INFORMATION SPILLOVER IN A BAYESIAN REPEATED SETTING: LACK OF INFORMATION ON TWO-SIDES

LUCAS PAHL

Abstract. In this paper we consider an infinitely repeated three-player Bayesian game with lack of information on two sides, in which an informed player plays two zero-sum games simultaneously at each stage against two uninformed players. In this game, under a correlated prior, the informed player faces the problem of how to optimally disclose information among the two other uninformed players in order to maximize his long term average payoffs. The objective is to understand the effects of the "information spillover" from one game to the other in the Nash-equilibrium payoff set of the informed player. The main results are a sufficient condition under which the "best possible payoff" can always be obtained by the informed player, and an example under which the "best possible payoff" is not attainable.

1. Introduction and Literature Review

The game that we consider in this paper is closely related to the theory of repeated two-person zero-sum games with lack of information on one side developed in the seminal work of Aumman, Maschler and Stearns (1967-1968). In the game that we study we consider not one, but two uninformed players who play at each stage of the repeated game a zero-sum game against one informed player. The informed player collects the payoff of each one of the stage games, and the uninformed players only care (payoffwise) about their own stage games. What is interesting about this setting is that players are allowed to observe all the actions (but not the payoffs) realized at each stage of the repeated game. This introduces new strategic problems for the informed player, since the use of information against one uninformed player in one game is observed by the other uninformed player. This is the

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phenomenon we call "information spillover": the ability of an uninformed player to learn payoff-relevant hidden information from payoff-irrelevant information.

Theoretically, we could argue that the modeling of the effects of the information spillover within a zero-sum game framework enables us to draw on preexisting techniques from the literature on zero-sum games in order to settle benchmark results: zero-sum games are usually easier to analyze than their non zero-sum counterparts, and this allows us to obtain clean results before developing a non zero-sum theory of this phenomenon.

Empirically, this framework allows us to model a number of relevant political phenomena. As an illustration - and following the tradition of Aumann-Maschler - we could consider a situation where a superpower - such as the USA - negotiates an agreement with two militarily smaller countries, Iran and Russia. Iran has an expansionist agenda in the region and the military power wants to block it. The optimal action for the USA is to reveal information about its military arsenal in order to threaten Iran. Russia, however, has access to the outcomes of the negotiation between the USA and Iran, and is simultaneously negotiating a reduction of some of its weapon's arsenal with the USA. Here the optimal action for the USA would be not to reveal anything about its arsenal. How should the military power play in this situation? What is the best possible payoff it could obtain? These are some of the questions that motivate our analysis.

1.1. **Quick Summary of the Results and Organization.** The main results of this paper are located in sections 6 and 7. Section 4 provides motivation for the results proved in 6 and 7. Section 6 proves that under certain assumptions, even under a correlated prior between both zero-sum games, the informed player can obtain in equilibrium more than the concavification of the sum of the nonrevealing values of each of the zero-sum games she is playing[1]; namely, she is able to obtain the sum of the concavifications of each of the nonrevealing values (which is the best possible payoff she can obtain)[2]. The assumptions for the results are general enough to encompass cases where the optimal strategies of the informed player, in each of the zero-sum games, are different - say, for example, "partially revealing" in one game and "nonrevealing" in the other, or "partially revealing" in both games.

[1]See section 3.

[2]See section 4.

In section 7 we give an example where the informed player cannot obtain the "best possible payoff" when the assumptions of section 6 are not met.

The paper is organized as follows: In section 2 we present the model formally. In section 3 we review preliminary results regarding zero-sum games with incomplete information that will be used later; in section 4 we present an analysis of a benchmark model (independent priors) in order to motivate the analysis of the more interesting case of correlated priors. In section 5 we prove existence of uniform equilibrium for finitely many types. In section 6 we explore the geometry of the nonrevealing value associated with incomplete information zero-sum games and how it allows to identify a sufficient condition for the construction of nonrevealing equilibria. In the last section we show what can happen when this geometric property is not satisfied.

1.2. **Brief Literature Review.** To the best of our knowledge, the model we present here is new. It can be seen as a multi-player generalization of the zero-sum model proposed by Aumann-Maschler-Stearns (1967-1968). The main motivation is Huangfu and Liu (2015). In this paper, the authors are concerned with the "information spillover" phenomenon but not exactly in a repeated setting: they consider a dynamic game in which a seller has two goods of two different qualities - high or low - that he wishes to sell to two short run buyers at each stage, in two different markets (each buyer interacts in only one of the markets and for only one stage). Once goods are sold, the game ends. The "information spillover" effect comes from the fact that the qualities of the goods are correlated and that the buyers are able to observe the past outcomes across markets and use it as a signal to learn about the type of the good being traded in the corresponding market. In what concerns the tools of the analysis, one of our main references is the work Hart (1985) as it provides a simple way to model players' payoff and posterior processes. In addition to it, the works of Sorin (1983) and Simon et al. (1995) provide theorems that immediately imply existence of equilibrium in our setting; they also suggest that the problem of showing when the "best possible payoff' is attainable lies in the geometry of the nonrevealing value of each of the zero-sum games. A related paper is Renault (2001), which analyzes an almost symmetric situation: 3 players, but two informed ones, play against 1 uninformed player. Renault, however, is concerned with non zero-sum types of games and with the problem of existence of equilibrium.

2. The Model

2.1. Objects.

- Players: 1 (informed), 2 (uninformed) and 3 (uninformed).
- Set of types: K_A, K_B (finite).
- Actions: I_A and I_B (for player 1), J_A (player 2) and J_B(player 3) (finite).
- Sets of matrices (stage payoffs): $\{A^{k_A}\}_{k_A \in K_A}$, A^{k_A} is a $|I_A| \times |J_A|$ matrix and $\{B^{k_B}\}_{k_B \in K_B}$, B^{k_B} is a $|I_B| \times |J_B|$ matrix with real entries.

2.2. Game Form.

- at stage 0, (k_1, k_2) is chosen according to probability p on $K_1 \times K_2$ and communicated to player 1 only.
- at stage 1, Player 1 chooses a move $(i_A, i_B) \in I_A \times I_B$, player 2 chooses a move $j_A \in J_A$, and player 3 chooses a move $j_B \in J_B$. Everyone is informed of the tuple $((i_A, i_B), j_A, j_B)$.
- inductively, at stage m, knowing the past history $h_m = ((i_A, i_B)_1, j_{A_1}, j_{B_1}, ..., (i_A, i_B)_{m-1}, j_{Am-1}, j_{Bm-1})$, player 1 chooses $(i_A, i_B)_m$, player 2 chooses j_{Am} and player 3 chooses j_{Bm} in their respective domains.
  Everyone is informed of $h_{m+1} = (h_m, (i_A, i_B)_m, j_{Am}, j_{Bm})$.

2.3. Strategies.

Taking into account Kuhn's theorem, there is no loss of generality in considering solely behavioral strategies:

(1) Player 1: $\sigma : K_A \times K_B \times \cup_{n \in \mathbb{N} \cup \{0\}} H_n \to \triangle(I_A \times I_B)$.
(2) Player 2: $\tau_A : \cup_n H_n \to \triangle(J_A)$.
(3) Player 3: $\tau_B : \cup_n H_n \to \triangle(J_B)$

2.4. Payoffs.

2.4.1. *Stage Payoffs.*

At each stage $t \in \mathbb{N}$, when $(k_A, k_B) \in K_A \times K_B$ is chosen and $((i^t_A, i^t_B), j^t_A, j^t_B)$ is played, the stage payoffs are defined by:

- Player 1: $A^{k_A}_{i^t_A, j^t_A} + B^{k_B}_{i^t_B, j^t_B}$.
- Player 2: $-A^{k_A}_{i^t_A, j^t_A}$.
- Player 3: $-B^{k_B}_{i^t_B, j^t_B}$.

2.4.2. *Long-Term Payoffs.* A Banach-Limit $\mathcal{L}$, a profile (σ, τ_A, τ_B), a prior $p \in \triangle(K_A \times K_B)$ are fixed. The payoffs for each one of the players are:

- Player 1: $\mathcal{L}(\mathbb{E}^{k_A,k_B}[\frac{1}{T}\sum_{t=1}^{T}(A^{k_A}_{i_t,j_t} + B^{k_B}_{i_t,j_t})])$, for each $(k_A, k_B) \in K_A \times K_B$
- Player 2: $\mathcal{L}(\mathbb{E}[\frac{1}{T}\sum_{t=1}^{T}(-A^{\kappa_A}_{i_t,j_t})])$.
- Player 3: $\mathcal{L}(\mathbb{E}[\frac{1}{T}\sum_{t=1}^{T}(-B^{\kappa_B}_{i_t,j_t})])$.

