

COSTLESS SIGNALING WITH COSTLY SIGNALS

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ABSTRACT. We study signaling environments with two common features: first, complete-information bliss points are heterogeneous across different types of sender; second, many choices are observed by the receiver. We demonstrate under relatively weak conditions that the incomplete-information signaling model has a fully separating equilibrium where the utility in the signaling equilibrium approaches that obtained in an analogous complete-information model as the number of signals increases. In other words, as the number of signals grows, the ratio of the cost of signaling to the benefit approaches 0. As an application, our main result suggests that greater transparency of decision making can reduce or eliminate signaling costs.

1. INTRODUCTION

The Spence signaling model (Spence [9]) seeks to explain how information can be credibly conveyed when reputational incentives are either not present or not sufficient to insure honesty. The private information of a sender (she) is conveyed through the inferences made by a receiver (he) based on an action (e.g., length of education) that the sender chooses. In order to credibly convey information, the sender must take an action that is more costly to her than her bliss point action, the choice the sender would make in a complete-information world (i.e., in the absence of the signaling incentive). This means, of course, that credible signaling requires welfare losses relative to the first-best world of complete information.

Given the importance of credibly conveying information, the insights derived from the Spence model may lead one to believe that virtually all of our decisions are distorted by the incentive to signal our types, and the resulting waste must amount to a significant cost on many of our interactions. We first consider a general setting where different types of senders have different bliss points and the cost of deviating from the bliss point action is parameterized abstractly through a weight attached to

signaling costs. We prove that so long as different types of senders have different bliss points, then the ratio of the cost of signaling to the benefit approaches 0 as the cost of deviating from the bliss points increases.

The assumption that different sender-types have different bliss points is necessary for our result. Although heterogeneous bliss points are not often highlighted in applications, many of the models analyzed in the signaling literature feature heterogeneous bliss-points. We now provide a handful of examples to emphasize this point:

- Spence [10] models students choosing a level of productive human capital to signal their ability, and lower cost students prefer to accrue more education even in a complete information world.
- Mailath [5] provides a model wherein firms playing a dynamic market competition game distort their choice of price in period 1 to signal their private information about their marginal costs, and firms with different marginal costs have different optimal prices in a complete-information world.
- Banks [1] analyzes a model in which politicians signal their policy preferences through their pre-election policy announcements and the politicians differ in their ideal policy choice.
- Miller and Rock [7] provide a model wherein firms signal the value of an earnings surprise through their dividend, and firms with different earnings surprises have differing levels of optimal investment (and hence dividends).
- Andreoni and Bernheim [3] study a model in which individuals signal a preference for fairness through a division in the dictator game, and dictators with a greater preference for fairness signal this preference by choosing a more equitable division.
- Bernheim and Bodoh-Creed [2] provide a model of politicians of differing levels of decisiveness that compete for election. The electorate prefers decisive politicians, so the politicians have an incentive to signal his or her decisiveness by making hasty decisions in lower office.

In the abstract, it is easy to add a parameter to any of these models to reflect the weight attached to deviations from a sender-type's bliss point. It may be less obvious what changes to such a parameter correspond to in practice. To this end,

we point out that if the receiver can observe multiple actions from the sender, then it is natural to assume that the cost of deviations from a bliss-point increases as the number of observed actions increases.

When the payoff for each action enters the sender's utility additively, we provide weak conditions under which the welfare optimal separating equilibrium is one in which the sender chooses the same realization for each of the actions. In such a symmetric equilibrium, increasing the number of signals has the effect of increasing the marginal cost of (symmetrically) deviating from the bliss point across all actions. Applying our primary result to this setting, we conclude that if agents of different types have different bliss points and send N signals, then the ratio of the costs of signaling to the benefits vanishes at a rate of $O\left(\frac{1}{N^{1-\alpha}}\right)$ for any $\alpha > 0$.

In order to prove that heterogeneous bliss points are necessary for our result to hold, we study an example in which all of the sender-types share the same bliss point. We derive the separating equilibrium as a function of the marginal cost of deviating from the bliss point, and we show that the signals of each type of sender must approach the bliss point as the marginal cost of deviating from the bliss point grows. However, the deviations from the bliss point do not vanish quickly enough to eliminate the total cost of the signals as N increases.

We consider two other settings with multiple signals where the cost of deviating from the bliss point of each action enters the sender's utility nonadditively. First we consider symmetric equilibria in a setting wherein the agent utility can be decomposed into a form analogous to the abstractly parameterized model studied initially. It turns out that a simple condition on the rate of change of the cost of signaling as the number of signals increases is sufficient for the total cost of signaling to vanish as the number of signals increases. We also provide a natural example that violates our condition on the rate of change and show that the total cost of signaling does not vanish as the number of signals increases.

Second, we consider settings with a general aggregator for the utility from the N signals. Our analysis is limited to a particular class of equilibria wherein agents choose the bliss points of higher types as signals, which is a natural extension of

both the one-dimensional model and the symmetric equilibria of the other multi-signal models we characterize. Our result implies that the total cost of signaling must vanish in a welfare optimal equilibrium (if such an equilibrium exists).

We are unaware of another paper that has pointed out the relationship between heterogeneous bliss points, the number of observed signals, and the total cost of signaling. However, there is a rich literature studying the properties of models wherein either (or both) the signal or the underlying private information are multi-dimensional. The majority of this theoretical literature is dedicated to providing conditions under which a separating equilibrium exists (e.g., [4], [8], and [12]). Our model is more specialized than the models considered in this literature, but this allows us to cleanly demonstrate our main points.

Section 2 introduces our benchmark model in which the sender's marginal cost of deviating from her bliss point is parameterized abstractly. Section 3 provides several examples of the benchmark model in order to build intuition for the main result, and Section 4 proves our main result. Section 5 provides a specialization of our benchmark model that interprets the marginal cost parameter as the number of signals observed by the receiver. We close in Section 6.

2. MODEL

We assume that there is a receiver observing the action of a sender. The sender's private information is represented by her type $t \in [\underline{t}, \bar{t}] = T \subset \mathbb{R}$. For our benchmark model we assume that the sender chooses an action $a \in \mathbb{R}_+$ that will serve as the signal used by the receiver to make inferences about the sender's type. Action a yields a direct utility $\lambda\pi(a, t)$ where π represents the costs and benefits of the action in the absence of a receiver (i.e., without any signaling incentive) and $\lambda > 0$ is the weight attached to the direct utility. We assume throughout that $\pi(a, t)$ is continuous. The *bliss point* for an agent of type t is

$$(1) \quad a_{BP}(t) = \underset{a}{\operatorname{arg\,max}} \pi(a, t)$$

$a_{BP}(t)$ is the action the sender would choose if her type was commonly known to both the sender and the receiver.

Having observed a , the receiver uses Bayes's rule to form beliefs about the sender's type. The beliefs of the receiver following an observation of action a is a measure $\delta(a) \in \Delta(T)$ where $\Delta(T)$ is the set of Borel measures over T , and we refer to $\delta(a)$ as the receiver's *perception* of the sender. When we focus our analysis on fully separating equilibria, the receiver's equilibrium beliefs place probability 1 on the sender having the type $\hat{t}(a)$. When convenient we will suppress the arguments of δ and \hat{t} and refer to the sender as "choosing" the receiver's perception.

Given the receiver's perception, the sender gets a benefit equal to $B(t, \delta)$.¹ The benefit to the sender, B , and the cost of signaling, $\lambda\pi(a, t)$, combine additively to yield sender utility function:

$$(2) \quad U(a, t; \lambda) = B(t, \delta(a)) + \lambda\pi(a, t)$$

In a fully separating equilibrium of the Spence [9] job market model, a represents a level of investment in human capital, $\pi(a, t)$ represents the payoff from these productive investments, and $B(t, \hat{t}(a))$ represents the wage the agent receives given her type t and the perception that she is of type $\hat{t}(a)$.

We now make several assumptions on our model that can be easily verified in applications. Throughout we assume that all derivatives referred to exist.

