2's profit, provided the recommendation is followed, is

$$\Pi_2 = \pi(Q(\bar{p}), \bar{p}) = (\bar{p} - c)\, \mathrm{Proj}_{[0,k]}(D(\bar{p}) - Q(\bar{p})). \tag{28}$$

The incentive constraint (F2-IC) requires that firm 2's profit $\pi(Q(p), p)$ after deviating to any other $p \in \mathcal{P}$ cannot be higher than Π_2. Graphically, it means that the graph of $Q(\cdot)$ must not intersect the region $\{(Q,p) \in \mathbb{R}_+ \times \mathcal{P} : \pi(Q,p) > \Pi_2\}$.

Note that the largest feasible firm 2's profit is $\pi(\max\{p^m, u'(k)\})$. Figure 3 shows firm 2's iso-profit curves, i.e., the level curves of $\pi(Q,p)$, for various level of $\Pi_2 \in (0, \pi(\max\{p^m, u'(k)\}))$. The shape of those iso-profit curves are guaranteed by Assumption 2, which implies that $\pi(Q,\cdot)$ is concave on $\{p : \pi(Q,p) > 0\}$ for every $Q \geq 0$. If firm 2 does not have capacity constraint (i.e., $k \geq q^e$), its iso-profit curves are the same as the level curves of $\pi(p) - (p-c)Q$, whose slopes are $(p-c)/(\pi'(p) - Q)$, as shown in Figure 3a. When firm 2 has capacity constraint (i.e., $k < q^e$), the iso-profit curves are horizontal when $Q < D(p) - k$, and coincide the level curves of $\pi(p) - (p-c)Q$ otherwise, as shown in Figure 3b.

To sum up, constraint (F2-IC) simply says that the graph of $Q(\cdot)$ must not cut into the left side of the iso-profit curve that passes through $(Q(\bar{p}), \bar{p})$.



(a) when $k \geq q^e$ (b) when $k < q^e$

Figure 3: Firm 2's Iso-profit Curves

6.3 Equilibrium

Using (25) and (27) to eliminate $T(\bar{p})$, firm 1's profit can be written as

$$\begin{aligned}\Pi_1 &= T(\bar{p}) - c \cdot Q(\bar{p}) \\ &= V(Q(\bar{p}), \bar{p}) - V(0, c) - \int_c^{\bar{p}} V_p(Q(p), p)dp - c \cdot Q(\bar{p}) \end{aligned} \tag{29}$$

With the introduction of Π_2 from (28), (OP1) can now be rewritten as

$$\underset{Q(\cdot), \bar{p}, \Pi_2}{\text{Maximize}} \ (29) \tag{OP1'}$$

subject to

$$(24)$$

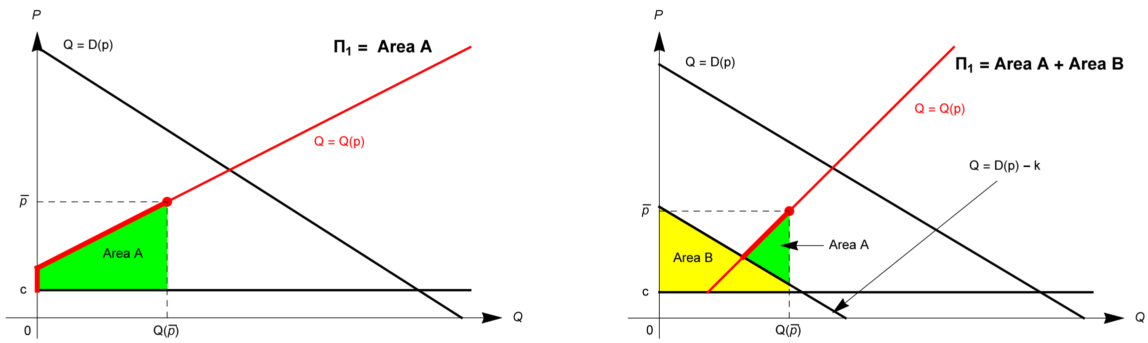
$$\Pi_2 \geq \pi(Q(p), p) \quad \forall p \in \mathcal{P} \tag{F2-IC}$$

$$\Pi_2 = \pi(Q(\bar{p}), \bar{p}). \tag{F2-Pro}$$

Our strategy of solving (OP1') is as follows. We decompose (OP1') into two stages: in the first stage, for any given Π_2, $Q(\cdot)$ and $\bar{p}$ are chosen contingent on Π_2; in the second stage, optimal Π_2 is chosen. Lemma 3 below solves the first stage for any feasible $\Pi_2 > 0$, and Lemma 4 solves the second stage to pin down Π_2.

To graphically show firm 1's profit, we use (7) and (8) to rewrite (29):

$$\begin{aligned}\Pi_1 &= \int_c^{\bar{p}} \left[V_p(Q(\bar{p}), p) - V_p(Q(p), p)\right] dp + \int_0^{Q(\bar{p})} [V_Q(Q, c) - c] dQ \\ &= \int_c^{\bar{p}} [\text{Proj}_{[0,k]}(D(p) - Q(p)) - \text{Proj}_{[0,k]}(D(p) - Q(\bar{p}))]dp + \int_0^{Q(\bar{p})} [\text{Proj}_{[u'(Q+k), u'(Q)]}(c) - c]dQ \\ &= \int_c^{\bar{p}} [\text{Proj}_{[D(p)-k, D(p)]}(Q(\bar{p})) - \text{Proj}_{[D(p)-k, D(p)]}(Q(p))]dp + \int_0^{Q(\bar{p})} [\text{Proj}_{[u'(Q+k), u'(Q)]}(c) - c]dQ \end{aligned} \tag{30}$$



(a) when $k \geq q^e$

(b) when $k < q^e$

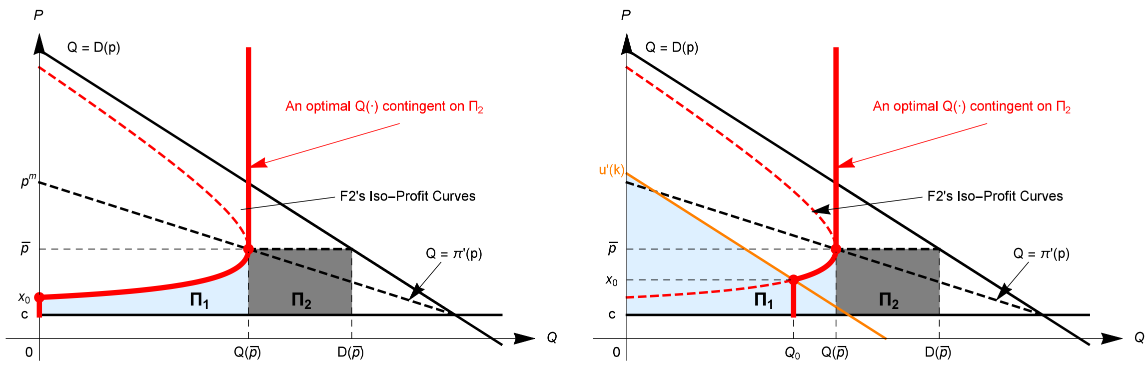
Figure 4: Firm 1's profit Π_1 contingent on $Q(\cdot)$ and $\bar{p}$

Figure 4 shows the area of Π_1 given by (30) for a given $Q(\cdot)$ and $\bar{p}$: Area A and Area B correspond to the first and the second integral in (30) respectively. It is worth noting that Figure 4 demonstrates that what really matters for Π_1 is *the part of* $Q(\cdot)$ *in region* Φ. Hence, we denote the intersection point of $Q(p)$ and $\max\{D(p) - k, 0\}$ as (Q_0, x_0), i.e.,

$$\max\{D(x_0) - k, 0\} = Q(x_0) \tag{31}$$

$$Q_0 = \max\{D(x_0) - k, 0\}. \tag{32}$$

It can be seen from Figures 3 and 4 that, given a Π_2 and hence a firm 2's iso-profit curve, in order to maximize (29) subject to (24) and (F2-IC), (i) the function $Q(\cdot)$ must lie on the iso-profit curve in region Φ, and (ii) the point $(Q(\bar{p}), \bar{p})$ must be chosen to be the most rightward point on the firm 2's iso-profit curve. Lemma 3 below formalizes these claims. Figures 5a and 5b graphically show the partial solutions contingent on Π_2 for two examples when firm 2's capacity is large and small, respectively.



