Collusion in Conflicts with Noise

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Abstract

We analyze the determinants of tacit collusion in an infinitely repeated conflict with noise in the contest success function. Sustaining collusion via Nash reversion strategies is easier the more noise there is, and is more difficult the larger is the contest's prize value. An increase in the contest's number of players can make sustaining collusion either more or less difficult.

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1 Introduction

In a contest, players compete by making irrecoverable expenditures or costly efforts to increase their probability of winning a prize. Lobbying, electoral competition, litigation, advertising competition, R&D competition, military conflict, and sporting competition are all examples of real-world contests. In many such contests, players' winning probabilities are determined not only by their expenditures, but also by pure chance or *noise*. For example, a military conflict could be decided not only by the sizes of the countries' armies, but also by the geography and prevailing weather where the conflict takes place. Many real-world contests are also repeated. Repeated contests could provide players opportunities and incentives to collude by mutually refraining from competing with one another. If players are sufficiently patient (or, equivalently, believe the contest will repeat with a sufficiently high probability), then long-term collusion could dominate short-term opportunism when players use strategies with implicit threats to punish deviations from collusion. Continuing with the military conflict example, the long-lived nature of interactions among countries could provide them incentives to alter their military expenditures or reach other agreements that have them refrain from engaging in costly conflicts. Because there are many repeated contests with noise, gaining insight into how noise affects incentives for collusion in repeated contests is important.

We analyze incentives for tacit collusion in an infinitely repeated contest with the contest success function (CSF) introduced by Amegashie (2006b) where players' winning probabilities are affected by their expenditures as well as by a noise parameter.[1] If players are sufficiently patient, they can sustain

[1]Dasgupta and Nti (1998) use a somewhat similar CSF specification in their study of optimal contest design, but interpret their parameterization as the probability that the contest does not award the prize, which is more like the contests with the possibility of a draw studied by Blavatskyy (2010) and Jia (2012). Existing studies of contests with noise in the CSF are either one-shot (e.g., Cason et al. (2013), Wasser (2013), and Grossmann (2014)) or are repeated but do not analyze players' incentives for collusion (e.g., Eggert

maximal collusion (i.e., mutual refraining from competing in the contest) by using Nash reversion strategies. We show that an increase in the contest's degree of noise makes sustaining collusion easier, while an increase in the contest's prize value makes sustaining collusion more difficult. An increase in the contest's number of players can make sustaining collusion either more or less difficult.

This paper contributes to a literature on collusion in repeated contests. The main message of this literature is that the long-lived nature of repeated contests can provide players incentives to collude, which typically leads to lowered contest expenditures. Yang (1993) and Leininger and Yang (1994) analyze contests in which players take turns choosing whether to increase or leave unchanged their current expenditures, which accumulate over the contest's horizon; when these alternating moves occur over an infinite horizon, a tit-for-tat-like strategy can enable players to keep their equilibrium expenditures low and possibly even minimal. Linster (1994) analyzes cooperative arrangements determined by the Nash bargaining solution when players' disagreement payoffs arise from reversion to Nash equilibrium play in an infinitely repeated contest. Amegashie (2006a) shows that increased prize value asymmetry between players makes sustaining collusion easier in an infinitely repeated contest. In an infinitely repeated game of investment with imperfect property rights, Amegashie (2011) shows that an equilibrium with overinvestment exists where the asset owner and the potential appropriator cooperate by not competing for the asset in a subsequent contest as long as the asset owner makes a transfer increasing in investment. Cheikbossian (2012) studies infinitely repeated contests between two groups of unequal size and shows that collusion (in the sense of a group overcoming its free-rider problem and increasing its expenditures) can be as easy to sustain in the larger group as it is in the smaller group.[2]

et al. (2011)).