Assumption 1. $a_{BP}(t)$ is the unique solution to equation 1. There exists $\beta > 0$ such that for any $t > t'$, we have $a_{BP}(t) - a_{BP}(t') \geq \beta(t - t')$.

Assumption 2. One of the following holds:

- (1) $\pi_{aa}(a, t) < 0$ and there is $C < \infty$ such that $\|\pi_{aaa}(a, t)\| \leq C$ for all pairs $(a_{BP}(t), t)$.
- (2) There is $C < \infty$ such that for all pairs $(a_{BP}(t), t)$ we have $\frac{\partial^i \pi(a, t)}{\partial a^i} = 0$ for $i \in \{2, \dots, k < \infty\}$, $\frac{\partial^{k+1} \pi(a, t)}{\partial a^{k+1}} \neq 0$, and $\left\| \frac{\partial^{k+2} \pi(a, t)}{\partial a^{k+2}} \right\| \leq C$.

Assumption 3. $B(t, \delta) \in [\underline{B}, \overline{B}]$ for all $\delta(a) \in \Delta(T)$ where $-\infty < \underline{B} \leq \overline{B} < \infty$.

Assumption 4. There is $\gamma > 0$ such that $0 \leq \frac{\partial B(t, \hat{t})}{\partial t} \leq \gamma$.

¹It is straightforward to allow the action a to influence B . If we were to include a in B , then we would require that the partial derivatives $B_a(t, \delta, a)$, $B_{aa}(t, \delta, a)$, and $B_{aaa}(t, \delta, a)$ be defined for all (t, δ, a) and that $B_{aaa}(t, \delta, a)$ be uniformly bounded from above.

Assumption 1 requires that different types have bliss points that are nontrivially different. Assumption 2 is a technical assumption that we require for our analysis of the effect of small deviations from the bliss points. Case 1 of Assumption 2 insures that we can analyze the effect of second order deviations from a_{BP} . In the event that $\pi(a, t)$ is not strictly convex in a , case 2 of Assumption 2 imposes an analogous assumption on the first nonzero higher-order term. Assumption 3 requires that the benefit from signaling a particular type must be bounded. Assumption 4 puts an upper bound on how rapidly the sender's benefit can change with her perceived type. Since T is compact and B is continuous, Assumption 4 implies Assumption 3, but some of our results hold under the weaker of the two assumptions.

Our main result constructs a separating equilibrium and proves that the total cost of following this separating equilibrium vanishes as $\lambda \rightarrow \infty$. Denote the strategy used in the separating equilibrium as $a_{SEP}(t)$. We can write the equilibrium utility of an agent of type t that mimics the action of an agent of type \hat{t} as

$$V(\hat{t}, t; \lambda) = B(t, \hat{t}) + \lambda \pi(a_{SEP}(\hat{t}; \lambda), t)$$

$a_{SEP}(t; \lambda)$ is described by the following differential equation

$$\left. \frac{\partial a_{SEP}(\circ; \lambda)}{\partial \hat{t}} \right|_{\hat{t}=t} = \frac{-1}{\lambda} \left. \frac{\partial B}{\partial \hat{t}} \right|_{(t, \hat{t})=(t, t)} \frac{1}{\left. \pi_a(a, t) \right|_{(a, t)=(a_{SEP}(t), t)}}$$

with the initial condition $a_{SEP}(\underline{t}; \lambda) = a_{BP}(\underline{t})$. Finally, Assumption 5 implies that $a_{SEP}(t; \lambda)$ is an equilibrium of our model. The single-crossing property is sufficient for Assumption 5 to hold, but is stronger than required for our results.

Assumption 5. $a_{SEP}(\circ; \lambda)$ forms a Bayes-Nash equilibrium.

3. EXAMPLES

We provide two examples in this section. First, we provide an example that has heterogeneous bliss points. Notice that the difference between the bliss-point and the equilibrium action vanishes at the rate $O(\lambda^{-1})$. This high speed of convergence of the equilibrium actions to the bliss points is crucial for the total cost of signaling to vanish.

Example 1. Suppose $B(t, \hat{t}) = \hat{t}$, $\pi(a, t) = -(a-t)^2$, and $T = [0, 1]$. The bliss points are (obviously) $a_{BP}(t) = t$. The ODE defining the fully separating equilibrium is

$$\left. \frac{\partial a_{SEP}}{\partial \hat{t}} \right|_{\hat{t}=t} = \frac{1}{2\lambda(a_{SEP}(t) - t)}$$

We use the change of variables $z(t) = a_{SEP}(t) - t$, and solving the inverse ODE yields

$$t = - \left[z + \frac{1}{2\lambda} \ln \left(\frac{1}{2\lambda} - z \right) \right] + C$$

The initial condition $z(0) = 0$ implies

$$t = - \left[z + \frac{1}{2\lambda} \ln (1 - 2\lambda z) \right]$$

Reversing our change of variables and rearranging, we find

$$z(t) = \frac{1 - e^{2\lambda a_{SEP}(t)}}{2\lambda}$$

The total cost of signaling is then

$$\lambda z(t)^2 \leq \lambda \left(\frac{1}{2\lambda} \right)^2 = \frac{1}{4\lambda}$$

The second example explores settings with homogenous bliss points, which reinforces the necessity of heterogeneous bliss points for our result. The agents share the bliss point of $a_{BP} = 0$, $a_{SEP}(t)$ converges to $a_{BP}(t)$ at the rate $O(\lambda^{-0.5})$, and the slow convergence causes the total cost of signaling is bounded away from 0. Intuitively, the progressively greater bunching of low t types around 0 puts pressure for higher t types to choose higher actions. This pressure is absent in Example 1 since the different types are attracted to different bliss points. In fact, we show that while $a_{SEP}(t; \lambda) \rightarrow t$ as $\lambda \rightarrow \infty$, the total cost of signaling is invariant to λ .

Example 2. Suppose $B(t, \hat{t}) = \hat{t}$, $\pi(a, t) = \frac{-a^2}{t+\lambda}$, $\lambda > 0$, and $T = [0, 1]$. The bliss points are $a_{BP}(t) = 0$ —in other words, the bliss points are homogenous. The ODE defining the fully separating equilibrium is

$$\left. \frac{\partial a_{SEP}}{\partial \hat{t}} \right|_{\hat{t}=t} = \frac{t + \lambda}{2\lambda a_{SEP}(t)}$$

We can write this in a more convenient form as

$$2\lambda a_{SEP}(t) \frac{\partial a_{SEP}}{\partial \widehat{t}} \Big|_{\widehat{t}=t} = t + \lambda$$

Integrating both sides and using our initial condition yields

$$\lambda a_{SEP}(t)^2 = \frac{1}{2}(t + \lambda)^2 - \frac{\lambda^2}{2}$$

The total cost of signaling, $\lambda a_{SEP}(t)^2$, is invariant with respect to λ .

4. MAIN RESULT

Our main result proves that the utility obtained by the sender in a fully separating equilibrium approaches the utility she receives in a complete information world when she chooses her bliss point action. If one thinks of $\lambda[\pi(a_{SEP}(t; \lambda), t) - \pi(a_{BP}(t), t)]$ as the total cost of signaling, then our main result implies that the total cost vanishes in the limit as $\lambda \rightarrow \infty$.

Before proceeding to our main result, we prove the following lemma. In addition to serving as the first step of the proof of our main result, the lemma is of additional interest since it characterizes *all* of the equilibria of our signaling model. Let $a(t; \lambda)$ denote an equilibrium of the signaling game, and let $\{a(t; \lambda_i)\}_{i=1}^{\infty}$, $\lambda_i \rightarrow \infty$, be a convergent sequence of equilibrium strategies. Lemma 1 implies that the limit must be $a_{BP}(t)$.