(a) when k is large
(b) when k is small

Figure 5: Optimal $Q(\cdot)$ contingent on Π_2

Lemma 3. *Contingent on any $\Pi_2 \in (0, \pi(\max\{p^m, u'(k)\}))$, there exist solutions $(Q(\cdot), \bar{p})$ of (OP1'). Any such contingent solution satisfies*

$$Q(p) = D(p) - \frac{\Pi_2}{p - c} \quad \forall p \in [x_0, \bar{p}], \tag{33}$$

and $\bar{p}$ is the unique solution of

$$\max\{D(\bar{p}) - k, \pi'(\bar{p})\} = D(\bar{p}) - \frac{\Pi_2}{\bar{p} - c}, \tag{34}$$

where x_0 and Q_0 are given by (31) and (32), respectively.

To solve (OP1'), it remains to pin down Π_2, which should be chosen to make the Π_1 area in Figure 5 as large as possible. It turns out the corresponding first-order condition can be simplified as (35) below. Once a solution $(Q(\cdot), \bar{p}, \Pi_2)$ of (OP1') is obtained, we can use the equivalence between (OP1) and (OP1') to obtain a solution $(Q(\cdot), T(\cdot), \bar{p})$ of (OP1).

Lemma 4. *(OP1') has at least one solution. For any such solution $(Q(\cdot), \bar{p}, \Pi_2)$, $(\bar{p}, x_0)$ are determined by*

$$\bar{p} - c = e \cdot (x_0 - c) > 0, \tag{35}$$

$$(\bar{p} - c)(D(\bar{p}) - \pi'(\bar{p})) = (x_0 - c) \min\{D(x_0), k\}, \tag{36}$$

and

$$\Pi_2 = (\bar{p} - c)(D(\bar{p}) - \pi'(\bar{p})), \tag{37}$$

$$\bar{Q} = \pi'(\bar{p}), \tag{38}$$

Q_0 *is given by (32).*

A solution of $Q(\cdot)$ *is given by*

$$Q(p) = \begin{cases} D(p) - \frac{\Pi_2}{p-c} & \text{if } p \in [x_0, \bar{p}] \\ Q_0 & \text{if } c \le p < x_0 \\ \bar{Q} & \text{if } \bar{p} < p \le u'(0) \end{cases}, \tag{39}$$

and the $T(\cdot)$ *satisfies*

$$T(p) = u(Q_0 + k) - u(k) + \int_{x_0}^{p} t dQ(t) \quad \forall p \in [x_0, \bar{p}]. \tag{40}$$

Finally, we can use the equivalence between solving SPE of the original game and solving (OP1) or (OP1') established in Theorem 1 to characterize the equilibrium outcome of the original game. Figure 6 illustrates the features of an equilibrium tariff offered by firm 1.

Theorem 2. (SPE Outcome) *There exists at least one SPE. In any SPE,* $(\Pi_2, \bar{p}, x_0, \bar{Q}, Q_0)$ *solves (32), (35)˜(38). Firm 2 chooses* $p = \bar{p}$*, and the buyer purchases* $\bar{Q}$ *units and* $D(\bar{p}) - \bar{Q} < k$ *units from firm 1 and firm 2 respectively. An equilibrium tariff* $\tau(\cdot)$ *offered by firm 1 can be constructed as*

$$\tau(Q) = \begin{cases} u(Q_0 + k) - u(k) + \int_{Q_0}^{Q} x(\tilde{Q}) d\tilde{Q} & \text{if } Q \in [Q_0, \bar{Q}] \\ 0 & \text{if } Q = 0 \\ \infty & \text{otherwise} \end{cases}, \tag{41}$$

where $x(\cdot)$ *on* $[Q_0, \bar{Q}]$ *is the inverse of* $Q(\cdot)$ *on* $[x_0, \bar{p}]$ *given by (33).*

Strictly speaking, equilibrium is never unique because $Q(p), T(p)$ for $p \notin [x_0, \bar{p}]$ and hence $\tau(Q)$ for $Q \notin [Q_0, \bar{Q}]$ are not unique. As demonstrated in Figure 4, only the part of $Q(p)$ for $p \in [x_0, \bar{p}]$ matters. In Theorem 2, equilibrium $\tau(Q)$ for $Q \notin [Q_0, \bar{Q}] \cup \{0\}$ do not affect the allocation, as long as they are sufficiently large.

We say the equilibrium is *essentially unique* if the equilibrium objects $\Pi_1, \Pi_2, \bar{p}, x_0, \bar{Q}, Q_0$ and hence $Q(p)$ for $p \in [x_0, \bar{p}]$ are unique. The following proposition provides a simple sufficient condition for the uniqueness.

Proposition 1. (Uniqueness) *The equilibrium is essentially unique if one of the following two equivalent conditions is satisfied:*

$$u'(q) - c \text{ is strictly log-concave in } q \text{ on } [0, q^e]; \tag{42}$$

$$-(p-c)D'(p) \text{ is strictly increasing in } p \text{ on } \mathcal{P}. \tag{43}$$

Note that $-(p-c)D'(p) = D(p) - \pi'(p)$. Thus, a graphical interpretation of condition (42) is that, for any k, the curve $\pi'(p) = Q$ (or $D(p) + (p-c)D'(p) = Q$) and the curve $D(p) - k = Q$ cross at most once, as shown in Figure 3.

7 Implications of the equilibrium

7.1 Other properties of the equilibrium

Corollary 2. (Increasing and Convex Tariff) *In any equilibrium, firm 1's tariff τ is strictly increasing and strictly convex on $[Q_0, \bar{Q}]$.*

A typical equilibrium tariff is shown in Figure 6. This is a stark contrast to a typical nonlinear tariff in Maskin and Riley (1984): in Maskin and Riley (1984), under some regularity conditions, a monopolist's optimal nonlinear tariff often involves quantity discount; nevertheless, under competition, we find that the dominant firm's optimal nonlinear tariff's marginal price in the effective supplying range $[Q_0, \bar{Q}]$ is always increasing, although it is still possible that the average price is decreasing in that range.

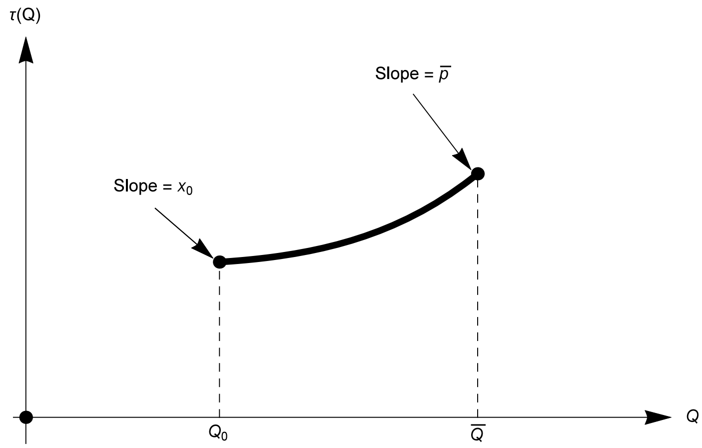


Figure 6: Equilibrium nonlinear pricing

Interestingly, as stated in the following corollary, what firm 2 and the buyer jointly earn in equilibrium is equal to their joint outside option under the counterfactual situation that firm 2's unit cost was raised to x_0, as stated in Corollary 3 below.

Corollary 3. *In any equilibrium,*

$$\begin{aligned} \Pi_2 + BS &= \int_{x_0}^{\infty} \min\{D(p), k\} dp \\ &= u(\min\{D(x_0), k\}) - x_0 \cdot \min\{D(x_0), k\} \\ &= u(D(x_0) - Q_0) - x_0 \cdot (D(x_0) - Q_0). \end{aligned} \tag{44}$$

By Lemma 3, constraint for firm 2 (33) is binding for any $p \in [x_0, \bar{p}]$, so firm 2 is indifferent between any prices in this range. However, in the proof of Lemma 2, we know that $V(Q(p), p) - T(p) - V(0, p)$ being non-decreasing in p. Hence, the buyer's most preferred deviation will be the one when firm 2 sets x_0 and the buyer buys Q_0 from firm 1.

