

# Bundling in Simple Games\*

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## Abstract

We extend the Baron-Ferejohn [3] bargaining protocol to model negotiations over multiple issues. The power structure is represented by simple games, with different winning coalitions over the different issues. We prove existence of a Markov perfect equilibrium, and provide sufficient conditions for the existence of efficient equilibria where all players make joint offers on the two issues. We also provide a sufficient condition for the existence of an equilibrium where players' limit equilibrium payoffs in the bargaining game are equal to the limit equilibrium payoffs when players negotiate separately on the two issues. This last condition is satisfied whenever one of the two simple games admits veto players.

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# 1 Introduction

Real world negotiations frequently entail a complex nexus of issues, with teams of negotiators discussing different aspects of the proposed agreements. Trade negotiations between countries, collective bargaining among unions and employers are good examples of complex negotiations dealing with a large number of issues. When bargaining involves multiple issues, a major question is whether negotiations should be organized item by item, or proceed only if a global agreement can be reached on all issues at once. When do players want to make separate offers or to bundle issues? How can efficient agreements be reached with multiple issues?

These questions have been asked ever since game theorists started to model negotiations as a bargaining game. Howard Raiffa, in his famous book on *The Art and Science of Negotiation*, initially published in 1982, [17] argues in favor of global agreements, as these enable negotiators to trade one issue against the other, thereby helping to build compromises and to reach satisfactory agreements. An important technical literature on multi-issue bargaining, extending Rubinstein [18]’s two-person bargaining game to allow for multiple issues, for the most part agrees with that conclusion. For example, Inderst [14] and In and Serrano [12] argue that joint offers on all the issues are better than separate offers for two reasons. They allow players to trade one issue against the other, extending the set of utility transfers, and economize on delay, increasing the total surplus to be shared. Hence, if players are free to choose whether to offer on separate issues or to make joint offers, they always choose to make joint offers in equilibrium.

In this paper, our objective is to compare item-by-item bargaining and global agreements and analyze the endogenous choice of negotiators over joint and separate offers, in the context of an  $n$ -player coalitional game. We restrict attention to "simple" games, where the worth of any coalition is either 0 or 1, so that every game is represented by a set of "winning coalitions" which can enforce agreements. Hence, by contrast to the existing literature on two-player multi-issues bargaining, agreement does not necessarily require unanimity. A proposal on one issue can be implemented if a subset of players (the "winning coalition") on this issue accepts the agreement. The power structure embodied by simple games may differ from issue to issue (for example some issues may require a super-majority while others can be settled by simple majority), giving rise to a very rich model of negotiations and a very complex study of players' incentives. In fact, even in the simplest case where all players have the same utility on all the issues – so that issues cannot be traded against one another – differences in power structure create a large heterogeneity among the players over the different issues, and a potential benefit for bundling or separating issues.

More precisely, we consider the celebrated bargaining protocol of Baron and Ferejohn [3], where players are recognized to make offers through a uniform probability distribution, in the context of a two-issue endogenous agenda game. Every player, when recognized, has the choice between making an offer on issue 1, making an offer on issue 2 or making a joint offer on both issues. An offer on issue 1 is implemented if a winning coalition forms on that issue. Similarly, an offer on issue 2 is implemented if a winning coalition forms on that issue. A joint offer can only be accepted if a winning coalition on issue 1 and on issue 2 is formed – typically a more demanding condition, requiring more players to accept the proposal. Whenever a proposal is rejected by one of the players, a new proposer is recognized at random after a period of time

elapses. If a proposal on one issue is settled, a period of time elapses, and players are recognized to make offers on the other issue, in a standard Baron-Ferejohn [3] single issue bargaining game.

We first prove existence of equilibrium in the multi-issue Baron-Ferejohn [3] bargaining game, when winning coalitions are given by an arbitrary simple game. The study of bargaining in the Baron-Ferejohn protocol when the underlying power structure is an arbitrary simple game rather than a simple majority game is due to Eraslan and Mc Lennan [8] who show existence of equilibrium and uniqueness of equilibrium payoffs. We also analyze two simple examples – two-player games where one player may be a dictator on one of the issues, and symmetric majority games with different super-majority, to show that equilibrium does not necessarily result in all players making joint offers. In two-player games where one player (the "strong" player) is a dictator on one of the issues whereas unanimous agreement is reached on the other, we show that the strong player has an incentive to choose the issue he dictates first and then bargain with the other player, resulting in an expected payoff converging to  $\frac{3}{2}$  when  $\delta$  goes to 1, whereas an agreement on both issues, treating the two player symmetrically, would result in an expected payoff of 1 for each player when  $\delta$  goes to 1. In symmetric majority games, players have an incentive to separate the two issues if the sizes of the supermajorities needed are very different. If one issue only requires the agreement of a small number of players, it may be beneficial to settle this issue among a small number of players first, rather than looking for a global agreement which requires payment to a larger set of players.

We then focus attention on two types of equilibria: no-delay equilibria where all players make joint offers and delay equilibria, where all players make offers on separate issues. No-delay equilibria are efficient for any choice of proposer, delay equilibria are inefficient whatever proposer is chosen to make the first offer. We show that no-delay equilibria arise whenever the set of winning coalitions on the two issues are identical. We also propose a sufficient condition for the existence of delay equilibria. However, a complete characterization of situations (pairs of simple games) under which no-delay and delay equilibria exist, remains an open question.