Lemma 1. *Let Assumptions 2 and either 3 or 4 hold. Then $|a(t; \lambda_i) - a_{BP}(t)| = O\left(\frac{1}{\sqrt{\lambda_i}}\right)$ as $\lambda_i \rightarrow \infty$. Moreover, $a(t; \lambda_i) > a_{BP}(t)$ for λ_i sufficiently large.*

Lemma 1 distills a somewhat obvious truth - if the cost of deviating from the bliss-point increases while the benefits remain fixed, then the deviations must shrink as the costs grow. Among the implications of this is that in any pooling equilibrium, the pools must vanish as λ_i increases.

Even given the convergence proven by Lemma 1, we cannot prove that the cost of the signal drops as λ grows. Consider a cost function of the form $\pi(a, t) = (a - t)^2$, which we studied in Example 2. If the convergence of $a(t)$ to $a_{BP}(t)$ proceeds at a rate of $O\left(\frac{1}{\sqrt{\lambda}}\right)$, the cost of the signal would remain nontrivial in the limit as $\lambda \rightarrow \infty$. In order to prove that the total cost of the signal vanishes, we need to prove that

the convergence of $a(t)$ to $a_{BP}(t)$ is faster, which is precisely what the proof of our main theorem establishes. Note that our theorem could have been restated in terms of the ratio of the benefits to the costs of signaling vanishing at the rate $O\left(\frac{1}{\lambda^{1-\alpha}}\right)$.

Theorem 1. *Let Assumptions 1, 2, and 4 hold. Then $\lambda [\pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t)] = O\left(\frac{1}{\lambda^{1-\alpha}}\right)$ for any $\alpha > 0$*

Before describing the intuition underlying our result, we would like to point out that our claim would be trivial in a model with a discrete set of types and actions if each type had a unique bliss point that was different from that of any other type. In such a discrete setting, there will be some value of λ sufficiently large that no agent would find it optimal to make a discrete deviation from her bliss point even if such a deviation created the best possible perception for the sender:

$$\underset{\hat{t}}{\operatorname{arg\,max}} B(t, \hat{t})$$

This might suggest that the truth of Theorem 1 “obvious.” However, consider a discrete world where the agents have homogenous bliss points (e.g., Example 2). In such a discrete model, as $\lambda \rightarrow \infty$ the agents pool on the common bliss point - signaling costs vanish and no information is transmitted - whereas in the continuous setting separation continues to occur as $\lambda \rightarrow \infty$ and the total cost of signaling remains nontrivial. This suggests that the discrete and continuous worlds are not clear analogs of one another, which in turn suggests our result on the vanishing costs is not as obvious as one might have thought.

The intuition underlying our theorem is remarkably straightforward. As a first step, Lemma 1 shows that $a_{SEP}(t) - a_{BP}(t) = O\left(\frac{1}{\sqrt{\lambda}}\right)$. When combined with our lower bound on the rate of change of $a_{BP}(t)$ (Assumption 1), we can bound the equilibrium inferences made following a deviation. The bounds on the equilibrium inferences combined with our bound on the rate of change of $B(t, \circ)$ (Assumption 4) yields an upper bound on the cost of deviating from $a_{SEP}(t)$ to $a_{BP}(t)$ in terms of the benefit of signaling, $B(t, \hat{t})$. Repeated applications of this bound allows us to prove that $a_{SEP}(t) - a_{BP}(t)$ converges to 0 at a rate of $O\left(\frac{1}{\lambda^{1-\alpha}}\right)$. Combined with the fact that deviations from $a_{BP}(t)$ cause only second order losses in utility for agents of

type t , we find that the total cost of the signal in our separating equilibrium vanishes as $O\left(\frac{1}{\lambda^{1-\alpha}}\right)$.

5. MULTIPLE SIGNALS

We now apply Theorem 1 to a setting where the receiver observes multiple signals from the sender simultaneously. We consider a sequence of economies indexed by $N \in \mathbb{N}$, where the N^{th} economy features a sender that takes N simultaneous actions $a \in \mathbb{R}$ yielding an action vector $\mathbf{a}^N \in \mathbb{R}^N$ with the i^{th} component denoted a_i^N . The utility for the sender from these N actions is $\pi_{\text{Agg}}(\mathbf{a}^N, t)$. We consider three separate cases for the aggregator function $\pi_{\text{Agg}}(\mathbf{a}^N, t)$. First, we consider the simplest case wherein $\pi_{\text{Agg}}(\mathbf{a}^N, t)$ is additively separable. This has the advantage of allowing us to characterize the optimal equilibrium. We then consider bilinear aggregators that bear a similarity to the abstractly parameterized model studied in Section 4, and we show that our results continue to hold for separating equilibria that are symmetric in the sense that $a_i^N = a_j^N$ for all i, j . Finally, we study general aggregator functions and show that our results hold for a particular class of separating equilibria. In the second and third cases we cannot characterize the optimal equilibrium, but our results imply that the signaling costs must vanish in an optimal equilibrium.

5.1. Additive Aggregators. Having observed \mathbf{a}^N , the receiver uses Bayes's rule to form beliefs about the sender's type. Since we focus on fully separating equilibria, in equilibrium the receiver has a degenerate belief placing probability 1 on the sender's type being $\hat{t}(\mathbf{a}^N)$. Given the receiver's perception, the sender receives a utility of $B(t, \hat{t}(\mathbf{a}^N))$. The total utility of the sender in the N^{th} economy can be written

$$U_N(\mathbf{a}^N, t) = B(t, \hat{t}(\mathbf{a}^N)) + \sum_{i=1}^N \pi(a_i^N, t)$$

Let $\mathbf{a}_{BP}(t) = (a_{BP}(t), a_{BP}(t), \dots, a_{BP}(t)) \in \mathbb{R}_+^N$ where $a_{BP}(t) = \underset{a}{\operatorname{arg\,max}} \pi(a, t)$ denotes the bliss point of the type t agent in the N -action economy.

Consider the symmetric signaling equilibrium where $a_1(t) = \dots = a_N(t) = a_{SEP}(t; N)$. We can write the equilibrium utility of an agent of type t that mimics the action of

an agent of type \hat{t} as

$$V_N(\hat{t}, t) = B(t, \hat{t}) + N\pi(a_{SEP}(\hat{t}), t)$$

$a_{SEP}(t)$ is defined by the following differential equation

$$\left. \frac{\partial a_{SEP}}{\partial \hat{t}} \right|_{\hat{t}=t} = \frac{-1}{N} \left. \frac{\partial B}{\partial \hat{t}} \right|_{(t, \hat{t})=(t, t)} \left. \frac{1}{\pi_a(a, t)} \right|_{(a, t)=(a_{SEP}(t), t)}$$

with the initial condition $a_{SEP}(t) = a_{BP}(t)$. The full vector of actions taken in the N^{th} economy is $\mathbf{a}_{SEP}^N(t) = (a_{SEP}(t), a_{SEP}(t), \dots, a_{SEP}(t)) \in \mathbb{R}^N$.

We now show that the symmetric separating equilibrium is the welfare optimal separating equilibrium for all types, which motivates our focus on this equilibrium.

Theorem 2. *Assume we have $\frac{\pi_{at}(a, t)}{\pi_a(a, t)}$ is weakly increasing in $a > a_{BP}(t)$ for all t and that $\pi(a, t)$ is supermodular in (a, t) . For any fixed N , \mathbf{a}_{SEP}^N maximizes the payoff of each type of sender relative to any other separating equilibrium.*

The multiple signals setting is a special case of the model of Section 4 where N plays the role of λ . We focus on this setting in particular because we view it as a natural environment in which counterfactuals on λ (i.e., N) can be considered. The following corollary to Theorem 1 implies that the ratio of the costs of the signaling to the benefits vanishes as $N \rightarrow \infty$ in the symmetric separating equilibrium.

