Constrained-Efficient Profit Division in a Dynamic Partnership*

Vi Cao

April 14, 2018

Preliminary and Incomplete

Professional service partnerships that value collegiality often use the lock-step system to compensate their members. The lock-step system distributes profit based solely on seniority, hence it fails to reward and encourage high performance. When there are two members and each member has two productivity types, we propose a profit division mechanism that screens members and offers a bigger profit share to a member who has a higher type. The proposed mechanism satisfies *constrained efficiency*, *periodic ex-post incentive compatibility*, and *periodically ex-post Pareto dominates* the lock-step system. In addition, a high-type member collects all welfare gain from replacing the lock-step system with the *constrained efficient* mechanism. The corresponding profit division rule is implemented in Nash equilibrium by a voting mechanism, in which each member is given several menus of partnership arrangements and is asked to vote. We suggest that, in each period, each member receive a compensation package of non-equity income (fixed wage payment) and equity share (share of current net profit, which is current profit net of current wage payments). Since wage payments are drawn from profit and all resulting profit (or loss) is fully distributed, budget is always balanced. For an n-member static partnership, we propose a mechanism that satisfies *constrained efficiency*, *Bayesian incentive compatibility*, and *Bayesian Pareto dominates* the lock-step system. Our mechanisms also apply to partnerships outside professional service industries.

Keywords: dynamic partnership, profit sharing, interdependent values, adverse selection, moral hazard

*I thank my advisor, Paulo Barelli, for his guidance and encouragement throughout this research. I also thank Srihari Govindan, William Thomson, Asen Kochov, Yu Awaya, and seminar participants at Rochester for helpful discussions and comments.

1 Introduction

Partnership is a business entity formed by two or more individuals. Partnership members contribute capital and effort in the knowledge that any future profit or loss will be divided according to the partnership agreement.[1] Compared with other business structures (such as limited liability company and corporation), forming a partnership requires the least cost and paperwork. In addition, a partnership bears less tax burden than a corporation.[2,3] Due to these advantages, the partnership structure is commonly chosen by small businesses. According to U.S. Census Bureau, in 2010, partnerships account for 50% of non-employer firms[4] owned by two or more individuals. In the Survey of Law Firm Economics (2014), 60 out of 162 participants are partnerships.[5] In the Medical Group Compensation and Productivity Survey (2015), the corresponding figures are 17 and 66.[6]

In many cases, partnership members benefit from complementary strengths. A functioning business typically requires a combination of technical skills, sales expertise, and social connections. Each member is assigned a role given his strength.[7] On the other hand, a professional partnership (that provides professional services, such as litigation, accounting, consulting, and medical care) might be preferred over a solo practice for the following reasons.[8] First, each member incurs smaller financial risks as his income depends on his own performance as well as his partners'. Second, a partnership offers a collegial environment for experience sharing, referrals, and cross-selling. As a result, each member receives access to richer information and a broader client base. Third, a larger firm usually has more capital and a stronger bargaining position in the market.[9]

When a partnership is formed, a partnership agreement specifies an equity share allocation

[1] In the absence of a partnership agreement, profit or loss is divided equally among members.

[2] Most corporations are subject to double taxation: a corporate pays corporate tax, then its shareholders pay income tax on dividends. Partnerships and limited liability companies are pass-through entities: profit is taxed only once (through owners' income tax).

[3] For references on business structures, see https://www.sba.gov/ (U.S. Small Business Administration) and https://www.irs.gov/ (Internal Revenue Service).

[4] The U.S. Census Bureau defines *non-employer firm* as "one that has no paid employees, has annual business receipts of $1,000 or more ($1 or more in the construction industries), and is subject to federal income taxes".

[5] The Survey of Law Firm Economics is conducted annually by the National Law Journal and ALM Legal Intelligence.

[6] The Medical Group Compensation and Productivity Survey is conducted annually by the American Medical Group Association.

[7] See Wasserman (2012) for some discussion on complementary strengths in a partnership.

[8] Professional partnerships benefit little from complementary strengths. In a single-specialty practice, all partners perform similar duties. In a multi-specialty practice, although partners might perform different tasks, gathering multiple specialties in one practice is unnecessary for its survival.

[9] For some comparison between group practice and solo practice, see http://www.apapracticecentral.org (the APA Practice Organization for promoting and supporting practicing psychologists), https://www.acponline.org (the American College of Physicians), and https://www.healio.com/ophthalmology (Ocular Surgery News).

that dictates how future profit will be divided. For instance, a partner who has 30% of equity will receive 30% of future profit.[10,11] According to Wasserman (2012), many startups divide equity equally among founders. In the lock-step system used by professional service firms, partners with the same level of seniority receive equal equity shares (Anderson 2013). The lock-step system is particularly popular among law firms in the UK, Europe, and Australia (Wesemann and Jarrett-Kerr 2015).[12] Such equal division rule encourages collaboration (via experience sharing, referrals, and cross-selling) as each partner's income depends on the group performance. However, it fails to punish shirkers and reward hard-working members. As a result, conflicts might arise, pushing the most talented partners to join other companies that reward them better. When only the least productive members remain, the partnership would struggle to compete and survive.[13]

This paper proposes a profit division mechanism that Pareto improves upon the equal division mechanism via rewarding hard work. We study a dynamic partnership in which each member has a privately known type that affects her marginal cost of working. Types are independently distributed across members and across time. A member has a high type if she has high ability, enjoys working, or incurs low marginal opportunity cost. We assume that members gain experience through working and share work experience among themselves. Hence, in each period, all members have the same experience level, which depends on past experience and past efforts.[14] Our objective is to incentivize members to reveal their types and give a bigger equity share to a member who has a higher type, thereby maximizing members' aggregate payoff.

The proposed mechanism suggests that, in each period, each member receive a compensation package of non-equity income (a fixed wage payment) and equity share (a share of current net profit, which is current profit net of current wage payments). Since wage payments are drawn from the partnership profit and any resulting net profit (or loss) is fully distributed, budget is always balanced. We note that an equity share and a wage payment play different

[10] In a limited liability partnership, the equity share allocation might not dictate how future loss will be divided (see https://www.sba.gov/). In this paper, we assume that a partnership never incurs a loss.

[11] For a professional partnership, since individual performances are usually recorded, there are several available profit sharing rules. Instead of distributing profit based on equity shares, a partnership can rely on individual performances and some subjective criteria. See Anderson (2013) for a description of some available profit sharing rules in a professional partnership.

[12] According to Wesemann and Jarrett-Kerr (2015), in Europe, 23% of law firms use the lock-step system and another 56% of law firms use some variations of the lock-step system. These figures are 18% and 54% in the UK.

[13] For some discussion on the weaknesses of the equal division rule/ the lock-step system, see Wasserman (2012), Anderson (2013), and Henderson (2006).

[14] The assumption of experience sharing is suitable for professional single-specialty partnerships. For professional multi-specialty partnerships, we assume that each member works and contributes to broadening the client base, which benefits everyone via referrals and cross-selling. For other types of partnerships, we assume that a firm with a bigger profit can afford more advanced technologies that reduce marginal costs of labor. In other words, in any partnership, an agent's current effort lowers everyone's future marginal cost.

roles. A member's effort contribution depends on his equity share, as a bigger equity share implies higher marginal benefit of effort. On the other hand, wage payments are independent of effort contributions and are designed to incentivize members to reveal their types. In particular, giving a member some fixed wage payment can deter her from attempting to acquire a higher equity share by lying. Hence, for each member, two different compensation packages create different incentives even if they are equivalent to the same income.

The aforementioned compensation packages are part of a partnership arrangement that enters the partnership agreement in each period. A partnership arrangement specifies (enforceable) equity shares, (enforceable) wage payments, and (unenforceable) effort recommendations for all members. Our mechanism design ensures that members voluntarily follow effort recommendations given the specified profit division.[15] For a two-member static partnership in which each member has two types, a welfare-maximizing arrangement is the outcome of a voting game that proceeds as follows. Each member is given several menus of arrangements and is asked to select one arrangement from each menu. The game outcome is an arrangement (across all menus) that receives the highest number of votes. In this voting game, members reveal their types through their voting decisions.

Here we highlight some features of the model. First, an equal division mechanism (such as the lock-step system) favors lower-type members and might push higher-type members to leave the partnership. Second, equity shares and wage payments are independent of individual efforts. In many cases, individual efforts are difficult to measure. Even when individual performances can be perfectly recorded, many partnerships choose to distribute the total output based on equity shares rather than rely on individual performances so as to avoid intra-firm competition (Anderson 2013). Third, a member's valuation of his equity share depends on the partnership profit, which is determined by members' effort contributions. Effort contributions, in turn, are endogenously determined (partly) by members' private types and equity shares. Hence, our model exhibits adverse selection, moral hazard, and interdependent valuations.

Although an optimal partnership arrangement maximizes members' aggregate payoff, it is *constrained-efficient* rather than *efficient*. Holmstrom (1982) studies a complete information static game in which agents jointly contribute efforts to a team whose profit is distributed according to a pre-determined division rule.[16] He shows that if agents are risk neutral and output is deterministic, then no budget-balanced profit division rule can induce the first-best efforts in Nash equilibrium. This impossibility result is due to the free-riding problem: since each rational agent does not internalize the positive externality of his effort

[15] Including effort recommendations simplifies our theoretical analysis. From a practical perspective, these recommended efforts might assist members in their decision making. A partnership agreement also usually has a section that specifies duties among members (see https://www.sba.gov/).

[16] Profit shares depend on the team output rather than individual efforts (moral hazard).

on his partners, he fails to supply the first-best effort. Since an *efficient* arrangement (that recommends first-best efforts) cannot be enforced, the proposed mechanism suggests a *constrained-efficient* arrangement that maximizes members' aggregate payoff subject to the constraint that members voluntarily follow the recommended efforts given the assigned compensation packages.

We note that for a self-managed partnership, a profit division mechanism that involves surplus destruction typically cannot be credible. Whenever the mechanism suggests that the partnership forgo some profit, its members would be better off re-negotiating and distributing that amount among themselves. According to Holmstrom (1982), if a team is managed by an outsider, then inducing the first-best efforts is possible: the manager threatens to confiscate all surplus if the first-best output is not achieved. These results imply that an *efficient* arrangement can be enforced by a mechanism that involves surplus destruction. However, as discussed earlier, we restrict our work to budget-balanced mechanisms for the case of self-managed partnerships.

The voting game described above is derived from a direct revelation mechanism that satisfies *constrained efficiency*, *periodic ex-post incentive compatibility*, and *periodically ex-post Pareto dominates the equal division mechanism*. In each period t, members report their types and receive an arrangement that is *constrained-efficient* if these reports are truthful.[17] In each period t, even after all current private types are revealed, each member optimizes by reporting his type truthfully and following effort recommendation from period t onwards (assuming that all other members report truthfully and follow effort recommendations from period t onwards).[18] In each period t, even after all current private types are revealed, all members weakly prefer to have the proposed mechanism implemented from period t onwards rather than the equal division mechanism; in addition, the preference is strict for some member and some current type profile. Hence, as long as members prefer the equal division mechanism to quitting the partnership, they would be willing to participate in our mechanism.

Here we show how wage payments are designed to incentivize truth-telling. For illustration, we assume that, in the last period, a low-type member prefers to over-report her type if her partner has a high type and no wage payments are made.[19] For preventing such over-reporting, a member receives some wage payment x if she reports low type and her partner reports high type. A member receives no wage payment if she reports high type and her

[17]We assume that, from period $t+1$ onwards, members report their types truthfully and *constrained efficient* arrangements are assigned.

[18]Under *periodic ex-post incentive compatibility*, it is possible that after a member learns some new information in period $t+1$, he regrets reporting truthfully and following effort recommendation in period t.

[19]Here we make the following additional assumptions. First, the high-type member reports truthfully and follows effort recommendation. Second, equity shares and effort recommendations are conditional on reports according to the direct mechanism above.

partner reports low type. When members report the same types, no wage payments are made. The wage payment x ensures that a low-type member is indifferent between telling truth and lying when her partner has a high type. Due to some *monotonicity* property, a low-type member prefers to tell truth if her partner has a low type. Due to some *increasing differences* property, the wage payment x is sufficiently small so as not to affect a high-type member's reporting decision (she prefers to tell truth regardless of her partner's type and regardless of whether x is paid). Wage payments for other periods are constructed by backward recursion.

We can verify that all members weakly prefer to have the proposed mechanism implemented in the last period rather than the equal division mechanism. When members have the same types, the two mechanisms give the same outcome: members receive equal equity shares and no wage payments. When members have different types, if the low-type member lies and the high-type member tells truth, then the proposed mechanism assigns the equal division outcome. The fact that the low-type member is indifferent between lying and telling truth (shown above) implies that she receives the same payoff under either mechanism. Budget balance then implies that the high-type member collects all welfare gain from replacing the equal division mechanism by the proposed *constrained efficient* mechanism. As a result, the high-type member strictly prefers the proposed mechanism.

For an n-member static partnership, we design a direct revelation mechanism that satisfies *constrained efficiency*, *Bayesian incentive compatibility*, and *Bayesian Pareto dominates* the equal division mechanism. For each member, given her belief about her partners' private types, (a) it is optimal to report truthfully and follow effort recommendation (assuming that all other members report truthfully and follow effort recommendations), (b) the proposed mechanism is at least as good as the equal division mechanism, and (c) the proposed mechanism is better than the equal division mechanism if she has a high type. Wage payments are sufficiently large to induce low-type members to reveal their types. They are also sufficiently small so as not to affect high-type members' reporting decisions. Such wage payments exist due to the *increasing differences* property.

2 Model

There are n agents seeking to form a partnership. The partnership lasts for T periods ($T < \infty$). In each period $t \in \{1, 2, \ldots, T\}$, each agent $i \in N \equiv \{1, 2, \ldots, n\}$ has a privately known type $\theta_{i,t}$ that is distributed according to a cumulative distribution function $F_{i,t}$ with support $\Theta_{i,t}$ (types are independently distributed across agents and across periods). We assume that $\Theta_{i,t} \subset \mathbb{R}_{++}$ and $\Theta_{i,t}$ is finite. Agent i's marginal cost of working for the partnership in period t is strictly decreasing in $\theta_{i,t}$.

Each period t consists of two stages.

Stage 1. A direct revelation mechanism (described in Section 4) or a voting mechanism (described in Section 6)

(a) chooses an equity share allocation $r_t \equiv (r_{1,t}, \dots, r_{n,t}) \in \Delta^{n-1} \equiv \{(r'_{1,t}, \dots, r'_{n,t}) \in \mathbb{R}^n_+ \mid \sum_{i \in N} r'_{i,t} = 1\}$, a wage budget $m_{0,t} \in \mathbb{R}_+$, and individual wages $(m_{1,t}, \dots, m_{n,t}) \in \mathbb{R}^n_+$ such that $m_{0,t} \geq \sum_{i \in N} m_{i,t}$ (the equity share allocation, wage budget, and individual wages are enforceable),

(b) recommends effort contributions $(e^r_{1,t}, \dots, e^r_{n,t}) \in \mathbb{R}^n_+$ (recommended efforts are unenforceable).

Stage 2. Agents simultaneously invest efforts $e_t \equiv (e_{1,t}, \dots, e_{n,t}) \in \mathbb{R}^n_+$ in the partnership. The cost of effort for each agent i is $c_{i,t}(e_{i,t}, \theta_{0,t}, \theta_{i,t}) = e^2_{i,t}/(2\theta_{0,t}\theta_{i,t})$, where $\theta_{0,t} \in \mathbb{R}_{++}$ is a publicly observed state. The partnership profit is $\pi(e_t) = \sum_{i \in N} e_{i,t}$. At the end of period t, each agent i receives his wage $m_{i,t}$ and net profit $\left[\pi(e_t) - m_{0,t}\right]$ is divided among partners according to the equity share allocation $r_t \equiv (r_{1,t}, \dots, r_{n,t})$. We note that the partnership treats wage budget $m_{0,t}$ as an expense.

We interpret the publicly observed state $\theta_{0,t}$ as the partnership's level of experience at the beginning of period t. Period-1 experience $\theta_{0,1}$ takes some given positive real value. Agents gain experience through working and share work experience among themselves. Hence, for each period $t > 1$, the current experience level $\theta_{0,t}$ depends on past experience $\theta_{0,t-1}$ and effort contributions e_{t-1} as follows: $\theta_{0,t} = \gamma\left(\theta_{0,t-1} + \sum_{i \in N} e_{i,t-1}\right)$, where $\gamma \in (0, 1)$. We can write $\theta_{0,t} = \gamma^{t-1}\theta_{0,1} + \sum_{\tau=1}^{t-1} \gamma^{t-\tau} \sum_{i \in N} e_{i,\tau}$. Since $\gamma \in (0, 1)$, more recent efforts contribute more to forming the current experience (as agents gradually forget what they learn farther back). Each agent i's marginal cost of effort $\partial c_{i,t}(e_{i,t}, \theta_{0,t}, \theta_{i,t})/\partial e_{i,t}$ is decreasing in the experience level $\theta_{0,t}$.

We refer to $\theta_t \equiv (\theta_{0,t}, \theta_{1,t}, \dots, \theta_{n,t})$ as the state of the partnership in period t, which consists of a publicly observed state $\theta_{0,t}$ and an unobservable state $\theta_{-0,t} \equiv (\theta_{1,t}, \dots, \theta_{n,t})$. Let $\Theta_t \equiv \mathbb{R}_{++} \times \prod_{i=1}^n \Theta_{i,t}$ be the period-t state space. In addition, let $\Theta_{-0,t} \equiv \prod_{i=1}^n \Theta_{i,t}$ and $\Theta_{-i,t} \equiv \mathbb{R}_{++} \times \prod_{j \neq i} \Theta_{j,t}$ for each $i \in N$.

Define $m_t \equiv (m_{0,t}, m_{1,t}, \dots, m_{n,t})$. Period-$t$ payoff of agent i is

$$u_{i,t}(r_t, m_t, e_t, \theta_{0,t}, \theta_{i,t}) \equiv r_{i,t}\left[\pi(e_t) - m_{0,t}\right] + m_{i,t} - c_{i,t}(e_{i,t}, \theta_{0,t}, \theta_{i,t}).$$

Period-t aggregate payoff is $u_t(r_t, m_t, e_t, \theta_t) \equiv \sum_{i \in N} u_{i,t}(r_t, m_t, e_t, \theta_{0,t}, \theta_{i,t})$.

Agents have a common discount factor $\delta \in (0, 1)$. The total payoff of each agent i is

$$\sum_{t=1}^{T} \delta^{t-1} u_{i,t}(r_t, m_t, e_t, \theta_{0,t}, \theta_{i,t}).$$

For notational convenience, we let $\tilde{u}_{i,t}(r_t, e_t, \theta_{0,t}, \theta_{i,t}) \equiv r_{i,t}\pi(e_t) - c_{i,t}(e_{i,t}, \theta_{0,t}, \theta_{i,t})$ be agent i's period-t payoff in the absence of wage budgets and wages. In addition, we let $\tilde{u}_t(r_t, e_t, \theta_t) \equiv \sum_{i \in N} \tilde{u}_{i,t}(r_t, e_t, \theta_{0,t}, \theta_{i,t})$ be the total surplus created in period t.