We also consider the following problem: When is the possibility of bundling issues relevant? Do equilibrium payments differ in an endogenous agenda game, where players have the opportunity of bundling issues and in an exogenous agenda game, where they bargain separately on the two issues following an exogenous sequence? To answer this question, we analyze when there exists an equilibrium of the endogenous agenda game in which limit equilibrium payoffs of the players, when  $\delta$  converges to 1, are equal to the sum of the limit equilibrium payoffs of the games played separately on the two issues. We provide a sufficient condition for the bargaining game to be "additive" in that sense. This condition is satisfied whenever one of the simple games admits veto players. This is the major finding of our analysis, showing that whenever one of the issues has a "collegial" game structure, admitting a college of veto players, then equilibrium payoffs are additive, and the possibility of bundling does not affect limit equilibrium payoffs.

## 1.1 Related literature

The literature on multi-issue two-player bargaining originates with Fershtman [9] 's analysis of a two-issue, exogenous agenda, extension of Rubinstein [18]'s alternating offers bargaining game. Busch and Horstmann [5] relate the outcome of the bargaining procedure to bargaining

frictions and differences in discount factors. Bac and Raff [2] and Busch and Horstmann [6] justify issue-by-issue bargaining as the outcome of a signaling game, where the strong player signals his type by selecting one specific issue. Inderst [14] and In and Serrano [12] and [13] propose models with endogenous agendas, where the proposer can select whether to propose on one issue or the other (In and Serrano [13]) or also on both issues at once (In and Serrano [12])). They conclude that, in equilibrium, players choose to make joint offers. Lang and Rosenthal [15] study a different model, where players can select any subset of issues and discuss conditions under which players prefer to make issue-by-issue proposals. Weinberger [19], Busch and Horstmann [7] and Flamini [10] analyze how issues are ordered when they are all implemented at the end or can be implemented as soon as agreement has been reached. information model. Heifetz and Ponsati [11] and Acharya and Ortner [1] propose alternative explanations for the prevalence of issue-by-issue bargaining. In Heifetz and Ponsati [11], when information is incomplete, global agreements may be detrimental because they involve bundling good and bad issues when gains may be negative. In Acharya and Ortner [1], issues arrive stochastically over time so partial, separate agreements on single issues may be preferable.

The study of additive bargaining games (when is bundling relevant or not?) is related to the study of additivity of solution concepts. The Shapley value is of course additive by construction. For other solution concepts, Peters [16] analyzes additivity of the Nash solution of bargaining problems, and Bloch and de Clippel [4] characterize those games for which the core is additive.

## 2 The model

In this Section, we introduce the underlying power structure of simple games and the extension of the Baron-Ferejohn bargaining protocol to multi-issues negotiations..

### 2.1 Simple games

We consider a set  $N$  of players,  $i = 1, 2, \dots, n$ . A coalition  $S$  is a nonempty subset of  $N$ . For any coalition  $S$ , we let  $\bar{S} = N \setminus S$  denote its complement of  $S$  and  $s$  the cardinality of the set  $S$ . A cooperative game  $\Gamma$  is a mapping  $v$  associating a real payoff to any coalition  $S$ . We will restrict attention to *simple games* where the worth of a coalition can only take on binary values and is equal to 0 or to 1. Alternatively, a simple game is characterized by the set of winning coalitions:  $\mathcal{W} = \{S | v(S) = 1\}$ . A simple game is *monotonic* if, whenever  $S \subseteq T$  and  $S \in \mathcal{W}$ ,  $T \in \mathcal{W}$ . When the game is monotonic, we can define a *minimal winning coalition* as a coalition such that  $v(S) = 1$  and  $v(T) = 0$  for any  $S \subset N$ . We let  $\mathcal{W}^*$  denote the set of minimal winning coalitions of the simple game  $\Gamma$ . A simple game is *proper* if whenever  $S \in \mathcal{W}$ ,  $\bar{S} \notin \mathcal{W}$ . If the game  $\Gamma$  is proper, two disjoint coalitions cannot both be winning. Player  $i$  is called a *veto player* for the game  $\Gamma$  if and only if  $i$  belongs to all winning coalitions of the game. A game is called *collegial* if it admits veto players and we let  $K$  denote the college (the set of veto players), with cardinality  $k$ . Finally for any player  $i$ , we let  $\mathcal{C}(i)$  denote the set of minimal winning coalitions containing player  $i$  and  $\mathcal{D}(i) = \{S | i \notin S, S \cup i \in \mathcal{C}(i)\}$  the set of coalitions which, together with player  $i$  form a minimal winning coalition in the game  $\Gamma$ .

## 2.2 Sum of simple games

As players interact in two distinct simple games  $\Gamma_1$  and  $\Gamma_2$ , it will be useful to define the *sum* of two simple games as the game played when the players interact jointly on the two issues. Formally, for any two games  $\gamma_1$  and  $\Gamma_2$ , we define the sum  $\Gamma_1 \oplus \Gamma_2$  as a game such that  $v(S) = 2$  if and only if  $v_1(S) = v_2(S) = 1$ .<sup>1</sup> A coalition is winning in  $\Gamma_1 \oplus \Gamma_2$  if and only if it is winning in both games  $\Gamma_1$  and  $\Gamma_2$ . Hence  $\mathcal{W} = \mathcal{W}_1 \cap \mathcal{W}_2$ . It is easy to check that when  $\Gamma_1$  and  $\Gamma_2$  are monotonic,  $\Gamma_1 \oplus \Gamma_2$  is monotonic and when  $\Gamma_1$  and  $\Gamma_2$  are proper,  $\Gamma_1 \oplus \Gamma_2$  is proper. We let  $\mathcal{W}^*$  denote the set of minimal winning coalitions of  $\gamma_1 \oplus \Gamma_2$ . We immediately obtain the following simple lemma:

**Lemma 1**  $S \in \mathcal{W}^* \leftrightarrow S = S_1 \cup S_2$  for some  $S_1 \in \mathcal{W}_1^*$  and some  $S_2 \in \mathcal{W}_2^*$ .