The impact of firm 2's capacity k on equilibrium is not monotone in general, as shown in Corollary 4 below.

Corollary 4. (Comparative Statics on k) *There is a unique $\hat{k} \in (D(p^m), q^e)$ such that $Q_0 = 0$ in equilibrium if and only if $k \geq \hat{k}$. The set of equilibria is independent* *of k on $[\hat{k}, \infty]$.*

The comparative statics for k are as follows.

(a) The equilibrium objects $\Pi_2, \bar{p}, x_0, \bar{p} - x_0$ (and also $D(\bar{p}) - \bar{Q}$ if we assume condition (42), or the equivalent one (43)) are increasing in k on $(0, \hat{k}]$.

(b) The equilibrium objects $\Pi_1, \bar{Q}, Q_0, TS$ are decreasing in k on $(0, \hat{k}]$.

(c) The equilibrium objects $\Pi_2 + BS$ and BS are increasing in k when k is small, and are decreasing in k when k is close to but below $\hat{k}$.

First, there exists a unique cutoff $\hat{k}$ above which the minimum quantity requirement from firm 1 shrinks to zero, and the equilibrium outcomes become independent of k. For $k \geq \hat{k}$, firm 2's competitive threat is so large that any further increase in its capacity will no longer affect equilibrium. This is because when $Q_0 = 0$, the buyer's best deviation is to buy nothing from firm 1 and single-source from firm 2. Due to its linear pricing, firm 2 would not want to sell to the single-sourcing buyer at its full capacity k when k is sufficiently large. Thus, the impact of k on equilibrium outcomes is limited to $0 \leq k \leq \hat{k}$.

For $k \leq \hat{k}$, an increase in firm 2's capacity benefits firm 2, and harms firm 1. This is not surprising because firm 2's capacity represents its competitive threat on firm 1. Social welfare decreases in k, because as firm 2's capacity increases, firm 1 has more to worry about the competitive constraint from its rival. Accordingly, it will have stronger incentives to use its nonlinear pricing schedule, as a latent strategy, to mitigate the competition with firm 2. Interestingly, as the potential competition becomes more intensive, both what firm 2 and the buyer as a joint earn and the buyer benefit from the increase in k when k is small, whereas get harmed from it when k is close to but below $\hat{k}$. This implies, even though firm 2 always has an incentive to increase its capacity as long as $k \leq \hat{k}$, firm 1 will not want this to happen, and the buyer will be on the same stance with firm 1 when k is close to but below $\hat{k}$.

7.2 Comparing with linear pricing

Consider a game that is similar to the one we presented in Section 2, except that firm 1 can only offer a unit price (linear pricing, or LP for short). First, firm 1 offers a unit price $p_1 \in \mathbb{R}_+$. Second, after observing p_1, firm 2 offers a unit price $p_2 \in \mathbb{R}_+$. Third, after observing p_1 and p_2, the buyer chooses the quantities $q_1 \in \mathbb{R}_+$ and $q_2 \in [0, k]$ he buys from firm 1 and from firm 2. Call it the *LP vs LP game*, and the game presented in Section 2 the *NLP vs LP game*. We use superscript "LP" to denote various variables for the LP vs LP game.

Proposition 2. (LP vs LP Equilibrium) *Consider the LP vs LP game. If $k < q^e$, then there is a unique SPE outcome, in which both firms offer $\bar{p}^{LP}$, where $\pi'(\bar{p}^{LP}) = k$, and the buyer purchases $q_1^{LP} = D(\bar{p}^{LP}) - k$ and $q_2^{LP} = k$ units from firm 1 and firm 2 respectively. If $k \geq q^e$, then there are multiple SPE outcome, in which the prevailing price can be any $\bar{p}^{LP} \in [c, p^m]$ (either $p_1 = p_2 = \bar{p}^{LP} \in [c, p^m]$ or $p_1 \geq p^m = p_2$) and firm 1 makes no sales.*

Corollary 5. (Comparative Statics for LP vs LP) *(a) $\bar{p}^{LP}, \Pi_1^{LP} + \Pi_2^{LP}, \Pi_1^{LP}$ (and also q_1^{LP} if we assume condition (43)) are decreasing in k on $(0, q^e)$.*

(b) $TS^{LP}, q_2^{LP}, q_1^{LP} + q_2^{LP}, BS^{LP}, \Pi_2^{LP} + BS^{LP}$ are increasing in k on $(0, q^e)$.

(c) Π_2^{LP} is increasing in k when k is small, and is decreasing in k when k is close to but below q^e.

Proposition 3. (Comparison) *Let $k \in (0, q^e)$ and compare any SPE outcome of the NLP vs LP game with the unique SPE outcome of the LP vs LP game.*

(a) $D(\bar{p}) - \bar{Q} < q_2^{LP} = k$, $\Pi_1 > \Pi_1^{LP}$, $\Pi_2 + BS < \Pi_2^{LP} + BS^{LP}$.

(b) $\bar{p} < \bar{p}^{LP}$, $D(\bar{p}) > D(\bar{p}^{LP}) = q_1^{LP} + q_2^{LP}$, $TS > TS^{LP}$, $\Pi_2 < \Pi_2^{LP}$ when k is small, and the opposite is true when $k \in [\hat{k}, q^e)$.

(c) $BS < BS^{LP}$ when k is small or $k \in [\hat{k}, q^e)$.

(d) $\bar{Q} > q_1^{LP} = D(\bar{p}^{LP}) - k$ when k is small or close to q^e.

7.3 A linear demand example

This subsection considers a linear demand example. Suppose that $u(q) = q - q^2/2$ and $c \in [0, 1)$. Then $D(p) = 1 - p$, $\pi(p) = (p - c)(1 - p)$, and $\pi'(p) = 1 + c - 2p$ for all $p \in \mathcal{P} = [c, 1]$. Assumptions 1, 2 and the conditions in Proposition 1 are satisfied, so that the equilibrium is essentially unique.

Pricing	x_0	Q_0	$\bar{p}$	$\bar{Q}$
	$c + \frac{1}{e^2}\min\{k, \hat{k}\}$	$\frac{1+e^2}{e^2}\max\{\hat{k} - k, 0\}$	$c + \frac{1}{e}\min\{k, \hat{k}\}$	$1 - c - \frac{2}{e}\min\{k, \hat{k}\}$
Surplus	Π_1		Π_2	
	$\frac{(1-c)^2}{2(1+e^2)} + \frac{1+e^2}{2e^2}(\max\{\hat{k} - k, 0\})^2$		$\frac{1}{e^2}(\min\{k, \hat{k}\})^2$	
	BS		TS	
	$(1-c)\min\{k, \hat{k}\} - \frac{4+e^2}{2e^2}(\min\{k, \hat{k}\})^2$		$\frac{(1-c)^2}{2} - \frac{1}{2e^2}(\min\{k, \hat{k}\})^2$	

Table 1: Linear Demand Example

Substituting (35) into (36), the latter becomes

$$p - c = \frac{1}{e} \cdot \min\left\{1 - c - \frac{p - c}{e}, k\right\}.$$

So

$$\bar{p} = c + \frac{1}{e}\min\{k, \hat{k}\},$$

where

$$\hat{k} = \frac{e^2(1-c)}{1+e^2}.$$

Other endogenous objects follow directly, and the solution is listed in Table 1.

As we claim generally in Corollary 4, all the above objects except $\Pi_2 + BS$ and BS are monotone in k. When $k < \frac{e^2(1-c)}{2+e^2}$, $\Pi_2 + BS$ is increasing in k. When $\frac{e^2(1-c)}{2+e^2} < k < \hat{k}$, $\Pi_2 + BS$ is decreasing in k. When $k < \frac{e^2(1-c)}{4+e^2}$, BS is increasing in k. When $\frac{e^2(1-c)}{4+e^2} < k < \hat{k}$, BS is decreasing in k. Figure 7 demonstrates these non-monotone patterns.