3 Constrained-Efficient Arrangement Rule

In this section, we define a *constrained-efficient* arrangement rule that assigns to each type profile a welfare-maximizing arrangement, which consists of a compensation plan and effort recommendations. This arrangement respects agents' effort choices: given the assigned compensation plan, it is in each agent's interest to follow the effort recommendation. The welfare-maximizing arrangement gives the highest aggregate payoff among all arrangements that respect agents' effort choices. We note that recommended efforts are not first-best; hence, such an arrangement is not *efficient*. We show that a *constrained-efficient* arrangement rule exists and provide some characterizations. In addition, we define an arrangement rule that Bayesian Pareto dominates the equal division rule (which is equivalent to the lock-step system in professional partnerships when partners have the same level of seniority).

3.1 Arrangement Rule

An arrangement rule is a triple $\alpha \equiv (s, w, \sigma)$, where

- $s \equiv \left\{s_t : \Theta_t \to \Delta^{n-1}\right\}_{t=1}^{T}$ is an equity sharing rule,
- $w \equiv \left\{w_t : \Theta_t \to \mathbb{R}_+^n\right\}_{t=1}^{T}$ is a wage rule, and
- $\sigma \equiv \left\{\sigma_t : \Theta_t \to \mathbb{R}_+^n\right\}_{t=1}^{T}$ is an effort recommendation rule.

For each period $t \in \{1, \ldots, T\}$, we refer to s_t, w_t, σ_t as period$-t$ equity sharing rule, period$-t$ wage rule, period$-t$ effort recommendation rule respectively, and refer to $\alpha_t \equiv (s_t, w_t, \sigma_t)$ as period$-t$ arrangement rule. For each state $\theta_t \in \Theta_t$, the period$-t$ arrangement rule α_t assigns equity shares $s_t(\theta_t) \equiv [s_{1,t}(\theta_t), \ldots, s_{n,t}(\theta_t)] \in \Delta^{n-1}$, wages $w_t(\theta_t) \equiv [w_{1,t}(\theta_t), \ldots, w_{n,t}(\theta_t)] \in \mathbb{R}_+^n$, and recommended efforts $\sigma_t(\theta_t) \equiv [\sigma_{1,t}(\theta_t), \ldots, \sigma_{n,t}(\theta_t)] \in \mathbb{R}_+^n$.

Let $\mathscr{A} \equiv \Delta^{n-1} \times \mathbb{R}_+^n \times \mathbb{R}_+^n$ be the set of feasible period-t arrangement rules. For each $t \in \{1, \ldots, T\}$, let $\mathcal{A}_t \equiv \{(\alpha_t, \ldots, \alpha_T) \mid \alpha_\tau : \Theta_\tau \to \mathscr{A}$ for each $\tau = t, \ldots T\}$ be the set of feasible sequences of arrangement rules.

3.2 Efficiency

Here we define *efficiency*, which will be used as a benchmark to define *constrained efficiency* in Section 3.3.

Definition 3.1 An arrangement rule $\alpha \equiv (s, w, \sigma)$ is **efficient** if for each $t \in \{1, \dots, T\}$ and each $\theta_t \in \Theta_t$,

$$\{\alpha_\tau\}_{\tau \geq t} \in \arg \max_{\{\alpha'_\tau\}_{\tau \geq t}} \mathbb{E}_{(\theta_{-0,t}, \dots, \theta_{-0,T})} \Big[\sum_{\tau=t}^{T} \delta^{\tau - t} u_\tau \big[\alpha'(\theta_\tau), \theta_\tau\big] \mid \theta_t \Big],$$

where for each $(\theta_{-0,t}, \dots, \theta_{-0,T}) \in \prod_{\tau \geq t} \Theta_{-0,\tau}$ and each $\tau \in \{t+1, \dots, T\}$.

$$\theta_{0,\tau} = \gamma \Big[\theta_{0,\tau-1} + \sum_{i \in N} \sigma'_{i,\tau-1}(\theta_{\tau-1})\Big].$$

If (s, σ) is an **efficient** equity-effort rule, then for each period t and each period-t state θ_t, the sequence of equity-effort rules $\{(s_\tau, \sigma_\tau)\}_{\tau=t}^{T}$ (applied from period t onwards) maximizes the expected sum of discounted aggregate payoffs conditional on θ_t.

3.3 Constrained Efficiency

We define *constrained efficiency* for a static partnership, then generalize this notion for a dynamic partnership. We suppress subscript t when discussing a static partnership. For each $\theta \in \Theta$, let

$$\mathcal{R}(\theta) \equiv \{(r, e) \in \Delta^{n-1} \times \mathbb{R}^n_+ \mid e_i \in \arg \max_{e'_i} \tilde{u}_i(r, e'_i, e_{-i}, \theta_0, \theta_i) \text{ for each } i \in N\}$$

be the set of equity-effort arrangements that respect agents' effort choices: for each $(r, e) \in \mathcal{R}(\theta)$ and each agent $i \in N$, conditional on the equity share allocation r, if the other agents follow effort recommendations e_{-i}, then agent i's optimal effort level is the recommended effort e_i. An equity-effort rule (s, σ) is *constrained-efficient* if for each $\theta \in \Theta$, we have $[s(\theta), \sigma(\theta)] \in \text{argmax}_{(r,e) \in \mathcal{R}(\theta)} \sum_{i \in N} \tilde{u}_i(r, e, \theta_0, \theta_i)$. That is, $[s(\theta), \sigma(\theta)]$ gives the highest aggregate payoff among all equity-effort arrangements that respect agents' effort choices.

We generalize this notion of *constrained efficiency* for a dynamic partnership as follows. For each $t \in \{1, \dots, T\}$ and each $\theta_t \in \Theta_t$, let $\mathcal{R}_t(\theta_t)$ be the collection of $\{s_\tau, w_\tau, \sigma_\tau\}_{\tau \geq t}$ such that for each $i \in N$, the sequence $\{\sigma_{i,\tau}\}_{\tau \geq t}$ solves

$$\max_{\{\sigma'_{i,\tau}\}_{\tau \geq t}} \mathbb{E}_{(\theta_{-0,t}, \dots, \theta_{-0,T})} \Big[\sum_{\tau=t}^{T} \delta^{\tau - t} u_{i,\tau} \big[s_\tau(\theta_\tau), w_\tau(\theta_\tau), \sigma'_{i,\tau}(\theta_\tau), \sigma_{-i,\tau}(\theta_\tau), \theta_{0,\tau}, \theta_{i,\tau}\big] \mid \theta_t \Big]$$

where for each $(\theta_{-0,t}, \ldots, \theta_{-0,T}) \in \prod_{\tau \geq t} \Theta_{-0,\tau}$ and each $\tau \in \{t+1, \ldots, T\}$,

$$\theta_{0,\tau} = \gamma\Big[\theta_{0,\tau-1} + \sigma'_{i,\tau-1}(\theta_{\tau-1}) + \sum_{j \neq i} \sigma_{j,\tau-1}(\theta_{\tau-1})\Big].$$

be the set of equity-effort rules (applied from period t onwards) that respect agents' effort choices: for each $\{(s_\tau, \sigma_\tau)\}_{\tau=t}^T \in \mathcal{R}_t(\theta_t)$ and each agent $i \in N$, conditional on the equity sharing rule $\{s_\tau\}_{\tau=t}^T$, if the other agents follow the effort recommendation rule $\{\sigma_{-i,\tau}\}_{\tau=t}^T$, then $\{\sigma_{i,\tau}\}_{\tau=t}^T$ maximizes agent i's expected sum of discounted payoffs.

Definition 3.2 An arrangement rule α is **constrained-efficient** if for each $t \in \{1, \ldots, T\}$ and each $\theta_t \in \Theta_t$,

$$\{\alpha_\tau\}_{\tau \geq t} \in \arg \max_{\{\alpha'_\tau\}_{\tau \geq t} \in \mathcal{R}_t(\theta_t)} \mathbb{E}_{(\theta_{-0,t}, \ldots, \theta_{-0,T})}\Big[\sum_{\tau=t}^{T} \delta^{\tau-t} u_\tau\big[\alpha'(\theta_\tau), \theta_\tau\big] \mid \theta_t\Big],$$

where for each $(\theta_{-0,t}, \ldots, \theta_{-0,T}) \in \prod_{\tau \geq t} \Theta_{-0,\tau}$ and each $\tau \in \{t+1, \ldots, T\}$.

$$\theta_{0,\tau} = \gamma\Big[\theta_{0,\tau-1} + \sum_{i \in N} \sigma'_{i,\tau-1}(\theta_{\tau-1})\Big].$$

That is, for each period t and each state θ_t, the equity-effort rule $\{(s_\tau, \sigma_\tau)\}_{\tau \geq t}$ (applied from period t onwards) maximizes the expected sum of discounted aggregate payoff among all equity-effort rules that respect agents' effort choices.

Recursive Form. Since period-t aggregate payoff depends only on period-t state (which is assumed to be Markov), the maximization problem in Definition 3.2 can be written in recursive form. Fix an equity-effort rule (s, σ). Fix an agent i and assume all other agents follow the effort recommendation rule σ. For each $t \in \{1, \ldots, T\}$ and each $\theta_t \in \Theta_t$, let

$$W_{i,t}(\theta_t, \alpha) \equiv \max_{e'_{i,t} \in \mathbb{R}_+} u_{i,t}\big[s_t(\theta_t), w_t(\theta_t), e'_{i,t}, \sigma_{-i,t}(\theta_t), \theta_{0,t}, \theta_{i,t}\big] + \delta \mathbb{E}\big[W_{i,t+1}(\theta_{t+1}, \alpha) | e'_{i,t}, \sigma_{-i,t}(\theta_t)\big].$$

be agent i's maximum expected payoff (starting from period t) conditional on the state θ_t.

For each $t \in \{1, \ldots, T\}$ and each $\theta_t \in \Theta_t$, the set of period-t equity-effort arrangements that respects agents' effort choices is

$$\tilde{\mathcal{R}}_t(\theta_t, \alpha) \equiv \Big\{(r_t, m_t, e_t) \in \Delta^{n-1} \times \mathbb{R}_+^n \times \mathbb{R}_+^n \mid \text{ for each } i \in N, \\ e_{i,t} \in \arg \max_{e'_{i,t} \in \mathbb{R}_+} u_{i,t}(r_t, m_t, e'_{i,t}, e_{-i,t}, \theta_{0,t}, \theta_{i,t}) + \delta \mathbb{E}\big[W_{i,t+1}(\theta_{t+1}, \alpha) \mid e'_{i,t}, e_{-i,t}\big]\Big\}.$$

Each equity-effort arrangement $(r_t, e_t) \in \tilde{\mathcal{R}}_t(\theta_t)$ satisfies the following property: for each agent i, given the assigned equity share allocation r_t, if all other agents follow the effort recommendations $e_{-i,t}$, then agent i's optimal effort level is $e_{i,t}$ (assuming that he optimizes from period $t+1$ onwards conditional on the equity sharing rule $\{s_\tau\}_{\tau>t}$ and the effort recommendation rule $\{\sigma_{-i,\tau}\}_{\tau>t}$ for other agents).

For each $t \in \{1, \ldots, T\}$ and each $\theta_t \in \Theta_t$, let

$$W_t(\theta_t, \alpha) \equiv \max_{(r_t, m_t, e_t) \in \tilde{\mathcal{R}}_t(\theta_t, \alpha)} \Big\{ u_t(r_t, m_t, e_t, \theta_t) + \delta \mathbb{E}\big[W_{t+1}(\theta_{t+1}, \alpha) \mid e_t\big] \Big\}.$$

be the maximum expected aggregate payoff (starting from period t) conditional on θ_t. By the principle of optimality, an equity-effort rule (s, σ) is *constrained-efficient* if and only if it is a solution to this recursive problem, as stated in the following lemma:

An arrangement rule α is *constrained-efficient* if and only if for each $t \in \{1, \ldots, T\}$ and each $\theta_t \in \Theta_t$,

$$\alpha_t(\theta_t) \in \arg \max_{(r_t, m_t, e_t) \in \tilde{\mathcal{R}}_t(\theta_t, \alpha)} \Big\{ u_t(r_t, m_t, e_t, \theta_t) + \delta \mathbb{E}\big[W_{t+1}(\theta_{t+1}, \alpha) \mid e_t\big] \Big\}.$$

The remaining of this section provides some characterizations of a *constrained-efficient* equity-effort rule for a static partnership.

3.4 Recommended Efforts

The following lemma claims that for each state θ and each equity share allocation r, there is a unique effort profile e such that the equity-effort arrangement (r, e) respects agents' effort choices.

Lemma 3.1 Fix $t \in \{1, \ldots, T\}$ and an arrangement rule α. Suppose

(a) $\{\alpha_\tau\}_{\tau>t}$ is *constrained-efficient*, and

(b) for each pair $\{i, j\} \subseteq N$ (not necessarily distinct), we have $\dfrac{\partial \mathbb{E}\big[W_{i,\tau}(\theta_\tau, \alpha) \mid \big]}{\partial}$.

a *constrained-efficient* equity-effort rule $(s^*_\tau, \sigma^*_\tau)_{\tau>t}$. For each $\theta_t \in \Theta_t$ and each $r_t \in \Delta^{n-1}$, there is a unique effort profile $e_t \equiv \tilde{\sigma}_t(\theta_t, r_t) \in \mathbb{R}^n_+$ such that $(r_t, e_t) \in \tilde{\mathcal{R}}_t(\theta_t)$.

By Lemma 3.1, a period$-t$ equity-effort rule (s^*_t, σ^*_t) is *constrained-efficient* if and only if, for each $\theta_t \in \Theta_t$, we have $\sigma^*_t(\theta_t) = \tilde{\sigma}_t\big[\theta_t, s^*_t(\theta_t)\big]$ and $s^*_t(\theta_t)$ solves

$$\max_{r_t \in \Delta^{n-1}} \sum_{i \in N} \Big(\tilde{u}_{i,t}\big[r_t, \tilde{\sigma}_t(\theta_t, r_t), \theta_{0,t}, \theta_{i,t}\big] + \delta \mathbb{E}\big[W_{i,t+1}(\theta_{t+1}) | \theta_t, \tilde{\sigma}_t(\theta_t, r_t)\big] \Big).$$

Hence, solving for a *constrained-efficient* equity-effort rule is reduced to finding an equity sharing rule that solves the maximization problem $(*)$.

The following lemma provides some characterizations of equity-effort arrangement rules that respect agents' effort choices. For each state θ, such rule assigns an equity-effort arrangement $(r, e) \in \tilde{\mathcal{R}}(\theta)$, where $e = \big[\tilde{\sigma}_i(\theta_0, \theta_i, r_i)\big]_{i \in N}$.

Lemma 3.2 For each $i \in N$, each $t \in \{1, \dots, T\}$, each $\theta_t \in \Theta_t$, and each $r_t \in \Delta^{n-1}$,

(a) $\dfrac{\partial \tilde{\sigma}_{i,t}(\theta_t, r_t)}{\partial r_{i,t}} > 0$ and $\dfrac{\partial^2 \tilde{\sigma}_{i,t}(\theta_t, r_t)}{\partial^2 r_{i,t}} = 0$,

(b) $\tilde{\sigma}_{i,t}(\theta_t, r_t) > \tilde{\sigma}_{i,t}(\theta'_{i,t}, \theta_{-i,t}, r_t)$ for $\theta_{i,t} > \theta'_{i,t}$, and

(c) $\dfrac{\partial \tilde{\sigma}_{i,t}(\theta_t, r_t)}{\partial r_{i,t}} > \dfrac{\partial \tilde{\sigma}_{i,t}(\theta'_{i,t}, \theta_{-i,t}, r_t)}{\partial r_{i,t}}$ for $\theta_{i,t} > \theta'_{i,t}$.

That is, an agent's recommended effort is strictly increasing in his equity share at a constant rate; and this rate is higher for a higher type. In addition, an agent's recommended effort is strictly increasing in his type.

3.5 Equity Shares

The following proposition claims that a *constrained-efficient* equity-effort rule exists and is unique. This rule specifies that an agent's equity share is increasing in his type.

Proposition 3.1

(a) For each $t \in \{1, \dots, T\}$, there exists a unique *constrained-efficient* equity-effort rule $(s^*_\tau, \sigma^*_\tau)_{\tau=t}^T$.

(b) For each $i \in N$, each $\tau \in \{t, \dots, T\}$, each $\theta_\tau \in \Theta_\tau$, and each $\theta'_{i,\tau} > \theta_{i,\tau}$, we have $s^*_{i,\tau}(\theta'_{i,\tau}, \theta_{-i,\tau}) \geq s^*_{i,\tau}(\theta_\tau)$.

For each $\theta \in \Theta$ and each $i \in N$, let $\sigma^*_i(\theta) = \tilde{\sigma}_i\big[\theta_0, \theta_i, s^*_i(\theta)\big]$. Then (s^*, σ^*) is a *constrained-efficient* equity-effort rule. It can be show that s^* always assigns an equity share of above $1/n$ to a high-type agent and an equity share of below $1/n$ to a low-type agent.

4 Direct Revelation Mechanism

In this section, we design a *constrained-efficient* direct revelation mechanism, in which agents report their types.

A **direct revelation mechanism** is a tuple $\mathcal{D} \equiv \big\{\Theta_{-0,t}, \alpha_t\big\}_{t=1}^{T}$, where for each period t,

(a) $\Theta_{i,t}$ is the message space for agent i, and

(b) $\alpha_t : \Theta_t \to \mathscr{A}$ is a period$-t$ arrangement rule.

We say that $\mathcal{D}$ is *constrained efficient* if α is *constrained efficient.* Likewise, we say that $\mathcal{D}$ *Bayesian Pareto dominates the equal division mechanism* if α *Bayesian Pareto dominates the equal division rule.*

4.1 Incentive Compatibility

Here we define another desirable property of a direct revelation mechanism. In order to assign *constrained-efficient* arrangements, agents must be incentivised to reveal their types. We say that a direct revelation mechanism $\mathcal{D}$ is *Bayesian incentive compatible* if each agent prefers to report his type truthfully and follow the effort recommendation given his belief about the other agents' types.