[2]There also exist a number of studies that analyze explicit collusion in one-shot contests (e.g., Alexeev and Leitzel (1991, 1996) and Huck et al. (2002)) and that develop models of

Our results on the factors affecting the sustainability of collusion differ markedly from the closely related analysis of Shaffer and Shogren (2008), who analyze incentives for collusion in an infinitely repeated contest with the Tullock (1980) CSF.[3] In the Tullock CSF, the exponent to which each player's expenditure is raised is the closest analogue to the contest noise parameter in the CSF we study in this paper. When players attempt to sustain collusion by using Nash reversion strategies, Shaffer and Shogren (2008) show that a decrease in this exponent (i.e., an increase in the level of noise in the contest) makes sustaining collusion more difficult by making the Nash reversion punishment less severe.[4] Shaffer and Shogren (2008) also show that an increase in the contest's prize value does not affect the sustainability of collusion and that an increase in contest's number of players makes sustaining collusion more difficult. Broadly speaking, our results differ from one another because of the nature of players' deviations from collusion. In Shaffer and Shogren (2008), each player's optimal deviation from collusion involves making an infinitesimally small expenditure and winning the contest with probability 1

infinitely repeated contests to analyze non-collusive behavior (e.g., Itaya and Sano (2003), Mehlum and Moene (2006), Krähmer (2007), Eggert et al. (2011), and Grossmann et al. (2011)).

[3]The Tullock CSF takes the form

$$p_{it}(x_{1t}, ..., x_{nt}) := \begin{cases} \frac{x_{it}^{\gamma}}{x_{it}^{\gamma} + \sum_{j \neq i} x_{jt}^{\gamma}} & \text{if } x_{it}^{\gamma} + \sum_{j \neq i} x_{jt}^{\gamma} \neq 0 \\ \frac{1}{n} & \text{otherwise.} \end{cases}$$

The exponent $\gamma > 0$ measures the CSF's sensitivity to expenditures in determining winning probabilities, and is commonly referred to in contest theory literature as the CSF's discriminatory power following Hillman and Riley (1989). For low levels of γ (i.e., high levels of noise), winning probabilities do not vary much among players with small expenditure differences; in the limit as $\gamma \to 0$, the CSF is completely insensitive to expenditures and each player has a uniform $1/n$ probability of winning no matter their expenditure choice. For high levels of γ (i.e., low levels of noise), winning probabilities vary widely among players with small expenditure differences; in the limit as $\gamma \to \infty$, the CSF becomes the all-pay auction CSF in which the player making the highest expenditures wins with probability 1.

[4]Shaffer and Shogren (2008) analyze the critical discount rate (r^*) sustaining collusion, which relates to the critical discount factor (δ^*) we analyze as $\delta^* = 1/(1 + r^*)$.

because the Tullock CSF lacks noise in this case. By contrast, the CSF in our model *always* has noise, thus each player's optimal deviation from collusion is more complex because making positive expenditures when all other players make 0 expenditures does not guarantee that a player wins the contest with probability 1.

The remainder of this paper is organized as follows. Section 2 describes the contest model that is the stage game played each period over an infinite number of repetitions. Section 3 derives our results on the factors affecting the sustainability of collusion in the repeated contest. Section 4 concludes the paper.

2 Model

Each period $t = 0, 1, 2, ...$ a finite set of players $I = \{1, ..., n\}, n \geq 2$ compete in a simultaneous-move contest to win a prize of value $v > 0$ to each player. Each player $i \in I$ makes irrecoverable expenditures $x_{it} \geq 0$ to increase its winning probability p_{it}, given by the CSF with noise parameter α from Amegashie (2006b):

$$p_{it}(x_{1t}, ..., x_{nt}) := \frac{x_{it} + \alpha}{x_{it} + \alpha + \sum_{j \neq i}(x_{jt} + \alpha)}. \tag{1}$$

Rai and Sarin (2009) provide an axiomatic foundation and Jia (2012) provides a stochastic foundation for CSFs of the form in (1). Increasing the degree of noise in the contest has the effect of discouraging expenditures. Thus, we assume that $\alpha \in [0, \overline{\alpha})$ where $\overline{\alpha} := (n-1)v/n^2$ so that the stage game's unique Nash equilibrium is interior; if instead $\alpha \geq \overline{\alpha}$, the stage game's unique Nash equilibrium has each player making an expenditure of 0 and there exists no form of collusion yielding players a Pareto improvement thus making the analysis of incentives for collusion moot.