Where $\kappa : K_A \times K_B \to K_A \times K_B, \kappa(k_A, k_B) = (k_A, k_B)$ is a random variable with distribution p, $\kappa_A := \pi_A \circ \kappa$, $\kappa_B := \pi_B \circ \kappa$, π_A is the projection onto K_A and π_B is the projection onto K_B. The expectation is defined with respect to the profile (σ, τ_A, τ_B).

The repeated game with prior p defined by the game form, strategies and payoffs above will be called $G(p)$.

2.5. **Equilibrium Concept.** Following the literature on undiscounted repeated games, we will use *uniform equilibrium* as our equilibrium concept.

A triple of strategies (σ, τ_A, τ_B) is a uniform equilibrium in $G(p)$ if:

(1) $\text{limsup}_{T\to\infty}(sup_{\sigma'}\mathbb{E}^{k_A,k_B}_{\sigma',\tau_A,\tau_B}[\frac{1}{T}\sum_{t=1}^{T}(A^{k_A}_{i_t,j_t}+B^{k_B}_{i_t,j_t})]) \leq \text{liminf}_{T\to\infty}\mathbb{E}^{k_A,k_B}_{\sigma,\tau_A,\tau_B}[\frac{1}{T}\sum_{t=1}^{T}(A^{k_A}_{i_t,j_t}+ B^{k_B}_{i_t,j_t})]$, for all $(k_A, k_B) \in K_A \times K_B$.

(2) $\text{limsup}_{T\to\infty}(sup_{\tau'_A}\mathbb{E}_{\sigma,\tau'_A,\tau_B,p}[\frac{1}{T}\sum_{t=1}^{T}(-A^{\kappa_A}_{i_t,j_t})] \leq \text{liminf}_{T\to\infty}\mathbb{E}_{\sigma,\tau_A,\tau_B,p}[\frac{1}{T}\sum_{t=1}^{T}(-A^{\kappa_A}_{i_t,j_t})]$.

(3) $\text{limsup}_{T\to\infty}(sup_{\tau_{B'}}\mathbb{E}_{\sigma,\tau_A,\tau_{B'},p}[\frac{1}{T}\sum_{t=1}^{T}(-B^{\kappa_B}_{i_t,j_t})] \leq \text{liminf}_{T\to\infty}\mathbb{E}_{\sigma,\tau_A,\tau_B,p}[\frac{1}{T}\sum_{t=1}^{T}(-B^{\kappa_B}_{i_t,j_t})]$.

3. Preliminary Results

In this section, we state some fundamental results for our analysis. This section is here for completeness, but could be jumped directly to section 4 and consulted if needed. Below, $ValA(\cdot)$[3] will denote the nonrevealing value function of the zero-sum game with lack of information on one side and $ValB(\cdot)$ the analogous object of game B.

Definition 3.1. Let $p \in \triangle(K_A \times K_B)$. We define the marginal distribution of p on K_A as $marg_{K_A}p^{k_A} := \sum_{k_B\in K_B} p^{k_A,k_B}$ and the marginal distribution of p on K_B as $marg_{K_B}p^{k_B} := \sum_{k_A\in K_A} p^{k_A,k_B}$.

[3] $ValA : \triangle(K_A) \to \mathbb{R}$ is defined as $ValA(p) := max_\sigma min_\tau \sigma A(p)\tau$, where $A(p) := \sum_{k\in K_A} p^k A^k$.

Theorem 3.2. *Given $p \in \triangle(K_A \times K_B)$, consider the infinitely repeated undiscounted* ***2-player zero-sum game with lack of information on one side****, where the informed player (the maximizer) has behavioral strategies $\sigma : K_A \times K_B \times \cup_{n\in\mathbb{N}\cup\{0\}} H_n \to \triangle(I_A \times I_B)$ and the uninformed player (the minimizer) has behavioral strategies of the form $\tau : \cup_{n\in\mathbb{N}\cup\{0\}} H_n \to \triangle(J_A \times J_B)$. The maximizer collects the sum of payoffs of matrices A and B at each stage, just as defined for player 1 in (2.4.1). Now restrict the strategies of the informed player to his non-revealing strategies and consider this non-revealing game. Then the informed player and the uninformed player can both guarantee $ValA(marg_{K_A}p) + ValB(marg_{K_B}p)$.*

Proof. When strategies of player 1 are restricted to nonrevealing strategies, player 1 can play independently the nonrevealing optimal strategies [4] of games $A(marg_{K_A}p)$ and $B(marg_{K_B}p)$ at each stage. Uninformed player can do the same for each of the games. □

Theorem 3.3. *Given $p \in \triangle(K_A \times K_B)$, consider the infinitely repeated undiscounted* ***2-player zero-sum game with lack of information on one side****, where the informed player (the maximizer) has behavioral strategies $\sigma : K_A \times K_B \times \cup_{n\in\mathbb{N}\cup\{0\}} H_n \to \triangle(I_A \times I_B)$ and the uninformed player (the minimizer) has behavioral strategies of the form $\tau : \cup_{n\in\mathbb{N}\cup\{0\}} H_n \to \triangle(J_A \times J_B)$. Let $\overline{h}(p) := ValA(marg_{K_A}p) + ValB(marg_{K_B}p)$. Both players can guarantee $Cav(\overline{h})(p)$[5] .*

Proof. From the theory of zero-sum games with lack of information on one side (see Sorin (2000) chapter 3), this game has a uniform value. This value can be shown to be the concavification of the game's non-revealing value. The previous theorem shows that the non-revealing value of this game is $\overline{h}(p) = ValA(marg_{K_A}p) + ValB(marg_{K_B}p)$. □

Remark 3.4. Without loss of generality (see Appendix), we can consider the optimal (behavioral) strategy of the uninformed player in the game of Theorem 3.3 to be a Blackwell strategy $\tau = (\tau_n)_{n\in N\cup\{0\}}$ with $\tau_n : H_n \to \triangle(J_A) \times \triangle(J_B)$.

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[4]See Sorin 2000 chapter 3.

[5]Recall that given a finite set K, a map $f : \triangle(K) \to \mathbb{R}$ the *concavification* of f is $Cavf(p) = sup\{\sum_i \lambda_i f(q^i) | \sum_i \lambda_i q^i = p \text{ and } \sum_i \lambda_i = 1\}$.

4. The Benchmark Case: No Correlation

To gain intuition let us start with the benchmark case of independent prior. Intuitively, the product structure of the prior allows the informed player to condition his strategies independently on each set of types, avoiding information spillover from his play from one game to the other.

Let $p = p_A \bigotimes p_B \in \triangle(K_A) \times \triangle(K_B)$ be a product prior of game $G(p)$. The results from zero-sum games with lack of information on one side imply that the informed player can guarantee an expected payoff of $CavValA(p_A)$ in the game against uninformed player 2 and can guarantee $CavValB(p_B)$ in the game against uninformed player 3. This is an immediate consequence from the fact that player 1 can use optimal strategies on each of the zero-sum games independently. On the other hand, each of the uninformed players have at their disposal the optimal strategies corresponding to their zero-sum games, which means that player 2 can hold the expected payoff of player 1 at a level not larger than $CavValA(p_A)$, and player 3 can hold player 1 at a level not larger than $CavValB(p_B)$.

These observations imply the following result:

Theorem 4.1. *If $p = p_A \bigotimes p_B$, then every ex-ante Nash-equilibrium payoff of the game $G(p)$ of informed player 1 equals $CavValA(p_A) + CavValB(p_B)$.*

No matter what case we analyze - independent prior or not - the optimal strategies of the zero-sum games corresponding to players 2 and 3 imply that an expected equilibrium payoff of the informed player can never be larger than $CavValA(p_A) + CavValB(p_B)$. This is the reason why we shall refer to $CavValA(p_A) + CavValB(p_B)$ as the "best possible payoff" to player 1.

Suppose now types are not independently chosen. **Could the informed player obtain the best possible payoff even when he is not able to condition his strategies independently in each of the type spaces of the two games he is playing?** We provide an answer to this question in section 7. But first, we settle the question of existence of uniform equilibrium in our model.

5. Uniform Equilibrium Existence

We prove existence of a special class of uniform equilibrium, namely, independent and safe joint-plan (Sorin 1983). This type of equilibrium has a very natural interpretation: the informed player uses the outcomes of his individual play, for a finite number of stages, to signal information about the underlying type of the game that is being played. Depending on the particular outcome realized, the players then play deterministically in every stage, according to a contract. Deviations are punished by minmax strategies in the corresponding zero-sum games, as usual.