Definition 4.1 A direct revelation mechanism $\mathcal{D} \equiv \big\{(\Theta_1, \ldots, \Theta_n), (s, w, \sigma)\big\}$ is **Bayesian incentive compatible** if for each $i \in N$ and each $\theta_i \in \Theta_i$,

$$(\theta_i, \sigma_i(\theta)) \in \arg\max_{(\hat{\theta}_i, e_i)} \mathbb{E}u_i\big[s(\hat{\theta}_i, \theta_{-i}), w_t(\hat{\theta}_i, \theta_{-i}), e_i, \sigma_{-i}(\hat{\theta}_i, \theta_{-i}), \theta_0, \theta_i\big].$$

We say that a direct revelation mechanism $\mathcal{D}$ is *ex post incentive compatible* if each agent prefers to report his type truthfully and follow the effort recommendation regardless other agents' types (as long as the other agents follow effort recommendations). Such a mechanism design is independent of the type distribution and agents' beliefs.

Definition 4.2 A direct revelation mechanism $\mathcal{D} \equiv \big\{(\Theta_1, \ldots, \Theta_n), (s, w, \sigma)\big\}$ is **ex post incentive compatible** if for each $i \in N$ and each $\theta \in \Theta$,

$$(\theta_i, \sigma_i(\theta)) \in \arg\max_{(\hat{\theta}_i, e_i)} u_i\big[s(\hat{\theta}_i, \theta_{-i}), w_t(\hat{\theta}_i, \theta_{-i}), e_i, \sigma_{-i}(\hat{\theta}_i, \theta_{-i}), \theta_0, \theta_i\big].$$

We generalize the *ex post incentive compatibility* property for a dynamic partnership as follows. Fix an arrangement rule $\alpha \equiv \big\{(s_t, w_t, \sigma_t)\big\}_{t=1}^{T}$. For each $i \in N$, each $t \in \{1, \ldots, T\}$, and each $\theta_t \in \Theta_t$, let

$$V_{i,t}(\theta_t) \equiv \max_{(\hat{\theta}_{i,t}, e_{i,t})} u_{i,t}\big[s_t(\hat{\theta}_{i,t}, \theta_{-i,t}), w_t(\hat{\theta}_{i,t}, \theta_{-i,t}), e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_{0,t}, \theta_{i,t}\big]$$

$$+\delta \mathbb{E}\big[V_{i,t+1}(\theta_{t+1}) \mid \theta_t, e_{i,t}, \sigma_{-i,t}\big(\hat{\theta}_{i,t}, \theta_{-i,t}\big)\big]^{20}$$

be agent i's maximum expected payoff (starting from period t) conditional on the current state θ_t (assuming that all other agents always follow effort recommendations).

Definition 4.3 A direct revelation mechanism $\mathcal{D} \equiv \big\{\Theta_{-0,t}, s_t, w_t, \sigma_t\big\}_{t=1}^T$ is **periodic ex-post incentive compatible** if for each $i \in N$, each $t \in \{1, \ldots, T\}$, and each $\theta_t \in \Theta_t$,

$$\big[\theta_{i,t}, \sigma_{i,t}(\theta_t)\big] \in \arg \max_{(\hat{\theta}_{i,t}, e_{i,t})} u_{i,t}\big[s_t(\hat{\theta}_{i,t}, \theta_{-i,t}), w_t(\hat{\theta}_{i,t}, \theta_{-i,t}), e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_{0,t}, \theta_{i,t}\big]$$

$$+\delta \mathbb{E}\big[V_{i,t+1}(\theta_{t+1}) \mid e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t})\big].$$

That is, for each agent and in each period, truth-telling is always a best response regardless of the current state, as long as other agents always follow effort recommendations.

4.2 Equal Division Mechanism

An **equal division mechanism** is a direct revelation mechanism $\mathcal{D}^e \equiv \big\{\Theta_{-0,t}, s_t^e, w_t^e, \sigma_t^e\big\}_{t=1}^T$, where for each $t \in \{1, \ldots, T\}$ and each $\theta_t \in \Theta_t$,

$$s_t^e(\theta_t) = (1/n, \ldots, 1/n) \qquad w_t^e(\theta_t) = (0, \ldots, 0) \qquad \big[s_t^e(\theta_t), \sigma_t^e(\theta_t)\big] \in \tilde{\mathcal{R}}_t\big[\theta_t, (s_\tau^e, \sigma_\tau^e)_{\tau > t}\big].$$

Let $\alpha_t^e \equiv (s_t^e, w_t^e, \sigma_t^e)$ for each $t \in \{1, \ldots, T\}$ and $\alpha^e \equiv (\alpha_1^e, \ldots, \alpha_T^e)$.

For each $i \in N$, each $(\theta_{-0,1}, \ldots, \theta_{-0,T}) \in \prod_{t=1}^T \Theta_{-0,t}$, and each arrangement rule $\alpha \equiv (\alpha_1, \ldots, \alpha_T) \equiv \big\{s_t, w_t, \sigma_t\big\}_{t=1}^T$, define

$$\mathcal{U}_i(\theta_{-0,1}, \ldots, \theta_{-0,T}, \alpha) \equiv \sum_{t=1}^T \delta^{t-1} u_{i,t}\big[\alpha_t(\theta_t), \theta_{0,t}, \theta_{i,t}\big],$$

where for each $t \in \{2, \ldots, T\}$, we have $\theta_{0,t} = \gamma\Big[\theta_{0,t-1} + \sum_{j \in N} \sigma_{j,t-1}(\theta_t)\Big]$.

Definition 4.4 Let $\mathcal{D} \equiv \big\{\Theta_{-0,t}, \alpha_t\big\}_{t=1}^T$ be an *incentive compatible* direct revelation mechanism. We say that $\mathcal{D}$ **ex-post Pareto dominates** the equal division mechanism $\mathcal{D}^e \equiv \big\{\Theta_{-0,t}, \alpha_t^e\big\}_{t=1}^T$ if

(a) for each $i \in N$ and each $(\theta_{-0,1}, \ldots, \theta_{-0,T}) \in \prod_{t=1}^T \Theta_{-0,t}$,

[20] $\mathbb{E}\big[V_{i,t+1}(\theta_{t+1}) \mid \theta_t, e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t})\big] \equiv \sum_{\theta_{t+1} \in \Theta_{t+1}} \mu_{t+1}[\theta_{t+1} | \theta_t, e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t})] \times V_{i,t+1}(\theta_{t+1}).$

$$\mathcal{U}_i(\theta_{-0,1},\ldots,\theta_{-0,T},\alpha) \geq \mathcal{U}_i(\theta_{-0,1},\ldots,\theta_{-0,T},\alpha^e),$$

(b) there exists some $i \in N$ and some $(\theta_{-0,1},\ldots,\theta_{-0,T}) \in \prod_{t=1}^{T}\Theta_{-0,t}$ such that

$$\mathcal{U}_i(\theta_{-0,1},\ldots,\theta_{-0,T},\alpha) > \mathcal{U}_i(\theta_{-0,1},\ldots,\theta_{-0,T},\alpha^e).$$

That is, everyone always prefers the arrangement rule (s,w,σ). In addition, there is a state θ at which some agent i strictly prefers (s,w,σ).

4.3 Some Preliminary Results

We discuss some properties of the payoff structure that allow us to design a direct revelation mechanism that satisfies *constrained efficiency*, *Bayesian/ex post incentive compatibility*, and *Bayesian Pareto dominates the equal division mechanism.*

Let (s^*,σ^*) be the *constrained-efficient* equity-effort rule in Section 3. For each $i \in N$, each $t \in \{1,\ldots,T\}$, each $\theta_t \in \Theta_t$, and each $\hat{\theta}_{i,t} \in \Theta_{i,t}$, let

$$\tilde{W}_{i,t}(\theta_t;\hat{\theta}_{i,t}) \equiv \max_{e_{i,t}\in\mathbb{R}_+} \tilde{u}_{i,t}\big[s_t^*(\hat{\theta}_{i,t},\theta_{-i,t}), e_{i,t}, \sigma^*_{-i,t}(\hat{\theta}_{i,t},\theta_{-i,t}), \theta_{0,t}, \theta_{i,t}\big] + \delta\mathbb{E}\big[\tilde{W}_{i,t+1}(\theta_{t+1};\theta_{i,t+1}) \mid \theta_t, e_{i,t}, \sigma^*_{-i,t}(\hat{\theta}_{i,t},\theta_{-i,t})\big]$$ [21]

be agent i's maximum expected payoff (starting from period t) once he reports $\hat{\theta}_{i,t}$, conditional on the true state θ_t (assuming that all other agents always report truthfully and follow effort recommendations).

Lemma 4.1 For each $i \in N$, each $t \in \{1,\ldots,T\}$, each $\theta_t \in \Theta_t$, and each $\hat{\theta}_{i,t} \in \Theta_{i,t}$,

(a) there is a unique solution $\tilde{\sigma}_{i,t}\big[\theta_t, s_t^*(\hat{\theta}_{i,t},\theta_{-i,t})\big]$ to the following maximization problem:

$$\max_{e_{i,t}\in\mathbb{R}_+} \tilde{u}_{i,t}\big[s_t^*(\hat{\theta}_{i,t},\theta_{-i,t}), e_{i,t}, \sigma^*_{-i,t}(\hat{\theta}_{i,t},\theta_{-i,t}), \theta_{0,t}, \theta_{i,t}\big] + \delta\mathbb{E}\big[W_{i,t+1}(\theta_{t+1}) \mid e_{i,t}, \sigma^*_{-i,t}(\hat{\theta}_{i,t},\theta_{-i,t})\big] \quad (*)$$

(b) $\tilde{\sigma}_{i,t}\big[\theta_t, s_t^*(\hat{\theta}_{i,t},\theta_{-i,t})\big] > \tilde{\sigma}_{i,t}\big[\theta'_{i,t},\theta_{-i,t}, s_t^*(\hat{\theta}_{i,t},\theta_{-i,t})\big]$ for $\theta'_{i,t} < \theta_{i,t}$.

For each $i \in N$, each $t \in \{1,\ldots,T\}$, and each $\theta_t \in \Theta_t$, define $s_t^e(\theta_t) \equiv (1/n,\ldots,1/n)$ and

[21] $\mathbb{E}\big[\tilde{W}_{i,t+1}\big(\theta_{t+1},\theta_{i,t+1},\{(s_\tau,\sigma_\tau)\}_{\tau=t+1}^{T}\big) \mid \theta_t, e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t},\theta_{-i,t})\big]$

$$\equiv \sum_{\theta_{t+1}\in\Theta_{t+1}} \mu_{t+1}[\theta_{t+1}|\theta_t, e_{i,t}, \sigma_{-i,t}(\hat{\theta}_{i,t},\theta_{-i,t})] \times \tilde{W}_{i,t+1}\big(\theta_{t+1},\theta_{i,t+1},\{(s_\tau,\sigma_\tau)\}_{\tau=t+1}^{T}\big).$$

$$W^e_{i,t}(\theta_t) = \tilde{u}_{i,t}\big(s^e_t(\theta_t), \tilde{\sigma}_t[\theta_t, s^e_t(\theta_t)], \theta_{0,t}, \theta_{i,t}\big) + \delta\mathbb{E}\big(W_{i,t+1}(\theta_{t+1}) \mid \tilde{\sigma}_t[\theta_t, s^e_t(\theta_t)]\big).$$

Lemma 4.2 (Increasing Differences) For each $i \in N$, each $t \in \{1, \dots, T\}$, each $\theta_t \in \Theta_t$, each $\theta'_{i,t} < \theta_{i,t}$, and each $\hat{\theta}_{i,t} \in \Theta_{i,t}$,

$$\tilde{W}_{i,t}(\theta_t; \hat{\theta}_{i,t}) - W^e_{i,t}(\theta_t) \geq \tilde{W}_{i,t}(\theta'_{i,t}, \theta_{-i,t}; \hat{\theta}_{i,t}) - W^e_{i,t}(\theta'_{i,t}, \theta_{-i,t})$$

if and only if $s^*_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) \geq 1/n$.

Increasing differences ensures that a higher type gains more from over-reporting his type. When there are two agents, the payoff structure satisfies an additional property (stated in Lemma 4.2) that allows us to design a *constrained-efficient* mechanism that is *ex post incentive compatible* and *Pareto dominates the equal division mechanism.*

5 A Direct Mechanism Design

In this section, we present a direct revelation mechanism that satisfies *constrained efficiency*, *Bayesian incentive compatibility*, and *Bayesian Pareto dominates the equal division mechanism.* Let (s^*, σ^*) be a *constrained-efficient* equity-effort rule characterized in Section 3. We construct a wage rule w^* such that the mechanism $\mathcal{D}^* \equiv \{(\Theta_1, \dots, \Theta_n), (s^*, w^*, \sigma^*)\}$ satisfies the aforementioned properties.

5.1 Two Members

Lemma 5.1 For each $i \in N$, each $t \in \{1, \dots, T\}$, and each $\alpha \in \mathcal{A}$,

$$\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_H; \alpha) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha) \geq \tilde{W}_{i,t}(\theta_L, \theta_L; \theta_H; \alpha) - \tilde{W}_{i,t}(\theta_L, \theta_L; \theta_L; \alpha).$$

Step 1. Let $t = T$. Fix an arbitrary arrangement rule $\alpha \equiv (s, w, \sigma)$. By Proposition 3.1, there is a unique period$-t$ equity-effort rule (s^*_t, σ^*_t) such that for each $\theta_t \in \Theta_t$,

$$\big[s^*_t(\theta_t), \sigma^*_t(\theta_t)\big] \in \arg \max_{(r_t, e_t) \in \tilde{\mathcal{R}}_t(\theta_t, \alpha)} \tilde{u}_t(r_t, e_t, \theta_t) + \delta\mathbb{E}\big[W_{t+1}(\theta_{t+1}, \alpha) \mid e_t\big].$$

Let α' be such that $\alpha'_t = (s^*_t, w_t, \sigma^*_t)$ and $\alpha'_\tau = \alpha_\tau$ for each $\tau \neq t$. We construct w^*_t as follows. Fix $\theta_t \in \Theta_t$. If $\theta_{-0,t} = (\theta_L, \theta_L)$ or $\theta_{-0,t} = (\theta_H, \theta_H)$, let $w^*_{i,t}(\theta_t) = 0$ for each $i \in N$. Otherwise, let i be the agent who has type θ_L and j be the agent who has type θ_H.

If $\tilde{W}^e_{i,t}(\theta_t) \geq \tilde{W}_{i,t}(\theta_t; \theta_L; \alpha')$, let $w^*_{j,t}(\theta_t) = 0$ and $w^*_{i,t}(\theta_t) = \dfrac{\tilde{W}^e_{i,t}(\theta_t) - \tilde{W}_{i,t}(\theta_t; \theta_L; \alpha')}{s^*_{j,t}(\theta_t)}.$

If $\tilde{W}^e_{i,t}(\theta_t) < \tilde{W}_{i,t}(\theta_t;\theta_L;\alpha')$, let $w^*_{i,t}(\theta_t) = 0$ and $w^*_{j,t}(\theta_t) = \dfrac{\tilde{W}_{i,t}(\theta_t;\theta_L;\alpha') - \tilde{W}^e_{i,t}(\theta_t)}{s^*_{i,t}(\theta_t)}$.

Let α^t be such that $\alpha^t_t = (s^*_t, w^*_t, \sigma^*_t)$ and $\alpha^t_\tau = \alpha_\tau$ for each $\tau \neq t$.

Step 2. If $t > 1$, then decrease t by 1. Return to Step 1 with $\alpha \equiv \alpha^{t+1}$.

Theorem 5.1 The direct revelation mechanism $\mathcal{D}^* = \{\Theta_{-0,t}, s^*_t, w^*_t, \sigma^*_t\}_{t=1}^T$ satisfies *constrained efficiency*, *periodic ex-post incentive compatibility*, and *periodically ex-post Pareto dominates* the equal division mechanism $\mathcal{D}^e$.

5.2 Multiple Members

For each $i \in N$ and each $\theta_{-i} \in \Theta_{-i}$, let $\eta(\theta_{-i})$ be the number of agents in the set $N \setminus \{i\}$ who have type θ_H. That is, $\eta(\theta_{-i}) \equiv |\{j \in N \setminus \{i\} \mid \theta_j = \theta_H\}|$. For each $n' \in \{0, \ldots, n-1\}$, let $\Psi_{-i}(n') \equiv \{\theta_{-i} \in \Theta_{-i} \mid \eta(\theta_{-i}) = n'\}$ and let $\xi_{-i}(n')$ be an element of $\Psi_{-i}(n')$. That is, $\xi_{-i}(n')$ is a type profile of agent i's partners, in which n' agents have type θ_H. Let $\rho_{-i}(n')$ be the cardinality of $\Psi_{-i}(n')$.

Fix an agent $i \in N$ and a type profile $\theta \in \Theta$. Let $w^*_i(\theta) = 0$ if either $\theta_i = \theta_H$ or $\theta = (\theta_L, \ldots, \theta_L)$. Suppose $\theta_i = \theta_L$ and $\theta_j = \theta_H$ for some $j \neq i$. Let $\theta' \in \Theta$ such that $\theta'_j = \theta_L$ and $\theta'_k = \theta_k$ for each $k \in N \setminus \{j\}$. Let $n^* \equiv \eta(\theta_{-i})$ and

$$\Big[\rho(n^*-1)(n-n^*)\mu(\theta_L) + \rho(n^*)n^*\mu(\theta_H)\Big] s^*_j(\theta) w^*_i(\theta) =$$
$$\mu(\theta_L)\max\big\{0, \quad \rho(n^*)\big[W^e_i(\theta) - W_i(\theta)\big], \quad \rho(n^*-1)\big[\tilde{W}_i(\theta';\theta_H) - W^e_i(\theta')\big]\big\}$$
$$+\mu(\theta_H)\min\big\{\rho(n^*)\big[W^e_i(\theta_H,\theta_{-i}) - \tilde{W}_i(\theta_H,\theta_{-i};\theta_L)\big], \quad \rho(n^*-1)\big[W_i(\theta_H,\theta'_{-i}) - W^e_i(\theta_H,\theta'_{-i})\big]\big\},$$

where $W^e_i(\theta) \equiv u_i\big[s^e(\theta), w^e(\theta), \sigma^e(\theta), \theta_0, \theta_i\big]$ for each $\theta \in \Theta$.

Theorem 5.2 The direct revelation mechanism $\mathcal{D}^*$ satisfies *constrained efficiency*, *Bayesian incentive compatibility*, and *Bayesian Pareto dominates the equal division mechanism.*

When agents have the same type, the mechanism $\mathcal{D}^*$ and the equal division mechanism give the same outcome: agents receive the same equity share and no wage payment. When agents have different types, the high-type agents receive no wage and the low-type agents receive some positive wage. This wage payment depends on the fraction of low-type agents. We ensure that these wage payments are sufficiently large for low-type agents to be truthful and better off compared with the equal division mechanism. They must also be sufficiently small for high-type agents to be truthful and better off compared with the equal division mechanism (we note that a high-type's income is strictly decreasing in the wage payment offered to a low-type). These wage payments exist due to the *increasing differences* property of the payoff structure. The wage rule w^* is demonstrated in the following examples.