When players make expenditures $(x_{1t}, ..., x_{nt})$ in period t, the expected profits of player $i \in I$ in period t are:

$$\pi_{it}(x_{1t}, ..., x_{nt}) := p_{it}(x_{1t}, ..., x_{nt})v - x_{it}$$

$$= \frac{x_{it} + \alpha}{x_{it} + \alpha + \sum_{j \neq i}(x_{jt} + \alpha)} v - x_{it}. \quad (2)$$

Each player discounts streams of future profits to their present value according to the discount factor $\delta \in (0, 1)$.

3 Incentives for Collusion

In the absence of collusion, we suppose that players make expenditures according to the stage game's Nash equilibrium. As Amegashie (2006b) shows, when $\alpha < (n-1)v/n^2$ the stage game has a unique Nash equilibrium in which each player makes expenditures

$$x_N = \frac{n-1}{n^2} v - \alpha$$

and earns expected profits

$$\pi_N = \frac{v}{n^2} + \alpha$$

each period.[5]

When $\alpha < (n-1)v/n^2$, players can improve upon the stage game's Nash equilibrium by collusion where each player makes an expenditure of

$$x_C = 0$$

[5] It is straightforward to show that the first derivative of (2) with respect to x_{it} is positive when $x_{jt} = 0$ for all $j \in I \setminus \{i\}$ and $\alpha < (n-1)v/n^2$, ruling out all players making 0 expenditures as a Nash equilibrium. It is also straightforward to show that (2) is strictly concave in x_{it}.

and earns expected profits

$$\pi_C = \frac{v}{n}$$

each period. We suppose that players use the following Nash reversion strategies of Friedman (1971) to sustain collusion tacitly as a subgame perfect Nash equilibrium of the infinitely repeated contest.[6] Nash reversion strategies prescribe that each player $i \in I$

- makes an expenditure of $x_C = 0$ in period $t = 0$;

- makes an expenditure of $x_C = 0$ in periods $t = 1, 2, ...$ as long as all players have made expenditures of $x_C = 0$ in all periods to date; otherwise, the player makes an expenditure of $x_N = (n-1)v/n^2 - \alpha$ forever.

As an alternative to Nash reversion strategies, we could analyze collusive behavior sustainable by the optimal punishment approach of Abreu (1986, 1988). Adopting such an approach would be preferable if Nash reversion strategies did not sustain maximal collusion because optimal punishments can support a wider range of collusive behavior in equilibrium. However, as we show below, since Nash reversion strategies *do* sustain maximal collusion, we opt to follow the Friedman (1971) approach.

The optimal deviation x_D of a player $i \in I$ from the collusive arrangement above solves

$$\max_{x_{it}} \pi_{it}(x_{it}, 0, ..., 0) = \frac{x_{it} + \alpha}{x_{it} + n\alpha} v - x_{it}. \tag{3}$$

The first-order condition of (3) is

$$\frac{(n-1)\alpha}{(x_{it} + n\alpha)^2} v - 1 = 0,^{7}$$

[6] Numerous studies of collusion in repeated contests follow a similar approach; see, for example, Linster (1994), Amegashie (2006a, 2011), Shaffer and Shogren (2008), and Cheikbossian (2012). Therefore, we adopt this approach so that our results on incentives for collusion are comparable to ones already existing in the literature.

[7] It is straightforward to show that (3) is strictly concave in x_{it}.

which is satisfied by

$$\tilde{x}_D = \pm\sqrt{(n-1)\alpha v} - n\alpha,$$

of which only $\tilde{x}_D = \sqrt{(n-1)\alpha v} - n\alpha > 0$ when $\alpha < (n-1)v/n^2$. Therefore, the optimal deviation is

$$x_D = \sqrt{(n-1)\alpha v} - n\alpha$$

and earns expected profits of

$$\pi_D = v - 2\sqrt{(n-1)\alpha v} + n\alpha.$$

Nash reversion strategies sustain collusion as a subgame perfect Nash equilibrium of the infinitely repeated contest if and only if the discounted profits from collusion exceed the discounted profits from deviation and reversion to the stage game's Nash equilibrium forever; that is, Nash reversion strategies sustain collusion if and only if

$$\sum_{t=0}^{\infty} \delta^t \pi_C \geq \pi_D + \sum_{t=1}^{\infty} \delta^t \pi_N,$$

