# Similarity-Based Learning and Similarity Equilibria (Preliminary)

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# Introduction

Equilibrium play in games comes about through repeated play and adaptive learning.

To learn equilibrium play that depends on external signals (types) agents needs to be repeatedly exposed to those signals.

When agents are exposed to a large number of external signals, they can only learn from past periods in which they received similar signals.

# Introduction

1. We propose an equilibrium concept (similarity equilibrium) for games in which players regard some signals as similar to other signals, and investigate:
   - the relation between similarity equilibrium and classical equilibrium concepts.

2. We propose a model of similarity-based learning, and investigate:
   - whether equilibrium will be reached;
   - whether some equilibria are more likely to be reached than others.

# Introduction

- Signals will be payoff irrelevant.
- Similarity will be exogenous.
- Similarity may be:
  - non-transitive
    (0 is similar to 0.01; 0.01 is similar to 0.02, ... ,
    999.99 is similar to 1000; but 0 is not similar to 1000.)
  - non-symmetric
    (Every dog reminds me of my dog;
    but my dog does not remind me of every other dog.)

# Introduction

Findings:

1. If similarity is symmetric and transitive, then similarity equilibria are the same as correlated equilibria.
2. If similarity-based learning converges, it converges to a similarity equilibrium.
3. In some classes of games, similarity-based learning favors signal-independent similarity equilibria (Nash equilibria) over signal-dependent similarity equilibria (e.g. correlated equilibria).

# Introduction

Related Literature:

- Itzhak Gilboa and David Schmeidler, Cased-Based Decision Theory, *Quarterly Journal of Economics* (1995), 605-639.
- Jakub Steiner and Colin Stewart, Contagion Through Learning, *Theoretical Economics* (2008), 431-458.
- Rossella Argenziano and Itzhak Gilboa, Similarity-Nash Equilibria in Statistical Games, unpublished, 2019.
- Philippe Jehiel, Analogy-Based Expectation Equilibrium, *Journal of Economic Theory* (2005), 81-104.
- Philippe Jehiel and Dov Samet, Valuation Equilibrium, *Theoretical Economics* (2007), 163-185.

# Introduction

Future Work:

- Extend to the case when signals are payoff relevant.
- Agents learn to behave "as if" they had infinite hierarchies of beliefs about other agents' types.
- Can equilibria that depend in complicated ways on higher order beliefs be learned?

# Similarity Equilibrium

$i \in \{1, 2\}$ players.

$A^i$ player $i$'s action set (finite).

$u^i : A^1 \times A^2 \to \mathbb{R}$ player $i$'s utility function.

$\theta^i \in \Theta^i$ player $i$'s signal (from a finite set of possible signal realization).

$f : \Theta^1 \times \Theta^2 \to [0, 1]$ probability distribution of pairs of signals.

($f$ has strictly positive marginals.)

# Similarity Equilibrium

$s^i : \Theta^i \to \mathcal{P}(\Theta^i) \setminus \{\emptyset\}$ similarity correspondence.

$\hat{\theta}^i \in s^i(\theta^i)$: signal realization $\hat{\theta}^i$ is similar to signal realization $\theta^i$.

- Reflexivity: $\theta^i \in s^i(\theta^i) \quad \forall i, \theta^i$ (assumed)
- Symmetry: $\hat{\theta}^i \in s^i(\theta^i) \Rightarrow \theta^i \in s^i(\hat{\theta}^i)$ (not assumed)
- Transitivity: $\hat{\theta}^i \in s^i(\theta^i) \wedge \tilde{\theta}^i \in s^i(\hat{\theta}^i) \Rightarrow \tilde{\theta}^i \in s^i(\theta^i)$ (not assumed)

# Similarity Equilibrium

Strategy of player $i$: $\sigma^i : \Theta^i \to A^i$.