Example 5.1 Suppose there are two partners.

For each $i \in N$, let $w_i^*(\theta_L, \theta_L) = w_i^*(\theta_H, \theta_H) = w_i^*(\theta_L, \theta_H) = 0$ and

$$s_i^*(\theta_L, \theta_H) w_i^*(\theta_H, \theta_L) = \mu(\theta_L) \max\Big\{0, \max_{\theta_{-i}} \Big[\tilde{W}_i(\theta_L, \theta_{-i}; \theta_H) - \tilde{W}_i(\theta_L, \theta_{-i}; \theta_L)\Big]\Big\}$$
$$+\mu(\theta_H) \min_{\theta_{-i}} \Big[\tilde{W}_i(\theta_H, \theta_{-i}; \theta_H) - \tilde{W}_i(\theta_H, \theta_{-i}; \theta_L)\Big].$$

Example 5.2 Suppose there are three partners. For each $i \in N$, let $w_i^*(\theta_L, \theta_L, \theta_L) = w_i^*(\theta_H, \theta_H, \theta_H) = w_i^*(\theta_H, \theta_{-i}) = 0$ for each θ_{-i},

$$2s_i^*(\theta_H, \theta_L, \theta_H) w_i^*(\theta_L, \theta_H, \theta_H) =$$
$$+\mu(\theta_L) \max\Big\{0,\ W_i^e(\theta_L, \theta_H, \theta_H) - W_i(\theta_L, \theta_H, \theta_H),\ 2\big[\tilde{W}_i(\theta_L, \theta_L, \theta_H; \theta_H) - W_i^e(\theta_L, \theta_L, \theta_H)\big]\Big\}$$
$$+\mu(\theta_H) \min\Big\{W_i^e(\theta_H, \theta_H, \theta_H) - \tilde{W}_i(\theta_H, \theta_H, \theta_H; \theta_L),\ 2\big[W_i(\theta_H, \theta_L, \theta_H) - W_i^e(\theta_H, \theta_L, \theta_H)\big]\Big\},$$

$$2s_i^*(\theta_H, \theta_L, \theta_L) w_i^*(\theta_L, \theta_L, \theta_H) =$$
$$+\mu(\theta_L) \max\Big\{0,\ 2\big[W_i^e(\theta_L, \theta_L, \theta_H) - W_i(\theta_L, \theta_L, \theta_H)\big],\ \tilde{W}_i(\theta_L, \theta_L, \theta_L; \theta_H) - W_i^e(\theta_L, \theta_L, \theta_L)\Big\}$$
$$+\mu(\theta_H) \min\Big\{2\big[W_i^e(\theta_H, \theta_L, \theta_H) - \tilde{W}_i(\theta_H, \theta_L, \theta_H; \theta_L)\big],\ W_i(\theta_H, \theta_L, \theta_L) - W_i^e(\theta_H, \theta_L, \theta_L)\Big\}.$$

When there are two agents, the property stated in Lemma 4.2 allows us to construct a mechanism $\mathcal{D}' \equiv \{(\Theta_1, \Theta_2), (s', w', \sigma')\}$ that is *ex post incentive compatible* and *Pareto dominates the equal division mechanism*. That is, each agent always prefers mechanism $\mathcal{D}'$ regardless of his partner's type; and a high-type agent strictly prefers $\mathcal{D}'$ when his partner has low type. Let (s', σ') be a *constrained-efficient* equity-effort rule for two agents (characterized in Section 3). We construct the wage rule w' as follows. For each $i \in N$, let $w_i'(\theta_L, \theta_L) = w_i'(\theta_H, \theta_H) = w_i'(\theta_H, \theta_L) = 0$, and

$$s_i'(\theta_H, \theta_L) w_i'(\theta_L, \theta_H) = \Big\{0,\ \ \tilde{W}_i(\theta_L, \theta_H; \theta_H) - \tilde{W}_i(\theta_L, \theta_H; \theta_L)\Big\}.$$

6 An Indirect Mechanism Design

In this section, we design a voting mechanism that implements the arrangement rule (s^*, w^*, σ^*) in Nash equilibrium.

6.1 Description

In a voting mechanism, each agent is given a few menus of some suggested arrangements and is asked to vote for one arrangement from each menu. These menus are designed so that if agents have type $\theta \in \Theta$, then $\left[s^*(\theta), w^*(\theta), \sigma^*(\theta)\right]$ is the unique arrangement (across all menus) that receives n votes (which is the highest number of votes that an arrangement can receive). We define a voting game formally below.

A **voting mechanism** is a tuple $\mathscr{V} \equiv \{(A_i, \mathcal{V}_i)_{i\in N}, \xi\}$, where A_i is a set of menus, $\mathcal{V}_i$ is a set of voting strategies, and ξ is an outcome function.

Menus.

In stage 1, each agent i receives a set of L menus, each of which consists of K *arrangements* $(1 \leq L, K < \infty)$. Each menu is represented by a column in the following matrix:

$$A_i \equiv \begin{bmatrix} a^i_{11} & a^i_{12} & \cdots & a^i_{1L} \\ a^i_{21} & a^i_{22} & \cdots & a^i_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ a^i_{K1} & a^i_{K2} & \cdots & a^i_{KL} \end{bmatrix}$$

where each entry a^i_{kl} is an *arrangement* in menu l. Let $\tilde{A}_i$ be the set of all entries in matrix A_i. We require that

(a) all entries in A_i are distinct (each agent can vote for/against an *arrangement* only once), and

(b) $\tilde{A}_i = \tilde{A}_j$ for $i \neq j$ (all agents can vote for/against any suggested *arrangement*).

Let $\tilde{A} \equiv \tilde{A}_i$ be the set of suggested *arrangements*.

Voting strategies.

Each agent is asked to vote for a unique *arrangement* in each menu. Formally, a voting strategy for agent i is a mapping $\nu_i : \tilde{A} \to \{0, 1\}$ such that $\sum_{k=1}^{K} \nu_i(a^i_{kl}) = 1$ for each $l \in \{1, 2, ..., L\}$. For instance, $\nu_i(a^i_{kl}) = 1$ implies that *arrangement* a^i_{kl} receives one vote and all other *arrangements* in menu l receive no vote from agent i ($\nu_i(a^i_{k'l}) = 0$ for each $k' \neq k$). Let $\mathcal{V}_i$ be the set of voting strategies for agent i and $\mathcal{V} \equiv \mathcal{V}_1 \times \cdots \times \mathcal{V}_n$ be the set of voting strategy profiles.

Outcome function.

An outcome function maps from each voting strategy profile to an *arrangement*. For each $\nu \in \mathcal{V}$,

$$\xi(\nu) \in \arg\max_{a \in \tilde{A}} \textstyle\sum_{i \in N} \nu_i(a).$$

Let $\xi^\pi(\nu) \in \Delta^{n-1} \times \mathbb{R}^n_+$ be the selected compensation plan and $\xi^e(\nu) \in \mathbb{R}^n_+$ be the selected effort recommendation.

Remark 6.1 If there are multiple arrangements that receive the same number of votes, then the outcome function ξ selects an arbitrary such arrangement. However, in equilibrium (defined in Section 6.2), there is a unique arrangement that receives the highest number of votes. $\square$

6.2 Implementation

A voting mechanism implements the arrangement rule $\alpha^* = (s^*, w^*, \sigma^*)$ in Bayesian Nash equilibrium if the corresponding voting game (defined below) has a Bayesian Nash equilibrium that induces the outcome $\big[s^*(\theta), w^*(\theta), \sigma^*(\theta)\big]$ for each type profile $\theta \in \Theta$. We formalize the concept of *implementation* as follow.

Fix a voting mechanism $\mathscr{V} \equiv \{(A_i, \mathcal{V}_i)_{i \in N}, \xi\}$. A **voting game** is a tuple

$$\mathcal{G}^{\mathscr{V}} \equiv \{N, \Theta, p, H, Z, (\varphi_i)_{i \in N}\},$$

where

- $H \equiv \{h^1\} \cup \mathcal{V}$ is the set of non-terminal histories (h^1 represents the root of the game tree and each $\nu \in \mathcal{V}$ describes the voting strategies chosen in stage 1),
- $Z \equiv \mathcal{V} \times \mathbb{R}^n_+$ is the set of terminal histories (each $(\nu, e) \in \mathcal{V} \times \mathbb{R}^n_+$ describes the chosen voting strategies and effort levels),
- $\varphi_i : Z \times \Theta_i \to \mathbb{R}$ is the payoff for agent i: if agent i has true cost parameter $\theta_i \in \Theta_i$ and the terminal history is $(\nu, e) \in Z$, then the selected compensation plan is $\xi^\pi(\nu)$ and the payoff for agent i is $\varphi_i(\nu, e, \theta_i) = v_i\big[\xi^\pi(\nu), e, \theta_i\big]$.

Strategies.

An agent's strategy is a mapping from each information state to an action available at that information state. For each agent i, a stage-one information state is his private information $\theta_i \in \Theta_i$ and a stage-two information state is a pair $(\theta_i, \nu) \in \Theta_i \times \mathcal{V}$, which consists of his private information θ_i and the voting strategies ν observed in stage 1. A strategy for agent i is some

$$\lambda_i : \Theta_i \ \cup \ \Theta_i \times \mathcal{V} \to \mathcal{V}_i \ \cup \ \mathbb{R}_+$$

such that

(a) $\lambda_i(\theta_i) \in \mathcal{V}_i$ for each $\theta_i \in \Theta_i$ (agent i chooses a voting strategy in stage 1), and

(b) $\lambda_i(\theta_i, \nu) \in \mathbb{R}_+$ for each $(\theta_i, \nu) \in \Theta_i \times \mathcal{V}$ (agent i chooses an effort level in stage 2).

Let Λ_i be the set of strategies for agent i and $\Lambda \equiv \Lambda_1 \times \cdots \times \Lambda_n$ be the set of strategy profiles. Suppose agents have true type $\theta \in \Theta$ and choose strategies $\lambda \in \Lambda$. Let $\zeta^\nu(\lambda, \theta)$ be the on-path voting strategies and $\zeta^e(\lambda, \theta)$ be the on-path efforts. Let $\zeta(\lambda, \theta) \equiv \big[\zeta^\nu(\lambda, \theta), \zeta^e(\lambda, \theta)\big]$ be the terminal history. Then the *ex-post* payoff for agent i is $\phi_i(\lambda, \theta) = \varphi_i\big[\zeta(\lambda, \theta), \theta_i\big]$.

Nash equilibrium.

A Nash equilibrium of voting mechanism $\mathscr{V}$ is a strategy profile $\lambda \in \Lambda$ such that for each agent $i \in N$, each $\theta_i \in \Theta_i$, and each strategy $\lambda'_i \in \Lambda_i$, we have

$$\phi_i(\lambda, \theta) \geq \phi_i(\lambda'_i, \lambda_{-i}, \theta).$$

Definition 6.1 The voting mechanism $\mathscr{V} \equiv \{(A_i, \mathcal{V}_i)_{i \in N}, \xi\}$ implements the arrangement rule α^* in Nash equilibrium if there is a Nash equilibrium λ of $\mathscr{V}$ such that $\xi\big[\zeta^\nu(\lambda, \theta)\big] = \alpha^*(\theta)$ for each $\theta \in \Theta^n$.

6.3 Design

Here we design a voting mechanism $\mathscr{V}^* \equiv \{(A_i^*, \mathcal{V}_i^*)_{i \in N}, \xi^*\}$ that implements the arrangement rule $\alpha^* = (s^*, w^*, \sigma^*)$ in Bayesian Nash equilibrium. For each $i \in N$ and each $\theta_{-i} \in \Theta^{n-1}$, let $\mu_i^*(\theta_{-i}) \equiv \big(\alpha^*(\theta_i, \theta_{-i})\big)_{\theta_i \in \Theta}$ be a menu of $|\Theta|$ *arrangements* ($\mu_i^*(\theta_{-i})$ is a column vector). Let $A_i^* = \big(\mu_i^*(\theta_{-i})\big)_{\theta_{-i} \in \Theta^{n-1}}$ be a set of $|\Theta^{n-1}|$ menus. By construction, A_i^* is a $|\Theta| \times |\Theta^{n-1}|$ matrix, whose each column is a menu of $|\Theta|$ *arrangements*. It is easy to verify that for each $i \in N$, all entries of A_i^* are distinct and $\tilde{A}_i^* = \tilde{A}_j^*$ for each $j \neq i$. The sets of menus $(A_i^*)_{i \in N}$ induce sets of voting strategies $(\mathcal{V}_i^*)_{i \in N}$ and an outcome function ξ^* as defined in Section 6.1.

Theorem 6.1 The voting mechanism $\mathscr{V}^* \equiv \{(A_i^*, \mathcal{V}_i^*)_{i \in N}, \xi^*\}$ implements the arrangement rule α^* in Nash equilibrium.

Hence, if agents have types $\theta \in \Theta$, the game outcome is $\alpha^*(\theta)$ in some Nash equilibrium. We recall that $\alpha^*(\theta)$ is a *constrained-efficient* arrangement that maximizes agents' aggregate welfare subject to the constraint that agents choose their efforts rationally conditional on their equity shares.

References

Michael J. Anderson. Partner compensation systems in professional services firms. In *Law Firm Compensation Systems*. The Remsen Group, 2013.

William D. Henderson. An empirical study of single-tier versus two-tier partnerships in the am law 200. Digital Repository @ Maurer Law, 2006. Indiana University Maurer School of Law.

Bengt Holmstrom. Moral hazard in teams. *The Bell Journal of Economics*, 13(2):324–340, 1982. ISSN 0361915X.

Patrick Legros and Hitoshi Matsushima. Efficiency in partnerships. *Journal of Economic Theory*, 55(2):296 – 322, 1991. ISSN 0022-0531.

Patrick Legros and Steven A. Matthews. Efficient and nearly-efficient partnerships. *The Review of Economic Studies*, 60(3):599–611, 1993. ISSN 00346527, 1467937X.

Eric Rasmusen. Moral hazard in risk-averse teams. *The RAND Journal of Economics*, 18 (3):428–435, 1987. ISSN 07416261.

Noam Wasserman. *The Founder's Dilemmas: Anticipating and Avoiding the Pitfalls That Can Sink a Startup*. The Kauffman Foundation series on innovation and entrepreneurship. Princeton University Press, 2012.

Ed Wesemann and Nick Jarrett-Kerr. Global partner compensation system survey. Edge International, 2015.

A Constrained-Efficient Arrangement Rule

Fix $t \in \{1, \ldots, T\}$ and a *constrained-efficient* equity-effort rule $(s^*_\tau, \sigma^*_\tau)_{\tau > t}$.

Assumption A.1 For each $\theta_t \in \Theta_t$ and each $r_t \in \Delta^{n-1}$,

(a) there is a unique effort profile $e_t \equiv \tilde{\sigma}_t(\theta_t, r_t) \in \mathbb{R}^n_+$ such that $(r_t, e_t) \in \tilde{\mathcal{R}}_t(\theta_t)$: for each $i \in N$, we have $e_{i,t} = f_{i,t}(r_{i,t})\theta_{0,t}\theta_{i,t}$ for some $f_{i,t} : [0,1] \to \mathbb{R}_+$ such that $f'_{i,t} = a > 0$,

(b) for each pair $\{i, j\} \subseteq N$ and each $e_t \in \mathbb{R}^n_+$, we have $\dfrac{\partial \mathbb{E}\big[W_{i,t+1}(\theta_{t+1})|e_t\big]}{\partial e_{j,t}} = g_i(t)$ for some $g_i : \{1, \ldots, T\} \to \mathbb{R}_+$.

Under Assumption A.1(a), a period$-t$ equity-effort rule (s^*_t, σ^*_t) is *constrained-efficient* if and only if, for each $\theta_t \in \Theta_t$, we have $\sigma^*_t(\theta_t) = \tilde{\sigma}_t\big[\theta_t, s^*_t(\theta_t)\big]$ and $s^*_t(\theta_t)$ solves

$$\max_{r_t} \sum_{i \in N} \Big(\tilde{u}_{i,t}\big[r_t, \tilde{\sigma}_t(\theta_t, r_t), \theta_{0,t}, \theta_{i,t}\big] + \delta \mathbb{E}\big[W_{i,t+1}(\theta_{t+1})|\tilde{\sigma}_t(\theta_t, r_t)\big] \Big)$$

$$\text{subject to } \textstyle\sum_{i \in N} r_{i,t} = 1 \text{ and } 0 \le r_{i,t} \le 1 \text{ for each } i \in N.$$

For each $r_t \in \Delta^{n-1}$, let

$$U_t(\theta_t, r_t) \equiv \textstyle\sum_{i \in N} \Big(\tilde{u}_{i,t}\big[r_t, \tilde{\sigma}_t(\theta_t, r_t), \theta_{0,t}, \theta_{i,t}\big] + \delta \mathbb{E}\big[W_{i,t+1}(\theta_{t+1})|\tilde{\sigma}_t(\theta_t, r_t)\big] \Big).$$

The Lagrangean function

$$\mathcal{L}[s_t(\theta_t)] = U_t\big[\theta_t, s_t(\theta_t)\big] + \sum_{i \in N} \lambda_{i,t} s_{i,t}(\theta_t) - \sum_{i \in N} \gamma_{i,t}\big[s_{i,t}(\theta_t) - 1\big] - \mu_t\big[\sum_{i \in N} s_{i,t}(\theta_t) - 1\big].$$

The Kuhn-Tucker conditions

$\dfrac{\partial U_t\big[\theta_t, s_t(\theta_t)\big]}{\partial s_{i,t}(\theta_t)} + \lambda_{i,t} - \gamma_{i,t} - \mu_t = 0$ for each $i \in N$,

$\lambda_{i,t} \ge 0,\ s_{i,t}(\theta_t) \ge 0,\ \lambda_{i,t} s_{i,t}(\theta_t) = 0$ for each $i \in N$,

$\gamma_{i,t} \ge 0,\ s_{i,t}(\theta_t) \le 1,\ \gamma_{i,t}\big[s_{i,t}(\theta_t) - 1\big] = 0$ for each $i \in N$,

$s_{1,t}(\theta_t) + s_{2,t}(\theta_t) + \ldots + s_{n,t}(\theta_t) = 1.$

Lemma A.1 Under Assumption A.1, for each $t \in \{1, \ldots, T\}$, each $r_t \in \Delta^{n-1}$, each $\theta_t \in \Theta_t$, and each $i \in N$,

(a) $\dfrac{\partial U_t(\theta_t, r_t)}{\partial r_{i,t}} = \left[1 - r_{i,t} + \delta \sum_{j \neq i} g_j(t)\right] a\theta_{0,t}\theta_{i,t},$

(b) for each $r'_t \in \Delta^{n-1}$ and each $\theta'_t \in \Theta_t$ such that $r'_{i,t} > r_{i,t}$, $\theta'_{0,t} = \theta_{0,t}$, and $\theta'_{i,t} \leq \theta_{i,t}$,

$$\frac{\partial U_t(\theta_t, r_t)}{\partial r_{i,t}} > \frac{\partial U_t(\theta'_t, r'_t)}{\partial r'_{i,t}}.$$

Proof.