which simplifies to

$$\frac{\pi_C}{1-\delta} \geq \pi_D + \frac{\delta}{1-\delta}\pi_N,$$

or, in terms of a critical discount factor δ^*,

$$\begin{aligned} \delta &\geq \frac{\pi_D - \pi_C}{\pi_D - \pi_N} \\ &= \frac{v - 2\sqrt{(n-1)\alpha v} + n\alpha - \frac{v}{n}}{v - 2\sqrt{(n-1)\alpha v} + n\alpha - \left(\frac{v}{n^2} + \alpha\right)} := \delta^*. \end{aligned} \tag{4}$$

Note that $\delta^* \in (0,1)$ because $\pi_D > \pi_C$ holds if and only if $[(n-1)v/n - n\alpha]^2 > 0$, which always holds, and $\pi_C > \pi_N$ holds if and only if $\alpha < (n-1)v/n^2$, which holds by assumption. Any factor increasing δ^* makes collusion more difficult to sustain and any factor decreasing δ^* makes collusion easier to sustain. The following proposition describes how δ^* varies with α, v, and n.

Proposition. *When players attempt to sustain maximal collusion by using Nash reversion strategies in an infinitely repeated contest with the contest success function in (1), (i) an increase in the contest's degree of noise makes sustaining collusion easier, (ii) an increase in the contest's prize value makes sustaining collusion more difficult, and (iii) an increase in the contest's number of players can make sustaining collusion either more or less difficult.*

Proof. (i) Differentiating (4) with respect to α, we have

$$\begin{aligned}\frac{\partial \delta^*}{\partial \alpha} &= \frac{-\frac{(n-1)v}{\sqrt{(n-1)\alpha v}}+n}{v-2\sqrt{(n-1)\alpha v}+n\alpha-\left(\frac{v}{n^2}+\alpha\right)} \\ &- \frac{\left[v-2\sqrt{(n-1)\alpha v}+n\alpha-\frac{v}{n}\right]\left[-\frac{(n-1)v}{\sqrt{(n-1)\alpha v}}+n-1\right]}{\left[v-2\sqrt{(n-1)\alpha v}+n\alpha-\left(\frac{v}{n^2}+\alpha\right)\right]^2} \\ &= -\frac{\sqrt{(n-1)\alpha v}\left\{(n-1)v-n\left[2\sqrt{(n-1)\alpha v}-n\alpha\right]\right\}}{\alpha\left[v-2\sqrt{(n-1)\alpha v}+n\alpha-\left(\frac{v}{n^2}+\alpha\right)\right]^2} < 0,\end{aligned}$$

which holds if and only if

$$(n-1)v-n\left[2\sqrt{(n-1)\alpha v}-n\alpha\right] > 0,$$

which holds if and only if

$$\left[(n-1)v-n^2\alpha\right]^2 > 0,$$

which always holds.

(ii) Differentiating (4) with respect to v, we have

$$\begin{aligned}\frac{\partial\delta^*}{\partial v} &= \frac{1-\frac{(n-1)\alpha}{\sqrt{(n-1)\alpha v}}-\frac{1}{n}}{v-2\sqrt{(n-1)\alpha v}+n\alpha-\left(\frac{v}{n^2}+\alpha\right)} \\ &- \frac{\left[v-2\sqrt{(n-1)\alpha v}+n\alpha-\frac{v}{n}\right]\left[1-\frac{(n-1)\alpha}{\sqrt{(n-1)\alpha v}}-\frac{1}{n^2}\right]}{\left[v-2\sqrt{(n-1)\alpha v}+n\alpha-\left(\frac{v}{n^2}+\alpha\right)\right]^2} \\ &= \frac{\sqrt{(n-1)\alpha v}\left\{(n-1)v-n\left[2\sqrt{(n-1)\alpha v}-n\alpha\right]\right\}}{v\left[v-2\sqrt{(n-1)\alpha v}+n\alpha-\left(\frac{v}{n^2}+\alpha\right)\right]^2}>0,\end{aligned}$$

which holds if and only if

$$(n-1)v-n\left[2\sqrt{(n-1)\alpha v}-n\alpha\right]>0,$$

which always holds, as we have shown above in (i).