**Proof:**  $S \in \mathcal{W}^*$  is by definition both in  $\mathcal{W}_1$  and in  $\mathcal{W}_2$ . If  $S \notin \mathcal{W}_j^*$  for some  $j$ , then pick a subset  $S_j \subset S$  in  $\mathcal{W}_j^*$ . We can then write  $S = S_1 \cup S_2$ .  $\square$ .

If player  $i$  is a veto player in  $\Gamma$ , he must be a veto player both in  $\Gamma_1$  and  $\Gamma_2$ . Hence, if  $K$  is the set of veto players in  $\Gamma_1 \oplus \Gamma_2$ ,  $K = K_1 \cap K_2$ .

## 2.3 The bargaining protocol

We consider a bargaining protocol which is natural extension to two issues of Baron and Ferejohn (1989)'s celebrated model of legislative bargaining. This is a multi-issues bargaining protocol with endogenous agenda: when both issues are on the table, players are free to select on which set of issues they want to make an offer. At any time  $t$ , one of the  $n$  players is selected with probability  $\frac{1}{n}$  to make an offer. If there is one issue left, an offer is a coalition  $S$  such that  $i \in S$ , and a distribution of the payoff among members of  $S$ ,  $(x_i)_{i \in S}$  such that  $\sum x_i = 1$ . If there are two issues left, player  $i$  can choose

- to make an offer on issue 1
- to make an offer on issue 2
- to make a joint offer on the two issues, in which case  $\sum x_i = 2$ .

Player in  $S$  respond simultaneously to the offer. If all players agree and the coalition  $S$  is winning, the agreement is implemented. If the agreement concludes the negotiations on both issues, the game ends. Otherwise one period elapses and a player is recognized at random to make an offer on the remaining issue. If one of the members of the coalition rejects the offer (or the coalition is not winning), then one period elapses and a player is to recognized to make an offer next period. All players have the same discount factor  $\delta$ . Notice that the main restriction implied by this protocol is that a player cannot simultaneously make two offers on two issues to different coalitions. We justify this assumption by the fact that players cannot make contingent agreements based on the behavior of other players in different coalitions.

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<sup>1</sup>Notice that this game differs from the sum  $v_1 + v_2$  defined by  $v(S) = v_1(S) + v_2(S)$ .

A strategy for player  $i$  after history  $h^t$  consists both of a proposal and response strategy. When a single issue is left to be decided, a proposal is a probability distribution over the set of coalitions  $S, i \in S$  and over the vectors  $(x_i)_{i \in S}, x_i \geq 0, \sum_{i \in S} x_i \leq 1$ . When the two issues are on the table, a proposal is a probability distribution over the pairs  $(S, l), l = 1, 2, \{12\}$ , where  $(S, l)$  is interpreted as a proposal to coalition  $S$  on issue  $l$ , and over the vectors  $(x_i)_{i \in S}, x_i \geq 0, \sum_{i \in S} x_i \leq 1$ . A response strategy at history  $h^t$  is a probability distribution over the binary set  $\{0, 1\}$  where 1 is interpreted as acceptance and 0 as rejection of the current offer.

We restrict attention to Markov strategies, which only depend on the current state of the game. A state of the game is described by the issue on which agreement has already been reached (if any) and the current offer. A *Markov Perfect Equilibrium* is a profile of Markov strategies such that, at any time  $t$ , after any history  $h^t$ , all players choose their optimal strategy.

### 3 Equilibrium

In this Section, we prove existence of a Markov perfect equilibrium and characterize several types of equilibria. The model with a single issue is identical to the model studied by Eraslan and Mc Lennan [8] who prove existence of a Markov Perfect Equilibrium and uniqueness of equilibrium payoffs. We will rely on their results to analyze the game with two issues and endogenous agenda.

#### 3.1 Existence

We first establish existence of a Markov Perfect Equilibrium.

**Proposition 1** *The coalitional game with endogenous agenda and multiple issues admits a Markov Perfect Equilibrium.*

Notice that, if one issues has already been resolved, the game is identical to the game studied by Eraslan and Mc Lennan [8] and hence existence of equilibrium is obtained following their argument. Hence we only consider a situation where both issues are on the table. In order to prove existence of a Markov perfect equilibrium, we first characterize optimal response strategies. For any player  $j \neq i$ , let  $V_j$  denote the expected payoff of player  $j$  at equilibrium (before any player has been recognized). By definition, player  $j$  accepts any offer  $x_j \geq \delta V_j$  and rejects any offer  $x_j < \delta V_j$ .

Next consider the optimal proposal strategies of player  $i$ . First notice that a player has no incentive to make an offer to a coalition which is not winning, as this would delay agreement without changing the state of the game. Player  $i$  also has no incentive to offer to any agent  $j$  in a winning coalition more than the minimal offer that player  $j$  will accept.