(a) By definition,

$$U_t(\theta_t, r_t) = \sum_{j\in N} \tilde{\sigma}_{j,t}(\theta_t, r_t) + \delta \sum_{j\in N} \mathbb{E}\big[W_{j,t+1}(\theta_{t+1})|\tilde{\sigma}_t(\theta_t, r_t)\big] - \sum_{j\in N} c_{j,t}\big[\tilde{\sigma}_{j,t}(\theta_t, r_t), \theta_{0,t}, \theta_{j,t}\big].$$

We have

$$\frac{\partial U_t(\theta_t, r_t)}{\partial r_{i,t}} = \left[1 + \delta \sum_{j\in N} \frac{\partial \mathbb{E}\big[W_{j,t+1}(\theta_{t+1})|\tilde{\sigma}_t(\theta_t, r_t)\big]}{\partial e_{i,t}} - \frac{\partial c_{i,t}\big[\tilde{\sigma}_{i,t}(\theta_t, r_t), \theta_{0,t}, \theta_{i,t}\big]}{\partial e_{i,t}}\right] \frac{\partial \tilde{\sigma}_{i,t}(\theta_t, r_t)}{\partial r_{i,t}}.$$

Since $r_{i,t} + \delta \dfrac{\partial \mathbb{E}\big[W_{i,t+1}(\theta_{t+1})|\tilde{\sigma}_t(\theta_t, r_t)\big]}{\partial e_{i,t}} = \dfrac{\partial c_{i,t}\big[\tilde{\sigma}_{i,t}(\theta_t, r_t), \theta_{0,t}, \theta_{i,t}\big]}{\partial e_{i,t}}$,

$$\frac{\partial U_t(\theta_t, r_t)}{\partial r_{i,t}} = \left[1 - r_{i,t} + \delta \sum_{j\neq i} g_j(t)\right] a\theta_{0,t}\theta_{i,t}.$$

(b) By Step (a), $\dfrac{\partial U_t(\theta'_t, r'_t)}{\partial r'_{i,t}} = \left[1 - r'_{i,t} + \delta \sum_{j\neq i} g_j(t)\right] a\theta_{0,t}\theta'_{i,t}.$

Since $1 - r_{i,T} > 1 - r'_{i,T}$ and $\theta_{i,t} \geq \theta'_{i,t}$, we have $\dfrac{\partial U_t(\theta_t, r_t)}{\partial r_{i,t}} > \dfrac{\partial U_t(\theta'_t, r'_t)}{\partial r'_{i,t}}$. ■

Lemma A.2 Under Assumption A.1, there is a unique *constrained-efficient* period$-t$ equity-effort rule (s^*_t, σ^*_t).

Proof. It suffices to show that, for each $\theta_t \in \Theta_t$, the following maximization problem has a unique solution:

$$\max_{r'_t\in\Delta^{n-1}} \sum_{i\in N} \Big(\tilde{u}_{i,t}\big[r'_t, \tilde{\sigma}_t(\theta_t, r'_t), \theta_{0,t}, \theta_{i,t}\big] + \delta \mathbb{E}\big[W_{i,t+1}(\theta_{t+1})|\theta_t, \tilde{\sigma}_t(\theta_t, r'_t)\big]\Big) \quad (*).$$

By Weierstrass Theorem, the maximization problem $(*)$ has a solution $s^*_t(\theta_t)$. Suppose there is some $r_t \neq s^*_t(\theta_t)$ that solves $(*)$. Since $r_t \neq s^*_t(\theta_t)$, there is a pair $\{i, j\} \subseteq N$ such that $r_{i,t} < s^*_{i,t}(\theta_t)$ and $r_{j,t} > s^*_{j,t}(\theta_t)$. By the Kuhn-Tucker conditions,

$$\frac{\partial U_t\big[\theta_t, s^*_t(\theta_t)\big]}{\partial s^*_{i,t}(\theta_t)} \geq \frac{\partial U_t\big[\theta_t, s^*_t(\theta_t)\big]}{\partial s^*_{j,t}(\theta_t)} \text{ and } \frac{\partial U_t(\theta_t, r_t)}{\partial r_{i,t}} \leq \frac{\partial U_t(\theta_t, r_t)}{\partial r_{j,t}} \quad (**).$$

By Lemma A.1(b), $\dfrac{\partial U_t(\theta_t, r_t)}{\partial r_{i,t}} > \dfrac{\partial U_t\big[\theta_t, s_t^*(\theta_t)\big]}{\partial s_{i,t}^*(\theta_t)}$ and $\dfrac{\partial U_t\big[\theta_t, s_t^*(\theta_t)\big]}{\partial s_{j,t}^*(\theta_t)} > \dfrac{\partial U_t(\theta_t, r_t)}{\partial r_{j,t}}$.

This contradicts $(**)$. Hence, the maximization problem $(*)$ has a unique solution $s_t^*(\theta_t)$. Let $\sigma_t^*(\theta_t) \equiv \tilde{\sigma}_t\big[\theta_t, s_t^*(\theta_t)\big]$. Then (s_t^*, σ_t^*) is the unique *constrained-efficient* period$-t$ equity-effort rule. ■

Lemma A.3 Under Assumption A.1, for each pair $\{\theta_t, \theta_t'\} \subseteq \Theta_t$ such that $\theta_{-0,t} = \theta_{-0,t}'$, we have $s_t^*(\theta_t) = s_t^*(\theta_t')$.

Proof. Suppose $s_t^*(\theta_t) \neq s_t^*(\theta_t')$. That is, there is a pair $\{i,j\} \subseteq N$ such that $s_{i,t}^*(\theta_t) < s_{i,t}^*(\theta_t')$ and $s_{j,t}^*(\theta_t) > s_{j,t}^*(\theta_t')$. We note that $s_{j,t}^*(\theta_t) > 0$. By the Kuhn-Tucker conditions and Lemma A.1,

$$\Big[1 - s_{i,t}^*(\theta_t) + \delta \textstyle\sum_{k\neq i} g_k(t)\Big] a\theta_{0,t}\theta_{i,t} \leq \Big[1 - s_{j,t}^*(\theta_t) + \delta \textstyle\sum_{k\neq j} g_k(t)\Big] a\theta_{0,t}\theta_{j,t},$$

which implies $\Big[1 - s_{i,t}^*(\theta_t') + \delta \sum_{k\neq i} g_k(t)\Big] a\theta_{0,t}'\theta_{i,t}' < \Big[1 - s_{j,t}^*(\theta_t') + \delta \sum_{k\neq j} g_k(t)\Big] a\theta_{0,t}'\theta_{j,t}'$. This is a contradiction. ■

Proof for Lemma 3.1 and Lemma 3.2. The proof is by induction. Fix $t \in \{1, \ldots, T\}$ and a *constrained-efficient* equity-effort rule $(s_\tau^*, \sigma_\tau^*)_{\tau > t}$. Fix $\tau \in \{t+1, \ldots, T\}$ and assume the following:

(a) for each $\theta_\tau \in \Theta_\tau$ and each $r_\tau \in \Delta^{n-1}$, there is a unique effort profile $e_\tau \in \mathbb{R}_+^n$ such that $(r_\tau, e_\tau) \in \tilde{\mathcal{R}}_\tau(\theta_\tau)$: for each $i \in N$, we have $e_{i,\tau} = f_{i,\tau}(r_{i,\tau})\theta_{0,\tau}\theta_{i,\tau}$ for some $f_{i,\tau} : [0,1] \to \mathbb{R}_+$ such that $f_{i,\tau}' = a > 0$,

(b) for each $\theta_\tau \in \Theta_\tau$, $s_\tau^*(\theta_\tau)$ is independent of $\theta_{0,\tau}$,

(c) for each $\{i,j\} \subseteq N$ and each $e_\tau \in \mathbb{R}_+^n$, $\dfrac{\partial \mathbb{E}\big[W_{i,\tau+1}(\theta_{\tau+1})|e_\tau\big]}{\partial e_{j,\tau}} = g_i(\tau)$ for some $g_i : \{1, \ldots T\} \to \mathbb{R}_+$.

We show that (a)$-$(c) hold for $\tau - 1$. Fix $\theta_{\tau-1} \in \Theta_{\tau-1}$ and $r_{\tau-1} \in \Delta^{n-1}$. An effort profile $e_{\tau-1} \in \mathbb{R}_+^n$ satisfies $(r_{\tau-1}, e_{\tau-1}) \in \tilde{\mathcal{R}}_{\tau-1}(\theta_{\tau-1})$ if and only if for each $i \in N$,

$$r_{i,\tau-1} + \delta \frac{\partial \mathbb{E}\big[W_{i,\tau}(\theta_\tau)|e_{\tau-1}\big]}{\partial e_{i,\tau-1}} = \frac{e_{i,\tau-1}}{\theta_{0,\tau-1}\theta_{i,\tau-1}}.$$

For each $e_{\tau-1} \in \mathbb{R}_+^n$,

$$\mathbb{E}\big[W_{i,\tau}(\theta_\tau)|e_{\tau-1}\big] = \mathbb{E}\left[s_{i,\tau}^*(\theta_\tau)\sum_{j\in N}\sigma_{j,\tau}^*(\theta_\tau) + \delta\mathbb{E}\big[W_{i,\tau+1}(\theta_{\tau+1})|\sigma_\tau^*(\theta_\tau)\big] - \frac{\big[\sigma_{i,\tau}^*(\theta_\tau)\big]^2}{2\theta_{0,\tau}\theta_{i,\tau}}\right],$$

where $\sigma^*_{j,\tau}(\theta_\tau) = \gamma\big(\theta_{0,\tau-1} + \sum_{j\in N} e_{j,\tau-1}\big)\theta_{j,\tau} f_{j,\tau}\big[s^*_\tau(\theta_\tau)\big]$ for each $j \in N$ and $\theta_\tau \in \Theta_\tau$.

For each $j \in N$,

$$\frac{\partial \mathbb{E}\big[W_{i,\tau}(\theta_\tau)|e_{\tau-1}\big]}{\partial e_{j,\tau-1}} = \mathbb{E}\Big(\big[s^*_{i,\tau}(\theta_\tau) + \delta g_i(\tau)\big] \sum_{j\neq i} \gamma\theta_{j,\tau} f_{j,\tau}\big[s^*_\tau(\theta_\tau)\big]\Big) \equiv g_i(\tau-1).$$

It follows that $e_{i,\tau-1} = \big[r_{i,\tau-1} + \delta g_i(\tau-1)\big]\theta_{0,\tau-1}\theta_{i,\tau-1} \equiv f_{i,\tau-1}(r_{i,\tau-1})\theta_{0,\tau-1}\theta_{i,\tau-1}$. By Lemma A.3, if $(s^*_{\tau-1}, \sigma^*_{\tau-1})$ is *constrained-efficient*, then $s^*_{\tau-1}(\theta_{\tau-1})$ is independent of $\theta_{0,\tau-1}$ for each $\theta_{\tau-1} \in \Theta_{\tau-1}$.

It is easy to verify that (a)$-$(c) hold for $\tau = T$. Hence, (a)$-$(c) hold for each $\tau \in \{t, \ldots, T\}$. It follows that for each $\theta_t \in \Theta_t$ and each $r_t \in \Delta^{n-1}$, there is a unique effort profile $e_t \in \mathbb{R}^n_+$ such that $(r_t, e_t) \in \tilde{\mathcal{R}}_t(\theta_t)$. Proving Lemma 3.2 is then straightforward. ■

Remark A.1 As shown in the proof for Lemma 3.1, Assumption A.1 holds for each $t \in \{1, \ldots, T\}$ and each *constrained-efficient* equity-effort rule $(s^*_\tau, \sigma^*_\tau)_{\tau>t}$.

Proof for Proposition 3.1.

(a) The proof is by contradiction. Fix $t \in \{1, \ldots, T\}$. Suppose there are two *constrained-efficient* equity-effort rules $(s^*_\tau, \sigma^*_\tau)^T_{\tau=t}$ and $(s'_\tau, \sigma'_\tau)^T_{\tau=t}$. Let

$$\tau^* \equiv \min\Big\{\tau \in \{t, \ldots, T\} \mid (s^*_\tau, \sigma^*_\tau) \neq (s'_\tau, \sigma'_\tau)\Big\}.$$

Then there are two *constrained-efficient* period$-\tau^*$ equity-effort rule $(s^*_{\tau^*}, \sigma^*_{\tau^*})$ and $(s'_{\tau^*}, \sigma'_{\tau^*})$ given $(s^*_\tau, \sigma^*_\tau)_{\tau>\tau^*}$. This contradicts Lemma A.2 and Remark A.1.

(b) Fix $i \in N$, $\tau \in \{t, \ldots, T\}$, and $\theta_\tau \in \Theta_\tau$. We show $s^*_{i,\tau}(\theta'_{i,\tau}, \theta_{-i,\tau}) \geq s^*_{i,\tau}(\theta_\tau)$ for $\theta'_{i,\tau} > \theta_{i,\tau}$. Let $\theta'_\tau \equiv (\theta'_{i,\tau}, \theta_{-i,\tau})$. If $s^*_{i,\tau}(\theta_\tau) = 0$, then it is clear that $s^*_{i,\tau}(\theta_\tau) \leq s^*_{i,\tau}(\theta'_\tau)$. Suppose $s^*_{i,\tau}(\theta_\tau) > 0$ and $s^*_{i,\tau}(\theta_\tau) > s^*_{i,\tau}(\theta'_\tau)$. Then there exists some $j \neq i$ such that $s^*_{j,\tau}(\theta_\tau) < s^*_{j,\tau}(\theta'_\tau)$. By the Kuhn-Tucker conditions,

$$\frac{\partial U_\tau\big[\theta_\tau, s^*_\tau(\theta_\tau)\big]}{\partial s^*_{i,\tau}(\theta_\tau)} \geq \frac{\partial U_\tau\big[\theta_\tau, s^*_\tau(\theta_\tau)\big]}{\partial s^*_{j,\tau}(\theta_\tau)} \text{ and } \frac{\partial U_\tau\big[\theta'_\tau, s^*_\tau(\theta'_\tau)\big]}{\partial s^*_{i,\tau}(\theta'_\tau)} \leq \frac{\partial U_\tau\big[\theta'_\tau, s^*_\tau(\theta'_\tau)\big]}{\partial s^*_{j,\tau}(\theta'_\tau)} \quad (*).$$

By Lemma A.1(b) and Remark A.1,

$$\frac{\partial U_\tau\big[\theta'_\tau, s^*_\tau(\theta'_\tau)\big]}{\partial s^*_{i,\tau}(\theta'_\tau)} > \frac{\partial U_\tau\big[\theta_\tau, s^*_\tau(\theta_\tau)\big]}{\partial s^*_{i,\tau}(\theta_\tau)} \text{ and } \frac{\partial U_\tau\big[\theta_\tau, s^*_\tau(\theta_\tau)\big]}{\partial s^*_{j,\tau}(\theta_\tau)} > \frac{\partial U_\tau\big[\theta'_\tau, s^*_\tau(\theta'_\tau)\big]}{\partial s^*_{j,\tau}(\theta'_\tau)}.$$

This contradicts $(*)$. Hence, $s^*_{i,\tau}(\theta_\tau) \leq s^*_{i,\tau}(\theta'_\tau)$. ■

B The Direct Revelation Mechanism

B.1 The Payoff Structure

Proof for Lemma 4.1. An effort level $e_{i,t}$ is a solution to the maximization problem $(*)$ if and only if

$$s^*_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) + \delta \frac{\partial \mathbb{E}\big[W_{i,t+1}(\theta_{t+1}) \mid e_{i,t}, \sigma^*_{-i,t}(\hat{\theta}_{i,t}, \theta_{-i,t})\big]}{\partial e_{i,t}} = \frac{e_{i,t}}{\theta_{0,t}\theta_{i,t}}.$$

By Remark A.1, $\tilde{\sigma}_{i,t}\big[\theta_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big] = \Big[s^*_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) + \delta g_i(t)\Big]\theta_{0,t}\theta_{i,t}$ is the unique solution to $(*)$. It is clear that $\tilde{\sigma}_{i,t}\big[\theta_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big] > \tilde{\sigma}_{i,t}\big[\theta'_{i,t}, \theta_{-i,t}, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big]$ for $\theta'_{i,t} < \theta_{i,t}$. ■

Proof for Lemma 4.2. Let $\theta'_t \equiv (\theta'_{i,t}, \theta_{-i,t})$. We have

$$\tilde{W}_{i,t}(\theta'_t; \hat{\theta}_{i,t}) = s^*_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) \sum_{j \in N} \tilde{\sigma}_{j,t}\big[\theta'_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big]$$
$$+ \delta \mathbb{E}\Big(W_{i,t+1}(\theta_{t+1}) \mid \tilde{\sigma}_t\big[\theta'_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big]\Big) - c_{i,t}\Big(\tilde{\sigma}_{i,t}\big[\theta'_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big], \theta_{0,t}, \theta'_{i,t}\Big).$$

Without loss of generality, assume $\theta_{i,t} - \theta'_{i,t} = \epsilon$ is sufficiently small. Then

$$\Big[\tilde{W}_{i,t}(\theta_t; \hat{\theta}_{i,t}) - \tilde{W}_{i,t}(\theta'_t, \hat{\theta}_{i,t})\Big]\epsilon^{-1} = \frac{\partial \tilde{W}_{i,t}(\theta'_t; \hat{\theta}_{i,t})}{\partial \theta'_{i,t}}$$
$$= \left[s^*_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) + \delta g_i(t) - \frac{\tilde{\sigma}_{i,t}\big[\theta'_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big]}{\theta_{0,t}\theta'_{i,t}}\right] \frac{\partial \tilde{\sigma}_{i,t}\big[\theta'_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big]}{\partial \theta'_{i,t}}$$
$$- \frac{\partial c_{i,t}\Big(\tilde{\sigma}_{i,t}\big[\theta'_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big], \theta_{0,t}, \theta'_{i,t}\Big)}{\partial \theta'_{i,t}}.$$

As shown in the proof for Lemma 4.1,

$$s^*_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) + \delta g_i(t) = \frac{\tilde{\sigma}_{i,t}\big[\theta'_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big]}{\theta_{0,t}\theta'_{i,t}}.$$

Hence, $\tilde{W}_{i,t}(\theta_t; \hat{\theta}_{i,t}) - \tilde{W}_{i,t}(\theta'_t, \hat{\theta}_{i,t}) = -\dfrac{\partial c_{i,t}\Big(\tilde{\sigma}_{i,t}\big[\theta'_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big], \theta_{0,t}, \theta'_{i,t}\Big)}{\partial \theta'_{i,t}}\epsilon.$