Corollary 1. *Let Assumptions 1, 2, and 4 hold. Then $N[\pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t)] = O\left(\frac{1}{N^{1-\alpha}}\right)$ for any $\alpha > 0$*

If one interprets transparency as the ability of the receiver to observe more signals, Corollary 1 implies that increasing transparency increases the utility of the sender. In a political context, this implies that political actors have an incentive to be more open to the public as this helps the actor credibly convey information at minimal cost. This incentive in favor of transparency must be weighted against the costs of transparency for the actor (e.g., transparency exposes corruption). In addition, it is not obvious that transparency is always good for the electorate, an issue discussed by Bernheim and Bodoh-Creed ?? in the context of politicians signaling decisiveness.

An astute reader may have observed by this point that one can interpret the one dimensional signal of Section 2 as an aggregate of many signals. One would hope that

our result does not turn on the seemingly innocuous choice of whether to treat this aggregate as a single scalar signal or a disaggregated vector of signals. Assumption 1 insures that disaggregation of a single signal into “small signals” will not reduce the net cost of signaling.

To see this, let us consider an aggregate signal a_{Agg} with a cost $\pi(a_{Agg}, t) = \frac{1}{N}(a_{agg} - t)^2$, which satisfies Assumption 1 with a choice of $\beta = 1$. Suppose that we instead studied a model where the agent chooses a vector of N signals (a_1, a_2, \dots, a_N) with $a_i = a_j$, $\pi(a_i = 1, t) = (a - \frac{1}{N}t)^2$, and $a_{agg} = \sum_{i=1}^N a_i = Na_i$. In the disaggregated formulation, each signal satisfies Assumption 1 with a choice of $\beta = \frac{1}{N}$. The bound on the signaling costs established in Theorem 1 has the form $O\left(\frac{1}{N\beta}\right)$ - in other words, the benefit of disaggregating the signals is exactly canceled by the reduction in β (as one would hope).

5.2. Bilinear Aggregators. So far in this section, we have assumed that the costs of each signal are aggregated additively to compute the total cost of signaling, and it is natural to wonder what would happen under other aggregators. The challenge of discussing general cost aggregators is that (1) one needs to identify the mathematical structure that is analogous to λ and (2) determine whether the analog to λ grows uniformly over a . When considering an additive cost aggregator over N signals, it was obvious that $\lambda = N$ and that N grows independently of (and hence uniformly over) a .

Consider the following utility function:

$$U_N(\mathbf{a}^N, t) = B(t, \hat{t}(\mathbf{a}^N)) + \boldsymbol{\pi}_{Agg}(a_1, \dots, a_N, t)$$

For analytical ease, we focus on symmetric separating equilibria. Suppose that instead of an additive cost aggregator our cost function had the following form:

$$\boldsymbol{\pi}_{Agg}(a_1, \dots, a_N, t) = \sqrt{\sum_{m=1}^N \pi(a_m, t)}$$

where $\sqrt{\pi(a, t)}$ satisfies Assumptions 1 and 2. Then in a separating equilibrium we can write

$$U_N(\mathbf{a}^N, t) = B(t, \hat{t}(\mathbf{a}^N)) + \sqrt{N} \sqrt{\pi(a_{SEP}(t), t)}$$

Theorem 1 implies $U_N(a_{BP}(t), t; \lambda) - U_N(a_{SEP}(t; \lambda), t; \lambda) = O\left(\frac{1}{N^{0.5-\alpha}}\right)$ for any $\alpha > 0$. This case was easy because the analog of λ , namely \sqrt{N} , grows uniformly with respect to a as $N \rightarrow \infty$.

More generally, suppose one can write the equilibrium utility for a symmetric separating equilibrium as:

$$U_N(\mathbf{a}_{SEP}(t), t) = B(t, t) + \lambda(a_{SEP}(t), t, N)g(a_{SEP}(t), t)$$

where $a_{BP}(t) = \underset{a}{\operatorname{arg\,max}} g(a, t)$ and $\mathbf{a}_{SEP}(t) = (a_{SEP}(t), a_{SEP}(t), \dots, a_{SEP}(t))$. We refer to this as a *blinear aggregator* since we can identify analogs to the λ and $\pi(a, t)$ and these analogs interact multiplicatively as per the model studied in Section 4.

Assumption 6. *The third derivative g with respect to a is bounded from above.*

Assumption 7. *There exists a sequence $\{\omega_N\}_{N=1}^\infty$, $\omega_N \rightarrow \infty$ as $N \rightarrow \infty$, and for all $a \geq 0$ and all t we have $\lambda(a_{SEP}(t), t, N) \geq \omega_N$.*

Assumption 6 is analogous to Assumption 2 and insures that our approximation argument is well defined. Assumption 7 requires that $\lambda(a, N)$ grows uniformly over a and t as $N \rightarrow \infty$. The additively separable model of Subsection 5.1, the example headlining this section, and the example leading this subsection satisfy these requirements.

If these assumptions are satisfied, an argument closely analogous to the proof of Theorem 1 shows that costs vanish as N grows.² For completeness, we state this result in the following corollary

Corollary 2. *Let Assumptions 1, 4, 6, and 7 hold. Then*

$$\lambda(a_{SEP}(t), t, N) [g(a_{BP}(t), t) - g(a_{SEP}(t), t)] = O\left(\frac{1}{\omega_N^{1-\alpha}}\right)$$

for any $\alpha > 0$.

Now let us consider a case where our theorem fails to apply.

²To see this, note that replacing λ with ω_N throughout the proof of Theorem 1, it is clear the same technical argument applies.

Example 3. Consider a utility function of the form

$$U_N(\mathbf{a}^N, t) = B(t, \hat{t}(\mathbf{a}^N)) - \prod_{m=1}^N (a_m - t)$$

where $t \in [0, 1]$ and $a_i \geq 0$. In a symmetric fully separating equilibrium, we can write the equilibrium utility as

$$\begin{aligned} B(t, \hat{t}(\mathbf{a}^N)) &= \lambda(a_{SEP}(t), t, N)g(a_{SEP}(t), t) \\ \lambda(a, t, N) &= (a - t)^{N-1} \\ g(a, t) &= (a - t) \end{aligned}$$

Assumption 7 does not hold since $\lambda(0, 0, N) = 0$. If we let $B(t, \hat{t}) = \hat{t}$, then the exact solution to the ODE defining the fully separating equilibrium is

$$a_{SEP}(t) = t^{1/N} + t$$

This implies that the cost of signaling is

$$(a_{SEP}(t) - t)^N = t$$

Despite the heterogeneous bliss points, the cost of signaling remains nontrivial as $N \rightarrow \infty$ in Example 3. The reason is that $(a - t)^{N-1}$, the analog for λ in this example, does not approach infinity as $N \rightarrow \infty$ since $a_{SEP}(t) \rightarrow t = a_{BP}(t)$ too quickly.

5.3. General Aggregator Functions. In this section we focus on a more general class of aggregators outside of the additive and separable classes studied above. We continue to consider a sequence of economies where the N^{th} economy allows the sender to choose N actions. The utility of the receiver remains

$$U_N(\mathbf{a}^N, t) = B(t, \hat{t}(\mathbf{a}^N)) + \pi_{Agg}(\mathbf{a}^N, t)$$

We define $\mathbf{a}_{BP}(t)$ as

$$(3) \quad \mathbf{a}_{BP}^N(t) = \arg \max_{\mathbf{a}^N \in \mathbb{R}^N} U_N(\mathbf{a}^N, t)$$

We continue to employ an analysis based on the speed of convergence of the equilibrium to the agents' bliss points combined - our main challenge is defining the notation that makes this argument precise.

We focus on a separating equilibrium that has the following form, which we assume for the duration is an equilibrium:³

$$(4) \quad \mathbf{a}_{SEP}^N(t) = \mathbf{a}_{BP}^N(t + \Delta^N(t)), \quad \Delta^N(t) \geq 0$$

In other words, senders signal by choosing the bliss point of an agent with a higher type, which we refer to as mimicking. For the very highest types, we must assume some extension of \mathbf{a}_{BP}^N so that we can define $\mathbf{a}_{BP}^N(\bar{t} + \Delta^N(\bar{t}))$, and this extension must satisfy Assumption 8 below. This separating equilibrium is a natural one since it captures the mimicking behavior present in the one-dimensional action setup of Section 4 or the symmetric equilibria of Subsections 5.1 and 5.2. Equilibria that have the form of Equation 4 have two important technical benefits, as well. First, we can prove our theorem based on the characterization of the one dimensional function $\Delta^N(t)$, which simplifies our argument. Second, since our argument revolves around the payoff of deviating to an agent's bliss-point, which is the equilibrium choice for lower type, we do not need to concern ourselves with off-path receiver beliefs following these deviations.