Below we define the concept of joint-plan and introduce some notation:

Definition 5.1. Let $h_m = (i^t_A, i^t_B, j^t_A, j^t_B)_{1\leq t\leq m-1}$ and $h^1_m := (i^t_A, i^t_B)_{1\leq t\leq m-1}$. We define $H^1_n := \{h^1_n\}$ and call it the set of individual histories of player 1.

Definition 5.2 (Aumann/Maschler/Stearns/Sorin)**.** An *independent 2-3-safe-joint-plan* is a triple (S, x, γ) where:

- (Signals) S is a set of signals, i.e. a subset of H^1_n, for some $n \in \mathbb{N}$.
- (Signaling strategy) x (is a $|K_A \times K_B|$-tuple where for each (k_A, k_B) in $K_A \times K_B$, $x^{(k_A,k_B)}$ is a probability on S.
- (Contracts) $\gamma = (\gamma_A, \gamma_B)$, where, $\gamma_i = (\gamma^s_i)_{s\in S}$, and $\gamma^s_i = \sigma_i \bigotimes \tau_i, \sigma_i \in \triangle(I_i)$ and $\tau_i \in \triangle(J_i)$, where τ_i is optimal for the corresponding uninformed player at game $i(marg_{K_i}p)$, for $i \in \{A, B\}$.

Here is the intuition of a joint-plan: player 1 uses his action during finitely many stages to signal about the underlying type of games A and B, which are his private information. After that, conditionally on the observed signal s, players play according to γ_s. γ_s can be defined as a deterministic sequence of actions that asymptotically approaches the product $\sigma_s \bigotimes \tau_s$ (see Sorin (1983), Lemma 2.).

Notation:

Given a joint plan, for each $s \in S$, $p(s)$ will stand for the conditional probability on types given s, defined by $p^{(k_A,k_B)}(s) = p^{(k_A,k_A)}x^{(k_A,k_B)}(s)/x(s)$, where $x(s) = \sum_{(k_A,k_B)} p^{(k_A,k_A)}x^{(k_A,k_B)}(s)$. Also, for each s in S and $(k_A, k_B) \in K_A \times K_B$:

(1) $\alpha^{k_A,k_B}(s) = \sum_{i_A,j_A,i_B,j_B}(A^{k_A}_{i_A,j_A}\gamma^s_{i_A,j_A} + B^{k_B}_{i_B,j_B}\gamma^s_{i_B,j_B})$ (the average payoff collected by the informed player after signal s).

(2) $\alpha^{k_A,k_B} = max_{t \in S} \alpha^{k_A,k_B}(t)$.

(3) $\alpha = (\alpha^{k_A,k_B})_{(k_A,k_B) \in K_A \times K_B}$.

(4) $\alpha^{k_A}(s) = \sum_{i_A,j_A} A^{k_A}_{i_A,j_A} \gamma^s_{i_A,j_A}$ (marginal payoff collected from game A^{k_A} given $s \in S$).

(5) $\alpha^{k_B}(s) = \sum_{i_B,j_B} B^{k_B}_{i_B,j_B} \gamma^s_{i_B,j_B}$ (marginal payoff collected from game B^{k_B} given $s \in S$).

(6) $\beta_A(s) = \sum_{k_A} marg_{K_A} p(s)^{k_A} \alpha^{k_A}(s)$.

(7) $\beta_B(s) = \sum_{k_B} marg_{K_B} p(s)^{k_B} \alpha^{k_B}(s)$.

Theorem 5.3. *A sufficient condition for a joint-plan payoff to be a Nash-equilibrium payoff in $G(p)$:*

(1) $\beta_A(s) \leq CavValA(marg_{K_A} p(s))$ *and* $\beta_B(s) \leq CavValB(marg_{K_B} p(s))$ *, for all* $s \in S$

(2) *For all* $(k_A, k_B) \in K_A \times K_B$ *, for all* $s \in S$, $p^{(k_A,k_B)} x^{(k_A,k_B)}(s) > 0$

(3) $\alpha \cdot q \geq \overline{h}(q)$, *for all* $q \in \triangle(K_A \times K_B)$ [6].

Proof. See Sorin (1983) □

(1) is an individual rationality for players 2 and 3 and (3) is an individual rationality condition for player 1.(2) is usually called "no cheating" condition and prevents the informed player from profitably deviating at the signalling stages. The theorem below provides technical conditions we will use in constructing a joint-plan that meets the conditions of Theorem 5.3.

Theorem 5.4. *[Simon, Spiez, Torunczyk, 1992] Let K be a finite set. Let $a : \triangle(K) \to \mathbb{R}$ and $h : \triangle(I) \times \triangle(K) \to \mathbb{R}^{|K|}$ be continuous functions such that:*

(1) *h is affine with respect to the variable $\sigma \in \triangle(I)$, for all $p \in \triangle(K)$.*

(2) *for all $p, q \in \triangle(K)$, there is $\sigma \in \triangle(I)$ such that $h(\sigma, p) \cdot q \geq a(q)$.*

Given $p_0 \in \triangle(K)$, there exist a set $P_0 \subset \triangle(K)$ of cardinality $\leq |K|$ and vectors $\sigma_p \in \triangle(I)(p \in P_0)$ and $\phi \in \mathbb{R}^{|K|}$ such that:

(3) *$\phi \cdot q \geq a(q)$ for all $q \in \triangle(K)$*

(4) *$p_0 \in conv(P_0)$*

[6]Condition 3 of Theorem 5.3 is the so called "approachability" condition. This condition is important because in case the informed player deviates from the equilibrium path of play, it allows the uninformed players to punish him, holding his payoff at a level lower than $\alpha^{k_A,k_B}, \forall (k_A, k_B) \in K_A \times K_B$. For more, please refer to the Appendix.

(5) *for all* $p \in P$, *for all* $k \in K$ *we have* $\phi^k \geq h^k(\sigma_p, p)$, *with equality occuring in place of inequality whenever* $p^k > 0$.

We will also make use of the following simple version of a topological lemma in Simon et al. (1992):

Lemma 5.5. *[Simon, Spiez, Torunczyk, 1992] For every* $\epsilon > 0$ *there exists a continuous map* $g_A : \triangle(K_A) \to \triangle(J_A)$ *such that* $\sigma_A A(p) g_A(p) \leq ValA(p) + \epsilon$, *for all* $(\sigma, p) \in \triangle(I_A) \times \triangle(K_A)$.

Remark 5.6. Of course, the above result works in the same way for game B.

Theorem 5.7. *(Existence of Joint-Plan Equilibrium Payoff) Assume* $p_0 \in int\triangle(K_A \times K_B)$ *There exists a joint-plan equilibrium payoff for the game* $G(p_0)$ *that is independent and 2-3-safe.*

Proof. First we shall write $p_A := marg_{K_A} p$ and $p_B := marg_{K_B} p$. Given $\epsilon > 0$, applying lemma 5.5 we have that $\sigma_A A(p_A) g_A(p_A) \leq ValA(p_A) + \epsilon$ and $\sigma_B B(p_B) g_B(p_B) \leq ValB(p_B) + \epsilon$ for all $(\sigma_A, p_A) \in \triangle(I_A) \times \triangle(K_A)$ and for all $(\sigma_B, p_B) \in \triangle(I_B) \times \triangle(K_B)$. Define $h(\sigma, p) = ((m_A\sigma) A^{k_A} g_A(p_A) + (m_B\sigma) B^{k_B} g_B(p_B))_{k_A \times k_B}$, where $m_A\sigma$ is the marginal of σ on game A and $m_B\sigma$ is the marginal on game B. Since the marginal operator is affine, the function h is affine on σ. It is also trivially continuous. Now, given $p, q \in \triangle(K_A \times K_B)$, let $\bar{\sigma_A}^p$ be the optimal non revealing strategy of the informed player in game A at prior p_A and let $\bar{\sigma_B}^p$ be the be the optimal non revealing strategy of the informed player in game B at prior p_B. Define $\sigma := \bar{\sigma_A}^p \bigotimes \bar{\sigma_B}^p$. Then we have that $h(\sigma, p) \cdot q \geq a(q) =: ValA(q_A) + ValB(q_B)$. Applying theorem 5.4, we have that there is $P_0 \subset \triangle(K_A \times K_B)$ with cardinality $\leq |K_A \times K_B|$, $(\sigma_p)_{p \in P_0}$ and $\phi \in \mathbb{R}^{|K_A \times K_B|}$, satisfying (3), (4) and (5). From this conditions we have that there exists a nonnegative collection $(\lambda_p)_{p \in P_0}$ such that $\sum_{p \in P_0} \lambda_p p = p_0$ and $\sum_{p \in P_0} \lambda_p = 1$, $\phi \cdot q \geq a(q)$ and $(m_A\sigma_p) A(p_A) g_A(p_A) \leq ValA(p_A) + \epsilon$ and $(m_B\sigma_p) B(p_B) g_B(p_B) \leq ValB(p_B) + \epsilon$ for $\sigma_p \in \triangle(I_A \times I_B)$ and $p \in P_0$.