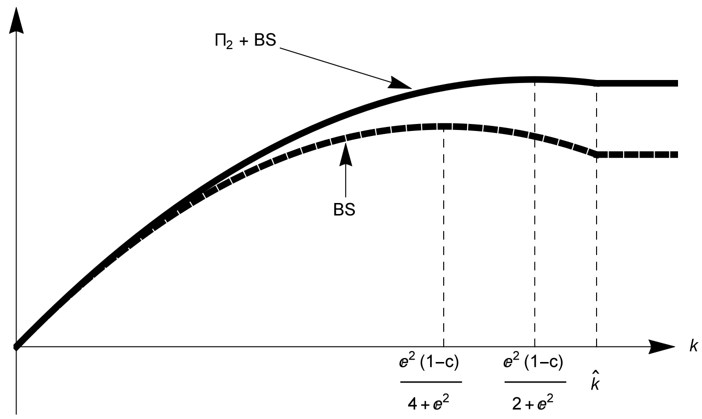


Figure 7: BS and $\Pi_2 + BS$ for Linear Demand Example

Figure 8 and Table 2 show various equilibrium objects in the linear demand example, when nonlinear pricing (NLP), quantity forcing (QF) (i.e. offering a take-it-or-leave-it quantity-payment bundle), or linear pricing (LP) is feasible to the dominant firm.

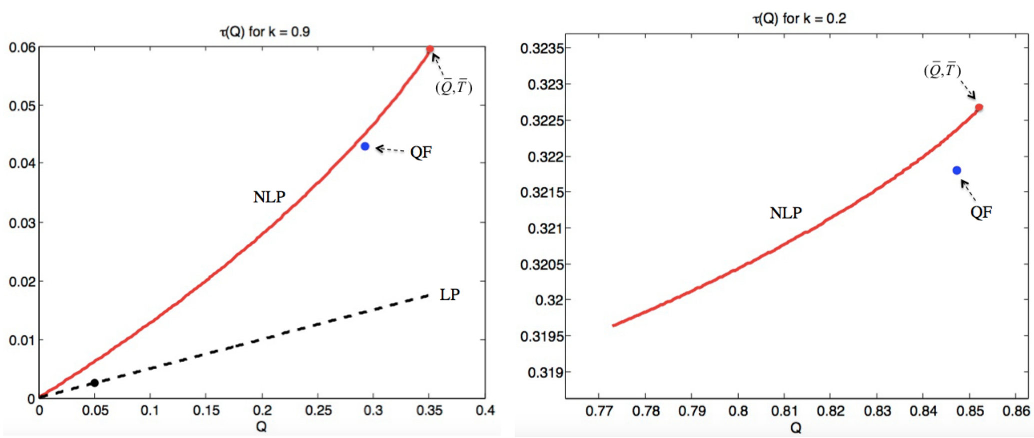


Figure 8: Dominant firm's equilibrium tariff schedules and the corresponding chosen purchases under assumptions $D(p) = 1 - p$, $c = 0$, and $k = 0.9$ or 0.2, when nonlinear pricing (NLP), quantity forcing (QF), or linear pricing (LP) is feasible to the dominant firm (the LP schedule is omitted in the right panel because its scale is far below that of the NLP schedule)

	q_1	q_2	Π_1	Π_2	BS	TS
$k = 0.2$						
LP	0.4	0.2	0.16	0.08	0.18	0.42
QF	0.8472	0.0764	0.3218	0.0058	0.1694	0.4971
NLP	0.8528	0.0736	0.3227	0.0054	0.1692	0.4973
$k = 0.9$						
LP	0.05	0.9	0.0025	0.045	0.4513	0.4987
QF	0.2929	0.3536	0.0429	0.125	0.2696	0.4375
NLP	0.3519	0.324	0.0596	0.105	0.2829	0.4475

Table 2: Equilibrium outputs (q_1, q_2), profits (Π_1, Π_2), buyer's surplus (BS), and total surplus (TS) under assumptions $D(p) = 1 - p$, $c = 0$, and $k = 0.9$ or 0.2, when nonlinear pricing (NLP), quantity forcing (QF), or linear pricing (LP) is feasible to the dominant firm

8 Concluding remarks

Recall that our model involves three kinds of asymmetries between the two firms: (1) the dominant firm is able to make nonlinear tariff schedules, while the rival firm can only choose linear pricing schemes; (2) the dominant firm commits to offering tariffs before its rival; and (3) relative to the demand size the dominant firm has no capacity limit while its rival is capacity-constrained. Our analysis above suggests that the asymmetry in capacity is not crucial for the equilibrium adoption of nonlinear pricing by the dominant firm, but is important for the results of partial foreclosure and harming the buyer welfare.

What would happen if we relax our assumptions about the asymmetry between the two firms by endogenizing the choices of timing and tariff options? One may consider a 4-stage extended game as follows. In Stage 0, each firm simultaneously decides whether to commit itself to use linear pricing. Any firm who makes this commitment can only offer a linear pricing scheme in later stages; and otherwise can more generally offer a nonlinear tariff schedule in later stages. In Stage 1, each firm can either offer a tariff (from the feasible set determined by its choice in stage 0), or wait until stage 2. In stage 2, any firm who chose waiting in stage 1 has to offer a tariff (again from the feasible set determined by its choice in stage 0).

Lastly, in stage 3, the buyer chooses the quantities she purchases from the two firms. We can show that this extended game has a subgame perfect equilibrium with the following properties: only the rival firm commits itself to linear pricing in stage 0, the dominant firm offers a nonlinear tariff in stage 1, the rival firm waits in stage 1 and offers a linear tariff in stage 2, and their offers and the buyer's choices are the same as those we characterized for our original 3-stage game. As a result, when both firms can choose their timing and pricing options the equilibrium outcome in the original 3-stage game remains to be part of the subgame perfect equilibrium outcome in the extended game. This demonstrates that our assumptions regarding the sequence of the moves and asymmetry in tariff options are not crucial for our main results. The asymmetry in capacity between the firms allows the unconstrained firm to take advantage of a menu of tariff offers in order to restrict the choices of the constrained firm and extract surpluses from the buyer.

Appendix

Proof of Lemma 1. Fix any $Q \in \mathbb{R}_+$. Note that the unique maximizer $\text{Proj}_{[0,k]}(D(p)-Q)$ of the value function $V(Q,p)$ is piecewise continuously differentiable. For any $p \in \mathcal{P}$ at which $\text{Proj}_{[0,k]}(D(p)-Q)$ is differentiable (i.e., $D(p)-Q \neq 0$ and $D(p)-Q \neq k$), clearly $V(Q,p)$ is also differentiable at p and the derivative $V_p(Q,p)$ computed from the Envelope Theorem is given by (7). Moreover, even for $p \in \mathcal{P}$ at which $\text{Proj}_{[0,k]}(D(p)-Q)$ is not differentiable (i.e., $D(p)-Q = 0$ or $D(p)-Q = k$), $\text{Proj}_{[0,k]}(D(p)-Q)$ is still continuous; it is clear that the left-derivative and right-derivative of $V(Q,\cdot)$ exist and both are equal to the r.h.s. of (7). Thus, $V(Q,\cdot)$ is differentiable and (7) holds. The same logic proves that $V(\cdot,p)$ is differentiable and (8) holds. From (7) and (8), we know $V_p(Q,\cdot)$, $V_p(\cdot,p)$, $V_Q(Q,\cdot)$, and $V_Q(\cdot,p)$ are all piecewise continuously differentiable. In particular, whenever differentiable (i.e., $D(p)-Q \neq 0$ and $D(p)-Q \neq k$), the cross derivatives V_{Qp} and V_{pQ} are given by (9). ■

The proof of Theorem 1 requires the following two lemmas.

Lemma A.1. *For any $Q : \mathcal{P} \to \mathbb{R}_+$, $T : \mathcal{P} \to \mathbb{R}$, and $\bar{p} \in \mathcal{P}$ that satisfy (B-IC), (B-IR), and (F2-IC), there is a $\tau \in \mathcal{T}$ and a SPE of the subgame after firm 1 offers τ such that*

(i) in this SPE of the subgame, firm 2 chooses $p = \bar{p}$, and the buyer, contingent on any firm 2's unit price $p \in \mathcal{P}$, chooses to buy $Q(p)$ and $\text{Proj}_{[0,k]}(D(p)-Q(p))$ units from firm 1 and firm 2 respectively, and

(ii) $\tau(Q(p)) = T(p)$ for all $p \in \mathcal{P}$.

