On the existence of subgame perfect equilibria in discontinuous perfect information games

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Abstract

We prove that for a large class of discontinuous perfect information games, called path secure games, a subgame perfect equilibrium exists. This is some counterpart of Reny's existence theorem (true for normal form games) for extensive form games. Roughly, a game is path secure if for every strategy profile s^* which is not a subgame perfect equilibrium, there is a deviation of some player i which improves strictly his payoff, even under perturbations of the paths involved.

Keywords: discontinuous game, path security, extensive-form game.

JEL Classification:

1. Introduction

Subgame perfect equilibrium concept is the cornerstone of dynamic strategic models, because it is a natural refinement of Nash equilbrium concept: in

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particular, it can be used as a prediction tool, for example in general equilibrium, game theory, or in network formation theory.

Its popularity also comes from the fact that its existence (in pure strategies) can be obtained through a standard backward induction scheme, in every extensive-form game with:

1. Perfect information [1],
2. A finite horizon,
3. A finite number of actions at each period.

Many papers have tried to extend such existence result. For example, Fudenberg and Levine [12] have proved that the existence proof can be extended to infinite horizon games assuming some continuity property of payoffs. But, as shown in Harris [15], their argument can not be extended to the case of an infinite numbers of actions at each period.

Yet, many naturel models in Economics require infinite number of actions at each period: e.g., Cournot competition model [2], Bertrand competition model [7] Rubinstein's competition model [27], Stochastic games [29]

The first extensions to the case of infinite sets of actions at each period have been provided by Harris [15], then Reny, Hellwig and Robson [31] and more recently Carmona [9]). All these paper give different proofs of the existence of a SPE, but this proof rest heavily on the fact that the payoff functions are assumed to be continuous.

Since then, there is no general existence proof which allows some kind of discontinuities of payoffs. It could seem astonishing, because many games introduced in Economics have discontinuous payoffs (e.g., timing games, price and spatial competitions, auctions, bargaining, preemption games or wars of attrition, etc.). In particular, a large literature connected to discontinuous games is developping for more than 20 years (see for example, [11, 10], [9], [25], [6], [21] , [26, 24], [5], [18], [4, 20, 30],[30],[17],[22]), but all the attention has been concentrated on normal form games, that is on the static case. In particular, one of the most used and analyzed result is Reny's theorem, which proves the existence of a pure Nash equilibrium for normal form games with possibly discontinuous payoffs. The class of games considered by Reny, called better-reply secure games, encompasses many standard examples of literature, as auction games or duopoly games.

A reason for this dissymetry between the literature on dynamic discontinuous games and static discontinuous games is probably technical. Indeed, the typical normal form game assumes compact and convex strategy space. These assumptions allows to apply classical fixed-point theorems (like Brouwer or Kakutani) in order to get the existence of a Nash equilibrium, at least for continuous payoffs, and when payoffs are discontinuous, one can hope to be able to approximate the game or the payoff to use a similar approach. On the contrary, for extensive-form games, there is, is in general, no natural topology on the strategy spaces for which the payoffs are continuous (with respect to profile of strategies) and strategy spaces compact, even when the payoff functions are continuous with respect to the sequence of actions.

In particular, given any well behaved extensive form game Γ, the normal form game G^{Γ} associated to this game do not possess, in general, sufficiently good property to apply standard existence result to get Nash equilibrium of G (thus SPE of Γ). Even for finite extensive form game, for which an immediate backward induction scheme can be applied to get a SPE, it is unclear what is the general topological argument that can be applied to G^{Γ} to provide the corresponding equilibrium in G^{Γ}.

In this paper, we provide an existence result of a SPE for a class of discontinuous extensive-form games. This class is called path-secure, and has some similarity, in terms of definition, with the class of better-reply secure games. But the connection between the two is unformal, since non of the two model implies the other, and our method of proof is completely different from Reny's one.[1]

[1] A natural question would be whether Reny's approach can be applied to extensive form games. A first idea would be to apply Reny's method to the normal form game associated to the exensive form game. But first, it would require some quasiconcavity assumption on the payoffs (an assumption that is necessary in Reny's paper), which seems unecessary for extensive form games (for example, [31], the author prove the existence of a subgame perfect equilibrium when the payoff are path-continuous, without any quasiconcavity assumption). Even if we can get rid of quasiconcavity (for example, using some extension of Reny's result), there still remain a big issue: what is the topology that should be defined of the space of strategies ?

2. Existence of a SPE in continuous or discontinuous games: some motivations

2.1 The continuous case

In [31], Hellwig, Leininger, Reny and Robson recall the backward induction method does not extend, in general, to the case where players can choose an infinite number of actions. As an illustration, they consider a game with two players, denoted 1 and 3. Player 1 plays $x_1 \in X_1 = [-1, 1]$, then player 3 plays $x_3 \in X_3 = [-1, 2]$. The payoffs are $u_1(x_1, x_3) = x_1 - x_3$, $u_3(x_1, x_3) = x_1 x_3$ for every $(x_1, x_3) \in X_1 \times X_3$.

For every action $x_1 \in X_1 = [-1, 1]$ of player 1, one can compute $B_3(x_1)$ the set of best-reponses of player 3 against x_1:

$$(2.1)\quad B_3(x_1) = \begin{cases} -1 & \text{if } x_1 \in [-1, 0[\\ [-1, 2] & \text{if } x_1 = 0 \\ 2 & \text{if } x_1 \in]0, 1] \end{cases}$$

Then, the backward-induction method precribes to consider any selection[2] $b_3 : X_1 \to X_3$ of B (which traduces that player 1 anticipates a rational behavior of player 3) and to solve the following maximization problem $(\mathcal{P})$ of player 1:

$$(\mathcal{P}) \max_{x_1 \in X_1} u_1(x_1, b_3(x_1)).$$

Contrarily to what happens when strategy sets are finite, Problem $(\mathcal{P})$ may have no solution, even when u_1 is assumed to be continuous, because b_3 can fail to be continuous. It comes from the undeterminacy of the best-responses of player 3 when player 1 chooses the strategy $x_1 = 0$. In the example above, $(\mathcal{P})$ has a solution if and only if $b_3(0) = -1$.