We make the following assumptions, which are satisfied by the additive and bilinear aggregators studied above.

Assumption 8. $\mathbf{a}_{BP}^N(t)$ is the unique solution to Equation 3. There exists $\beta, \beta_{UB} > 0$ such that for all N and any $t > t'$, we have

$$\beta_{UB}(t - t')\mathbf{1}^N \geq \mathbf{a}_{BP}^N(t) - \mathbf{a}_{BP}^N(t') \geq \beta(t - t')\mathbf{1}^N$$

where $\mathbf{1}^N = (1, 1, \dots, 1) \in \mathbb{R}^N$.

Assumption 9. $D_{\mathbf{a}}\pi_{Agg}(\mathbf{a}, t) = \left(\frac{\partial \pi_{Agg}(\mathbf{a}, t)}{\partial \mathbf{a}_1}, \dots, \frac{\partial \pi_{Agg}(\mathbf{a}, t)}{\partial \mathbf{a}_N} \right)$ exists for all pairs $(\mathbf{a}_{BP}(t), t)$ and all of the following hold:

- (1) There exists a sequence $\{\phi_N\}_{N=1}^{\infty}$ such that $\phi_N < 0$, $\sum_{i,j=1}^N \frac{\partial^2 \pi_{Agg}(\mathbf{a}_{BP}^N(t), t)}{\partial \mathbf{a}_i \partial \mathbf{a}_j} < \phi_N$, and $\phi_N \rightarrow -\infty$.

³Engers [4] proves that if we assume supermodularity between (a_i, t) then this strategy (amongst potentially many others) is an equilibrium.

- (2) *The ratio of third to second order effects is bounded in the sense that there exists $C < \infty$ such that in an open neighborhood of the pairs $(a_{BP}(t), t)$ we have:*

$$\left\| \frac{\sum_{i,j,k=1}^N \frac{\partial^3 \pi_{Agg}(\mathbf{a}^N, t)}{\partial \mathbf{a}_i \partial \mathbf{a}_j \partial \mathbf{a}_k}}{\sum_{i,j=1}^N \frac{\partial^2 \pi_{Agg}(\mathbf{a}^N, t)}{\partial \mathbf{a}_i \partial \mathbf{a}_j}} \right\| < C$$

Assumption 8 resembles Assumption 1 from the one-dimensional case. The lower bound on the rate of change helps determine how the receiver's inferences change if the sender deviates from $\mathbf{a}_{SEP}^N(t)$ to $\mathbf{a}_{BP}^N(t)$. Since we work with $\Delta_N(t)$, the upper bound on the rate of change of $\mathbf{a}_{BP}^N(t)$ helps us to convert bounds on $\Delta_N(t)$ into bounds on actions and utilities.

Assumption 9 includes several properties that are necessary for our Taylor expansion argument. Part (1) insures that deviations from the bliss point grow more costly as N grows - in other words the second order terms are significantly nonzero. In particular, when combined with Assumption 8, we know that it grows increasingly costly to mimic the bliss point of an agent with a higher type as N grows. Part (2) requires that the higher order effects in the Taylor expansion can be neglected as $a_{SEP}(t) \rightarrow a_{BP}(t)$.

We now prove that the signaling costs vanish when the sender's use equilibrium strategies within the class we consider. The basic logic of our argument is the same as that of Theorem 1 in that we argue that the $\mathbf{a}_{SEP}^N(t)$ converges to $\mathbf{a}_{BP}^N(t)$ at a rate of $O\left(\frac{1}{\phi_N^{1-\alpha}}\right)$ and then convert this into a statement on the convergence of the utility. We include a formal proof because the additional steps required to handle the multidimensionality of the action space are delicate.

Theorem 3. $\pi_{Agg}(\mathbf{a}_{BP}^N(t), t) - \pi_{Agg}(\mathbf{a}_{SEP}^N(t), t) = O\left(\frac{1}{\phi_N^{1-\alpha}}\right)$ for any $\alpha > 0$.

6. CONCLUSION

The common message of most signaling models is that credibly conveying information entails costs. This note makes the simple point that if (1) agents' bliss point actions are heterogeneous and (2) the marginal cost of deviating from the bliss point increases, then the total costs of signaling can be surprisingly small. These two conditions are plausibly satisfied in many real world markets, which would suggest

that perhaps signaling costs are not as pervasive a source of economic waste as the prior literature suggests. We close by discussing how two of the applications in the introduction could be modified so that our result pertains and what our results mean for the applications.

Consider the oligopolistic signaling model of Mailath [5] and assume that each firm competes in several geographically distinct markets. Within each market, the firm chooses a price that reflects its marginal cost, the actions of its competitors, and the desire to signal a high marginal cost to slacken competition in future periods. If each firm can observe all of the prices set by a competitor, this is analogous to the multiple signal model of Section 5. Theorem 1 then implies that as the competitors can observe a given firm's behavior in more markets, the firm's behavior approaches that predicted in a complete information model. In other words, the signaling incentive vanishes.

Second, consider the model of political decisiveness proposed in Bernheim and Bodoh-Creed [2]. In this model, politicians make hasty decisions in order to signal to the electorate that the politician is decisive. Define the transparency of a political system as the number of decisions that the electorate can observe. Theorem 1 implies that politicians will become less and less hasty as transparency increases and the cost of signaling to the public will vanish. If the political system is sufficiently transparent, then the politicians will be hesitant and make decisions more slowly than the median voter would prefer. This suggests that there may be costs to transparency that push against the usual benefit of increased accountability.

On a closing note, the obvious direction to push our results is to refine the bound on how quickly the marginal cost of signaling has to increase for some version of Theorem 1 to hold. Example 3 suggests that any such refinement of our result will require joint conditions on $g(a, t)$ and $\lambda(a, t, N)$ to insure $a_{SEP}(t)$ approaches $a_{BP}(t)$ sufficiently quickly. We leave such a refinement for future work.

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APPENDIX A. PROOF APPENDIX

Lemma 1. *Let Assumptions 2 and either 3 or 4 hold. Then $|a(t; \lambda_i) - a_{BP}(t)| = O\left(\frac{1}{\sqrt{\lambda_i}}\right)$ as $\lambda_i \rightarrow \infty$. Moreover, $a(t; \lambda_i) > a_{BP}(t)$ for λ_i sufficiently large.*

Proof. The bound on $a_{SEP}(t) - a_{BP}(t)$ will be derived from the inequality

$$(5) \quad \lambda_i [\pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t)] \leq \bar{B} - \underline{B}$$

We first prove our result for the case where $\pi_{aa}(a_{BP}(t), t) < 0$, and then consider what occurs when $\pi_{aa}(a_{BP}(t), t) = 0$.