(iii) Differentiating (4) with respect to n, we have

$$\begin{aligned}\frac{\partial\delta^*}{\partial n} &= \frac{-\frac{\alpha v}{\sqrt{(n-1)\alpha v}}+\alpha+\frac{v}{n^2}}{v-2\sqrt{(n-1)\alpha v}+n\alpha-\left(\frac{v}{n^2}+\alpha\right)} \\ &- \frac{\left[v-2\sqrt{(n-1)\alpha v}+n\alpha-\frac{v}{n}\right]\left[-\frac{\alpha v}{\sqrt{(n-1)\alpha v}}+\alpha+\frac{2v}{n^3}\right]}{\left[v-2\sqrt{(n-1)\alpha v}+n\alpha-\left(\frac{v}{n^2}+\alpha\right)\right]^2}\gtreqless 0.\end{aligned}$$

Figure 1 illustrates the nonmonotonic relationship between n and δ^* by graphing (4) for $\alpha=0.9$, $v=100$, and $n\in[2,100]$. Figure 1 shows that an increase in n initially makes collusion more difficult to sustain and eventually makes collusion easier to sustain. □

The properties of δ^* are fairly intuitive. An increase in α decreases π_D, decreasing incentives to deviate from collusion, while it increases π_N, increas-

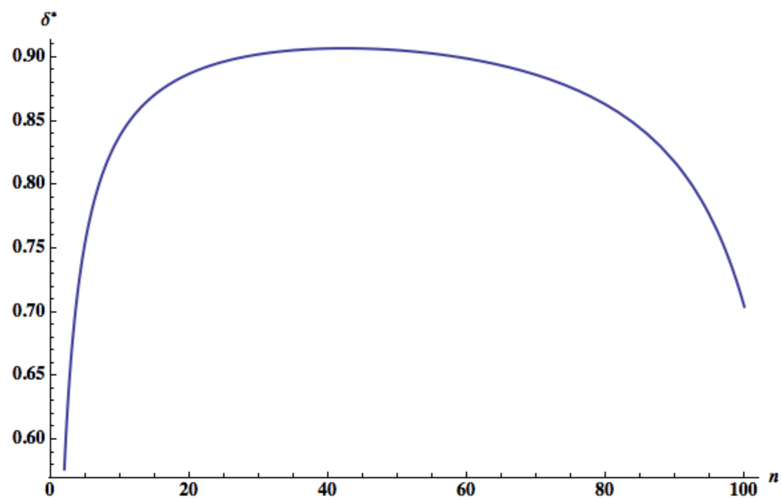

Figure 1: Critical Discount Factor Sustaining Collusion ($\alpha = 0.9$, $v = 100$, $n \in [2, 100]$)

ing deviation incentives because of the less severe Nash reversion punishment; the former effect dominates, thus an increase in α makes sustaining collusion easier. An increase in v has three effects on δ^*: it increases π_D and π_N, both of which decrease incentives to deviate from collusion, and it increases π_C, increasing incentives for collusion; the first two effects dominate, thus an increase in v makes sustaining collusion more difficult. An increase in n also has three effects on δ^*: it decreases π_D and π_N, both of which decrease incentives to deviate from collusion, and it decreases π_C, decreasing incentives for collusion; which of the three effects dominate depends upon the levels of α, v, and n, thus an increase in n can make sustaining collusion either more or less difficult.

4 Conclusion

Both expenditures and noise affect players' performance in many real-world contests. Many of these same contests are repeated, and therefore could offer players incentives to collude. We analyze incentives for tacit collusion in an infinitely repeated contest with noise where players can sustain collusion by using Nash reversion strategies if players are sufficiently patient. We show that an increase in the contest's degree of noise makes sustaining collusion easier, an increase in the contest's prize value makes sustaining collusion more difficult, and an increase in the contest's number of players can make sustaining collusion either more or less difficult.

The analysis in this paper suggests a number of avenues for future research. Empirical research in the form of a carefully designed experiment could investigate the results in this paper. Analysis of alternative mechanisms sustaining collusion in repeated contests with noise or extending the analysis in this paper to the case of players asymmetrically affected by noise could test the generalizability of our results.

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