By Eraslan and Mac Lennan [8], the games on the two issues  $\Gamma_1$  and  $\Gamma_2$  admit unique equilibrium payoffs. We denote by  $y_j$  and  $z_j$  these payoffs. Player  $i$  then considers the following alternatives: if she proposes on issue 1, the minimal offer that will be accepted by any player  $j$  is  $\delta(V_j - z_j)$ . Hence the payoff of player  $i$  on an offer on issue 1 is

$$\pi_1(i) = \max_{D \in \mathcal{D}_1(i)} (1 - \delta \sum_{j \in D} (V_j - z_j) + \delta z_i).$$

Similarly, the payoff of player  $i$  on an offer on issue 2 is

$$\pi_2(i) = \max_{D \in \mathcal{D}_2(i)} (1 - \delta \sum_{j \in D} (V_j - y_j) + \delta y_i).$$

If player  $i$  makes an offer on both issues, she must give  $\delta V_j$  to any of the other players in the winning coalition, so that the payoff becomes

$$\pi(i) = \max_{D \in \mathcal{D}(i)} (2 - \delta \sum_{j \in D} V_j).$$

In the endogenous agenda game, player  $i$  selects an offer  $O(i)$  which gives her the maximal payoff among the three alternatives,  $O(i) = \max\{\pi_1(i), \pi_2(i), \pi(i)\}$ . Finally, in order to compute the expected payoff of player  $i$ , let  $\rho_j^i$ ,  $\tau_j^i$  and  $v_j^i$  denote the probability that player  $j$  makes an offer to player  $i$ , makes an offer on issue 2 excluding player  $i$  and makes an offer on issue 1 excluding player  $i$  respectively. Hence  $\rho_j^i = \sum_{S|i \in S} (\sigma_j(S, 1) + \sigma_j(S, 2))$ ,  $\tau_j^i = \sum_{S|i \notin S} \sigma_j(S, 1)$  and  $v_j^i = \sum_{S|i \notin S} \sigma_j(S, 2)$ . The expected payoff of player  $i$  is given by:

$$V_i = \frac{1}{n} [O_i + \sum_j \sigma_j^i \delta V_i + \tau_j^i \delta y_i + v_j^i \delta z_i] \quad (1)$$

**Proof:** We now prove Proposition 1 using Kakutani's fixed point theorem. For any player  $j$ , we consider the pair  $(V_j, \sigma_j)$  where  $V_j$  is the expected payoff of the player and  $\sigma_j$  his mixed strategy. Notice that values  $V_j$  are bounded,  $0 \leq V_j \leq 2$  and that  $\sigma_j$  is a point in the simplex  $(\Delta^{2^{n-1}-1})^3$ . Hence  $(V - j, \sigma_j)$  lies in a compact interval for all  $j$ . Now define a correspondence  $\phi_i$  for player  $i$  from  $([0, 2] \times ((\Delta^{2^{n-1}-1})^3)^{n-1})$  to  $[0, 2] \times (\Delta^{2^{n-1}-1})^3$  by letting  $\sigma_i(S, l) = 0$  if  $\pi_l(i) \neq O(i)$  or if  $S \notin \{D \in \mathcal{D}_1(i), \max_{D \in \mathcal{D}_1(i)} (1 - \delta \sum_{j \in D} (V_j - z_j) + \delta z_i) = \pi_1(i)\}, S \notin \{D \in \mathcal{D}_2(i), (1 - \delta \sum_{j \in D} (V_j - y_j) + \delta y_i) = \pi_2(i)\} \text{ and } S \notin \{D \in \mathcal{D}(i), (2 - \delta \sum_{j \in D} V_j) = \pi(i)\}$ . Compute also  $V(i)$  as in equation 1. Let  $\phi = \times_i \phi_i$ .

The correspondence  $\phi$  is nonempty values because there always exists an issue  $l \in \{1, 2, 12\}$  and a coalition  $(S, l)$  which maximizes the payoff of player  $i$ . Setting  $\sigma(S, l) = 1$  and computing  $V_i$  accordingly is a point in  $\phi_i$ . The correspondence is also convex-valued because, if there exists a set  $\mathcal{S}$  of coalition and issues  $(S, l)$  which maximize the payoff of player  $i$ , then the set of mixed strategies in  $\phi_i$  is equal to the simplex over  $\mathcal{S}$ , a convex set, and all points in  $\mathcal{S}$  give rise to the same equilibrium payoff  $V_i$ .

We now show that the correspondence  $\phi_i$  is upper hemi continuous. (Notice however that it is not continuous, as small changes in  $V_j$  may change the set  $\mathcal{S}$  of optimal coalition-issues pairs. Pick a sequence  $(V_j^n, \sigma_j^n)_{j \neq i}$  converging to  $(\bar{V}_j, \bar{\sigma}_j)$  and a sequence  $(V_i^n, \sigma_i^n)$  converging to  $(\bar{V}_i, \bar{\sigma}_i)$  such that  $(V_i^n, \sigma_i^n) \in \phi_i((V_j^n, \sigma_j^n))$  for all  $n$  and suppose by contradiction that  $(\bar{V}_i, \bar{\sigma}_i) \notin \phi_i((\bar{V}_j, \bar{\sigma}_j))$ . This must imply that there exists an  $(S, l)$  such that  $\sigma(S, l) > 0$  and  $(S, l)$  is not the optimal choice of player  $i$  at  $\bar{V}_j$ . Let  $W(S, l) = (1 - \delta \sum_{j \in S} (\bar{V}_j - z_j) + \delta z_i)$  if  $l = 1$ ,

$W(S, l) = (1 - \delta \sum_{j \in S} (\bar{V}_j - y_j) + \delta y_i)$  if  $l = 2$  and  $W(S, l) = (2 - \delta \sum_{j \in D} \bar{V}_j)$  if  $l = 12$ . If  $(S, l)$  is not the optimal choice of player  $i$ , there exists another coalition-issues pair  $(S', l')$  such that  $W(S', l') > W(S, l)$ . Now as  $\sigma_i^n$  converges to  $\bar{\sigma}_i$ , there exists  $N_1$  such that  $\sigma_i^n(S, l) > 0$  for all  $n > N_1$ . As  $V_j^n$  converge to  $\bar{V}_j$  there exists  $N_2$  such that  $|v_j^n - \bar{V}_j| \leq \epsilon$  for all  $n > N_2$ . But, as the mappings  $W(S, l)$  are continuous in  $S$  for all  $l$ , this implies that  $(S, l)$  is an optimal choice of player  $i$  for  $(\bar{V}_j)$ , yielding a contradiction.