Consider, therefore, $\gamma_A^{p_A} = m_A\sigma_p \bigotimes g_A(p_A)$, $\gamma_B^{p_B} = m_B\sigma_p \bigotimes g_B(p_B)$ for $p \in P_0$. Notice that the solutions given by the application of theorem 5.4 are all indexed by $\epsilon > 0$. We therefore consider cluster points of these solutions to obtain the result. Notice that property (5) of theorem 5.4 guarantees that the sequence of vectors ϕ is bounded, so it will also have an accumulation point. □

Remark 5.8. The assumption for $p_0 \in int\triangle(K_A \times k_B)$ is innocuous. If the prior were to be at the relative interior of any proper face of the simplex, we could restate the problem with p_0 in a simplex with one dimension less than the dimension of $\triangle(K_A \times K_B)$.

Corollary 5.9. *The above equilibrium has an associated ex-ante payoff for the informed player of $Cav(\overline{h})(p_0)$* [7].

Proof. Let τ_p^A , τ_p^B, with $p \in P_0$ be the equilibrium strategies of the uninformed players, whose existence is guaranteed by the above theorem. By construction, $max_\sigma \sigma A(p_A)\tau_p^A = ValA(p_A)$ and $max_\sigma \sigma B(p_B)\tau_p^B = ValB(p_B)$. Therefore, taking σ as the informed player strategy inducing the joint-plan Nash-equilibrium and defining $\sigma^A := marg_A \sigma$, $\sigma^B := marg_B \sigma$, it implies that $\sigma_p^A A(p_A)\tau_p^A \leq ValA(p_A)$ and $\sigma_p^B B(p_B)\tau_p^B \leq ValB(p_B)$, for each $p \in P_0$. Now, at equilibrium, applying condition (5) of Theorem 5.4, we have that $\sum_{p\in P_0} \lambda_p (h(\sigma_p, p) \cdot p) = \phi \cdot p_0 = Cav\overline{h}(p_0)$. Rewriting the expression for h, $\sum_{p\in P_0} \lambda_p(\sigma_p^A A(p)\tau_p^A + \sigma_p^B B(p)\tau_p^B) = Cav\overline{h}(p_0)$. □

$Cav(\overline{h})(p_0)$ is the worst possible ex-ante payoff the informed player can obtain: as showed in Theorem 3.3, the informed player could guarantee this level of payoff. The natural question now is whether he is able to obtain more than this level.

6. When can the informed player obtain the best possible payoff?

Having proved the existence of equilibrium for the finite type case in our model, we established as a corollary the payoff associated with the equilibrium. In this section we approach the question of whether the best-possible payoff - defined in the end of section 4 - can be obtained as an equilibrium of a game where there is correlation among types. Somewhat surprisingly, we show that correlation is not enough to preclude the informed player in obtaining the best possible payoff in the game. There is a certain geometry of the game, which is captured by the shape of the nonrevealing value function of each zero-sum game, that in certain cases allow for a play of the informed player that attains the best possible payoff.

Let us first start with a useful result:

[7] Recall that $\overline{h}(p) := ValA(marg_{K_A}(p)) + ValB(marg_{K_B}(p))$

Theorem 6.1. *The ex-ante Nash-equilibrium payoff set is convex and the ex-ante payoffs of the informed player lie between*

$$I(p_0) = [Cav(\overline{h})(p_0), CavValA(marg_{K_A}p_0) + CavValB(marg_{K_B}p_0)]$$

Proof. Consider two ex-ante Nash-Equilibrium payoffs of the game, say γ_1 and γ_2, for the informed player. Since each of the uninformed players can play their optimal strategies of their respective games, it implies that $CavValA(marg_{K_A}p_0) + CavValB(marg_{K_B}p_0) \geq \gamma_1$ as well as $CavValA(marg_{K_A}p_0) + CavValB(marg_{K_B}p_0) \geq \gamma_2$. Now, the availability of the optimal strategy for the informed player in the infinitely repeated zero-sum game with lack of information on one side, defined in Theorem 3.3, implies that $\gamma_1 \geq Cav(ValA(marg_{K_A}p_0) + ValB(marg_{K_B}p_0))$ as well as $\gamma_2 \geq Cav(ValA(marg_{K_A}p_0) + ValB(marg_{K_B}p_0))$. Now, let $\alpha \in (0,1)$. Consider a jointly controlled lottery (Aumann et al., p. 274) that implements the equilibrium profile associated with γ_1 with probability α and the equilibrium associated with γ_2 with probability $1-\alpha$. By the properties of the jointly controlled lottery, there cannot be profitable undetectable deviations at the stages where the jointly controlled lottery is ran. For detectable deviations of one of the uninformed players at the lottery stages, the informed player plays the optimal strategy of the zero-sum game to punish. For detectable deviations of the informed player at the lottery stages,uniformed players play the Blackwell strategy of remark 3.4 to punish. The strategy profile where a jointly controlled lottery is played at initial stages and, after that, the corresponding strategy profile paying γ_1 or γ_2 drawn for the lottery, is trivially an equilibrium of the game. The payoff of this equilibrium is $\alpha\gamma_1 + (1-\alpha)\gamma_2$. □

The example below shows that the interval $I(p)$ could indeed be non-degenerate for a certain prior $p \in \triangle(K_A \times K_B)$.

Example 6.2. Consider the following example:

$$p_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

$$A^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B^1 = \begin{bmatrix} 4 & 0 & 2 \\ 4 & 0 & -2 \end{bmatrix}; B^2 = \begin{bmatrix} 0 & 4 & -2 \\ 0 & 4 & 2 \end{bmatrix}$$

In p_0, the sum of rows entries correspond to probabilities of types of game A; columns correspond to probabilities of types of game B. Since the prior assigns perfect correlation between the types of each one of the games, we will consider *wlog* p to be the probability of type 1 in both games and $(1-p)$ the probability of type 2 in both games.

For this example we have that:

$$ValA(p) = p(1-p), \text{ for all } p \in [0,1]$$

$$ValB(p) = \begin{cases} 4p & \text{if } p \in [0,1/4) \\ -4+2 & \text{if } p \in [1/4,1/2) \\ 4p-2 & \text{if } p \in [1/2,3/4) \\ -4p+4 & \text{if } p \in [3/4,1] \end{cases}$$

These imply that:

$$CavValA(p) = ValA(p) = p(1-p), \text{ for all } p \in [0,1]$$

$$CavValB(p) = \begin{cases} 4p & \text{if } p \in [0,1/4) \\ 1 & \text{if } p \in [1/4,3,4) \\ -4p+4 & \text{if } p \in [3/4,1] \end{cases}$$

$$Cav(ValB+ValA)(p) = \begin{cases} 4p+p(1-p) & \text{if } p \in [0,1/4) \\ 1+3/16 & \text{if } p \in [1/4,3,4) \\ -4p+4+p(1-p) & \text{if } p \in [3/4,1] \end{cases}$$

In figures 1 and 2 below we depict the graphs of the nonrevealing value in a continuous line. Since the nonrevealing value in game A is concave, it equals its least concave majorant. In figure 2, we depict in dotted line the region where the graphs of the concavification and the nonrevealing value differ in game B. Figures 3 and 4 follow the same notation.

Figure 1. Graphs of CavValA and ValA for a "nonrevealing" game A

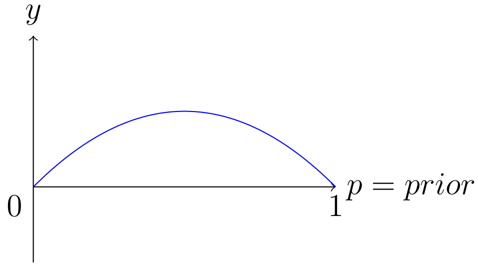
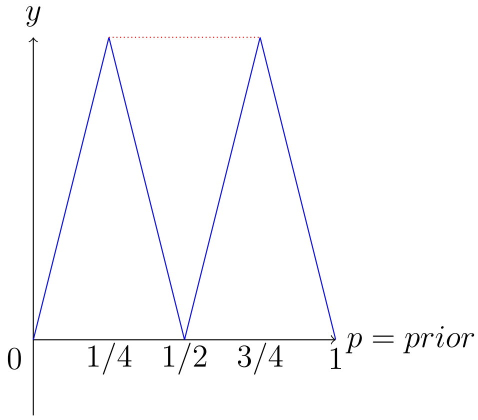


Figure 2. Graphs of CavValB and ValB for a "partially revealing" game B



Notice that $CavValA(1/2) + CavValB(1/2) = 1/4 + 1 > 1 + 3/16 = Cav(ValA + ValB)(1/2)$. A natural question is whether we can obtain the whole interval I(1/2) as ex-ante equilibrium payoffs for the informed player. It turns out, as we will see, that games as the one above satisfy a general geometric property that allows the informed player to obtain the whole interval in ex-ante equilibrium payoffs.