Definition

Strategies $(\sigma^1, \sigma^2)$ form a similarity equilibrium if

$$\sigma^i(\theta^i) \in \arg\max_{a_i \in A_i} \sum_{\hat{\theta}^i \in s^i(\theta^i)} \sum_{\theta^j \in \Theta^j} u^i(a^i, \sigma^j(\theta^j)) f(\hat{\theta}^i, \theta^j) \quad \forall i, \theta^i.$$

# Similarity Equilibrium

Given $(\sigma^1, \sigma^2)$ the implied distribution $\rho$ over actions is:

$$\rho(a^1, a^2) = \sum_{\substack{(\theta^1,\theta^2) \in \Theta^1 \times \Theta^2 \\ (\sigma^1(\theta^1),\sigma^2(\theta^2))=(a^1,a^2)}} f(\theta^1, \theta^2) \quad \forall (a^1, a^2).$$

## Proposition

*If both players' similarity correspondences $s^i$ are symmetric and transitive, then the action distribution implied by a similarity equilibrium is a correlated equilibrium.*

# Similarity Equilibrium (Example)

Game:

| | A | B | C |
|---|---|---|---|
| A | 1,1 | 0,0 | 0,-2 |
| B | 0,-2 | 2,0 | 0,1 |

Signals:

| | $\theta^2$ | $\hat{\theta}^2$ | $\tilde{\theta}^2$ |
|---|---|---|---|
| $\theta^1$ | $1/3$ | 0 | 0 |
| $\hat{\theta}^1$ | 0 | $1/3$ | 0 |
| $\tilde{\theta}^1$ | 0 | 0 | $1/3$ |

similarity correspondences:
$s^i(\theta^i) = s^i(\hat{\theta}^i) = \{\theta^i, \hat{\theta}^i\}, \quad s^i(\tilde{\theta}^i) = \{\hat{\theta}^i, \tilde{\theta}^i\} \quad \forall i.$
Not symmetric nor transitive.

similarity equilibrium:
$\sigma^i(\theta^i) = \sigma^i(\hat{\theta}^i) = A,\ \sigma^i(\tilde{\theta}^i) = B \quad \forall i.$

# Similarity Equilibrium (Example)

Game:

| | A | B | C |
|---|---|---|---|
| A | 1,1 | 0,0 | 0,-2 |
| B | 0,-2 | 2,0 | 0,1 |

Implied Action Distribution:

| | A | B | C |
|---|---|---|---|
| A | $2/3$ | 0 | 0 |
| B | 0 | $1/3$ | 0 |

Not a correlated equilibrium.

# Similarity-Based Learning

$t = 1, 2, \ldots$ time periods.

$(\theta^1_t, \theta^2_t)$ i.i.d. across time, in each period distributed according to $f$.

$(a^1_t, a^2_t)$ actions taken in period $t$.

$h_t = \big((\theta^1_1, a^1_1, \theta^2_1, a^2_1), (\theta^1_2, a^1_2, \theta^2_2, a^2_2), \ldots, (\theta^1_t, a^1_t, \theta^2_t, a^2_t)\big)$
history of length $t$.

Players observes both players' actions and their own signal.

Initial history of length $t_0 \in \mathbb{N}$ such hat for every $i \in \{1, 2\}$ and every $\theta^i \in \Theta^i$ there is at least one $\tau \leq t_0$ with $\theta^i_\tau \in s^i(\theta^i)$.

# Similarity-Based Learning

The Learning Algorithm: In every period $t > t_0$ each player $i$:

- observes $\theta^i_t$;
- considers the past $N$ periods $\tau \leq t$ in which $\theta^i_\tau \in s^i(\theta^i_t)$;
- best responds to the frequency distribution of the other players' actions in those periods.