Because the correspondence  $\phi$  satisfies the conditions of Kakutani's Theorem, it admits a fixed point. It is easy to check that the mixed strategies at the fixed point form a Markov Perfect equilibrium of the bargaining game and that the expected payoffs form the equilibrium values of the game.  $\square$

### 3.2 Uniqueness of equilibrium

Eraslan and Mc Lennan ?? prove uniqueness of equilibrium payoffs in the single issue game. When the game has multiple issues, uniqueness is unlikely to obtain – even in two-player multi issue bargaining games with endogenous agenda, multiplicity of equilibria occurs very easily. We still need to construct examples where the MPE is not unique.

### 3.3 Limit equilibrium payoffs

We will study equilibria when players become infinitely patient,  $\delta \rightarrow 1$ . In the games  $\Gamma_1, \Gamma_2$ , we define unique limit equilibrium payoffs as

We compute *limit equilibrium payoffs* when  $\delta \rightarrow 1$ :

- $\bar{y}_i$  on issue 1
- $\bar{z}_i$  on issue 2

The game  $\Gamma = \Gamma_1 \oplus \Gamma_2$  is also a simple game and hence admits unique equilibrium payoffs. We let  $w_i$  denote these payoffs and  $\bar{w}_i$  the limit equilibrium payoffs as  $\delta$  converges to 1. For a particular equilibrium of the multi-issue bargaining game, we let  $\bar{V}_i$  denote the limit equilibrium payoff when  $\delta$  converges to 1.

### 3.4 Efficient and inefficient equilibria

When players become perfectly patient, an equilibrium is efficient if and only if all players make joint offers on the two issues so that no delay occurs. Identifying games for which all equilibria are efficient is not an easy task, but we are able to obtain two sufficient conditions under which all equilibria of the game are efficient:

**Proposition 2** *If  $\mathcal{D}_1(i) = \mathcal{D}_2(i)$  for all  $i$ , then all equilibria are efficient.*



**Proof:** If  $\mathcal{D}_1(i) = \mathcal{D}_2(i)$ , then the same coalitions are winning on both issues,  $\mathcal{D}_1(i) = \mathcal{D}_2(i) = \mathcal{D}(i)$ . Consider then  $S \in \mathcal{D}(i)$  such that  $\sum_{j \in S} V_j \leq \sum_{l \in S'} V_l$  for all  $S' \in \mathcal{D}(i)$ . Now, as  $\delta < 1$ ,

$$\begin{aligned} 2 - \delta \sum V_j &\geq 2 - \delta \sum V'_k \\ &> 1 - \delta \sum V'_k + \delta(\sum y_k + y_i), \\ &> 1 - \delta \sum V'_k + \delta(\sum z_k + z_i) \end{aligned}$$

so that player  $i$  always prefers to make a joint offer.  $\square$

Proposition 2 shows that when the power structure is identical on both issues, players have no incentive to separate issues and delay agreement. In that case, the outcome of the bargaining game is the same as if a single issue was negotiated.

**Proposition 3** *If  $\max_{D \in \mathcal{D}(i)} (2 - \sum_{j \in D} \bar{w}_j) > \max_{D \in \mathcal{D}_\infty(i)} (1 - \sum_{j \in D} (\bar{w}_j - \bar{z}_j) + \bar{z}_i)$ ,  $\max_{D \in \mathcal{D}_\infty(i)} (1 - \sum_{j \in D} (\bar{w}_j - \bar{y}_j) + \bar{y}_i)$  for all  $i$ , then there exists  $\underline{\delta}$  such that for all  $\delta \geq \underline{\delta}$ , an efficient equilibrium exists.*

**Proof:** Consider the payoffs when all players make joint offers, i.e. the solution to the system of equations:

$$V_i = \frac{1}{n} \left[ \max_{D \in \mathcal{D}(i)} (2 - \delta \sum_{j \in D} V_j) + \delta \sum_j \sigma_j^i V_i \right].$$

Suppose by contradiction that one of the players, say player  $i$ , has an incentive to deviate and to make a separate offer on one of the issues (wlog suppose it is issue 1). Then:

$$\max_{D \in \mathcal{D}_1(i)} (1 - \delta \sum_{j \in D} (V_j - z_j) + \delta z_i) > (2 - \delta \sum_{j \in D} V_j).$$

As  $\delta$  converges to 1,  $V_j$  converges to  $\bar{w}_j$  and  $z_j$  converges to  $\bar{z}_j$  so that in the limit,

$$\max_{D \in \mathcal{D}_1(i)} (1 - \delta \sum_{j \in D} (\bar{w}_j - \bar{z}_j) + \delta \bar{z}_i) \geq \max_{D \in \mathcal{D}(i)} (2 - \delta \sum_{j \in D} \bar{w}_j),$$

contradicting the assumption of the proposition.  $\square$

Proposition 3 provides a sufficient condition on limit equilibrium payoffs of the games  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_1 \oplus \Gamma_2$  under which an efficient equilibrium exists when the discount factor is sufficiently high. The intuition underlying Proposition 3 is easy to grasp: as payoffs are continuous in  $\delta$ , if at the limit players strictly prefer to make joint offers, they must also prefer to make joint offers for discount factors close to 1. This Proposition is interesting as it provides a condition which can easily be checked by studying separately the limit equilibrium payoffs of the single-issue games  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma$ .