Below we define some important technical objects to the proof of the theorem. The notation is borrowed from Sorin (1983):

For all $\tau \in \triangle(J_A), q \in \triangle(K_A)$, we define

- $f^A_\tau(q) := max_\sigma \sigma A(q)\tau, \tau \in \triangle(J_A), q \in \triangle(K_A)$.
- $C_A(\tau) := Epi(f^A_\tau) := \{(q,t) \in \triangle(K_A) \times \mathbb{R} | t \geq f^A_\tau(q)\}$
- $D_A := \{(q,t) \in \triangle(K_A) \times \mathbb{R} | t \leq CavValA(q)\}$

Lemma 6.3. *Let $p_0 \in \triangle(K_A)$. Then there are $\sigma \in \triangle(I_A)$ and $\tau_{p_0} \in \triangle(J_A)$, with τ_{p_0} optimal at $A(p_0)$, such that $\sigma A(q)\tau_{p_0} \geq ValA(q), q \in \triangle(K_A)$ if and only if $int(C_A(\tau_{p_0}) \cap D_A) = \emptyset$.*

Proof. See Sorin (1983), p. 200. □

As Sorin showed in 1983, one of the ways of constructing Nash-Equilibria in non-zero sum games with 2 players is to investigate the (possible) separation between the epigraph $C_A(\tau)$ and the hypograph D_A (this is the idea in theorem 5.4). The specific type of equilibrium contructed by Sorin is called a "2-safe joint-plan". The "safe" qualification comes from the fact that the uninformed player, after the signalling stages of the joint-plan, plays optimally at each nonrevealling game with given posterior (see property 3 of definition 5.2). Having these observations in mind, the sets $X_A^\star$ and $X_B^\star$ defined below comprise the sets of priors at which there is a nonrevealing, safe joint-plan for each of the zero-sum games A and B, respectively, considered independently. The existence of a nonrevealing, safe joint-plan of games A and B at prior p immediately implies that there is a nonrevealing, safe joint-plan of the game $G(p)$. Game A of Example 6.2 above is a game for which there are nonrevealing and safe joint-plans associated to each prior. In game B of the same example, however, considering a prior $p \in (1/4, 3/4)$, there will not be any nonrevealing safe joint-plan associated to p, since, for τ_A optimal at $marg_{K_A}p$, separation of $C_A(\tau)$ and D_A is impossible (Lemma 6.3).

Definition 6.4. Let $X_A^\star := \{p \in \triangle(K_A)|$ there exists $\tau_p \in \triangle(J_A)$ optimal at $A(p)$ with $int(C_A(\tau_p) \cap D_A) = \emptyset\}$ and $X_B^\star := \{p \in \triangle(K_B)|$ there exists $\tau_p \in \triangle(J_B)$ optimal at $B(p)$ with $int(C_B(\tau_p) \cap D_B) = \emptyset\}$

Definition 6.5. Let $p_0 \in \triangle(K_A \times K_B)$. Then $F(marg_{K_A}p_0)$ will denote a face of the convex set D_A that contains $(marg_{K_A}(p_0), CavValA(marg_{K_A}(p_0))$. $F(marg_{K_B}p_0)$ is analogously defined for D_B.

Definition 6.6. ($X(p_0)$-Property) Let $p_0 \in \triangle(K_A \times K_B)$. Let $G(p_0)$ satisfy the conditions below:

(1) $marg_{K_A}p_0$ and $marg_{K_B}p_0$ are non-degenerate.
(2) If $marg_{K_A}p_0 \notin X_A^\star$ then there exists a face $F(marg_{K_A}p_0)$ and $\overline{p} \in int(\triangle(K_A))$ such that $(\overline{p}, ValA(\overline{p})) \in F(marg_{K_A}p_0)$
(3) If $marg_{K_B}p_0 \notin X_B^\star$ then there exists a face $F(marg_{K_B}p_0)$ and $\tilde{p} \in int(\triangle(K_B))$ such that $(\tilde{p}, ValB(\tilde{p})) \in F(marg_{K_B}p_0)$

$G(p_0)$ is said to satisfy the $X(p_0)$-property.

If the prior p is such that either of $marg_{K_A}p_0$ or $marg_{K_B}p_0$ is degenerate, then there is no information spillover problem, since the informed player has to be concerned only with the game with a non-degenerate prior. Hence condition (1) above specifies the case of interest.

Example 6.2 gives a specification of $G(p)$ for which the $X(p)$-property is satisfied at every prior p. For example, in game A, with strictly concave nonrevealing value, $F(marg_{K_A}p_0)$ is degenerate and $(marg_{K_A}p_0, ValA(marg_{K_A}p_0)) \in F(marg_{K_A}p_0)$. In game B, for prior 1/2 at that game, $F(1/2)$ corresponds to the dotted line in Figure 2. This face has its extremum points at interior priors at which the $CavValA$ is equal to $ValA$. On the other hand, the game defined in Example 7.5 is an example of a game that does not satisfy the $X(p_0)$-property.

Theorem 6.7. *Let $p_0 \in \triangle(K_A \times K_B)$, p_A and p_B its corresponding marginals and assume that $G(p_0)$ satisfies the $X(p_0)$-property. Then we have that $CavValA(p_A) + CavValB(p_B)$ can be obtained as an ex-ante equilibrium payoff for the informed player.*

The idea of the proof of this theorem is to explore the geometric condition stated above in order to construct a particular kind of nonrevealing joint-plan that attains the best-possible payoff for the informed player. For an intuition of how the proof proceeds, let us first think about the case $|K_A| = |K_B| = 2$. An implication of this assumption is that the regions of the simplex $\triangle(K_A)$ for which the $CavValA$ is strictly larger than the $ValA$ are open intervals and $CavValA$ is linear on them.[8] Let now p_A be the marginal prior over game A and assume it is contained in one of the open intervals (this is the interesting case). We can associate to this open interval a hyperplane whose intersection with the graph of $CavValA$ is an a nondegenerate face of D_A. This hyperplane is given by a normal vector and intersects $Graph(CavValA)$ at the endpoints of the interval, one of them being interior to the simplex, by the geometric assumption. These endpoints of the interval are points where $CavValA$ is equal to $ValA$, by definition. The idea is now to show that the normal vector to the hyperplane can be described by strategies of the informed players 1 and uninformed player 2 in the sense that there exists $\sigma_A \in \triangle(I_A), \tau_A \in \triangle(J_A)$ such that $(\sigma_A A^{k_A} \tau_A)_{k_A \in K_A}$

[8]The set $\{q \in \triangle(K_A) | ValA(q) < CavValA(q)\}$ is a semi-algebraic set and therefore is a union of finitely many open connected components. These components, for the two types environment we consider, are therefore open intervals. The definition of the concavification operator implies that $CavValA$ is linear in each of this open intervals.

is the normal vector. This will imply by construction $(\sigma_A A^{k_A}\tau_A)_{k_A\in K_A}\cdot p_A = CavValA(p_A)$ and $(\sigma_A A^{k_A}\tau_A)_{k_A\in K_A}\cdot q \geq ValA(p_A), \forall q \in \triangle(K_A)$, which are the sufficient conditions for a nonrevealing joint-plan to be an equilibrium. If $ValA(\cdot)$ is differentiable at one of the endpoints of the interval, then, as we will show in the proof below, it is easy to obtain these strategies. However, in general $ValA(\cdot)$ is not differentiable at these endpoints. The suitable tool to study the local geometry of $ValA$ is then the Clarke superdifferential of $ValA$ at the endpoint of the interval. Essentially, the Clarke superdifferential will allow us to conclude that the normal vector to the hyperplane can be obtained as a convex combination of vectors which are limit points of normal vectors corresponding to points of differentiability of $ValA$. A symmetric reasoning applied to the game between players 1 and 3 allows us to obtain strategies $\sigma_B \in \triangle(I_B), \tau_B \in \triangle(J_B)$ with $(\sigma_B B^{k_B}\tau_B)_{k_B\in K_B}\cdot p_B = CavValA(p_B)$.