Likewise, $W^e_{i,t}(\theta_t) - W^e_{i,t}(\theta'_t) = -\dfrac{\partial c_{i,t}\Big(\tilde{\sigma}_{i,t}\big[\theta'_t, s^e_t(\theta'_t)\big], \theta_{0,t}, \theta'_{i,t}\Big)}{\partial \theta'_{i,t}}\epsilon.$

We have $\Big[\tilde{W}_{i,t}(\theta_t; \hat{\theta}_{i,t}) - W^e_{i,t}(\theta_t)\Big] - \Big[\tilde{W}_{i,t}(\theta'_t; \hat{\theta}_{i,t}) - W^e_{i,t}(\theta'_t)\Big]$

$$= \Big[\tilde{W}_{i,t}(\theta_t; \hat{\theta}_{i,t}) - \tilde{W}_{i,t}(\theta'_t; \hat{\theta}_{i,t})\Big] - \Big[W^e_{i,t}(\theta_t) - W^e_{i,t}(\theta'_t)\Big]$$

$$= \left[-\frac{\partial c_{i,t}\big(\tilde{\sigma}_{i,t}\big[\theta'_t, s^*_t(\hat{\theta}_{i,t}, \theta_{-i,t})\big], \theta_{0,t}, \theta'_{i,t}\big)}{\partial \theta'_{i,t}} + \frac{\partial c_{i,t}\big(\tilde{\sigma}_{i,t}\big[\theta'_t, s^e_t(\theta'_t)\big], \theta_{0,t}, \theta'_{i,t}\big)}{\partial \theta'_{i,t}} \right] \epsilon$$

$$= \frac{\theta_{0,t}}{2}\Big(\big[f_{i,t}\big(s^*_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t})\big)\big]^2 - \big[f_{i,t}(1/n)\big]^2 \Big) \geq 0$$

if and only if $s^*_{i,t}(\hat{\theta}_{i,t}, \theta_{-i,t}) \geq 1/n$ (since $f_{i,t}$ is strictly increasing). ■

B.2 A Special Case

Lemma B.1 Fix $i \in N$ and $t \in \{2, \ldots, T\}$. Suppose that for each $\theta_t \in \Theta_t$ and each $e_t \in \mathbb{R}^n_+$, we have $\mathbb{E}\big[W_{i,t+1}(\theta_{t+1}, \alpha^*) \mid \theta_t, e_t\big] = \gamma(\theta_{0,t} + \sum_{j\in N} e_{j,t})h_i(t+1)$. Then for each $\theta_{t-1} \in \Theta_{t-1}$ and each $e_{t-1} \in \mathbb{R}^n_+$,

$$\mathbb{E}\big[W_{i,t}(\theta_t, \alpha^*) \mid \theta_{t-1}, e_{t-1}\big] = \gamma(\theta_{0,t-1} + \textstyle\sum_{j\in N} e_{j,t-1})h_i(t).$$

Proof. For each $\theta_t \in \Theta_t$,

$$\begin{aligned} W_{i,t}(\theta_t, \alpha^*) &= \max_{e_{i,t}\in\mathbb{R}_+} u_{i,t}\big[s^*_t(\theta_t), w^*_t(\theta_t), e_{i,t}, \sigma^*_{-i,t}(\theta_t), \theta_{0,t}, \theta_{i,t}\big] \\ &\qquad + \delta\mathbb{E}\big[W_{i,t+1}(\theta_{t+1}, \alpha^*) \mid \theta_t, e_{i,t}, \sigma^*_{-i,t}(\theta_t)\big] \\ &= \max_{e_{i,t}\in\mathbb{R}_+} \tilde{u}_{i,t}\big[s^*_t(\theta_t), e_{i,t}, \sigma^*_{-i,t}(\theta_t), \theta_{0,t}, \theta_{i,t}\big] - s^*_{i,t}(\theta_t)\sum_{j\in N} w^*_{j,t}(\theta_t) + w^*_{i,t}(\theta_t) \\ &\qquad + \delta\mathbb{E}\big[W_{i,t+1}(\theta_{t+1}, \alpha^*) \mid \theta_t, e_{i,t}, \sigma^*_{-i,t}(\theta_t)\big] \\ &= \tilde{W}_{i,t}(\theta_t; \theta_{i,t}; \alpha^*) - s^*_{i,t}(\theta_t)\sum_{j\in N} w^*_{j,t}(\theta_t) + w^*_{i,t}(\theta_t). \end{aligned}$$

By Lemma 4.1, $\sigma^*_{i,t}(\theta_t) \equiv \tilde{\sigma}_{i,t}\big[\theta_t, s^*_t(\theta_t)\big]$ solves the maximization problem above. Hence,

$$\tilde{W}_{i,t}(\theta_t; \theta_{i,t}; \alpha^*) = s^*_{i,t}(\theta_t)\sum_{j\in N} \sigma^*_{j,t}(\theta_t) - \frac{\big[\sigma^*_{i,t}(\theta_t)\big]^2}{2\theta_{0,t}\theta_{i,t}} + \delta\mathbb{E}\big[W_{i,t+1}(\theta_{t+1}, \alpha^*) \mid \theta_t, \sigma^*_t(\theta_t)\big]$$

$$= s^*_{i,t}(\theta_t)\theta_{0,t}\sum_{j\in N} f_{j,t}\big[s^*_{j,t}(\theta_t)\big]\theta_{j,t} - \frac{1}{2}\Big(f_{i,t}\big[s^*_{i,t}(\theta_t)\big]\Big)^2\theta_{0,t}\theta_{i,t}$$

$$+\delta\gamma\Big(\theta_{0,t} + \theta_{0,t}\sum_{j\in N} f_{j,t}\big[s^*_{j,t}(\theta_t)\big]\theta_{j,t}\Big)h_i(t+1)$$

$$= \theta_{0,t}\left[\Big[s^*_{i,t}(\theta_t) + \delta\gamma h_i(t+1)\Big]\sum_{j\in N} f_{j,t}\big[s^*_{j,t}(\theta_t)\big]\theta_{j,t} - \frac{1}{2}\Big(f_{i,t}\big[s^*_{i,t}(\theta_t)\big]\Big)^2\theta_{i,t} + \delta\gamma h_i(t+1)\right].$$

Likewise,

$\tilde{W}^e_{i,t}(\theta_t; \alpha^*)$

$$= \theta_{0,t}\left[\Big[s^e_{i,t}(\theta_t) + \delta\gamma h_i(t+1)\Big]\sum_{j\in N} f_{j,t}\big[s^e_{j,t}(\theta_t)\big]\theta_{j,t} - \frac{1}{2}\Big(f_{i,t}\big[s^e_{i,t}(\theta_t)\big]\Big)^2\theta_{i,t} + \delta\gamma h_i(t+1)\right].$$

Case 1. Suppose $\theta_{-0,t} = (\theta_L, \theta_L)$ or $\theta_{-0,t} = (\theta_H, \theta_H)$. Then

$$W_{i,t}(\theta_t, \alpha^*) = \tilde{W}_{i,t}(\theta_t; \theta_{i,t}; \alpha^*) \equiv \theta_{0,t}\tilde{h}_i(t, \theta_t).$$

Case 2. Suppose $\theta_{-0,t} = (\theta_L, \theta_H)$.

If $\tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*) \geq \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)$, then

$W_{i,t}(\theta_{0,t}, \theta_L, \theta_H, \alpha^*) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) + \big[1 - s^*_{i,t}(\theta_{0,t}, \theta_L, \theta_H)\big]w^*_{i,t}(\theta_{0,t}, \theta_L, \theta_H)$

$= \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) + \tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)$

$= \tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*)$.

If $\tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*) < \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)$, then

$W_{i,t}(\theta_{0,t}, \theta_L, \theta_H, \alpha^*) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) - s^*_{i,t}(\theta_{0,t}, \theta_L, \theta_H)w^*_{i,t}(\theta_{0,t}, \theta_H, \theta_L)$

$= \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) - \big[\tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*)\big]$

$= \tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*)$.

In either case, $W_{i,t}(\theta_{0,t}, \theta_L, \theta_H, \alpha^*) = \tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*) \equiv \theta_{0,t}\tilde{h}_i(t, \theta_{0,t}, \theta_L, \theta_H)$.

Case 3. Suppose $\theta_{-0,t} = (\theta_H, \theta_L)$.

If $\tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*) \geq \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)$, then

$W_{i,t}(\theta_{0,t}, \theta_H, \theta_L, \alpha^*) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) - s^*_{i,t}(\theta_{0,t}, \theta_H, \theta_L)w^*_{i,t}(\theta_{0,t}, \theta_L, \theta_H)$

$= \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) - \big[\tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)\big]$.

If $\tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*) < \tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*)$, then

$W_{i,t}(\theta_{0,t}, \theta_H, \theta_L, \alpha^*) = \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) + \big[1 - s^*_{i,t}(\theta_{0,t}, \theta_H, \theta_L)\big]w^*_{i,t}(\theta_{0,t}, \theta_H, \theta_L)$

$= \tilde{W}_{i,t}(\theta_{0,t}, \theta_H, \theta_L; \theta_H; \alpha^*) + \big[\tilde{W}_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}^e_{i,t}(\theta_{0,t}, \theta_L, \theta_H; \alpha^*)\big]$.

In either case, $W_{i,t}(\theta_{0,t}, \theta_H, \theta_L, \alpha^*) \equiv \theta_{0,t}\tilde{h}_i(t, \theta_{0,t}, \theta_H, \theta_L)$.

Hence, $\mathbb{E}\big[W_{i,t}(\theta_t, \alpha^*) \mid \theta_{t-1}, e_{t-1}\big] = \mathbb{E}\big[\theta_{0,t}\tilde{h}_i(t, \theta_t) \mid \theta_{t-1}, e_{t-1}\big]$

$= \gamma(\theta_{0,t-1} + \sum_{j\in N} e_{j,t-1})h_i(t)$. ■

Proof for Lemma 5.1. Let $s_H \equiv s_{i,t}(\theta_{0,t}, \theta_H, \theta_L)$ and $s_L \equiv s_{i,t}(\theta_{0,t}, \theta_L, \theta_H)$. We have

$$\tilde{W}_{i,t}(\theta_{0,t},\theta_L,\theta_H;\theta_H;\alpha)-\tilde{W}_{i,t}(\theta_{0,t},\theta_L,\theta_H;\theta_L;\alpha)$$
$$=\frac{1}{n}\Big[f_{i,t}(1/n)\theta_{0,t}\theta_L+f_{-i,t}(1/n)\theta_{0,t}\theta_H\Big]-\frac{1}{2}\big[f_{i,t}(1/n)\big]^2\theta_{0,t}\theta_L$$
$$+\delta\gamma\Big(\theta_{0,t}+f_{i,t}(1/n)\theta_{0,t}\theta_L+f_{-i,t}(1/n)\theta_{0,t}\theta_H\Big)h_i(t+1)$$
$$-s_L\Big[f_{i,t}(s_L)\theta_{0,t}\theta_L+f_{-i,t}(s_H)\theta_{0,t}\theta_H\Big]+\frac{1}{2}\big[f_{i,t}(s_L)\big]^2\theta_{0,t}\theta_L$$
$$-\delta\gamma\Big(\theta_{0,t}+f_{i,t}(s_L)\theta_{0,t}\theta_L+f_{-i,t}(s_H)\theta_{0,t}\theta_H\Big)h_i(t+1)$$
$$=\frac{1}{n}\Big[f_{i,t}(1/n)\theta_{0,t}\theta_L+f_{-i,t}(1/n)\theta_{0,t}\theta_L\Big]-\frac{1}{2}\big[f_{i,t}(1/n)\big]^2\theta_{0,t}\theta_L$$
$$+\delta\gamma\Big(\theta_{0,t}+f_{i,t}(1/n)\theta_{0,t}\theta_L+f_{-i,t}(1/n)\theta_{0,t}\theta_L\Big)h_i(t+1)$$
$$-s_L\Big[f_{i,t}(s_L)\theta_{0,t}\theta_L+f_{-i,t}(s_H)\theta_{0,t}\theta_L\Big]+\frac{1}{2}\big[f_{i,t}(s_L)\big]^2\theta_{0,t}\theta_L$$
$$-\delta\gamma\Big(\theta_{0,t}+f_{i,t}(s_L)\theta_{0,t}\theta_L+f_{-i,t}(s_H)\theta_{0,t}\theta_L\Big)h_i(t+1)$$
$$+\Big[\frac{1}{n}+\delta\gamma h_i(t+1)\Big]f_{-i,t}(1/n)\theta_{0,t}(\theta_H-\theta_L)-\Big[s_L+\delta\gamma h_i(t+1)\Big]f_{-i,t}(s_H)\theta_{0,t}(\theta_H-\theta_L),\text{ and}$$

$$\tilde{W}_{i,t}(\theta_{0,t},\theta_L,\theta_L;\theta_H;\alpha)-\tilde{W}_{i,t}(\theta_{0,t},\theta_L,\theta_L;\theta_L;\alpha)$$
$$=s_H\Big[f_{i,t}(s_H)\theta_{0,t}\theta_L+f_{-i,t}(s_L)\theta_{0,t}\theta_L\Big]-\frac{1}{2}\big[f_{i,t}(s_H)\big]^2\theta_{0,t}\theta_L$$
$$-\delta\gamma\Big(\theta_{0,t}+f_{i,t}(s_H)\theta_{0,t}\theta_L+f_{-i,t}(s_L)\theta_{0,t}\theta_L\Big)h_i(t+1)$$
$$-\frac{1}{n}\Big[f_{i,t}(1/n)\theta_{0,t}\theta_L+f_{-i,t}(1/n)\theta_{0,t}\theta_L\Big]+\frac{1}{2}\big[f_{i,t}(1/n)\big]^2\theta_{0,t}\theta_L$$
$$-\delta\gamma\Big(\theta_{0,t}+f_{i,t}(1/n)\theta_{0,t}\theta_L+f_{-i,t}(1/n)\theta_{0,t}\theta_L\Big)h_i(t+1).$$

Hence,

$$\Big[\tilde{W}_{i,t}(\theta_{0,t},\theta_L,\theta_H;\theta_H;\alpha)-\tilde{W}_{i,t}(\theta_{0,t},\theta_L,\theta_H;\theta_L;\alpha)\Big]$$
$$-\Big[\tilde{W}_{i,t}(\theta_{0,t},\theta_L,\theta_L;\theta_H;\alpha)-\tilde{W}_{i,t}(\theta_{0,t},\theta_L,\theta_L;\theta_L;\alpha)\Big]$$
$$=W_t(\theta_{0,t},\theta_L,\theta_L,\alpha)-\dots$$
$$+\Big[\frac{1}{n}+\delta\gamma h_i(t+1)\Big]f_{-i,t}(1/n)\theta_{0,t}(\theta_H-\theta_L)-\Big[s_L+\delta\gamma h_i(t+1)\Big]f_{-i,t}(s_H)\theta_{0,t}(\theta_H-\theta_L).$$

Lemma B.2 The direct revelation mechanism $\mathcal{D}^*$ is *periodic ex-post incentive compatible.*

Proof. Fix $i\in N$, $t\in\{1,\dots,T\}$, and $\theta_t\in\Theta_t$. For each $\hat{\theta}_{i,t}\in\Theta_{i,t}$ and each $e_{i,t}\in\mathbb{R}_+$,

$$u_{i,t}\big[s_t^*(\hat{\theta}_{i,t},\theta_{-i,t}),w_t^*(\hat{\theta}_{i,t},\theta_{-i,t}),e_{i,t},\sigma^*_{-i,t}(\hat{\theta}_{i,t},\theta_{-i,t}),\theta_{0,t},\theta_{i,t}\big]$$
$$+\delta\mathbb{E}\big[W_{i,t+1}(\theta_{t+1},\alpha^*)\mid e_{i,t},\sigma^*_{-i,t}(\hat{\theta}_{i,t},\theta_{-i,t})\big]$$

$$= \tilde{u}_{i,t}\big[s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), e_{i,t}, \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_{0,t}, \theta_{i,t}\big] - s_{i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}) \sum_{j\in N} w_{j,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t})$$
$$+ w_{i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}) + \delta \mathbb{E}\big[W_{i,t+1}(\theta_{t+1}, \alpha^*) \mid e_{i,t}, \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t})\big].$$

By Lemma 4.1,

$$\tilde{\sigma}_{i,t}\big[\theta_t, s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t})\big] = \arg\max_{e_{i,t}} u_{i,t}\big[s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), w_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), e_{i,t}, \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_{0,t}, \theta_{i,t}\big]$$
$$+ \delta \mathbb{E}\big[W_{i,t+1}(\theta_{t+1}, \alpha^*) \mid e_{i,t}, \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t})\big].$$

In the following, we show

$$\theta_{i,t} \in \arg\max_{\hat{\theta}_{i,t}} u_{i,t}\Big[s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), w_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), \tilde{\sigma}_{i,t}\big[\theta_t, s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t})\big], \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_{0,t}, \theta_{i,t}\Big]$$
$$+ \delta \mathbb{E}\Big[W_{i,t+1}(\theta_{t+1}, \alpha^*) \mid \tilde{\sigma}_{i,t}\big[\theta_t, s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t})\big], \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t})\Big] \quad (1),$$

which implies

$$\big[\theta_{i,t}, \sigma_{i,t}^*(\theta_t)\big] \in \arg\max_{(\hat{\theta}_{i,t}, e_{i,t})} u_{i,t}\big[s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), w_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), e_{i,t}, \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_{0,t}, \theta_{i,t}\big]$$
$$+ \delta \mathbb{E}\big[W_{i,t+1}(\theta_{t+1}, \alpha^*) \mid e_{i,t}, \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t})\big],$$

as required by *periodic ex-post incentive compatibility* $\Big[$ note that $\sigma_{i,t}^*(\theta_t) \equiv \tilde{\sigma}_{i,t}\big[\theta_t, s_t^*(\theta_t)\big]$ $\Big]$.

For each $\hat{\theta}_{i,t} \in \Theta_{i,t}$, define $Z_{i,t}(\theta_t; \hat{\theta}_{i,t}; \alpha^*) \equiv$

$$u_{i,t}\Big[s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), w_t^*(\hat{\theta}_{i,t}, \theta_{-i,t}), \tilde{\sigma}_{i,t}\big[\theta_t, s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t})\big], \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}), \theta_{0,t}, \theta_{i,t}\Big]$$
$$+ \delta \mathbb{E}\Big[W_{i,t+1}(\theta_{t+1}, \alpha^*) \mid \tilde{\sigma}_{i,t}\big[\theta_t, s_t^*(\hat{\theta}_{i,t}, \theta_{-i,t})\big], \sigma_{-i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t})\Big]$$
$$= \tilde{W}_{i,t}(\theta_t; \hat{\theta}_{i,t}; \alpha^*) - s_{i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}) \sum_{j\in N} w_{j,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}) + w_{i,t}^*(\hat{\theta}_{i,t}, \theta_{-i,t}).$$

Then showing (1) is equivalent to showing $Z_{i,t}(\theta_t; \theta_{i,t}; \alpha^*) \geq Z_{i,t}(\theta_t; \hat{\theta}_{i,t}; \alpha^*)$ for each $\hat{\theta}_{i,t}$.