$N < \infty$: finite memory

$N = \infty$: infinite memory

$a^i_{t,N}(\theta^i)$: the action player $i$ with memory size $N$ would take in period $t$ if she observed $\theta^i$
(a random variable)

# Similarity-Based Learning and Similarity Equilibrium

**Proposition ($N = \infty$)**

*Suppose for every $i \in \{1, 2\}$ and every $\theta^i \in \Theta^i$ there exists an action $\sigma^i(\theta^i) \in A^i$ such that*

$$\mathbb{P}\left[\exists T : a^i_{t,\infty}(\theta^i) = \sigma^i(\theta^i)\ \forall t \geq T\right] = 1.$$

*Then $(\sigma^1, \sigma^2)$ is a similarity equilibrium.*

# Similarity-Based Learning and Similarity Equilibrium

### Conjecture ($N < \infty$)

*Assume there are no indifferences. Then there exists an $\varepsilon > 0$ such that for every $N \in \mathbb{N}$ the following is true:*
*If for every $i \in \{1, 2\}$ and every $\theta^i \in \Theta^i$ there exist an action $\sigma^i(\theta^i) \in A^i$ and a period $T \in \mathbb{N}$ such that*

$$\forall t \geq T : \quad \mathbb{P}\left[a^i_{t,N}(\theta^i) = \sigma^i(\theta^i)\ \right] \geq 1 - \varepsilon,$$

*then $(\sigma^1, \sigma^2)$ is a similarity equilibrium.*

# Similarity-Based Learning and Equilibrium Selection

Game:

| | A | B |
|---|---|---|
| A | 2,7 | 6,6 |
| B | 0,0 | 7,2 |

or any other game with two strict Nash equilibria: $(A, A)$ and $(B, B)$.

## Assumption (1)

*There are* $(\theta^1, \theta^2)$, *and* $\hat{\theta}^i \in \Theta^i$ *for some* $i$ *such that:*

$$f(\theta^1, \theta^2) > 0, f(\hat{\theta}^i, \theta^j) > 0, \text{ and } \theta^i \notin s^i(\hat{\theta}^i).$$

## Definition

$(\theta^1, \theta^2) \to (\hat{\theta}^1, \hat{\theta}^2)$ ($(\theta^1, \theta^2)$ is connected to $(\hat{\theta}^1, \hat{\theta}^2)$) if

$$f(\theta^1, \theta^2) > 0, \ f(\hat{\theta}^1, \hat{\theta}^2) > 0 \text{ and } \theta^i \in s^i(\hat{\theta}^i) \ \forall i.$$

## Assumption (2)

*For all* $(\theta^1, \theta^2)$ *with* $f(\theta^1, \theta^2) > 0$, *for every* $i \in \{1, 2\}$, *and for every* $\hat{\theta}^i$ *with* $\hat{\theta}^i \neq \theta^i$, *there is a sequence* $\left(\theta^1_\nu, \theta^2_\nu\right)_{\nu=1,2,\ldots,n}$ *such that:*

(i) $(\theta^1_1, \theta^2_1) = (\theta^1, \theta^2)$;

(ii) $(\theta^1_\nu, \theta^2_\nu) \to (\theta^1_{\nu+1}, \theta^2_{\nu+1})$ *for every* $\nu = 1, 2, \ldots, n-1$;

(iii) $\theta^i_n = \hat{\theta}^i$.

## Proposition

*For every $N \in \mathbb{N}$, under Assumptions (1) and (2),*

$$\begin{aligned} & \mathbb{P}\left[\exists T: \; a^i_{t,N}(\theta^i) = A \quad \forall i, \theta^i, t \geq T\right] \\ + \quad & \mathbb{P}\left[\exists T: \; a^i_{t,N}(\theta^i) = B \quad \forall i, \theta^i, t \geq T\right] = 1 \end{aligned}$$

- This rules out convergence to correlated equilibria.

Game:

| | A | B |
|---|---|---|
| A | 2,7 | 6,6 |
| B | 0,0 | 7,2 |

Correlated Equilibrium (Aumann, 1974):

| | A | B |
|---|---|---|
| A | 1/3 | 1/3 |
| B | 0 | 1/3 |

- Every similarity structure is "close" to a similarity structure for which Assumptions (1) and (2) hold.