Proof. (Theorem 6.4) Let $y_0 = marg_{K_A}p_0$ and $w_0 = marg_{K_B}p_0$. Assume first $y_0 \notin X_A^{\star}$.

We now focus our analysis on proving the existence of a nonrevealing joint-plan with associated expected payoff $CavValA(y_0)$ in the marginal game with matrix A. The reader will be able to immediately check that these results can be equally reproduced to the "marginal game" B, which will imply the existence of a joint-plan with associated payoff $CavValA(y_0)+CavValB(w_0)$.

Notice that if $y_0 \notin X_A^{\star}$, then there exists, by the $X(p_0)$-property, a face $F(y_0) = D_A \cap H$, where H is a hyperplane of $\mathbb{R}^{K_A}\times\mathbb{R}$, for which we can pick $\overline{p} \in int(\triangle(K_A))$ where $(\overline{p}, ValA(\overline{p})) \in F(y_0)$. Let $\Phi = (\phi, 1) \in \mathbb{R}^{K_A}\times\mathbb{R}$ be a normal vector (interpreted here as a row vector) to hyperplane H such that $\phi\cdot q \geq ValA(q), \forall q \in \triangle(K_A)$.

Let $e_i^{K_A-1} = (0,..,1,...,0) \in \mathbb{R}^{K_A-1}$ with one in the i-th position. Analogously, $e_i^{K_A} = (0,..,1,...,0) \in \mathbb{R}^{K_A}$. Define $S : \mathbb{R}^{K_A-1} \longrightarrow \mathbb{R}^{K_A}$ as the linear transformation that maps $e_i^{K_A-1} \mapsto e_{i+1}^{K_A}$, for $i \in \{1,2,...,K_A-1\}$. We also use S to denote the canonical matrix representation of transformation S. Now define the affine transformation $T := Sx + e_1^{K_A}$. Notice that T is injective. Let now $P = conv\{\{e_1^{K_A-1},...,e_{K_A}^{K_A-1}\}\cup\{0\}\}$, $T^{\star} := T \upharpoonright P$ and $f := ValA\circ T^{\star} : P \to \mathbb{R}$. Notice that $T^{\star}$ is also a homeomorphism with its image - which implies it is a parametrization of the simplex $\triangle(K_A)$. Now since $ValA(\cdot)$ is Lipschitz and $T^{\star}$ is affine, f is also Lipschitz. Now extend f to $\mathbb{R}^{K_A-1}$ by putting $f(x) = -M, x \notin P$, where $M = max\{|f(x)| : x \in P\}$.

It follows by the classical Radamacher's theorem that f is almost everywhere differentiable around an open neighborhood V of $x_0 \in intP$ such that $T^\star x_0 = \overline{p}$. Let $x_0+h, x_0+h+v \in V$ with $x_0 + h$ a point of differentiability of f: then we have that $f(x_0 + h + v) = f(x_0 + h) + \nabla f(x_0 + h) \cdot v + o(||v||)$.

Notice now that ϕS is a supergradient of f at x_0, therefore is in the generalized (Clarke) superdifferential $\partial f(x_0) := conv\{lim \nabla f(x + h_i)$, as $h_i \to 0$ with $i \to \infty\}$ [Clarke, (1975)].

Since $\partial f(x_0)$ is convex, Caratheodory's theorem allows us to write ϕS as a convex combination of $|K_A|$ points $\mathcal{N}(x_0) := \{d_1, ..., d_{|K_A|}\} \subset \{lim \nabla f(x + h_i)$, as $h_i \to 0$ as $i \to \infty\}$.

Now define $g_\tau := f_\tau^A \circ T^\star$. It follows that $g_{\tau_{x_0+h}}(x_0+h+v) \geq g_{\tau_{x_0+h}}(x_0+h) + \nabla f(x_0+h) \cdot v + o(||v||)$, τ_x is optimal at the zero-sum game $A(T^\star(x))$. This implies that $\nabla f(x_0 + h)$ is a subgradient of the convex function $g_{\tau_{x_0+h}}$ at point $x_0 + h$. Since the epigraph $\mathcal{B}$ of $g_{\tau_{x_0+h}}$ is a polyhedral set, every maximal proper face of $\mathcal{B}$ is contained in a hyperplane H_i of $\mathbb{R}^{K_A-1} \times \mathbb{R}$ given by $\{(x,t) \in \mathbb{R}^{K_A-1} \times \mathbb{R} | t = (s_i A^{k_A} \tau_{x_0+h})_{k_A \in K_A} Sx\}$, where $s_i \in I_A$.

On the other hand the graph $\mathcal{G}$ of the affine function $r(v) = f(x_0 + h) + \nabla f(x_0 + h) \cdot v$ is a hyperplane that supports $\mathcal{B}$ at $\overline{x} = (x_0 + h, f(x_0 + h))$. Let I_0 be the set of all pure strategies corresponding to maximal proper faces of $\mathcal{B}$ that contain $\overline{x}$. Now, an application of Farkas' lemma will give us that there exists $\sigma_{x_0+h} \in \triangle(I_0)$ such that $\nabla f(x_0 + h) = \sum_{s \in I_0} \sigma_{x_0+h}(s)(sA^{k_A}\tau_{x_0+h})_{k_A \in K_A} S$. [9]

Therefore, considering the definition of the Clarke-superdifferential, for each $d_k \in \mathcal{N}(x_0)$, there exists $\{\overline{h}_i^k\}_{k \in \mathbb{N}}$ such that $\overline{h}_i^k := \nabla f(x_0 + h_i^k) = (\sigma_{x_0+h_i^k} A^{k_A} \tau_{x_0+h_i^k})_{k_A \in K_A} S$ such that $\overline{h}_i^k \to d_k$ as $i \to \infty$. Now, passing to a subsequence if necessary, we can assume that $\tau_{x_0+h_i^k} \to \tau_{x_0} \in \triangle(J_A)$ optimal at p_0, and $\sigma_{x_0+h_i^k} \to \sigma_{x_0} \in \triangle(I_A)$. So $(\sigma_{x_0}^k A^{k_A} \tau_{x_0}^k)_{k_A \in K_A} S = d_k$. This implies that $\phi S = \sum_k \lambda_k (\sigma_{x_0}^k A^{k_A} \tau_{x_0}^k)_{k_A \in K_A} S$ for $\lambda_k \geq 0$ and $\sum_k \lambda_k = 1$.

The strategy of the players will proceed as follows: for a finite number of initial rounds, the players will run a jointly controlled lottery with probability λ_k for each of the $k \in \mathcal{O}$, where $\mathcal{O}$ is the set of possible outcomes of the lottery and $|\mathcal{O}| = |K_A|$. If the outcome drawn is k, then players will implement the deterministic path of play that asymptotically approximates the distribution $\sigma_{x_0}^k \bigotimes \tau_{x_0}^k$ (see Sorin, 1983, Lemma 2). If the outcome of the lottery is k, the payoffs corresponding to the implementation of the deterministic sequence of moves approximating asymtoptically $\sigma_{x_0}^k \bigotimes \tau_{x_0}^k$ is $(\sigma_{x_0}^k A^{k_A} \tau_{x_0}^k)_{k_A \in K_A}$. The expected payoffs of

[9] See proof of claim (15) of Simon et al. (1995). The application of Farkas' lemma is the same as there.

the implementation of this strategy for each of the types are $\sum_k \lambda_k(\sigma^k_{x_0} A^{k_A} \tau^k_{x_0})_{k_A \in K_A}$. Now, for any $q \in \triangle(K_A)$, we have that there exists $x \in P$ such that $Tx = q$, and $\phi S \cdot (x - x_0) = \sum_k \lambda_k(\sigma^k_{x_0} A^{k_A} \tau^k_{x_0})_{k_A \in K_A} S(x - x_0) \geq ValA(T^\star x) - ValA(T^\star x_0)$, which is evidently equivalent to $\sum_k \lambda_k(\sigma^k_{x_0} A^{k_A} \tau^k_{x_0})_{k_A \in K_A} q \geq ValA(q)$. Since q was taken arbitrarily we have that the previous inequality holds for every $q \in \triangle(K_A)$. This implies that the payoffs associated to the strategy described above are individually rational for the informed player. By construction, now, $CavValA(y_0) = \phi \cdot y_0 = \sum_k \lambda_k(\sigma^k_{x_0} A^{k_A} \tau^k_{x_0})_{k_A \in K_A} p_0$, which implies that the vector of payoffs is also individually rational for uninformed player 2. As usual, if either of the players deviate at the deterministic path of play, they are punished by minmax strategies of the zero-sum game. At the jointly controlled lottery stage, by construction, no undetectable deviation is profitable and detectable ones are punished also by minmax strategies. Notice that there is no revelation of information.