The uniqueness of the bliss point for type t implies that $\pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t) > 0$, and the continuity of $\pi(\circ, t)$ implies $a_{SEP}(t) - a_{BP}(t) \rightarrow 0$ as $\lambda \rightarrow \infty$. The Taylor expansion of $\pi(a_{SEP}(t), t)$ around $(a_{BP}(t), t)$:

$$(6) \quad \pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t) = \frac{-1}{2} \pi_{aa}(a_{BP}(t), t) (a_{SEP}(t) - a_{BP}(t))^2 - \frac{\pi_{aaa}(\xi)}{6} (a_{SEP}(t) - a_{BP}(t))^3$$

Suppose $\pi_{aaa}(\xi) < 0$. Then

$$\pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t) \geq \frac{-1}{2} \pi_{aa}(a_{BP}(t), t) (a_{SEP}(t) - a_{BP}(t))^2$$

Suppose $\pi_{aaa}(\xi) > 0$. Then since $\|\pi_{aaa}(a, t)\| \leq C$ and $a_{SEP}(t) - a_{BP}(t) \rightarrow 0$ as $\lambda_i \rightarrow \infty$, we can choose λ_i^* such that for all $\lambda_i > \lambda_i^*$

$$\left\| \frac{\pi_{aaa}(\xi)}{6} (a_{SEP}(t) - a_{BP}(t)) \right\| \leq \frac{-1}{4} \pi_{aa}(a_{BP}(t), t)$$

which means we have

$$(7) \quad \pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t) \geq \frac{-1}{4} \pi_{aa}(a_{BP}(t), t) (a_{SEP}(t) - a_{BP}(t))^2$$

In either case, using equation 5 we can write

$$(a_{SEP}(t) - a_{BP}(t))^2 \left(\frac{-1}{4} \pi_{aa}(a_{BP}(t), t) \right) \leq \frac{\bar{B} - \underline{B}}{\lambda}$$

which in turn yields

$$\|a_{SEP}(t) - a_{BP}(t)\| \leq \frac{1}{\sqrt{\lambda}} \sqrt{\frac{-4(\bar{B} - \underline{B})}{\pi_{aa}(a_{BP}(t), t)}}$$

In the case where $\pi_{aa}(a_{BP}(t), t) = 0$, the first higher-order partial derivative that is nonzero must be of an even-numbered order. suffices to prove our claim.⁴

To prove the final part of our lemma, first note that by definition a_{SEP} is continuous. If $a_{SEP}(t) < a_{BP}(t)$ for any $t > \underline{t}$, we must have $a_{SEP}(t) \leq a_{BP}(\underline{t}) < a_{BP}(\bar{t})$ for all $t > \underline{t}$. However, this contradicts the fact that $|a(\bar{t}; \lambda_i) - a_{BP}(\bar{t})| = O\left(\frac{1}{\sqrt{\lambda_i}}\right)$ as $\lambda_i \rightarrow \infty$. From this contradiction, we conclude that $a_{SEP}(t) \geq a_{BP}(t)$. \square

Theorem 1. *Let Assumptions 1, 2, and 4 hold. Then $\lambda [\pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t)] = O\left(\frac{1}{\lambda^{1-\alpha}}\right)$ for any $\alpha > 0$*

Proof. The goal of this proof is to tighten the bound provided by Lemma 1. To that end, suppose agent t deviates from $a_{SEP}(t)$ to $a_{BP}(t)$. The type t' such that $a_{SEP}(t') = a_{BP}(t)$ defines the inference made by the receiver following the deviation

⁴In fact, if $\pi_{aa}(a_{BP}(t), t) = 0$ and the k -th order term is the first nonzero partial derivative, we can actually show that $|a(t; \lambda_i) - a_{BP}(t)| = O\left(\frac{1}{\lambda_i^{k/2}}\right)$

by type t . Lemma 1 implies

$$a_{SEP}(t') = a_{BP}(t) \leq a_{BP}(t') + \frac{1}{\sqrt{\lambda}} \sqrt{\frac{-4(\bar{B} - \underline{B})}{\pi_{aa}(a_{BP}(t), t)}}$$

From Assumption 1 we have

$$\begin{aligned} \beta(t - t') &\leq a_{BP}(t) - a_{BP}(t') \leq \frac{1}{\sqrt{\lambda}} \sqrt{\frac{-4(\bar{B} - \underline{B})}{\pi_{aa}(a_{BP}(t), t)}} \\ t' &\geq t - \frac{1}{\beta} \frac{1}{\sqrt{\lambda}} \sqrt{\frac{-4(\bar{B} - \underline{B})}{\pi_{aa}(a_{BP}(t), t)}} \end{aligned}$$

Therefore deviating from $a_{SEP}(t)$ to $a_{BP}(t)$ leads to an inference that the sender's type is at least $t - \frac{1}{\beta} \frac{1}{\sqrt{\lambda}} \sqrt{\frac{-4(\bar{B} - \underline{B})}{\pi_{aa}(a_{BP}(t), t)}}$. This allows us to bound the effect on the signaling incentive $B(t, \hat{t})$ more tightly:

$$\begin{aligned} \pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t) &\leq \frac{1}{\lambda} \left[B(t, t) - B \left(t, t - \frac{1}{\beta} \frac{1}{\sqrt{\lambda}} \sqrt{\frac{-4(\bar{B} - \underline{B})}{\pi_{aa}(a_{BP}(t), t)}} \right) \right] \\ &\leq \frac{\gamma}{\lambda^{1.5}} \left[\frac{1}{\beta} \sqrt{\frac{-4(\bar{B} - \underline{B})}{\pi_{aa}(a_{BP}(t), t)}} \right] \end{aligned}$$

Repeating our argument, we can then write

$$\begin{aligned} \frac{-1}{4} \pi_{aa}(a_{BP}(t), t) [a_{SEP}(t) - a_{BP}(t)]^2 &\leq \pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t) \\ &\leq \frac{1}{\lambda^{1.5}} \frac{\gamma}{\beta} \sqrt{\frac{-4(\bar{B} - \underline{B})}{\pi_{aa}(a_{BP}(t), t)}} \end{aligned}$$

which implies

$$a_{SEP}(t) - a_{BP}(t) \leq \frac{1}{\lambda^{3/4}} \sqrt{\frac{\gamma}{\beta}} \left(\frac{-4}{\pi_{aa}(a_{BP}(t), t)} \right)^{3/4} (\bar{B} - \underline{B})^{1/4}$$

Iterating this process K times yields

$$a_{SEP}(t) - a_{BP}(t) \leq \frac{C_K}{\lambda^{1-0.5K}}$$

When we use this in our Taylor expansion we get

$$\begin{aligned} \pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t) &= (a_{SEP}(t) - a_{BP}(t))^2 \left(\frac{-1}{2} \pi_{aa}(a_{BP}(t), t) - \right. \\ &\quad \left. \frac{\pi_{aaa}(\xi)}{6} (a_{SEP}(t) - a_{BP}(t)) \right) \\ &= \frac{C_K^2}{\lambda^{2-0.5K-1}} \left(\frac{-1}{2} \pi_{aa}(a_{BP}(t), t) - \frac{\pi_{aaa}(\xi)}{6} \left(\frac{C_K}{\lambda^{1-0.5K}} \right) \right) \end{aligned}$$

Using the negligibility of the third order terms, we find

$$\begin{aligned} \lambda [\pi(a_{BP}(t), t) - \pi(a_{SEP}(t), t)] &\leq \frac{C_K^2}{\lambda^{1-0.5K-1}} \left(\frac{-1}{2} \pi_{aa}(a_{BP}(t), t) - \frac{\pi_{aaa}(\xi)}{6} \left(\frac{C_K}{\lambda^{1-0.5K}} \right) \right) \\ &= O\left(\frac{1}{\lambda^{1-0.5K-1}}\right) \end{aligned}$$

as desired. \square

Theorem 2. *Assume we have $\frac{\pi_{at}(a,t)}{\pi_a(a,t)}$ is weakly increasing in $a > a_{BP}(t)$ for all t and that $\pi(a, t)$ is supermodular in (a, t) . For any fixed N , \mathbf{a}_{SEP}^N maximizes the payoff of each type of sender relative to any other separating equilibrium.*

Proof. Suppose we have a separating equilibrium with action functions $\mathbf{a}(t) = (a_1(t), \dots, a_N(t))$. Defining

$$(8) \quad \Gamma(\mathbf{a}, \mathbf{t}) \equiv \sum_{i=1}^N \pi(a_i, t)$$

we can write the first-order condition for type t 's optimal choice as:

$$(9) \quad \frac{\partial B(t, \hat{t})}{\partial \hat{t}} \Big|_{\hat{t}=t} + \sum_{i=1}^N \frac{\partial \pi(a_i, t)}{\partial a_i} \frac{da_i(t)}{dt} = 0$$