Case 1. Suppose $\tilde{W}_{i,t}^e(\theta_L, \theta_H; \alpha^*) \geq \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)$ and $\theta_t = (\theta_L, \theta_L)$.

By construction of w_t^*,

$Z_{i,t}(\theta_L, \theta_L; \theta_H; \alpha^*) - Z_{i,t}(\theta_L, \theta_L; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_L, \theta_L; \theta_H; \alpha^*) - s_{i,t}^*(\theta_H, \theta_L) w_{-i,t}^*(\theta_H, \theta_L) - \tilde{W}_{i,t}(\theta_L, \theta_L; \theta_L; \alpha^*)$

$= \big[\tilde{W}_{i,t}(\theta_L, \theta_L; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_L; \theta_L; \alpha^*)\big] - \big[\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big]$

≤ 0 (by Lemma 5.1).

Case 2. Suppose $\tilde{W}_{i,t}^e(\theta_L, \theta_H; \alpha^*) \geq \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)$ and $\theta_t = (\theta_H, \theta_H)$.

By construction of w_t^*,

$Z_{i,t}(\theta_H, \theta_H; \theta_H; \alpha^*) - Z_{i,t}(\theta_H, \theta_H; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_H, \theta_H; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_H, \theta_H; \theta_L; \alpha^*) - \big[1 - s_{i,t}^*(\theta_L, \theta_H)\big] w_{i,t}^*(\theta_L, \theta_H)$

$= \big[\tilde{W}^e_{i,t}(\theta_H, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_H, \theta_H; \theta_L; \alpha^*)\big] - \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big]$

> 0 (by Lemma 4.2).

Case 3. Suppose $\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) \geq \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)$ and $\theta_t = (\theta_L, \theta_H)$.

By construction of w^*_t,

$Z_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - Z_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \big[1 - s^*_{i,t}(\theta_L, \theta_H)\big] w^*_{i,t}(\theta_L, \theta_H)$

$= \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big] - \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big] = 0,$

and

$Z_{i,t}(\theta_H, \theta_L; \theta_H; \alpha^*) - Z_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_H, \theta_L; \theta_H; \alpha^*) - s^*_{i,t}(\theta_H, \theta_L) w^*_{i,t}(\theta_L, \theta_H) - \tilde{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_H, \theta_L; \theta_H; \alpha^*) - \tilde{W}^e_{i,t}(\theta_H, \theta_L; \alpha^*) - \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big]$

$= \big[\tilde{W}_{i,t}(\theta_H, \theta_L; \theta_H; \alpha^*) + \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big] - \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) + \tilde{W}^e_{i,t}(\theta_H, \theta_L; \alpha^*)\big]$

$= V_t(\theta_L, \theta_H; \alpha^*) - V^e_t(\theta_L, \theta_H; \alpha^*) \geq 0$

(since $V_t(\theta_L, \theta_H; \alpha^*)$ is the aggregate payoff given by the *constrained-efficient* arrangement rule α^* when agents have types (θ_L, θ_H) in period t).

Case 4. Suppose $\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) < \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)$ and $\theta_t = (\theta_L, \theta_L)$.

By construction of w^*_t,

$Z_{i,t}(\theta_L, \theta_L; \theta_H; \alpha^*) - Z_{i,t}(\theta_L, \theta_L; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_L, \theta_L; \theta_H; \alpha^*) + \big[1 - s^*_{i,t}(\theta_H, \theta_L)\big] w^*_{i,t}(\theta_H, \theta_L) - \tilde{W}_{i,t}(\theta_L, \theta_L; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_L, \theta_L; \theta_H; \alpha^*) + \big[\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*)\big] - \tilde{W}_{i,t}(\theta_L, \theta_L; \theta_L; \alpha^*)$

$= \big[\tilde{W}_{i,t}(\theta_L, \theta_L; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_L; \theta_L; \alpha^*)\big] - \big[\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big]$

≤ 0 (by Lemma 5.1).

Case 5. Suppose $\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) < \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)$ and $\theta_t = (\theta_H, \theta_H)$.

By construction of w^*_t,

$Z_{i,t}(\theta_H, \theta_H; \theta_H; \alpha^*) - Z_{i,t}(\theta_H, \theta_H; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_H, \theta_H; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_H, \theta_H; \theta_L; \alpha^*) + s^*_{i,t}(\theta_L, \theta_H) w^*_{i,t}(\theta_H, \theta_L)$

$= \big[\tilde{W}^e_{i,t}(\theta_H, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_H, \theta_H; \theta_L; \alpha^*)\big] + \big[\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*)\big]$

$= \big[\tilde{W}^e_{i,t}(\theta_H, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_H, \theta_H; \theta_L; \alpha^*)\big] - \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big]$

> 0 (by Lemma 4.2).

Case 6. Suppose $\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) < \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)$ and $\theta_t = (\theta_L, \theta_H)$.

By construction of w^*_t,

$Z_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - Z_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) + s^*_{i,t}(\theta_L, \theta_H) w^*_{i,t}(\theta_H, \theta_L)$

$= \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big] + \big[\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*)\big]$

$= \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big] - \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big] = 0,$

and

$Z_{i,t}(\theta_H, \theta_L; \theta_H; \alpha^*) - Z_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_H, \theta_L; \theta_H; \alpha^*) + \big[1 - s^*_{i,t}(\theta_H, \theta_L)\big] w^*_{i,t}(\theta_H, \theta_L) - \tilde{W}_{i,t}(\theta_H, \theta_L; \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_H, \theta_L; \theta_H; \alpha^*) + \big[\tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*) - \tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*)\big] - \tilde{W}^e_{i,t}(\theta_H, \theta_L; \alpha^*)$

$= \tilde{W}_{i,t}(\theta_H, \theta_L; \theta_H; \alpha^*) - \tilde{W}^e_{i,t}(\theta_H, \theta_L; \alpha^*) - \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) - \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big]$

$= \big[\tilde{W}_{i,t}(\theta_H, \theta_L; \theta_H; \alpha^*) + \tilde{W}_{i,t}(\theta_L, \theta_H; \theta_L; \alpha^*)\big] - \big[\tilde{W}^e_{i,t}(\theta_L, \theta_H; \alpha^*) + \tilde{W}^e_{i,t}(\theta_H, \theta_L; \alpha^*)\big]$

$= V_t(\theta_L, \theta_H; \alpha^*) - V^e_t(\theta_L, \theta_H; \alpha^*) \geq 0$

(since $V_t(\theta_L, \theta_H; \alpha^*)$ is the aggregate payoff given by the *constrained-efficient* arrangement rule α^* when agents have types (θ_L, θ_H) in period t). ■

Lemma B.3 The direct revelation mechanism $\mathcal{D}^*$ *periodically ex-post Pareto dominates* the equal division mechanism $\mathcal{D}^e$.

Proof. Suppose that the following holds for some $t \in \{2, \ldots, T\}$: for each $i \in N$, each $\theta_t \in \Theta_t$, and each $(\theta_{-0,\tau})_{\tau>t}$, we have $\mathcal{U}_{i,t}\big[(\theta_{-0,\tau})_{\tau>t}, \alpha^* \mid \theta_t\big] \geq \mathcal{U}_{i,t}\big[(\theta_{-0,\tau})_{\tau>t}, \alpha^e \mid \theta_t\big]$.

Step 1. We show that for each $i \in N$, each $\theta_{t-1} \in \Theta_{t-1}$, and each $(\theta_{-0,\tau})_{\tau>t-1}$,

$$\mathcal{U}_{i,t-1}\big[(\theta_{-0,\tau})_{\tau>t-1}, \alpha^* \mid \theta_{t-1}\big] \geq \mathcal{U}_{i,t-1}\big[(\theta_{-0,\tau})_{\tau>t-1}, \alpha^e \mid \theta_{t-1}\big].$$

Case 1. Suppose $\theta_{t-1} = (\theta_L, \theta_H)$ and $\tilde{W}^e_{i,t-1}(\theta_{t-1}) \geq \tilde{W}_{i,t-1}(\theta_{t-1}; \theta_L; \alpha^*)$. Let

$$\theta^*_{0,t} \equiv \gamma[\theta_{0,t-1} + \textstyle\sum_{j\in N} \sigma^*_{t-1}(\theta_{t-1})] \text{ and } \theta^e_{0,t} \equiv \gamma[\theta_{0,t-1} + \textstyle\sum_{j\in N} \sigma^e_{t-1}(\theta_{t-1})].$$

Then

$\mathcal{U}_{i,t-1}\big[(\theta_{-0,\tau})_{\tau>t-1}, \alpha^* \mid \theta_{t-1}\big] - \mathcal{U}_{i,t-1}\big[(\theta_{-0,\tau})_{\tau>t-1}, \alpha^e \mid \theta_{t-1}\big]$

$= u_{i,t-1}\big[\alpha^*_{t-1}(\theta_{t-1}), \theta_{0,t-1}, \theta_{i,t-1}\big] + \delta \mathcal{U}_{i,t}\big[(\theta_{-0,\tau})_{\tau>t}, \alpha^* \mid \theta^*_{0,t}, \theta_{-0,t}\big]$

$$-u_{i,t-1}\big[\alpha^e_{t-1}(\theta_{t-1}),\theta_{0,t-1},\theta_{i,t-1}\big]-\delta\mathcal{U}_{i,t}\big[(\theta_{-0,\tau})_{\tau>t},\alpha^e\mid\theta^e_{0,t},\theta_{-0,t}\big]$$
$$=u_{i,t-1}\big[s^*_{t-1}(\theta_{t-1}),\sigma^*_{t-1}(\theta_{t-1}),\theta_{0,t-1},\theta_{i,t-1}\big]+\delta\mathcal{U}_{i,t}\big[(\theta_{-0,\tau})_{\tau>t},\alpha^*\mid\theta^*_{0,t},\theta_{-0,t}\big]$$
$$-u_{i,t-1}\big[s^e_{t-1}(\theta_{t-1}),\sigma^e_{t-1}(\theta_{t-1}),\theta_{0,t-1},\theta_{i,t-1}\big]-\delta\mathcal{U}_{i,t}\big[(\theta_{-0,\tau})_{\tau>t},\alpha^e\mid\theta^e_{0,t},\theta_{-0,t}\big]$$
$$+\tilde{W}^e_{i,t-1}(\theta_{t-1})-\tilde{W}_{i,t-1}(\theta_{t-1};\theta_L;\alpha^*)$$
$$=u_{i,t-1}\big[s^*_{t-1}(\theta_{t-1}),\sigma^*_{t-1}(\theta_{t-1}),\theta_{0,t-1},\theta_{i,t-1}\big]+\delta\mathcal{U}_{i,t}\big[(\theta_{-0,\tau})_{\tau>t},\alpha^*\mid\theta^*_{0,t},\theta_{-0,t}\big]$$
$$\delta\theta^e_{0,t}h_i(t)-\delta\mathcal{U}_{i,t}\big[(\theta_{-0,\tau})_{\tau>t},\alpha^e\mid\theta^e_{0,t},\theta_{-0,t}\big]$$
$$+\tilde{W}^e_{i,t-1}(\theta_{t-1})-\tilde{W}_{i,t-1}(\theta_{t-1};\theta_L;\alpha^*)$$

B.3 Finite Horizon

Lemma B.4 Fix $i\in N$ and $\theta\in\Theta$ such that $\theta_i=\theta_L$ and $\theta_j=\theta_H$ for some $j\neq i$. Let $\theta'\in\Theta$ such that $\theta'_j=\theta_L$ and $\theta'_k=\theta_k$ for each $k\in N\setminus\{j\}$. Let $n^*\equiv\eta(\theta_{-i})$. We have

$$\rho(n^*-1)\big[W_i(\theta_H,\theta'_{-i})-W^e_i(\theta_H,\theta'_{-i})\big]\geq\rho(n^*)\big[W^e_i(\theta)-W_i(\theta)\big].$$

Proof. We have

$$\rho(n^*-1)\big[W_i(\theta_H,\theta'_{-i})-W^e_i(\theta_H,\theta'_{-i})\big]-\rho(n^*)\big[W^e_i(\theta)-W_i(\theta)\big]$$
$$=\rho(n^*-1)\Bigg(\big[W_i(\theta_H,\theta'_{-i})-W^e_i(\theta_H,\theta'_{-i})\big]-\frac{n-n^*}{n^*}\big[W^e_i(\theta)-W_i(\theta)\big]\Bigg)$$
$$=\frac{\rho(n^*-1)}{n^*}\Big(n^*\big[W_i(\theta_H,\theta'_{-i})-W^e_i(\theta_H,\theta'_{-i})\big]-(n-n^*)\big[W^e_i(\theta)-W_i(\theta)\big]\Big)$$
$$=\frac{\rho(n^*-1)}{n^*}\Big(\big[n^*W_i(\theta_H,\theta'_{-i})+(n-n^*)W_i(\theta)\big]-\big[n^*W^e_i(\theta_H,\theta'_{-i})+(n-n^*)W^e_i(\theta)\big]\Big)\geq 0.$$

We note that n^* agents have type θ_H and $n-n^*$ agents have type θ_L in the type profile θ. The inequality comes from the fact that $\big[n^*W_i(\theta_H,\theta'_{-i})+(n-n^*)W_i(\theta)\big]$ is the aggregate payoff when the constrained-efficient equity-effort rule (s^*,σ^*) is used, whereas $\big[n^*W^e_i(\theta_H,\theta'_{-i})+(n-n^*)W^e_i(\theta)\big]$ is the aggregate payoff when the equal sharing rule is used. ■

The following lemma claims that an agent who has type θ_L always prefers to report truthfully in expectation.

Lemma B.5 For each $i\in N$, we have $\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_L,\theta_{-i};\theta_L)\geq\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_L,\theta_{-i};\theta_H)$.

Proof. Agent i's expected payoff from reporting truthfully is

$\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_L,\theta_{-i};\theta_L)=$

$$\sum_{n'=0}^{n-1} \rho(n')\mu(\theta_L)^{n-n'-1}\mu(\theta_H)^{n'} \Big[W_i\big[\theta_L,\xi_{-i}(n')\big] + \Big(1-(n-n')s_i^*\big[\theta_L,\xi_{-i}(n')\big]\Big)w_i^*\big[\theta_L,\xi_{-i}(n')\big]\Big].$$

Agent i's expected payoff from reporting θ_H is

$$\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_L,\theta_{-i};\theta_H) = \sum_{n'=0}^{n-1} \rho(n')\mu(\theta_L)^{n-n'-1}\mu(\theta_H)^{n'} \Big[\tilde{W}_i\big[\theta_L,\xi_{-i}(n');\theta_H\big] - (n-n'-1)s_i^*\big[\theta_H,\xi_{-i}(n')\big]w_i^*\big[\theta_L,\xi_{-i}(n'+1)\big]\Big].$$

For each $n^* \in \{1,\ldots,n-1\}$, let

$$f(n^*) = \sum_{n'=0}^{n^*} \rho(n')\mu(\theta_L)^{n-n'-1}\mu(\theta_H)^{n'} \Big[W_i\big[\theta_L,\xi_{-i}(n')\big] - \tilde{W}_i\big[\theta_L,\xi_{-i}(n');\theta_H\big]$$
$$+\Big(1-(n-n')s_i^*\big[\theta_L,\xi_{-i}(n')\big]\Big)w_i^*\big[\theta_L,\xi_{-i}(n')\big] + (n-n'-1)s_i^*\big[\theta_H,\xi_{-i}(n')\big]w_i^*\big[\theta_L,\xi_{-i}(n'+1)\big]\Big].$$

Then $\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_L,\theta_{-i};\theta_L) - \mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_L,\theta_{-i};\theta_H) = f(n-1)$. We show $f(n-1)\geq 0$ by induction.

Step 1. We have

$$f(1) = \mu(\theta_L)^{n-1}\Big(W_i\big[\theta_L,\xi_{-i}(0)\big] - \tilde{W}_i\big[\theta_L,\xi_{-i}(0);\theta_H\big] + (n-1)s_i^*\big[\theta_H,\xi_{-i}(0)\big]w_i^*\big[\theta_L,\xi_{-i}(1)\big]\Big)$$
$$+\rho(1)\mu(\theta_L)^{n-2}\mu(\theta_H)\Big(W_i\big[\theta_L,\xi_{-i}(1)\big] - \tilde{W}_i\big[\theta_L,\xi_{-i}(1);\theta_H\big]$$
$$+s_i^*\big[\theta_H,\xi_{-i}(0)\big]w_i^*\big[\theta_L,\xi_{-i}(1)\big] + (n-2)s_i^*\big[\theta_H,\xi_{-i}(1)\big]w_i^*\big[\theta_L,\xi_{-i}(2)\big]\Big)$$
$$= \mu(\theta_L)^{n-2}\Big\{\big[(n-1)\mu(\theta_L) + \rho(1)\mu(\theta_H)\big]s_i^*\big[\theta_H,\xi_{-i}(0)\big]w_i^*\big[\theta_L,\xi_{-i}(1)\big]$$
$$-\mu(\theta_L)\Big(\tilde{W}_i\big[\theta_L,\xi_{-i}(0);\theta_H\big] - W_i^e\big[\theta_L,\xi_{-i}(0)\big]\Big) - \rho(1)\mu(\theta_H)\Big(W_i^e\big[\theta_L,\xi_{-i}(1)\big] - W_i\big[\theta_L,\xi_{-i}(1)\big]\Big)\Big\}$$
$$+\rho(1)\mu(\theta_L)^{n-2}\mu(\theta_H)\Big\{(n-2)s_i^*\big[\theta_H,\xi_{-i}(1)\big]w_i^*\big[\theta_L,\xi_{-i}(2)\big] - \Big(\tilde{W}_i\big[\theta_L,\xi_{-i}(1);\theta_H\big] - W_i^e\big[\theta_L,\xi_{-i}(1)\big]\Big)\Big\}.$$

Step 2. Suppose $n\geq 3$. Fix $n^*\in\{1,\ldots,n-2\}$.