Now, if $marg_{K_B} p_0 \in X^*_B$, there is nothing to prove. If $marg_{K_B} p_0 \notin X^*_B$, then we can follow the exact same steps above to construct a similar nonrevealing joint-plan between players 1 and 3 with associated ex-ante payoff of $CavValB(w_0)$ for the uninformed player. In any case, the strategies constructed in both games are nonrevealing and have an associated ex-ante payoff of $CavValA(y_0) + CavValB(w_0)$ for the informed player. □

We should point out that when there are only two types of matrices for each of the zero-sum games, i.e., $\{A^1, A^2\}$ and $\{B^1, B^2\}$, then if precisely items (2) and (3) of the $X(p_0)$-property are violated then $CavValA$ and $CavValB$ are affine. is violated condition is violated, say for zero-sum game A, we have that $CavValA(\cdot)$ is affine on $\triangle(K_A)$. This means the optimal strategy of the zero-sum games are both completely revealing. Therefore, for the two-types case, when both conditions (2) and (3) are not met in Definition 6.6, we have that the optimal strategies of both zero-sum games are completely revealing. In this case, the informed player can completely reveal the underlying types of both games, by playing the optimal strategies of both zero-sum games and therefore guarantee the best-possible payoff under any prior. On the other hand, the uninformed players can guarantee that the informed player will not obtain more than $\overline{h}(p)$. This implies that $\overline{h}(p) = CavValA(p_A) + CavValB(p_B)$, which implies that $I(p)$ is degenerate. These cases are trivial examples of games where the $X(p)$-property is violated under any prior p but $I(p)$ is degenerate. Are there examples under which $I(p)$ is nondegenerate, the $X(p)$-property is violated and the best-possible payoff is still attained as an ex-ante equilibrium payoff of the informed player? The answer is yes and here is the example:

Example 6.8. Let

$$p_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Consider the payoff matrices:

$$A^1 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}; A^2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

FIGURE 3. Graphs of $CavValA(\cdot)$ and $ValA(\cdot)$ for a "fully revealing" game A

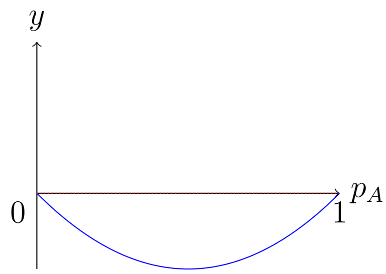
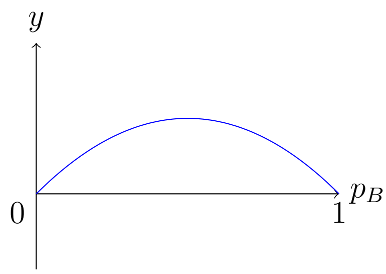


FIGURE 4. Graphs of $CavValB(\cdot)$ and $ValB(\cdot)$ for the "nonrevealing" game B



First notice that $I(p_0)$ is nondegenerate and property (2) of definition 6.6 is not satisfied at p_0. Since the nonrevealing value of game B is concave, for a τ_B optimal in $B(p_0)$, $int(C_B(\tau_B) \cap D_B) = \emptyset$, which implies, by lemma 6.3, the existence of $\sigma_B \in \triangle(I_B)$ such that $\sigma_B B(p_0)\tau_B \geq ValB(q) = CavValA(q), q \in \triangle(K_B)$. Consider, now strategies $\sigma_A = (1, 0) \in$

$\triangle(I_A)$ and $\tau_A = (0,1)^T \in \triangle(J_A)$. Then, $\sigma_A A(q)\tau_A = CavValA(q) = 0, \forall q \in \triangle(K_A)$. Let $\sigma := \sigma_A \bigotimes \sigma_B$. The nonrevealing joint-plan induced by strategies (σ, τ_A, τ_B) - with contracts $\sigma_A \bigotimes \tau_A$ and $\sigma_B \bigotimes \tau_B$ and punishments for deviationsof the informed player given by the Balckwell strategy of remark 3.4, and of an uninformed player given by the optimal strategy of his corresponding zero-sum game - is an equilibrium (since it satisfies the conditions of theorem 5.3) and has an associated ex-ante equilibrium payoff of $CavValA(1/2) + CavValB(1/2)$.

7. Geometry, Signaling and what happens when the X-property fails to be satisfied

TWe now construct an example of a game where the best possible payoff cannot be attained. In order to do that we take another view on $G(p)$ and analyze the stochastic process of payoffs and posteriors associated with that game. This section uses some of the machinery introduced on the work Hart (1985).

From now on we fix a Banach limit L to compute payoffs.

7.1. The Basic Probability Space.

We reproduce here some of the content of section 4.1 in Hart (1983) adapting it to our setting:

For $t \in \mathbb{N}$, we define $H_t := (I_A \times I_B \times J_A \times J_B)^{t-1}$, the set of histories before stage t. We also define the set of infinite histories $H_\infty = \Pi_{t=1}^{\infty}(I_A \times I_B \times J_A \times J_B)$. On H_∞ we define for each $t \in \mathbb{N}$ the finite field generated by H_t and call it $\mathcal{H}_t$. The basic probability space will also include the choice of $\kappa \in K_A \times K_B$ by chance. Thus, let $\Omega = H_\infty \times K$ be endowed with the σ-field $\mathcal{H} \bigotimes 2^{K_A \times K_B}$. Each profile of strategies (σ, τ_A, τ_B) and each probability vector $p \in \triangle(K_A \times K_B)$ determine a probability distribution on this space.

For each $t \in \mathbb{N}$, let $H_{t+1/2} = (I_A \times I_B \times J_A \times J_B)^{t-1} \times (I_A \times I_B) = H_t \times (I_A \times I_B)$ and denote by $\mathcal{H}_{t+1/2}$ the finite field it generates. This defines the collection $(\mathcal{H}_s)_{s \in N_2}$, with $N_2 = \{1, 1+1/2, 2, 2+1/2, ...\}$. We will also denote $\mathcal{H}_s$ the probability space generated by generated by $H_s \times \{k_A\} \times \{k_B\}$ on Ω.

7.2. Payoffs and Posteriors Processes.

(1) $\alpha_s := L(\mathbb{E}[\alpha_T|\mathcal{H}_s])$, where $\alpha_T = \frac{1}{T}\sum_{t=1}^{T} A^{\kappa_A}_{i_t,j_t}$ (process of ex-ante payoffs in game A).

(2) $\beta_s := L(\mathbb{E}[\beta_T|\mathcal{H}_s])$, where $\beta_T = \frac{1}{T}\sum_{t=1}^{T} B^{\kappa_B}_{i_t,j_t}$ (process of ex-ante payoffs in game B).

(3) $p_s^{(k_A,k_B)} := \mathbb{P}_{(\sigma,\tau_A,\tau_B,p_0)}([\kappa = (k_A,k_B)]|\mathcal{H}^s)$ (the martingale of posteriors).

By the properties of Banach-Limits (see Hart, (1985), p.129), the processes in (1) and (2) are martingales. The next result is intuitive: since the posteriors work as state variables for the uninformed players at each stage of the game, the uninformed players can play their Blackwell strategies, which guarantee the corresponding concavification of the nonrevealing value as payoff.

Proposition 7.1. *if* (σ, τ_A, τ_B) *is a Nash-equilibrium profile then it must satisfy:*
$\alpha_s \leq CavValA(marg_{K_A} p_s)$ *for all* $s \in \mathbb{N}_2$
$\beta_s \leq CavValB(marg_{K_B} p_s)$ *for all* $s \in \mathbb{N}_2$

Proof. This result is already in Hart (1985) and the proof is straightforward by the existence of Blackwell strategies for the uninformed players. □

Here we present an example of a game $G(p)$ where $I(p)$ is nondegenerate and the highest possible payoff is not attainable by the informed player. Indeed, only the lower bound of $G(p)$ is attainable by the informed player.