Proof of Lemma A.1. Suppose that $Q : \mathcal{P} \to \mathbb{R}_+$, $T : \mathcal{P} \to \mathbb{R}$, and $\bar{p} \in \mathcal{P}$ satisfy (B-IC), (B-IR), and (F2-IC). Define

$$\tau(Q) = \begin{cases} T(p) & \text{if } \exists p \in \mathcal{P} \text{ s.t. } Q(p) = Q \\ 0 & \text{if } Q = 0 \text{ and } \nexists p \in \mathcal{P} \text{ s.t. } Q(p) = 0 \\ \infty & \text{otherwise} \end{cases}. \quad \text{(A1)}$$

Note that the above τ is well defined because (B-IC) implies $T(p) = T(\tilde{p})$ whenever $Q(p) = Q(\tilde{p})$. Clearly, (ii) holds. To see that $\tau(0) \leq 0$, note that if $\nexists p \in \mathcal{P}$ s.t.

$Q(p) = 0$, then $\tau(0) = 0$; if $Q(\hat{p}) = 0$ for some $\hat{p} \in \mathcal{P}$, then $\tau(0) = T(\hat{p}) \leq 0$, where the inequality follows from (B-IR). Thus, $\tau(0) \leq 0$.

Given this τ and any $p \in \mathcal{P}$, (B-IC) and (B-IR) imply that a buyer's optimal action is to buy $Q(p)$ and $\text{Proj}_{[0,k]}(D(p) - Q(p))$ units from firm 1 and firm 2 respectively. Given τ and that the buyer uses the above strategy, (F2-IC) implies that a firm 2's optimal action is to choose $p = \bar{p}$. Therefore, the strategies in (i) constitute a SPE of the subgame after firm 1 offers τ. It follows that τ is regular and hence $\tau \in \mathcal{T}$. ■

Lemma A.2. *For any $\tau \in \mathcal{T}$ and any SPE of the subgame after firm 1 offers τ, if $Q : \mathcal{P} \to \mathbb{R}_+$, $T : \mathcal{P} \to \mathbb{R}$, and $\bar{p} \in \mathcal{P}$ satisfy (i) and (ii) in Lemma A.1, then $Q(\cdot), T(\cdot), \bar{p}$ also satisfy (B-IC), (B-IR), and (F2-IC).*

Proof of Lemma A.2. Take any $\tau \in \mathcal{T}$ and any SPE of the subgame after firm 1 offers τ. Suppose that $Q : \mathcal{P} \to \mathbb{R}_+$, $T : \mathcal{P} \to \mathbb{R}$, and $\bar{p} \in \mathcal{P}$ satisfy (i) and (ii) in Lemma A.1. Since the strategies described in (i) constitute a SPE of the subgame after firm 1 offers τ, we have (F2-IC) and

$$V(Q(p), p) - \tau(Q(p)) \geq V(Q, p) - \tau(Q) \quad \forall (Q, p) \in \mathbb{R}_+ \times \mathcal{P}. \tag{A2}$$

To see (B-IC), take $Q = Q(\tilde{p})$ for arbitrary $\tilde{p} \in \mathcal{P}$ in (A2) and use (ii). To see (B-IR), take $Q = 0$ in (A2) and use $\tau(0) \leq 0$ and (ii). ■

Proof of Theorem 1. *("only if" part)* Suppose that $(Q^*(\cdot), T^*(\cdot), \bar{p}^*)$ is a solution of (OP1). Then $Q^*(\cdot), T^*(\cdot), \bar{p}^*$ satisfy (B-IC), (B-IR), and (F2-IC). From Lemma A.1, there is a $\tau^* \in \mathcal{T}$ (defined by (A1) with $\tau(\cdot), Q(\cdot), T(\cdot)$ replaced by $\tau^*(\cdot), Q^*(\cdot), T^*(\cdot)$) such that (22) holds and a SPE $(p^*(\tau^*), q^*(\tau^*, \cdot))$ of the subgame after firm 1 offers τ^* is described by (20), (21), and (23).

In the subgame after firm 1 offers this τ^*, we let firm 2 and the buyer play the SPE $(p^*(\tau^*), q^*(\tau^*, \cdot))$, so that firm 1's profit is $T^*(\bar{p}^*) - c \cdot Q^*(\bar{p}^*)$. In the subgame after firm 1 offers any other $\tau \in \mathcal{T} \backslash \{\tau^*\}$, we let firm 2 and the buyer play any SPE $(p^*(\tau), q^*(\tau, \cdot))$, which exists because every $\tau \in \mathcal{T}$ is regular. By such constructions, p^*, q^* satisfy (1) and (2).

From Lemma A.2, the SPE outcome of the subgame after firm 1 offers an arbitrary $\tau \in \mathcal{T}$ must be characterized by some $Q(\cdot), T(\cdot), \bar{p}$ that satisfy (B-IC), (B-IR), and (F2-IC), and the associated firm 1's profit is $T(\bar{p}) - c \cdot Q(\bar{p})$. Since $(Q^*(\cdot), T^*(\cdot), \bar{p}^*)$ is a solution of (OP1), firm 1 cannot make strictly higher profit than $T^*(\bar{p}^*) - c \cdot Q^*(\bar{p}^*)$ by offering any $\tau \in \mathcal{T}$. That is, (τ^*, p^*, q^*) satisfies (3) and hence is a SPE of the whole game.

("if" part) Suppose that (τ^*, p^*, q^*) is a SPE and $Q^*(\cdot), T^*(\cdot), \bar{p}^*$ satisfy (20), (21), (22), and (23). From Lemma A.2, $Q^*(\cdot), T^*(\cdot), \bar{p}^*$ satisfy (B-IC), (B-IR), and (F2-IC). Suppose, by way of contradiction, that $(Q^*(\cdot), T^*(\cdot), \bar{p}^*)$ is not a solution of (OP1). Then, there is some $(Q^0(\cdot), T^0(\cdot), \bar{p}^0)$ satisfying (B-IC), (B-IR), and (F2-IC), such that $T^0(\bar{p}^0) - c \cdot Q^0(\bar{p}^0) > T^*(\bar{p}^*) - c \cdot Q^*(\bar{p}^*)$. We shall show that firm 1 then can offer a tariff in $\mathcal{T}$ that guarantees itself a profit arbitrarily close to $T^0(\bar{p}^0) - c \cdot Q^0(\bar{p}^0)$ in every SPE of the firm 2-buyer subgame that follows. Once this is proved, offering such a tariff is a firm 1's profitable deviation in the SPE (τ^*, p^*, q^*), which is a contradiction.

To do that, we perturb the solution $(Q^0(\cdot), T^0(\cdot), \bar{p}^0)$ so that firm 2 would have to lower its price a bit more if it wishes to increase its sales by any given amount. We can keep $\bar{p}^0$ unchanged and, for any $\varepsilon > 0$, let

$$Q_\varepsilon(p) = \begin{cases} Q^0(p) & \text{if } p \geq \bar{p}^0 \\ Q^0(\bar{p}^0) & \text{if } \bar{p}^0 - \varepsilon < p < \bar{p}^0 \\ Q^0(p+\varepsilon) & \text{if } p \leq \bar{p}^0 - \varepsilon \end{cases},$$

and

$$T_\varepsilon(p) = V(Q_\varepsilon(p), p) - V(0, c) - \int_c^p V_p(Q_\varepsilon(t), t)dt.$$

Note that $(Q_\varepsilon(\cdot), T_\varepsilon(\cdot), \bar{p}^0)$ satisfies all the constraints of (OP1); the (F2-IC) constraint holds strictly at every $p \neq \bar{p}^0$; the value of (OP1) evaluated at $(Q_\varepsilon(\cdot), T_\varepsilon(\cdot), \bar{p}^0)$ is arbitrarily close to the maximum value $T^0(\bar{p}^0) - c \cdot Q^0(\bar{p}^0)$ when ε is made arbitrarily small.