Let $g(n^*) \equiv$

$$\sum_{n'=0}^{n^*-1} \mu(\theta_L)^{n-n'-2}\mu(\theta_H)^{n'}\Big\{\big[\rho(n')(n-n'-1)\mu(\theta_L) + \rho(n'+1)(n'+1)\mu(\theta_H)\big]s_i^*\big[\theta_H,\xi_{-i}(n')\big]w_i^*\big[\theta_L,\xi_{-i}(n'+1)\big]$$
$$-\rho(n')\mu(\theta_L)\Big(\tilde{W}_i\big[\theta_L,\xi_{-i}(n');\theta_H\big] - W_i^e\big[\theta_L,\xi_{-i}(n')\big]\Big)$$
$$-\rho(n'+1)\mu(\theta_H)\Big(W_i^e\big[\theta_L,\xi_{-i}(n'+1)\big] - W_i\big[\theta_L,\xi_{-i}(n'+1)\big]\Big)\Big\}.$$

Suppose

$$f(n^*) = g(n^*) + \rho(n^*)\mu(\theta_L)^{n-n^*-1}\mu(\theta_H)^{n^*}\Big\{(n-n^*-1)s_i^*\big[\theta_H,\xi_{-i}(n^*)\big]w_i^*\big[\theta_L,\xi_{-i}(n^*+1)\big]$$
$$-\Big(\tilde{W}_i\big[\theta_L,\xi_{-i}(n^*);\theta_H\big] - W_i^e\big[\theta_L,\xi_{-i}(n^*)\big]\Big)\Big\}.$$

We have

$$f(n^*+1) = g(n^*) + \rho(n^*)\mu(\theta_L)^{n-n^*-1}\mu(\theta_H)^{n^*}\Big\{(n-n^*-1)s_i^*\big[\theta_H,\xi_{-i}(n^*)\big]w_i^*\big[\theta_L,\xi_{-i}(n^*+1)\big]$$
$$-\Big(\tilde{W}_i\big[\theta_L,\xi_{-i}(n^*);\theta_H\big] - W_i^e\big[\theta_L,\xi_{-i}(n^*)\big]\Big)\Big\}$$

$$+\rho(n^*+1)\mu(\theta_L)^{n-n^*-2}\mu(\theta_H)^{n^*+1}\Big[W_i\big[\theta_L,\xi_{-i}(n^*+1)\big]-\tilde{W}_i\big[\theta_L,\xi_{-i}(n^*+1);\theta_H\big]$$

$$+\Big(1-(n-n^*-1)s_i^*\big[\theta_L,\xi_{-i}(n^*+1)\big]\Big)w_i^*\big[\theta_L,\xi_{-i}(n^*+1)\big]+(n-n^*-2)s_i^*\big[\theta_H,\xi_{-i}(n^*+1)\big]w_i^*\big[\theta_L,\xi_{-i}(n^*+2)\big]\Big]$$

$$=g(n^*+1)+\rho(n^*+1)\mu(\theta_L)^{n-n^*-2}\mu(\theta_H)^{n^*+1}\Big\{(n-n^*-2)s_i^*\big[\theta_H,\xi_{-i}(n^*+1)\big]w_i^*\big[\theta_L,\xi_{-i}(n^*+2)\big]$$

$$-\Big(\tilde{W}_i\big[\theta_L,\xi_{-i}(n^*+1);\theta_H\big]-W_i^e\big[\theta_L,\xi_{-i}(n^*+1)\big]\Big)\Big\}.$$

Step 3. Note that $\tilde{W}_i\big[\theta_L,\xi_{-i}(n-1);\theta_H\big]=W_i^e\big[\theta_L,\xi_{-i}(n-1)\big]$. It follows from Steps 1 and 2 that $f(n-1)=g(n-1)$. By construction of w^*, we have $f(n-1)\geq 0$. ■

The following lemma claims that an agent who has type θ_H always prefers to report truthfully in expectation.

Lemma B.6 For each $i\in N$, we have $\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_H,\theta_{-i};\theta_H)\geq\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_H,\theta_{-i};\theta_L)$.

Proof. Agent i's expected payoff from reporting truthfully is

$$\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_H,\theta_{-i};\theta_H)=$$

$$\sum_{n'=0}^{n-1}\rho(n')\mu(\theta_L)^{n-n'-1}\mu(\theta_H)^{n'}\Big[W_i\big[\theta_H,\xi_{-i}(n')\big]-(n-n'-1)s_i^*\big[\theta_H,\xi_{-i}(n')\big]w_i^*\big[\theta_L,\xi_{-i}(n'+1)\big]\Big].[22]$$

Agent i's expected payoff from reporting θ_L is

$$\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_H,\theta_{-i};\theta_L)=$$

$$\sum_{n'=0}^{n-1}\rho(n')\mu(\theta_L)^{n-n'-1}\mu(\theta_H)^{n'}\Big[\tilde{W}_i\big[\theta_H,\xi_{-i}(n');\theta_L\big]+\Big(1-(n-n')s_i^*\big[\theta_L,\xi_{-i}(n')\big]\Big)w_i^*\big[\theta_L,\xi_{-i}(n')\big]\Big].$$

For each $n^*\in\{1,\dots,n-1\}$, let

$$f(n^*)=\sum_{n'=0}^{n^*}\rho(n')\mu(\theta_L)^{n-n'-1}\mu(\theta_H)^{n'}\Big[W_i\big[\theta_H,\xi_{-i}(n')\big]-\tilde{W}_i\big[\theta_H,\xi_{-i}(n');\theta_L\big]$$

$$-(n-n'-1)s_i^*\big[\theta_H,\xi_{-i}(n')\big]w_i^*\big[\theta_L,\xi_{-i}(n'+1)\big]-\Big(1-(n-n')s_i^*\big[\theta_L,\xi_{-i}(n')\big]\Big)w_i^*\big[\theta_L,\xi_{-i}(n')\big]\Big].$$

Then $\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_H,\theta_{-i};\theta_H)-\mathbb{E}_{\theta_{-i}}\tilde{W}_i(\theta_H,\theta_{-i};\theta_L)=f(n-1)$. We show $f(n-1)\geq 0$ by induction.

Step 1. We have

$$f(1)=\mu(\theta_L)^{n-1}\Big(W_i\big[\theta_H,\xi_{-i}(0)\big]-\tilde{W}_i\big[\theta_H,\xi_{-i}(0);\theta_L\big]-(n-1)s_i^*\big[\theta_H,\xi_{-i}(0)\big]w_i^*\big[\theta_L,\xi_{-i}(1)\big]\Big)$$

$$+\rho(1)\mu(\theta_L)^{n-2}\mu(\theta_H)\Big(W_i\big[\theta_H,\xi_{-i}(1)\big]-\tilde{W}_i\big[\theta_H,\xi_{-i}(1);\theta_L\big]$$

$$-s_i^*\big[\theta_H,\xi_{-i}(0)\big]w_i^*\big[\theta_L,\xi_{-i}(1)\big]-(n-2)s_i^*\big[\theta_H,\xi_{-i}(1)\big]w_i^*\big[\theta_L,\xi_{-i}(2)\big]\Big)$$

$$=\mu(\theta_L)^{n-2}\Big\{-\big[(n-1)\mu(\theta_L)+\rho(1)\mu(\theta_H)\big]s_i^*\big[\theta_H,\xi_{-i}(0)\big]w_i^*\big[\theta_L,\xi_{-i}(1)\big]$$

[22]When $n'=n-1$, $\xi_{-i}(n')$ is undefined. However, for notational convenience, let $w_i^*\big[\theta_L,\xi_{-i}(n)\big]=0$.

$$+\mu(\theta_L)\Big(W_i\big[\theta_H,\xi_{-i}(0)\big]-W_i^e\big[\theta_H,\xi_{-i}(0)\big]\Big)+\rho(1)\mu(\theta_H)\Big(W_i^e\big[\theta_H,\xi_{-i}(1)\big]-\tilde{W}_i\big[\theta_H,\xi_{-i}(1);\theta_L\big]\Big)\Big\}$$

$$+\rho(1)\mu(\theta_L)^{n-2}\mu(\theta_H)\Big\{-(n-2)s_i^*\big[\theta_H,\xi_{-i}(1)\big]w_i^*\big[\theta_L,\xi_{-i}(2)\big]+\Big(W_i\big[\theta_H,\xi_{-i}(1)\big]-W_i^e\big[\theta_H,\xi_{-i}(1)\big]\Big)\Big\}.$$

Step 2. Suppose $n\geq 3$. Fix $n^*\in\{1,\dots,n-2\}$.

Let $g(n^*)\equiv$

$$\sum_{n'=0}^{n^*-1}\mu(\theta_L)^{n-n'-2}\mu(\theta_H)^{n'}\Big\{-\big[\rho(n')(n-n'-1)\mu(\theta_L)+\rho(n'+1)(n'+1)\mu(\theta_H)\big]s_i^*\big[\theta_H,\xi_{-i}(n')\big]w_i^*\big[\theta_L,\xi_{-i}(n'+1)\big]$$

$$+\rho(n')\mu(\theta_L)\Big(W_i\big[\theta_H,\xi_{-i}(n')\big]-W_i^e\big[\theta_H,\xi_{-i}(n')\big]\Big)$$

$$+\rho(n'+1)\mu(\theta_H)\Big(W_i^e\big[\theta_H,\xi_{-i}(n'+1)\big]-\tilde{W}_i\big[\theta_H,\xi_{-i}(n'+1);\theta_L\big]\Big)\Big\}.$$

Suppose

$$f(n^*)=g(n^*)+\rho(n^*)\mu(\theta_L)^{n-n^*-1}\mu(\theta_H)^{n^*}\Big\{-(n-n^*-1)s_i^*\big[\theta_H,\xi_{-i}(n^*)\big]w_i^*\big[\theta_L,\xi_{-i}(n^*+1)\big]$$
$$+\Big(W_i\big[\theta_H,\xi_{-i}(n^*)\big]-W_i^e\big[\theta_H,\xi_{-i}(n^*)\big]\Big)\Big\}.$$

We have

$$f(n^*+1)=g(n^*)+\rho(n^*)\mu(\theta_L)^{n-n^*-1}\mu(\theta_H)^{n^*}\Big\{-(n-n^*-1)s_i^*\big[\theta_H,\xi_{-i}(n^*)\big]w_i^*\big[\theta_L,\xi_{-i}(n^*+1)\big]$$
$$+\Big(W_i\big[\theta_H,\xi_{-i}(n^*)\big]-W_i^e\big[\theta_H,\xi_{-i}(n^*)\big]\Big)\Big\}$$
$$+\rho(n^*+1)\mu(\theta_L)^{n-n^*-2}\mu(\theta_H)^{n^*+1}\Big[W_i\big[\theta_H,\xi_{-i}(n^*+1)\big]-\tilde{W}_i\big[\theta_H,\xi_{-i}(n^*+1);\theta_L\big]$$
$$-(n-n^*-2)s_i^*\big[\theta_H,\xi_{-i}(n^*+1)\big]w_i^*\big[\theta_L,\xi_{-i}(n^*+2)\big]-(n^*+1)s_i^*\big[\theta_H,\xi_{-i}(n^*)\big]w_i^*\big[\theta_L,\xi_{-i}(n^*+1)\big]\Big]$$
$$=g(n^*+1)+\rho(n^*+1)\mu(\theta_L)^{n-n^*-2}\mu(\theta_H)^{n^*+1}\Big\{-(n-n^*-2)s_i^*\big[\theta_H,\xi_{-i}(n^*+1)\big]w_i^*\big[\theta_L,\xi_{-i}(n^*+2)\big]$$
$$+\Big(W_i\big[\theta_H,\xi_{-i}(n^*+1)\big]-W_i^e\big[\theta_H,\xi_{-i}(n^*+1)\big]\Big)\Big\}.$$

Step 3. Note that $W_i\big[\theta_H,\xi_{-i}(n^*+1)\big]=W_i^e\big[\theta_H,\xi_{-i}(n^*+1)\big]$. It follows from Steps 1 and 2 that $f(n-1)=g(n-1)$. By construction of w^*, we have $f(n-1)\geq 0$. ■

C A Voting Mechanism

For each $i\in N$, let $A_i^*\equiv\begin{bmatrix} a_{11}^{i*} & a_{12}^{i*} & \cdots & a_{1L}^{i*} \\ a_{21}^{i*} & a_{22}^{i*} & \cdots & a_{2L}^{i*} \\ \vdots & \vdots & \ddots & \vdots \\ a_{K1}^{i*} & a_{K2}^{i*} & \cdots & a_{KL}^{i*}\end{bmatrix}$.

Lemma C.1 For each $i\in N$, all entries of A_i^* are distinct and $\tilde{A}_i^*=\tilde{A}_j^*$ for each $j\neq i$.

Proof.

Step 1. We show that all entries of A_i^* are distinct. Take two entries a_{kl}^{i*} and $a_{k'l'}^{i*}$. By construction, $a_{kl}^{i*} \equiv \alpha^*(\theta)$ and $a_{k'l'}^{i*} \equiv \alpha^*(\theta')$ for some $\theta \neq \theta'$. Let $j \in N$ such that $\theta_j \neq \theta'_j$. Without loss of generality, assume $\theta_j > \theta'_j$. If $s^*(\theta) \neq s^*(\theta')$, then it is clear that $\alpha^*(\theta) \neq \alpha^*(\theta')$. If $s^*(\theta) = s^*(\theta')$, then $\sigma_j^*(\theta) < \sigma_j^*(\theta')$ (by Lemma 3.2), which implies $\alpha^*(\theta) \neq \alpha^*(\theta')$.

Step 2. By construction, $\tilde{A}_i^* = \{\alpha^*(\theta) \mid \theta \in \Theta^n\}$ for each $i \in N$, which implies $\tilde{A}_i^* = \tilde{A}_j^*$ for $i \neq j$. ■

Proof for Theorem 6.1.

We construct a Nash equilibrium λ^* of voting mechanism $\mathscr{V}^*$ as follows. For each $i \in N$ and each $\theta_i \in \Theta_i$,

- $\lambda_i^*(\theta_i)\big[\alpha^*(\theta_i, \theta_{-i})\big] = 1$ for each $\theta_{-i} \in \Theta_{-i}$,
- $\lambda_i^*(\theta_i, \nu) \in \arg\max_{e_i \in \mathbb{R}_+} v_i\big[\xi^{\pi*}(\nu), e_i, \xi_{-i}^{e*}(\nu), \theta_i\big]$ for each $\nu \in \mathcal{V}$.

Step 1. Fix $\theta \in \Theta$. We show that $\xi^*\big[\lambda_1^*(\theta_1), \ldots, \lambda_n^*(\theta_n)\big] = \alpha^*(\theta)$. By construction, $\tilde{A}^* = \{\alpha^*(\theta') \mid \theta' \in \Theta^n\}$ and

$$\sum_{i \in N} \lambda_i^*(\theta_i)\big[\alpha^*(\theta')\big] = |\{i \in N \mid \theta_i = \theta'_i\}|.$$

It follows that $\alpha^*(\theta) = \arg\max_{a \in \tilde{A}^*} \sum_{i \in N} \lambda_i^*(\theta_i)\big[a\big]$.

Step 2. We show that λ^* is a Nash equilibrium of $\mathscr{V}^*$. Let $\theta \in \Theta$ be true type profile. Fix $i \in N$.

Let $\nu^* \equiv [\lambda_1^*(\theta_1), ..., \lambda_n^*(\theta_n)]$. By definition, agent i's payoff given strategy profile λ^* is

$$\phi_i(\lambda^*, \theta) = \varphi_i\big[\nu^*, \big(\lambda_j^*(\theta_j, \nu^*)\big)_{j \in N}, \theta_i\big] = u_i\big[\xi^{\pi*}(\nu^*), \big(\lambda_j^*(\theta_j, \nu^*)\big)_{j \in N}, \theta_i\big].$$

By Step 1, $\xi^{\pi*}(\nu^*) = [s^*(\theta), w^*(\theta)]$. By construction, for each $j \in N$,

$$\lambda_j^*(\theta_j, \nu^*) \in \arg\max_{e_j \in \mathbb{R}_+} u_j\big[s^*(\theta), w^*(\theta), e_j, \sigma_{-j}^*(\theta), \theta_j\big].$$

By definition, $\lambda_j^*(\theta_j, \nu^*) = \sigma_j^*(\theta)$. It follows that

$$\phi_i(\lambda^*, \theta) = u_i\big[s^*(\theta), w^*(\theta), \sigma^*(\theta), \theta_i\big].$$

Suppose agent i chooses some strategy $\lambda'_i \in \Lambda_i$ while other agents choose λ_{-i}^*. Let $\nu' \equiv \big[\lambda'_i(\theta_i), \big(\lambda_j^*(\theta_j)\big)_{j \neq i}\big]$. By construction, for each $\theta'_i \in \Theta$,

$$\sum_{j \in N} \nu'_j\big[\alpha^*(\theta'_i, \theta_{-i})\big] = n - 1 + \lambda'_i(\theta_i)\big[\alpha^*(\theta'_i, \theta_{-i})\big].$$

Since $\lambda'_i(\theta_i)\big[\alpha^*(\theta'_i, \theta_{-i})\big] \in \{0, 1\}$ for each $\theta'_i \in \Theta$ and $\sum_{\theta'_i \in \Theta} \lambda'_i(\theta_i)\big[\alpha^*(\theta'_i, \theta_{-i})\big] = 1$, there is a unique $a \in \{\alpha^*(\theta'_i, \theta_{-i}) \mid \theta'_i \in \Theta\}$ such that $\sum_{j \in N} \nu'_j(a) = n$. For each $\theta' \in \Theta^n$ such that $\theta'_{-i} \neq \theta_{-i}$,

$$\sum_{j \in N} \nu'_j\big[\alpha^*(\theta')\big] = |\{j \in N \setminus \{i\} \mid \theta_j = \theta'_j\}| + \lambda'_i(\theta_i)\big[\alpha^*(\theta')\big] \leq n - 1.$$

Hence, $\xi^*(\nu') = \alpha^*(\theta'_i, \theta_{-i})$ for some $\theta'_i \in \Theta$. By construction of λ^*, for each $j \in N \setminus \{i\}$,

$$\lambda^*_j(\theta_j, \nu') \in \arg\max_{e_j \in \mathbb{R}_+} u_j\big[s^*(\theta'_i, \theta_{-i}), w^*(\theta'_i, \theta_{-i}), e_j, \sigma^*_{-j}(\theta'_i, \theta_{-i}), \theta_j\big].$$

By definition of σ^*, $\lambda^*_j(\theta_j, \nu') = \sigma^*_j(\theta'_i, \theta_{-i})$. It follows that

$$\begin{aligned}\phi_i(\lambda'_i, \lambda^*_{-i}, \theta) &= \varphi_i\big[\nu', \lambda'_i(\theta_i, \nu'), \big(\lambda^*_j(\theta_j, \nu')\big)_{j \neq i}, \theta_i\big] \\ &= u_i\big[s^*(\theta'_i, \theta_{-i}), w^*(\theta'_i, \theta_{-i}), \lambda'_i(\theta_i, \nu'), \sigma^*_{-i}(\theta'_i, \theta_{-i}), \theta_i\big].\end{aligned}$$

By *ex post incentive compatibility*, for each $\theta_i \in \Theta$, we have $\phi_i(\lambda^*, \theta) \geq \phi_i(\lambda'_i, \lambda^*_{-i}, \theta)$. ■