7.3. **Example.**

$$p_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Consider games:

$$A^1 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}; A^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$B^1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

FIGURE 5. Graphs of $CavValA$(dotted) and $ValA$(continuous) for a "fully revealing" game A

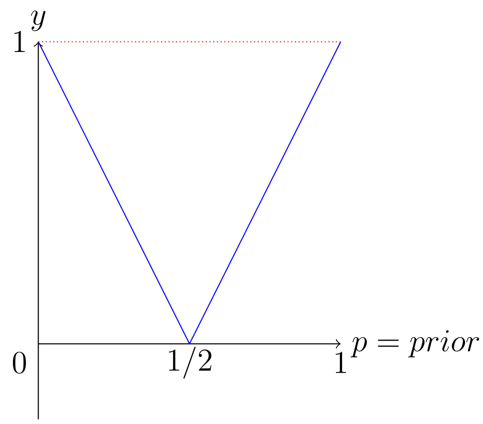
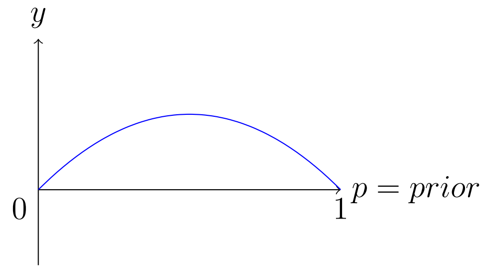


FIGURE 6. Graphs of $CavValB$ and $ValB$ for the "nonrevealing" game B



Assume by way of contradiction that (σ, τ_A, τ_B) is an equilibrium that pays ex-ante $CavValA(p_0^A) + CavValB(p_0^B)$, where $marg_{K_A} p_0 = p_0^A$ and $marg_{K_b} p_0 = p_0^B$, for the informed player, in the example. Let $V_A : \triangle(K_A) \to \mathbb{R}, V_A(p) := max_{\sigma,\tau}\{\sigma A(p)\tau | \sigma A(p)\tau \leq CavValA(p)\}$. For the example, $V_A(p) = ValA(p), \forall p \in \triangle(K_A)$. For each $s \in \mathbb{N}_2$, $\alpha_s \leq V_A(p_s)$ a.s.. Therefore, $\alpha_s \leq ValA(p_s)$ a.s.. Letting $s \to \infty$, we have, by the martingale convergence theorem, $\alpha_s \to \alpha_\infty, p_s^A \to p_\infty^A$ a.s. and, by the Dominated Convergence Theorem, $\mathbb{E}[\alpha_\infty] \leq \mathbb{E}[ValA(p_\infty^A)]$. Since, by assumption, $\mathbb{E}[\alpha_\infty] = \mathbb{E}[ValA(p_\infty^A)] = CavValA(marg_{K_A} p_0)$. We have that the distribution of p_∞^A has to be concentrated at the boundary of $\triangle(K_A)$. Also, $marg_{K_B} p_0 = marg_{K_B} p_\infty$ a.s.. Indeed, if not, since $\beta_s \leq CavValB(p_s^B) a.s.$, then $\mathbb{E}[\beta_\infty] \leq \mathbb{E}[CavValA(p_\infty^B)] < CavValA(p_0^B)$, by Jensen's Inequality, a contradiction.

Hence, we have that for any history $h_\infty \in H_\infty$ outside a set of $\mathbb{P}_{\sigma,\tau_A,\tau_B}$-measure zero, the bistochastic matrix representation of $p_\infty(h_\infty)$ has either the first or the second row filled with zeros. Since, $marg_{K_B}p_\infty(h_\infty) = marg_{K_B}p_0(h_\infty)$ a.s., then we have that the bistochastic representation of $p_\infty(h_\infty)$ is either:

$$\begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix} or \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix}$$

Now the process of posteriors is a martingale which implies that the expectation of p_∞ has to be p_0. In bistochastic representation, we then have the following equality for $\lambda \in [0,1]$:

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} = \lambda \begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix} + (1-\lambda) \begin{bmatrix} 1/2 & 1/2 \\ 0 & 0 \end{bmatrix}.$$

This equation has no solution in $\lambda \in [0,1]$, which finally implies a contradiction. There is no equilibrium paying ex-ante to the informed player $CavValA(p_0^A) + CavValB(p_0^B)$. The arguments above indeed give us more: remember we had $\alpha_s \leq V_A(p_s^A) = ValA(p_s^A)$ a.s. and $\beta_s \leq CavValB(p_s^B) = V_B(p_s^B) = ValB(p_s^B)$ a.s.. This implies that $\alpha_s + \beta_s \leq ValA(p_s) + ValB(p_s)$ a.s. and therefore that $\mathbb{E}[\alpha_\infty + \beta_\infty] \leq \mathbb{E}[ValA(p_\infty^A) + ValB(p_\infty^B)] \leq Cav(\overline{h})(p_0)$. $Cav(\overline{h})(p_0)$ is the lowest possible ex-ante payoff to the informed player, the reasoning implies that every equilibrium of the example pays $Cav(\overline{h})(p_0)$ to the informed player.

8. Appendix B: Blackwell Strategies

The environment is as follows: Let $G^*(p)$ be a zero-sum two-player game with lack of information on one side, where player 1 is the maximizer and player 2 the minimizer. $(C^{k_A,k_B})_{(k_A,k_B)\in K_A\times K_B} = (a^{k_A}_{(i_{A,n},j_{A,n})} + b^{k_B}_{(i_{B,n},j_{B,n})})_{(k_A,k_B)\in K_A\times K_B}$ are the payoffs of player 1 when he takes actions $(i_A, i_B) \in I_A \times I_B$, and player 2 takes actions $(j_A, j_B) \in J_A \times J_B$, for each of the types in $K_A \times K_B$. It is known from the theory of zero-sum games with lack of information on one-side that $G^*(p)$ has a value. Below is a theorem that describes an optimal strategy of player 2 in this game.

Theorem 8.1. *There is an optimal (behavioral) strategy for player 2 in the repeated game, which we call τ_0, that satisfies the following property: at each stage $n \in \mathbb{N}, \tau_{0,n} : H_n \to \triangle(J_A) \times \triangle(J_B)$.*

Proof. Consider the two-player zero-sum vector-payoff game given by payoffs $C_{(i_A,i_B),(j_A,j_B)} = (a^{k_A}_{(i_A,j_A)} + b^{k_B}_{(i_B,j_B)})_{(k_A,k_B)\in K_A\times K_B}$. Let $z \in \mathbb{R}^{|K_A\times K_B|}$ satisfy $z \cdot q \geq \overline{h}(q), \forall q \in \triangle(K_A \times K_B)$. Then $z - \mathbb{R}^{|K_A\times K_B|}_+$ is approachable by player 2 (the uninformed player) and therefore has the B-set property [10]. Call $\overline{g}_n$ the average vector-payoff obtained up to stage n in this game and let τ be the B-set strategy associated with the B-set property and $\overline{g}_n$. Now consider $C\tau := co\{\sum_{j_A,j_B} C_{(i_A,i_B),(j_A,j_B)}\tau_{(j_A,j_B)} | (i_A,i_B) \in I_A \times I_B\}$. Let $\tau^A = marg_{J_A}\tau$ and $\tau^B = marg_{J_B}\tau$. Let $x \in C\tau$. Then,

$$x = \sum_{(i_A,i_B)} \lambda^{i_A,i_B} \sum_{j_A,j_B} C_{(i_A,i_B),(j_A,j_B)}\tau_{(j_A,j_B)} = \sum_{(i_A,i_B)} \lambda^{i_A,i_B} \sum_{j_A,j_B} [a^{k_A}_{(i_A,j_A)} + b^{k_B}_{(i_B,j_B)}]_{(k_A,k_B)}\tau_{j_A,j_B} = \sum_{(i_A,i_B)} \lambda^{i_A,i_B} [\sum_{j_A} a^{k_A}_{(i_A,j_A)}\tau^A_{j_A} + \sum_{j_B} b^{k_B}_{(i_B,j_B)}\tau^B_{j_B}]_{(k_A,k_B)} = \sum_{(i_A,i_B)} \lambda^{i_A,i_B} \sum_{j_A,j_B} [a^{k_A}_{(i_A,j_A)} + b^{k_B}_{(i_B,j_B)}]_{(k_A,k_B)}\tau^A_{j_A}\tau^B_{j_B} = \sum_{(i_A,i_B)} \lambda^{i_A,i_B} \sum_{j_A,j_B} C_{(i_A,i_B),(j_A,j_B)}\tau^A_{j_A}\tau^B_{j_B}.$$

This shows $C\tau = C(\tau^A \bigotimes \tau^B)$

So, if player 2 uses at each stage n such that $\overline{g}_n \notin z - \mathbb{R}^{|K_A\times K_B|}$ the strategy $\tau_n := \tau_A \bigotimes \tau_B$, then τ_n is also a B-set strategy.

□

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[10]See Theorem B3 in Sorin (2000).

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Department of Economics, University of Rochester, Rochester, NY 14627, USA.

E-mail address: pahl.lucas@gmail.com