We are interested in determining type t 's total payoff in equilibrium. If we let $V(t, \hat{t})$ denote the payoff of a type t sender having chosen the action of type \hat{t} , we have by definition:

$$(10) \quad V(t, \hat{t}) = B(t, \hat{t}) + \Gamma(\mathbf{a}(\hat{t}), t)$$

and the Envelope Theorem yields:

$$\frac{dV(t, t)}{dc} = \frac{dB(t, t)}{dt} + \frac{\partial \Gamma(\mathbf{a}(t), t)}{\partial t}$$

Notice that only the final term depends on the particular separating equilibrium. Let \mathbf{a}^0 denote the symmetric separating equilibrium with payoffs V^0 , and \mathbf{a}^A denote an asymmetric separating equilibrium with payoffs V^A . To demonstrate that payoffs in the symmetric separating equilibrium are strictly higher than in the asymmetric separating equilibrium, we will establish the following Property (capitalized for clarity of subsequent references): if it were the case for some t that either (i) $V^0(t, t) = V^A(t, t)$ and $\mathbf{a}^0(t) \neq \mathbf{a}^A(t)$, or (ii) $V^0(t, t) < V^A(t, t)$, then we would have $\frac{dV^0(t, t)}{dt} > \frac{dV^A(t, t)}{dt}$.⁵

To understand why this Property delivers the desired conclusion, note that $V^A(t', t') - V^0(t', t')$ would shrink as t' rises over $[\underline{t}, t]$ if the property holds. But then we would have a violation of the boundary condition $V^0(\underline{t}, \underline{t}) = V^A(\underline{t}, \underline{t}) = B(\underline{t}, \underline{t}) - \Gamma(\mathbf{a}_{BP}(\underline{t}), \underline{t})$ where $\mathbf{a}_{BP}(t) = (a_{BP}(t), \dots, a_{BP}(t)) \in \mathbb{R}^N$. In light of Equation 10, we can rewrite the Property as follows: if it were the case for some t that either (i)' $\Gamma(\mathbf{a}^0(t), t) = \Gamma(\mathbf{a}^A(t), t)$ and $\mathbf{a}^0(t) \neq \mathbf{a}^A(t)$, or (ii)' $\Gamma(\mathbf{a}^0(t), t) > \Gamma(\mathbf{a}^A(t), t)$, then we would have $\frac{\partial \Gamma(\mathbf{a}^0(t), t)}{\partial t} > \frac{\partial \Gamma(\mathbf{a}^A(t), t)}{\partial t}$.

We now establish the Property. Supposing condition (i)' were satisfied for some $t > \underline{t}$, we would begin by defining:⁶

$$\bar{a}_m = \begin{cases} a_m^A(t) & \text{if } a_m^A(t) \geq a_{BP}(t) \\ a \geq a_{BP}(t) \text{ s.t. } \pi(a, t) = \pi(a_m^A(t), t) & \text{otherwise} \end{cases}$$

Let $Q \equiv \{m \mid a_m^A(t) < a_{BP}(t)\}$. Then:

$$\frac{\partial \Gamma(\bar{\mathbf{a}}, t)}{\partial t} - \frac{\partial \Gamma(\mathbf{a}^A(t), t)}{\partial t} = \sum_{m \in Q} \pi_t(\bar{a}_m, t) - \pi_t(a_m^A, t) \geq 0$$

with strict inequality if Q is non-empty.

⁵Suppose our claim is true. Then if either condition (i) or (ii) holds for t , then condition (ii) must hold for all $t' \in (\underline{t}, t)$.

⁶This step sets computes a cost-equivalent signal to \mathbf{a}^A that has the intuitive property that $a_m^A \geq a_{BP}(t)$.

If $\mathbf{a}^0(t) = \bar{\mathbf{a}}$, we are done. If not, then since $\Gamma(\mathbf{a}^A(t), t) = \Gamma(\bar{\mathbf{a}}, t)$ by construction, there must exist i and j such that $\bar{a}_i > a^0(t) > \bar{a}_j$. Define the function $\tilde{\mathbf{a}}(a_i)$ as follows: $\tilde{a}_i(a_i) = a_i$, $\tilde{a}_k(a_i) = \bar{a}_k$ for $k \neq i, j$, and $\Gamma(\tilde{\mathbf{a}}(a_i), t) = \Gamma(\bar{\mathbf{a}}, t)$. In other words, $\tilde{a}_j(a_i)$ indicates how a_j must vary in response to changes in a_i to keep the value of Γ constant at its equilibrium value. Implicit differentiation reveals that:

$$\left. \frac{d\tilde{a}_j}{da_i} \right|_{a_i=\bar{a}_i} = -\frac{\pi_a(\bar{a}_i, t)}{\pi_a(\tilde{a}_j(\bar{a}_i), t)} < 0$$

Plainly, there exists a unique value $a_i^e > a_{BP}(t)$ such that $\tilde{a}_j(a_i^e) = a_i^e$. For $a_i \in [a_i^e, \bar{a}_i(t))$, so we have:

$$\begin{aligned} \frac{d}{da_i} \left(\frac{\partial \Gamma(\tilde{\mathbf{a}}(a_i), t)}{\partial t} \right) &= \frac{d}{da_i} \left(\sum_{i=1}^N \pi_t(\tilde{a}_i(a_i), t) \right)_{a_i=\bar{a}_i} \\ &= \pi_{at}(a_i, t) + \pi_{at}(\tilde{a}_j(a_i), t) \left. \frac{d\tilde{a}_j}{da_i} \right|_{a_i=a_i} \\ &= \pi_{at}(a_i, t) - \pi_{at}(\tilde{a}_j(a_i), t) \frac{\pi_a(a_i, t)}{\pi_a(\tilde{a}_j(a_i), t)} < 0 \end{aligned}$$

where we have used the fact that since $\bar{a}_i \geq a_i \geq a_i^e \geq \tilde{a}_j(a_i) > a_{BP}(t)$ (which implies $\pi_a(\bar{a}_i, t) < 0$) and our assumption that:

$$\frac{\pi_{at}(a_i, t)}{\pi_a(a_i, t)} > \frac{\pi_{at}(\tilde{a}_j(\bar{a}_i), t)}{\pi_a(\tilde{a}_j(\bar{a}_i), t)}$$

It follows that $\frac{\partial \Gamma(\tilde{\mathbf{a}}(a_i^e), t)}{\partial t} > \frac{\partial \Gamma(\bar{\mathbf{a}}, t)}{\partial t}$ since $\bar{a}_i > a^0(t)$ is being reduced in this equalization step. Through repeated application of this equalization argument, we conclude that $\frac{\partial \Gamma(a^0(c), t)}{\partial t} > \frac{\partial \Gamma(\bar{\mathbf{a}}, t)}{\partial t} \geq \frac{\partial \Gamma(a^A(t), t)}{\partial t}$, as desired.

Next, supposing condition (ii)' were satisfied for some $t > \underline{t}$, we would begin by defining a' s.t. $a'_1 = a'_2 = \dots = a'_N > a_{BP}(t)$ and $\Gamma(a', t) = \Gamma(a^A(t), t)$. By the same argument as for condition (i)', we infer $\frac{\partial \Gamma(a', t)}{\partial t} \geq \frac{\partial \Gamma(a^A(t), t)}{\partial t}$.⁷ Because $\Gamma(a^0(t), t) > \Gamma(a^A(t), t) = \Gamma(a', t)$ by assumption, we have $a_m^0(t) > a'_m$. From our

⁷The inequality is weak because we include the possibility that $a' = a^A(t)$.

assumption of supermodularity we conclude:

$$\frac{\partial \Gamma(a^0(c), t)}{\partial t} - \frac{\partial \Gamma(a', t)}{\partial t} = \sum_{m=1}^N \pi_t(a_m^0, t) - \pi_t(a'_m, t) \geq 0$$

It follows that $\frac{\partial \Gamma(a^0(c), t)}{\partial t} > \frac{\partial \Gamma(a^A(c), t)}{\partial t}$, as desired.