Using the same intuition, we can characterize situations where all players prefer to make offers on separate issues by looking at limit equilibrium payoffs. When the condition of limit equilibrium payoffs is satisfied, all equilibria are inefficient, as they involve delay. (Notice however that as the discount factor converges to one, the inefficiency linked to delay goes to zero). We obtain the following proposition:

**Proposition 4** *If  $\max_{D \in \mathcal{D}_\infty(i)} (1 - \sum_{j \in D} (\bar{y}_j + \bar{z}_i)) > \max_{D \in \mathcal{D}(i)} (2 - \sum_{j \in D} (\bar{y}_j + \bar{z}_j))$  for all  $i$  or  $\max_{D \in \mathcal{D}_\infty(i)} (1 - \sum_{j \in D} (\bar{z}_j) + \bar{y}_i) > \max_{D \in \mathcal{D}(i)} (2 - \sum_{j \in D} (\bar{y}_j + \bar{z}_j))$  for all  $i$ , then there exists  $\underline{\delta}$  such that for all  $\delta \geq \underline{\delta}$ , an inefficient equilibrium exists.*

**Proof:** The proof is omitted as it follows the same argument as the proof of Proposition 3.  $\square$

## 4 Two examples

In this Section, we illustrate the model by considering two examples. The first example is an exhaustive study of all two-player games. The second example considers general symmetric majority games.

### 4.1 Two-player games

Consider two simple games played by two players. For each simple game, there are only two possibilities: either both players are needed in the winning coalition or a single player has the power to dictate agreement. We thus obtain three situations (up to a relabeling of the players and the issues): either both players are needed on both issues, or each one is a dictator on one issue, or both players are needed on one issue and both are needed on the other. In the last case we call the player who can dictate one issue strong and the other one weak. The following table summarizes these three cases

case $\mathcal{W}_1$	$\mathcal{W}_2$	$\mathcal{W}$
1 (both needed) 12	12	12
2 (specialized) 1, 12	2, 12	12
3 (weak and strong player) 1, 12	12	12

In cases 1 and 2 (both needed and specialized), the only equilibrium is for all players to make joint proposals. Players have an incentive to make a joint proposal because they avoid delay and cannot strategically benefit from reaching agreement on one issue. As  $\delta \rightarrow 1$ , equilibrium payoffs converge to  $(1, 1)$  in both cases.

In case 3 (the weak/strong player case), there is no equilibrium where both players make joint offers. If they did, as  $\delta \rightarrow 1$ , each player gets a payoff of 1. This cannot be an equilibrium: the strong player has an incentive to conclude on issue 1 and then bargain on issue 2, as this results in a payoff of  $1 + \frac{1}{2}$  when  $\delta \rightarrow 1$ . Hence player 1 has a strategic incentive to separate the issues, resulting in inefficient delay.

**Proposition 5** *In the strong/weak player case, in the unique equilibrium, player 1 concludes agreement on issue 1 first and player 2 makes a joint offer on the two issues.*

**Proof:** The equilibrium payoffs of the two players are

$$V_1 = \frac{2 + \delta}{2(2 - \delta)}$$

$$V_2 = \frac{4 - 2\delta - \delta^2}{2(2 - \delta)}$$

We check that  $2 - \delta V_1 > \frac{\delta}{2}$  (the weak player prefers to make a joint proposal) and  $1 + \frac{\delta}{2} > 2 - \delta V_2$  (the strong player prefers to settle first on issue 1). Notice that, as  $\delta \rightarrow 1$ ,  $V_1 \rightarrow \frac{3}{2}$  and  $V_2 \rightarrow \frac{1}{2}$ . These limit equilibrium payoffs are the sum of the limit equilibrium payoffs if players bargained separately on the two issues.,  $\bar{V}_1 = \bar{y}_1 + \bar{z}_1$  and  $\bar{V}_2 = \bar{y} + 2 + \bar{z}_2$

## 4.2 Symmetric majority games

Consider two symmetric majority games with supermajorities  $l_1$  and  $l_2$ , with  $l_2 \geq l_1$ . In game  $\Gamma_1$ , a coalition is winning if and only if  $s \geq l_1$  and in game  $\Gamma_2$  a coalition is winning if and only if  $s \geq l_2$ . Clearly, because the game is proper,  $l_2 \geq l_1 \geq \frac{n+1}{2}$ . Because players are symmetric, the limit equilibrium payoffs are  $\bar{y}_i = \bar{z}_i = \frac{1}{n}$  and  $\bar{w}_i = \frac{2}{n} \forall i$ . We consider the payoff of a player when he makes a separate offer on issue 1:

$$1 - (l_1 - 1)\bar{V} + l_1\bar{z} = \frac{n - l_1 + 2}{n}.$$

when he makes a separate offer on issue 2:

$$1 - (l_2 - 1)\bar{V} + l_2\bar{y} = \frac{n - l_2 + 2}{n}.$$

when he makes a joint offer on both issues

$$2 - (l_2 - 1)\bar{V} = \frac{2n - 2l_2 + 2}{n}.$$

We observe that making a separate offer on issue 2 is always dominated.

**Proposition 6** *When  $\delta \rightarrow 1$ , in equilibrium, all players make a separate offer on issue 1 if  $2l_2 - l_1 \geq n$  and a joint offer on both issues if  $2l_2 - l_1 < n$ .*

This Proposition shows that the equilibrium is efficient only when the difference between the two thresholds  $l_1$  and  $l_2$  is sufficiently small. If this threshold is too large, players prefer to first settle on issue 1, where a small majority is needed, because it reduces the payment to the other players. A joint agreement involves a large number of players to which the worth of the coalition must be distributed. By making a joint offer, a player benefits from a proposer's advantage on both issues rather than one. However, if the size of the coalition  $l_2$  is too large with respect to the size of the coalition  $l_1$ , the proposer will prefer to offer on issue 1 rather and forgo the proposer's advantage on issue 2 rather than making a joint offer.