Define $\tau_\varepsilon(\cdot)$ by the r.h.l. of (A1) with $Q(\cdot)$ and $T(\cdot)$ replaced by $Q_\varepsilon(\cdot)$ and $T_\varepsilon(\cdot)$.

Now, if firm 1 offers τ_ε, the best responses of the buyer and firm 2 are unique. In particular, firm 2 would surely offer $\bar{p}^0$; the buyer would surely purchase $Q_\varepsilon(\bar{p}^0)$ from firm 1; firm 1's profit would surely be the value of (OP1) evaluated at $(Q_\varepsilon(\cdot), T_\varepsilon(\cdot), \bar{p}^0)$. Therefore, offering τ_ε with small enough $\varepsilon > 0$ is a firm 1's profitable deviation as desired. ■

Proof of Lemma 2. We shall first show that (B-IC) is equivalent to (24) and (25), then establish that, given (24), (B-IR) is equivalent to (26). Let $U(p) \equiv V(Q(p), p) - T(p)$. Then (B-IC) can be written as

$$U(p) - U(\tilde{p}) \geq V(Q(\tilde{p}), p) - V(Q(\tilde{p}), \tilde{p}) \quad \forall p, \tilde{p} \in \mathcal{P}, \tag{A3}$$

and (25) can be written as

$$U(p) - U(c) = \int_c^p V_p(Q(t), t)dt \quad \forall p \in \mathcal{P}. \tag{A4}$$

Step 1: (B-IC) implies (24) and (25).

Suppose (B-IC) is satisfied. Then (A3) implies that, for any $p_1, p_2 \in \mathcal{P}$,

$$V(Q(p_1), p_2) - V(Q(p_1), p_1) \leq U(p_2) - U(p_1) \leq V(Q(p_2), p_2) - V(Q(p_2), p_1). \tag{A5}$$

If (24) does not hold, then there exist $p_1, p_2 \in \mathcal{P}$ such that $p_1 < p_2$ and $Q(p_1) > Q(p_2)$ and $D(p_1) > Q(p_2)$ and $Q(p_1) > D(p_2) - k$. But then (A5) implies

$$\begin{aligned} 0 &\geq [V(Q(p_1), p_2) - V(Q(p_1), p_1)] - [V(Q(p_2), p_2) - V(Q(p_2), p_1)] \\ &= \int_{p_1}^{p_2} \int_{Q(p_2)}^{Q(p_1)} V_{pQ}(Q, p)dQdp > 0, \end{aligned}$$

which is a contradiction. The above equality holds because, from Lemma 1, $V(Q, \cdot)$ is continuously differentiable and $V_p(\cdot, p)$ is piecewise continuously differentiable (and hence they are absolutely continuous on any compact interval). The last inequality holds because, first, $V_{pQ} \geq 0$ almost everywhere and $V_{pQ} = 1$ on the interior of Φ;

second, in the Q-p space, the point $(Q(p_2), p_1)$ is strictly below the curve $Q = D(p)$ (from $D(p_1) > Q(p_2)$) and the point $(Q(p_1), p_2)$ is strictly above the curve $Q = D(p) - k$ (from $Q(p_1) > D(p_2) - k$), so the rectangle $[Q(p_2), Q(p_1)] \times [p_1, p_2]$ must intersect the interior of Φ, on which $V_{pQ} > 0$. Therefore, (24) must hold.

Moreover, (A5) implies (A4). Therefore, (25) holds.

Step 2: (24) and (25) imply (B-IC).

First, (24) implies that, for all $p_1, p_2 \in \mathcal{P}$ with $p_1 \leq p_2$, we have

$$\mathrm{Proj}_{[0,k]}(D(p_2) - Q(p_1)) \geq \mathrm{Proj}_{[0,k]}(D(p_2) - Q(p_2)), \tag{A6}$$

$$\mathrm{Proj}_{[0,k]}(D(p_1) - Q(p_1)) \geq \mathrm{Proj}_{[0,k]}(D(p_1) - Q(p_2)). \tag{A7}$$

Indeed, $p_1 \leq p_2$ and (24) imply either (i) $Q(p_1) \leq Q(p_2)$, or (ii) $D(p_1) \leq Q(p_2)$, or (iii) $Q(p_1) \leq D(p_2) - k$. In case (i), clearly (A6) and (A7) hold. In case (ii), we have $D(p_2) \leq D(p_1) \leq Q(p_2)$ so that the right-hand sides of (A6) and (A7) are 0. In case (iii), we have $Q(p_1) + k \leq D(p_2) \leq D(p_1)$ so that the left-hand sides of (A6) and (A7) are $k > 0$. Therefore, (A6) and (A7) hold in each case.

Recall that (25) is equivalent to (A4). Therefore, for any $p_1, p_2 \in \mathcal{P}$ (no matter whether $p_1 \leq p_2$ or not), we have

$$\begin{aligned} U(p_2) - U(p_1) &= \int_{p_1}^{p_2} V_p(Q(p), p)dp \\ &= -\int_{p_1}^{p_2} \mathrm{Proj}_{[0,k]}(D(p) - Q(p))dp \\ &\geq -\int_{p_1}^{p_2} \mathrm{Proj}_{[0,k]}(D(p) - Q(p_1))dp \; (\because \text{(A6) when } p_1 \leq p_2\text{; and (A7) when } p_1 \geq p_2) \\ &= \int_{p_1}^{p_2} V_p(Q(p_1), p)dp \\ &= V(Q(p_1), p_2) - V(Q(p_1), p_1), \end{aligned}$$

where the inequality is from (*A6*) when $p_1 \leq p_2$ and from (*A7*) when $p_1 \geq p_2$. It proves (A3) and hence (B-IC).

Step 3: Given (B-IC) (in fact, (25) only), (B-IR) is equivalent to (26).

It suffices to show that $V(Q(p),p)-T(p)-V(0,p) = U(p)-V(0,p)$ is non-decreasing in p on $\mathcal{P}$. Indeed, from (25), which is equivalent to (A4), and Lemma 1, we know both $U(\cdot)$ and $V(0,\cdot)$ are differentiable, and $U'(p) = V_p(Q(p),p) = -\operatorname{Proj}_{[0,k]}(D(p)-Q(p)) \geq -\operatorname{Proj}_{[0,k]}(D(p)) = V_p(0,p)$. Therefore, (B-IR) is equivalent to (26). ■

Proof of Corollary 1. It is implied by (A6) in the proof of Lemma 2. ■

Proof of Lemma 3. Fix any $\Pi_2 \in (0, \pi(\max\{p^m, u'(k)\}))$ and hence a firm 2's iso-profit curve in the Q-p space (see Figure 3). Note that $\Pi_2 > 0$ implies $\bar{p} > c$ and $D(\bar{p}) > Q(\bar{p})$. Moreover, from Assumption 2, firm 2's iso-profit curves are (horizontally) single-peaked, so each iso-profit curve has a unique most rightward point.

Here we prove $(Q(\bar{p}), \bar{p})$ must be the most rightward point (the unique horizontal peak) on the iso-profit curve by contradiction. Suppose that $(Q(\bar{p}), \bar{p})$ is not the horizontal peak on the iso-profit curve. Consider the case where $(Q(\bar{p}), \bar{p})$ lies on the strictly decreasing portion of the iso-profit curve (which implies $D(\bar{p}) - Q(\bar{p}) < k$). Then, to satisfy (F2-IC), for small $\varepsilon > 0$, we have $Q(\bar{p} - \varepsilon) > Q(\bar{p})$. But then, from Lemma 1, (24) is violated. Now consider the case where $(Q(\bar{p}), \bar{p})$ lies on the non-decreasing portion of the iso-profit curve. Then, Π_1 can be raised by increasing both $Q(\bar{p})$ and $\bar{p}$ along the iso-profit curve toward the horizontal peak (see Figures 3 and 4). (34) follows immediately.