Having established that the Property holds, the Proposition follows for the reasons given above. \square

Theorem 3. $\pi_{Agg}(\mathbf{a}_{BP}^N(t), t) - \pi_{Agg}(\mathbf{a}_{SEP}^N(t), t) = O\left(\frac{1}{\phi_N^{1-\alpha}}\right)$ for any $\alpha > 0$.

Proof. First we prove an analog to Lemma 1. We begin by using the fact that

$$(11) \quad \pi_{Agg}(\mathbf{a}_{BP}^N(t), t) - \pi_{Agg}(\mathbf{a}_{SEP}^N(t), t) \leq \bar{B} - \underline{B}$$

Note that $\mathbf{a}_{SEP}^N(t) \gg \mathbf{a}_{BP}^N(t)$ for all t and N from our definition of $\mathbf{a}_{SEP}^N(t)$ and the monotonicity and uniqueness of $\mathbf{a}_{BP}^N(t)$. Suppose there exists $\eta > 0$ such that $\mathbf{a}_{SEP}^N(t) \gg \mathbf{a}_{BP}^N(t) + \eta \mathbf{1}^N \gg \mathbf{a}_{BP}^N(t)$ for all N . Assumption 9 part 1 implies that $\pi_{Agg}(\mathbf{a}_{BP}^N(t), t) - \pi_{Agg}(\mathbf{a}_{BP}^N(t) + \eta \mathbf{1}^N, t)$ diverges to $+\infty$, which violates Equation 11. From this contradiction, we can conclude that for any $\eta > 0$ there exists N sufficiently large that $\mathbf{a}_{SEP}^N(t) - \mathbf{a}_{BP}^N(t) \ll \eta \mathbf{1}^N$.

Consider a Taylor expansion around $\mathbf{a}_{BP}^N(t)$:

$$\begin{aligned} \pi_{Agg}(\mathbf{a}_{BP}^N(t), t) - \pi_{Agg}(\mathbf{a}_{SEP}^N(t), t) = & \\ & - \frac{1}{2} \sum_{i,j=1}^N \frac{\partial^2 \pi_{Agg}(\mathbf{a}_{BP}^N(t), t)}{\partial a_i \partial a_j} (a_{BP,i}^N(t) - a_{SEP,i}^N(t)) (a_{BP,j}^N(t) - a_{SEP,j}^N(t)) \\ & - \frac{1}{6} \sum_{i,j,k=1}^N \frac{\partial^3 \pi_{Agg}(\mathbf{a}_{BP}^N(t), t)}{\partial a_i \partial a_j} (a_{BP,i}^N(t) - a_{SEP,i}^N(t)) (a_{BP,j}^N(t) - a_{SEP,j}^N(t)) \\ & \quad (a_{BP,k}^N(t) - a_{SEP,k}^N(t)) \end{aligned}$$

where $a_{BP,i}^N$ (for example) refers to the i^{th} component of \mathbf{a}_{BP}^N and ξ is on the line connecting $\mathbf{a}_{BP}^N(t)$ and $\mathbf{a}_{SEP}^N(t)$. Assumption 9 part 2 insures that the third order terms vanish relative to the first order terms as $\mathbf{a}_{SEP}^N(t) \rightarrow \mathbf{a}_{BP}^N(t)$, so for N sufficiently

large we can write

$$\begin{aligned}
 & \frac{-1}{4} \sum_{i,j=1}^N \frac{\partial^2 \pi_{Agg}(\mathbf{a}_{BP}^N(t), t)}{\partial a_i \partial a_j} (a_{BP,i}^N(t) - a_{SEP,i}^N(t)) (a_{BP,j}^N(t) - a_{SEP,j}^N(t)) \\
 & \leq \pi_{Agg}(\mathbf{a}_{BP}^N(t), t) - \pi_{Agg}(\mathbf{a}_{SEP}^N(t), t) \\
 & \leq \bar{B} - \underline{B}
 \end{aligned}$$

Since $\mathbf{a}_{SEP}^N(t) = \mathbf{a}_{BP}^N(t + \Delta^N(t))$, Assumption 8 implies that $a_{SEP,i}^N(t) - a_{BP,i}^N(t) \geq \beta \Delta^N(t)$. Therefore we can write

$$(12) \quad \frac{-1}{4} \sum_{i,j=1}^N \frac{\partial^2 \pi_{Agg}(\mathbf{a}_{BP}^N(t), t)}{\partial a_i \partial a_j} (\beta \Delta^N(t))^2 \leq \bar{B} - \underline{B}$$

Manipulating Equation 12 and using Assumption 9 part 1, we can write

$$(13) \quad \Delta^N(t) \leq \frac{2}{\beta} \sqrt{\frac{\bar{B} - \underline{B}}{-S_N(t)}} \leq \frac{1}{\beta} \sqrt{\frac{\bar{B} - \underline{B}}{-\phi_N/4}} \quad \text{where } S_N(t) = \sum_{i,j=1}^N \frac{\partial^2 \pi_{Agg}(\mathbf{a}_{BP}^N(t), t)}{\partial a_i \partial a_j}$$

Using Equation 13, we know that if a sender of type t deviates from $\mathbf{a}_{SEP}^N(t)$ to $\mathbf{a}_{BP}^N(t)$, the receiver will place probability 1 on the sender having type $\hat{t} \geq t - \frac{2}{\beta} \sqrt{\frac{\bar{B} - \underline{B}}{-\phi_N}}$. Therefore, the benefit gained by sending $\mathbf{a}_{SEP}^N(t)$ relative to $\mathbf{a}_{BP}^N(t)$ is

$$B(t, t) - B(t, \hat{t}) \leq \frac{\gamma}{\beta} \sqrt{\frac{\bar{B} - \underline{B}}{-\phi_N/4}}$$

We can then refine Equation 12 to

$$\begin{aligned}
 \frac{-\phi_N}{4} (\beta \Delta^N(t))^2 & \leq \frac{-1}{4} S_N(t) (\beta \Delta^N(t))^2 \\
 & \leq \frac{\gamma}{\beta} \sqrt{\frac{\bar{B} - \underline{B}}{-\phi_N/4}}
 \end{aligned}$$

which then yields a tighter bound of

$$\Delta^N(t) \leq \frac{\sqrt{\gamma}}{\beta^{3/2}} \left(\frac{-4}{\phi_N} \right)^{3/4} (\bar{B} - \underline{B})^{1/4}$$

For any $\alpha > 0$, repeating this process sufficiently often yields:

$$\Delta^N(t) \leq \frac{1}{\beta} \left(\frac{\gamma}{\beta} \right)^{1-2\alpha} \left(\frac{-4}{\phi_N} \right)^{1-\alpha} (\overline{B} - \underline{B})^\alpha]$$

Using this upper bound, we find that for any $\alpha > 0$

$$\begin{aligned} \mathbf{a}_{SEP}^N(t) - \mathbf{a}_{BP}^N(t) &= \mathbf{a}_{BP}^N(t + \Delta^N(t)) - \mathbf{a}_{BP}^N(t) \\ &\leq \beta_{UB} \Delta^N(t) \\ &= O\left(\frac{1}{\phi_N^{1-\alpha}}\right) \end{aligned}$$

Using the negligibility of the third order terms, we can use our Taylor expansion to write

$$\begin{aligned} \boldsymbol{\pi}_{Agg}(\mathbf{a}_{BP}^N(t), t) - \boldsymbol{\pi}_{Agg}(\mathbf{a}_{SEP}^N(t), t) &\leq -S_N(t) (\beta_{UB} \Delta^N(t))^2 \\ &\leq -\phi_N \frac{C}{\phi_N^{2-2\alpha}} \\ &= \frac{C}{\phi_N^{1-2\alpha}} \end{aligned}$$

which proves our result. □