## 5 Additive games

In this Section, we study additive bargaining games, i.e. games for which the possibility of bundling does not affect limit equilibrium payoffs. We call a bargaining game *additive* if there exists an  $\epsilon$ -equilibrium of the multi-issue bundling game such that  $\bar{V}_i = \bar{y}_1 + \bar{z}_i$  for all  $i \in N$ . If the bargaining game is additive, we find an equilibrium where the payoffs of the players are the

same in the endogenous agenda protocol as in a situation where both issues are settled separately. In order to study additive bargaining games, we first need to introduce some notation on limit equilibrium payoffs and strategies.

Let  $\bar{a}_i = \bar{w}_i - \bar{y}_i - \bar{z}_i$  denote the deviation between limit equilibrium payoffs in the game  $\Gamma$  and the sum of equilibrium payoffs in  $\Gamma_1$  and  $\Gamma_2$ . This deviation denotes differences in power structures between the two separate games  $\Gamma_1$  and  $\Gamma_2$  and the sum of the games  $\Gamma_1 \oplus \Gamma_2$ .

Next compare, for any player  $i$ ,

- $\max_{D \in \mathcal{D}_1(i)} (1 - \sum_{j \in D} \bar{y}_j + \bar{z}_i)$
- $\max_{D \in \mathcal{D}_2(i)} (1 - \sum_{j \in D} \bar{z}_j + \bar{y}_i)$
- $\max_{D \in \mathcal{D}(i)} (2 - \sum_{j \in D} (\bar{y}_j + \bar{z}_j))$

Let  $I_1, I_2, I$  be the sets of agents who prefer to make offers on issue 1, issue 2 or both issues. If a player is indifferent between any two alternatives when  $\delta = 1$ , we break ties arbitrarily. Notice that this partition of the set of players only depends on the primitives of the model and not on the equilibrium of the multi-issue bargaining game. To simplify the analysis, we suppose that each player has a unique optimal winning coalition to which she makes an offer and let  $D_1(i)$ ,  $D_2(i)$  and  $D(i)$  denote this coalition. Let  $\sigma_j^i, \tau_j^i$  and  $\omega_j^i$  denote the probability (equal to 0 or 1) that  $j$  makes an offer to a coalition including  $i$  in games  $\Gamma_1, \Gamma_2$  and  $\Gamma$  as  $\delta \rightarrow 1$ .

We are now ready to provide a sufficient condition for the bargaining game to be additive:

**Proposition 7** *The bargaining game is additive if the following conditions hold:*

- For all  $i \in I_1$ ,  $\sum_{j \notin I_1} \sigma_j^i \bar{y}_i + (n - i_1) \bar{z}_i - \sum_{j \in I_2} \tau_j^i \bar{z}_i + i_2 \bar{y}_i - \sum_{j \in I} \omega_j^i (\bar{y}_i + \bar{z}_i) = 0$ ,
- For all  $i \in I_2$ ,  $\sum_{j \notin I_2} \tau_j^i \bar{z}_i + (n - i_2) \bar{z}_i - \sum_{j \in I_1} \sigma_j^i \bar{y}_i + i_1 \bar{z}_i - \sum_{j \in I} \omega_j^i (\bar{y}_i + \bar{z}_i) = 0$ ,
- For all  $i \in I$ ,  $\sum_{j \notin I} \omega_j^i (\bar{y}_i + \bar{z}_i) - i_1 \bar{z}_i - i_2 \bar{y}_i - \sum_{j \in I_1} \sigma_j^i \bar{y}_i - \sum_{j \in I_2} \tau_j^i \bar{z}_i + a_i + \sum_{j \in D(i)} a_j - \sum_{j \in N} \omega_j^i a_i = 0$ .

**Proof:** The idea of the proof is to show that when the conditions hold, the system of equations defining equilibrium payoffs is satisfied when  $\bar{V}_i = \bar{y}_i + \bar{z}_i$ . To see this, we know that the system of equations defining equilibrium payoffs in the game  $\Gamma_1$  gives

$$n\bar{y}_i = 1 - \sum_{j \in D_1(i)} \bar{y}_j + \sum_j \sigma_j^i \bar{y}_i. \quad (2)$$

Similarly, in game  $\Gamma_2$ ,

$$n\bar{z}_i = 1 - \sum_{j \in D_2(i)} \bar{z}_j + \sum_j \tau_j^i \bar{z}_i. \quad (3)$$

And in game  $\Gamma$ ,

$$n\bar{w}_i = 2 - \sum_{j \in D(i)} \bar{w}_j + \sum_j \omega_j^i \bar{w}_i. \quad (4)$$

Now, construct an equilibrium where all players in  $I_1$  make an offer on issue 1, all players in  $I_2$  make an offer on issue 2 and all players in  $I$  make a joint offer. Limit equilibrium payoffs for players in  $I_1$  are given by

$$\begin{aligned} nV_i &= (1 - \sum_{j \in D_1(i)} (V_j + \bar{z}_j) + \bar{z}_i) + (\sum_{j \in I_1} \sigma_j^i + \sum_{j \in I_2} \tau_j^i + \sum_{j \in I} \omega_j^i) V_i \\ &+ \sum_{j \in I_1} (1 - \sigma_j^i) \bar{z}_i + \sum_{j \in I_2} (1 - \tau_j^i) \bar{y}_i. \end{aligned}$$