From Figure 5, it is easy to see $Q(\cdot)$ on $[x_0, \bar{p}]$ must coincide the iso-profit curve, which satisfies (24) and (F2-IC), otherwise Π_1 can be improved by shifting the part of $Q(\cdot)$ on $[x_0, \bar{p}]$ that does not match with the iso-profit curve toward the latter. Thus, we have (33). ■

Proof of Lemma 4. Lemma 3 has characterized the optimal $(Q(\cdot), \bar{p})$ and maximum Π_1 contingent on any $\Pi_2 \in (0, \pi(\max\{p^m, u'(k)\}))$. Clearly, the maximum Π_1 contingent on $\Pi_2 = 0$ is equal to the limiting contingent maximum Π_1 as $\Pi_2 \downarrow 0$ (which is equal to $u(\max\{q^e - k, 0\}) - c \cdot \max\{q^e - k, 0\}$), and the maximum Π_1 contingent on $\Pi_2 = \pi(\max\{p^m, u'(k)\})$ is equal to the limiting contingent maximum Π_1 as $\Pi_2 \uparrow \pi(\max\{p^m, u'(k)\})$ (which is equal to 0). After reducing the second stage (where $(Q(\cdot), \bar{p})$ is chosen contingent on Π_2), (OP1') has only one choice variable, Π_2, and the reduced objective function is continuous in Π_2 on $[0, \pi(\max\{p^m, u'(k)\})]$. Thus, (OP1') has at least one solution.

If $\Pi_2 = 0$, then the contingent maximum can be raised by increasing Π_2 (contemplating an upward-and-leftward shift of $Q(\cdot)$ to a higher firm 2's iso-profit curve in Figure 5). Thus, at any optimum, $\Pi_2 > 0$. On the other hand, if Π_2 is $\pi(\max\{p^m, u'(k)\})$ or is so large that the contingent solution exhibits $D(\bar{p}) - \bar{Q} = k$, then the contingent maximum can be raised by decreasing Π_2 (contemplating a downward-and-rightward shift of $Q(\cdot)$ to a lower firm 2's iso-profit curve in Figure 5 again). Thus, at any optimum, $0 < \Pi_2 < \pi(\max\{p^m, u'(k)\})$ and $D(\bar{p}) - \bar{Q} < k$. So (36)˜(39) follow from Lemma 3.

Next, we show (35). From Figure 4, (30) can be rewritten as

$$\begin{aligned}
\Pi_1 &= \int_0^{Q_0} u'(Q+k)dQ + x_0 \cdot \left(\bar{Q} - Q_0\right) + \int_{x_0}^{\bar{p}} \left(\bar{Q} - Q(p)\right) dp - c\bar{Q} \\
&= \int_0^{Q_0} \left[u'(Q+k) - x_0\right] dQ + (\bar{p} - c)\bar{Q} - \int_{x_0}^{\bar{p}} Q(p)dp \\
&= \int_{x_0}^{\infty} \max\{D(p) - k, 0\}dp + (\bar{p} - c)\bar{Q} - \int_{x_0}^{\bar{p}} \left[D(p) - \frac{\Pi_2}{p - c}\right] dp \ (\because (32) \text{ and } (33)) \\
&= \int_{x_0}^{\infty} \max\{D(p) - k, 0\}dp + TS(\bar{p}) - \Pi_2 - \int_{x_0}^{\infty} D(p)dp + \Pi_2 \cdot \ln \frac{\bar{p} - c}{x_0 - c} \\
&\quad (\because (\bar{p} - c)\bar{Q} = TS(\bar{p}) - \Pi_2 - \int_{\bar{p}}^{\infty} D(p)dp) \\
&= TS(\bar{p}) - \int_{x_0}^{\infty} \min\{D(p), k\}dp + \left(\ln \frac{\bar{p} - c}{x_0 - c} - 1\right) \Pi_2, \qquad \text{(A8)}
\end{aligned}$$

where

$$TS(\bar{p}) \equiv u(D(\bar{p})) - cD(\bar{p}) = \int_{\bar{p}}^{\infty} D(p)dp + (\bar{p} - c)D(\bar{p}) \qquad \text{(A9)}$$

denotes the total surplus.

The partial derivatives of (A8) are

$$\frac{\partial \Pi_1}{\partial \bar{p}} = (\bar{p} - c)D'(\bar{p}) + \frac{\Pi_2}{\bar{p} - c} = \pi'(\bar{p}) - \bar{Q} \ (\because (33)),$$

$$\frac{\partial \Pi_1}{\partial x_0} = \min\{D(x_0), k\} - \frac{\Pi_2}{x_0 - c},$$

$$\frac{\partial \Pi_1}{\partial \Pi_2} = \ln \frac{\bar{p} - c}{x_0 - c} - 1.$$

Note that (36)˜(38) imply that $\partial \Pi_1 / \partial \bar{p} = \partial \Pi_1 / \partial x_0 = 0$. Therefore, the total derivative of (A8) with respect to Π_2 is

$$\frac{d\Pi_1}{d\Pi_2} = \ln \frac{\bar{p} - c}{x_0 - c} - 1. \tag{A10}$$

Therefore, the first-order condition $d\Pi_1 / d\Pi_2 = 0$ implies (35).

Last, we derive (40). From (25) and (27),

$$\begin{aligned}
T(p) &= V(Q(p), p) - V(0, c) - \int_c^p V_p(Q(t), t)dt \\
&= \int_0^{Q(p)} V_Q(Q, c)dQ + \int_c^p [V_p(Q(p), t) - V_p(Q(t), t)]dt \\
&= \int_0^{Q(p)} \mathrm{Proj}_{[u'(Q+k), u'(Q)]}(c)dQ + \int_c^p [\mathrm{Proj}_{[0,k]}(D(t) - Q(t)) - \mathrm{Proj}_{[0,k]}(D(t) - Q(p))]dt \\
&= \int_0^{Q_0} u'(Q+k)dQ + x_0 \cdot (Q(p) - Q_0) + \int_{x_0}^p (Q(p) - Q(t))\, dt \\
&= u(Q_0 + k) - u(k) + \int_{x_0}^p t dQ(t)
\end{aligned}$$

where the third equality follows from (7) and (8), the fourth one follows from $\mathrm{Proj}_{[u'(Q+k), u'(Q)]}(c) = u'(Q+k)$ for $Q < Q(p)$ when $p \in [x_0, \bar{p}]$,

$$\mathrm{Proj}_{[0,k]}(D(t) - Q(t)) - \mathrm{Proj}_{[0,k]}(D(t) - Q(p)) = \begin{cases} 0 & \text{if } c \le t < u'(Q(p)+k) \\ Q(p) + k - D(t) & \text{if } u'(Q(p)+k) \le t \le x_0 \\ Q(p) - Q(t) & \text{if } x_0 < t \le \bar{p} \end{cases},$$

$\int_{u'(Q(p)+k)}^{x_0} [Q(p) + k - D(t)]\, dt = \int_{Q_0}^{Q(p)} [x_0 - u'(Q + k)]\, dQ$ through integration by substitution, and the last equality follows from integration by parts. ■

Proof of Theorem 2. The results are from Lemma 4 and Theorem 1. (41) is derived from (40) through changing of variable: $x(\cdot)$ for $Q(\cdot)$.

Suppose that firm 1's tariff τ is given by (41). Then, firm 2's profit would be Π_2 if it chooses any $p \in [x_0, \bar{p}]$. One can see from Figure 5 that, firm 2's profit would be lower than Π_2 if it chooses any $p > \bar{p}$ (so that the buyer would still purchase $\bar{Q}$ units from firm 1) or any $p < x_0$ (so that the buyer would purchase Q_0 units from firm 1). ■

Proof of Proposition 1. From Theorem 2, the equilibrium is essentially unique if and only if the solution of $(\bar{p}, x_0)$, i.e., (35)˜(36), is unique.

Using (35) to eliminate x_0 and dividing both sides of (36) by $\bar{p} - c$, (36) becomes

$$-(\bar{p} - c)D'(\bar{p}) = \frac{1}{e} \min \left\{ D\left(c + \frac{\bar{p} - c}{e}\right), k \right\}. \tag{A11}$$

(A11) can be solved for $\bar{p}$. Therefore, the equilibrium is essentially unique if and only if (A11) has at most one solution. Under condition (43), the left-hand side of (A11) is strictly increasing in $\bar{p}$, and the right-hand side is non-increasing in $\bar{p}$. Therefore, (A11) has at most one solution under (42).