Similarly, limit equilibrium payoffs for players in  $I_2$  are given by

$$\begin{aligned} nV_i &= (1 - \sum_{j \in D_2(i)} (V_j + \bar{y}_j) + \bar{y}_i) + (\sum_{j \in I_1} \sigma_j^i + \sum_{j \in I_2} \tau_j^i + \sum_{j \in I} \omega_j^i) V_i \\ &+ \sum_{j \in I_1} (1 - \sigma_j^i) \bar{z}_i + \sum_{j \in I_2} (1 - \tau_j^i) \bar{y}_i. \end{aligned}$$

and for players in  $I$

$$\begin{aligned} nV_i &= (2 - \sum_{j \in D(i)} V_j + (\sum_{j \in I_1} \sigma_j^i + \sum_{j \in I_2} \tau_j^i + \sum_{j \in I} \omega_j^i) V_i \\ &+ \sum_{j \in I_1} (1 - \sigma_j^i) \bar{z}_i + \sum_{j \in I_2} (1 - \tau_j^i) \bar{y}_i. \end{aligned}$$

Now replace  $V_i$  by  $\bar{y}_i + \bar{z}_i$  to obtain, for players in  $I_1$

$$\begin{aligned} n(\bar{y}_i + \bar{z}_i) &= (1 - \sum_{j \in D_1(i)} \bar{y}_j) + \bar{z}_i + (\sum_{j \in I_1} \sigma_j^i + \sum_{j \in I_2} \tau_j^i + \sum_{j \in I} \omega_j^i) V_i \\ &+ \sum_{j \in I_1} (1 - \sigma_j^i) \bar{z}_i + \sum_{j \in I_2} (1 - \tau_j^i) \bar{y}_i. \end{aligned}$$

and use equation (2) to replace  $1 - \sum_{j \in D_1(i)} \bar{y}_j$  by  $n\bar{y}_i - \sum_j \sigma_j^i \bar{y}_i$ . This leads to the equation

$$\sum_{j \in I_1} \sigma_j^i \bar{y}_i + (n - i_1) \bar{z}_i - \sum_{j \in I_2} \tau_j^i \bar{z}_i + i_2 \bar{y}_i - \sum_{j \in I} \omega_j^i (\bar{y}_i + \bar{z}_i) = 0.$$

Similarly, for  $i \in I_2$ , replacing  $V_i$  by  $\bar{y}_i + \bar{z}_i$  and using equation (3),

$$\sum_{j \in I_2} \tau_j^i \bar{z}_i + (n - i_2) \bar{y}_i - \sum_{j \in I_1} \sigma_j^i \bar{y}_i + i_1 \bar{z}_i - \sum_{j \in I} \omega_j^i (\bar{y}_i + \bar{z}_i) = 0.$$

Finally for  $i \in I$ , replacing  $V_i$  by  $\bar{y}_i + \bar{z}_i$ , using equation (4) and the fact that  $\bar{w}_i = \bar{y}_i + \bar{z}_i + a_i$ , we obtain

$$\sum_{j \in I} \omega_j^i (\bar{y}_i + \bar{z}_i) - i_1 \bar{z}_i - i_2 \bar{y}_i - \sum_{j \in I_1} \sigma_j^i \bar{y}_i - \sum_{j \in I_2} \tau_j^i \bar{z}_i + a_i + \sum_{j \in D(i)} a_j - \sum_{j \in N} \omega_j^i a_i = 0.$$

Hence, when the conditions in Proposition 7 hold, the system of equations where players in  $I_1$  make an offer on issue 1, players in  $I_2$  an offer on issue 2 and players in  $I$  an offer on both issues results in equilibrium payoffs equal to  $\bar{y}_i + \bar{z}_i$  for all the players. To check that this forms an equilibrium, we need to check that players do not want to deviate and make an offer on any other issue. If the condition guaranteeing that they belong to  $I_1$ ,  $I_2$  or  $I$  is strict, this is not an issue. If not, players are indifferent between making any offer in the limit and equilibrium conditions hold at the limit. It is an  $\epsilon$ -equilibrium for players to propose on any issue which guarantees the highest equilibrium at the limit, as the benefit of any deviation goes to 0 when  $\delta$  converges to 1.  $\square$

The sufficient condition of Proposition 7 is not easy to interpret. Notice however that the condition holds whenever all players agree to make offers on the same issue, i.e.  $N = I_1$  or  $N = I_2$ . We show that the condition also holds when one of the two games admits veto players.

**Corollary 1** *Suppose that one of the two games,  $\Gamma_1$  or  $\Gamma_2$  admits veto players. Then the bargaining game is additive.*

**Proof:** Suppose that game  $\Gamma_1$  admits veto players. Then  $\bar{y}_i = \frac{1}{k_1}$  for all veto players in  $K_1$  and  $\bar{y}_i = 0$  whenever  $i$  is not a veto player. In addition all veto players in  $K_1$  must be included in the winning coalitions  $D_1(i)$  and  $D(i)$  for all  $i$ . This implies that  $1 - \sum_{j \in D_1(i)} \bar{y}_j = \bar{y}_i$ . As  $1 \geq \bar{z}_i + \sum_{j \in D_2(i)} \bar{z}_j$ , player  $i$  must weakly prefer to make an offer on issue 2 than on issue 1. We also have that  $2 - \sum_{j \in D(i)} (\bar{y}_j + \bar{z}_j) = 1 - \sum_{j \in D(i)} \bar{z}_j$ , so that player  $i$  must weakly prefer to make an offer on issue 2 than on both issues. Hence we can break ties by setting  $N = I_2$  and the condition of Proposition 7 holds.

We also note that when all players prefer to make joint offers, the condition of Proposition 7 takes a simple form:

**Corollary 2** *Suppose that all players weakly prefer to make joint offers so that  $N = I$ . Then if all deviations are equal to zero,  $a_i = 0$  for all  $i$ , then the bargaining game is additive.*

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