It is easy to see that (42) and (43) are equivalent, so the proposition follows. ■

Proof of Corollary 2. From (40) and $T(p) = \tau(Q(p))$, the first line of (41) must hold for any equilibrium τ. Thus, $\tau'(\cdot) = x(\cdot)$ on $[Q_0, \bar{Q}]$. Since $x(\cdot)$ on $[Q_0, \bar{Q}]$ is the inverse of $Q(\cdot)$ on $[x_0, \bar{p}]$, and the latter is positive and strictly increasing on $[Q_0, \bar{Q}]$. The corollary follows. ■

Proof of Corollary 3. In equilibrium, the total output is $D(\bar{p})$ and firm 2's output is $D(\bar{p}) - \bar{Q} < k$. So the buyer's surplus

$$\begin{aligned} BS &\equiv u(D(\bar{p})) - \bar{p}(D(\bar{p}) - \bar{Q}) - \tau(\bar{Q}) = TS - \Pi_1 - \Pi_2 \\ &= TS - \Pi_1 - \Pi_2 \ (\because (A9)) \end{aligned} \tag{A12}$$

(35) and (A8) give a simple formula to compute Π_1, i.e.,

$$\Pi_1 = TS - \int_{x_0}^{\infty} \min\{D(p), k\} dp. \tag{A13}$$

So the corollary follows. ■

Proof of Corollary 4. Let $\hat{x}_0$ be the minimum equilibrium x_0 when $k = \infty$, given by (35) and (36) with $\min\{D(x_0), k\} = D(x_0)$ in (36). Define $\hat{k} \equiv D(\hat{x}_0)$. From Theorem 2, $\hat{k}$ satisfies the first two claims (see Figure 5).

The rest of the proof considers comparative statics for $k \in (0, \hat{k}]$. Following the proof of Lemma 4, we regard $\Pi_1, \bar{p}, \bar{Q}, x_0, Q_0, x(\cdot), BS, TS$ as functions of Π_2. Here we also regard them as functions of k. In particular, we write $\Pi_1(\Pi_2; k)$.

Fix Π_2 and let k increase on $(0, \hat{k}]$. Note that $Q_0 = \max\{D(x_0) - k, 0\} > 0$ before the increase, so that we have $D(x_0) > k$ before the increase. The $\bar{p}$ and $\bar{Q}$ determined by (37) and (38) do not change. The x_0, Q_0, and Π_1 determined by (36), (32), and (A8) decrease as k increases (see Figure 5).

In equilibrium, $\Pi_1 = \max_{\Pi_2} \{\Pi_1(\Pi_2; k)\}$ decreases, because $\Pi_1(\cdot; k)$ shifts down as k increases. From (A10), we see that $\partial \Pi_1(\Pi_2; k)/\partial \Pi_2$ increases, because $\bar{p}$ is unchanged whereas x_0 decreases when we fix Π_2 and let k increase. In other words, $\Pi_1(\Pi_2; k)$ satisfies strict increasing differences. Therefore, the Π_2 that maximizes Π_1 must increase when k increases. Then, from (37) and Assumption 2, $\bar{p}$ must increase, and hence $\bar{Q}$ decreases follows from (38). Then from (A9), TS decreases. From (35), x_0 increases. From (32), Q_0 decreases. Also, $\bar{p} - x_0$ increases because (35) can be written as $\bar{p} - x_0 = (e-1)(x_0 - c)$. The result for $D(\bar{p}) - \bar{Q}$ can be immediately seen from $D(\bar{p}) - \bar{Q} = -(\bar{p} - c)D'(\bar{p})$. This completes the proof of parts (a) and (b).

Last, we prove part (c). To see the first half of part (c), note that both Π_2 and BS are positive and tend to zero as $k \to 0$. To see the second half of part (c), first note that, as shown above, we have $\min\{D(x_0), k\} = k$ when $k \le \hat{k}$. From Proposition 44, $\Pi_2 + BS = u(k) - x_0 k$ whenever $k \le \hat{k}$. Hence,

$$\left.\frac{d(\Pi_2 + BS)}{dk}\right|_{k \nearrow \hat{k}} = u'(\hat{k}) - x_0 - \hat{k} \cdot \left.\frac{dx_0}{dk}\right|_{k \nearrow \hat{k}} < 0.$$

The last inequality follows from $u'(\hat{k}) - x_0 \leq u'(\hat{k}) - \hat{x}_0 = 0$ and $\frac{dx_0}{dk}|_{k \nearrow \hat{k}} > 0$. Therefore, $\Pi_2 + BS$ is decreasing in k when k is close to but below $\hat{k}$. This is true for BS as well, because Π_2 is increasing in k. ■

Proof of Proposition 2. Straightforward and omitted. ■

Proof of Corollary 5. Straightforward and omitted. ■

Proof of Proposition 3. In the proof of Lemma 4, we have shown $D(\bar{p}) - \bar{Q} < k$. Clearly, $\Pi_1 > \Pi_1^{LP}$ hold.

$$\begin{aligned}
\Pi_2^{LP} + BS^{LP} &= v(p^{LP}) + \left(p^{LP} - c\right) k \\
&> \int_c^{\infty} \min\{D(p), k\}\, dp \\
&\geq \int_{x_0}^{\infty} \min\{D(p), k\}\, dp \ (\because x_0 > c) \\
&= \Pi_2 + BS \ (\because (44)).
\end{aligned}$$

This completes the proof of part (a).

Compare $\pi'(\bar{p}) = \bar{Q}$ with $\pi'(\bar{p}^{LP}) = k$ and note that $\bar{Q} > k$ when k is small, and $\bar{Q} < k$ when $k \geq \hat{k}$ because $\bar{Q} < D(\bar{p}) < D(x_0) < D(\hat{x}_0) = \hat{k} \leq k$. It proves the result for $\bar{p}, \bar{p}^{LP}$. The results for $D(\bar{p}), D(\bar{p}^{LP})$ and TS, TS^{LP} follows.

Clearly, both Π_2 and Π_2^{LP} tend to zero as $k \searrow 0$. Since $\Pi_2^{LP} = (\bar{p}^{LP} - c)k$,

$$\left.\frac{d\Pi_2^{LP}}{dk}\right|_{k \searrow 0} = \bar{p}^{LP}|_{k \searrow 0} - c = p^m - c > 0.$$

Since $\Pi_2 = (\bar{p} - c)(D(\bar{p}) - \bar{Q})$, and both $\bar{p} - c$ and $D(\bar{p}) - \bar{Q}$ tend to zero as $k \searrow 0$ (contemplating$(\bar{Q}, \bar{p})$ moves along the curve $Q = \pi'(p)$ toward (q^e, c) in Figure 5b),

$$\left.\frac{d\Pi_2}{dk}\right|_{k \searrow 0} = 0.$$

It proves the result for Π_2, Π_2^{LP} when k is small.

When $k \in [\hat{k}, q^e)$, (36) implies $\Pi_2 = (\hat{x}_0 - c)k$, where $\hat{x}_0$ is (as in the proof of Corollary 4) the minimum equilibrium x_0 when $k = \infty$. Therefore, $\Pi_2^{LP} = (\bar{p}^{LP} - c)k < \Pi_2$ since $\bar{p}^{LP} < u'(k) \leq \hat{x}_0$. It proves the result for Π_2, Π_2^{LP} when $k \in [\hat{k}, q^e)$. It completes the proof of part (b).

Compare $BS = TS - \Pi_1 - \Pi_2$ and $BS^{LP} = TS^{LP} - \Pi_1^{LP} - \Pi_2^{LP}$. When $k \in [\hat{k}, q^e)$, our previous results that $TS < TS^{LP}$, $\Pi_1 > \Pi_1^{LP}$, and $\Pi_2 > \Pi_2^{LP}$ together imply $BS < BS^{LP}$. As $k \searrow 0$, from (44), BS tends to zero but BS^{LP} is positive. Therefore, we also have $BS < BS^{LP}$ when k is small. It completes the proof of part (c).

For any k, $\bar{Q} + k > D(\bar{p})$ (see Figure 5). It, together with part (b), implies that $\bar{Q} > D(\bar{p}^{LP}) - k$ when k is small. As $k \nearrow q^e$, $D(\bar{p}^{LP}) - k$ tends to zero and $\bar{Q}$ tends to $q^e > 0$. It proves part (d). ■

References

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