Example 2.1.— *Graph of* $u_1(x_1, b_3(x_1))$, $x_1 \in [-1, 1]$, *when* $b_3(0) = 0$.

A first possible answer to this problem, mentioned in [31], is to consider the following "improved backward-induction" principle: choose a particular selection b_3 which not only maximizes the payoff of player 3 (which is true

[2] which means that $b_3(x_1) \in B_3(x_1)$ for every $x_1 \in X_1$

by definition of b_3), but also maximizes the payoff of the player just before player 3 (here, player 1). Formally, this is equivalent to consider a particular selection $b_3 : X_1 \to X_3$ of B_3 such that for every $x_1 \in X_1$, $b_3(x_1)$ is a solution of

$$(2.2) \quad \max_{b \in B_3(x_1)} u_1(x_1, b).$$

This implies that when $x_1 = 0$, we "solve" the undeterminacy of player's 3 maximization problem in favor of player 1, by fixing $x_3 = -1$. In particular, this implies that player 3 is benevolent to player 1. If we impose this additional condition, then it is easy to see that despite the discontinuity of b_3 at $x_1 = 0$, the problem $(\mathcal{P})$ has a solution: it can be easily seen that $u_1(x_1, b_3(x_1))$ is upper semicontinuous, thus has a maximum (here $x_1 = 0$) on the compact set X_1. In particular, this provides the unique subgame perfect equilibrium of the game: player 1 plays $x_1 = 0$, player 3 plays -1 if $x_1 \in [-1, 0]$ and plays $x_3 = 2$ if $x_1 > 0$.

Unfortunately, this "improved backward-induction" approach fails with more than 2 players, as illustrated with the following modification of the previous game, proposed by the same authors in [31]. Player 2 is introduced in the new game, he moves between players 1 and 3 and his payoff is $u_2(x_2, x_3) = x_2 x_3$ with $x_2 \in X_2 := [-1, 2]$.

Then, the "improved Backward induction scheme" decribed above would induce player 3 to play a best reply $b_3(x_1, x_2)$ defined as follows:

$$(2.3) \quad b_3(x_1, x_2) = \begin{cases} -1 & \text{if} \quad x_1 \in [-1, 0[\\ [-1, 2] & \text{if} \quad x_1 = x_2 = 0 \\ 2 & \text{if} \quad x_1 = 0, x_2 > 0 \\ -1 & \text{if} \quad x_1 = 0, x_2 < 0 \\ 2 & \text{if} \quad x_1 \in]0, 1] \end{cases}$$

Which would drive player 2 to choose the following best replies:

$$(2.4) \quad b_2(x_1) = \begin{cases} -1 & \text{if} \quad x_1 < 0 \\ 2 & \text{if} \quad x_1 \geq 0 \end{cases}$$

and player 1 maximization problem would have no solution :

Example 2.2.— *Graph of* $u_1(x_1, b_2(x_1), b_3(x_1, b_2(x_1)),\ x_1 \in [-1, 1]$.

Yet, this game has a (unique) subgame perfect equilibrium, for which player 1 should play 0, player 3 should play 2 if $x_1 > 0$, -1 if $x_1 \leq 0$, and player 2 should play -1 if $x_1 \leq 0$ and 2 if $x_1 > 0$. At this subgame perfect equilibrium, player 3 is not benevolent to the player just before, but to player 1, and it is unclear how some general "benevolent" rule could drive to the existence of a subgame perfect equilibrium.

To summarize, the "improved backward induction" rule above is unefficient to provide a subgame perfect-equilibrium in this game, because if player 3 wants to play in favor of player 2 when he is indifferent between several strategies, he could induce a behavior of player 2 which prevent player 1 to play at equilibrium.

In their paper, Reny et al. [31] propose the following answer[3]: they ap-

[3]Other papers have proposed methods to prove the existence of a subgame-perfect equilibrium in finite-horizon extensive-form games with infinite action sets. In his seminal paper, Selten [28] shows that every finite extensive game with perfect information has at least one perfect equilibrium point which is a Subgame perfect equilibrium. Harris [16] shows the existence of perfect equilibrium for a class of games with payoff functions are continuous, and their choice sets are compact sets that depend continuously on the choices made previously. He supposes that the couple of histories and payoffs is a game of perfect information and the game admits a perfect information point which is the equilibrium paths of all subgames of histories and payoffs, Harris [15] considers a class of infinite action games and he shows that the perfect equilibrium points can be characterized as limits of sequences of perfect approximate equilibrium points of finite action approximations to the game, he indicates that the strong topology is useful to show that a game possess a unique perfect equilibrium point. Perfect equilibria have been refined in several directions: Carmona [9] that shows that for every game with perfect information has an ϵ-perfect equilibrium assuming that the payoff functions are bounded and continuous at infinity. The payoff function of each player in [9] is approximated by a simple sequence of simple functions that converges uniformly to the payoff function of that player in the original game. Fudenberg and Levine [12] demonstrate that a subgame-perfect of infinite horizon games has a perfect equilibrium points which is calculated by a perfect approximate equilibrium points of finite horizon. Borgers [8] presents the existence of a perfect equilibrium of infinite horizon games through the existence of perfect epsilon-equilibria of finite horizon truncations of such games. Maitra and Sudderth [19] provide sufficient conditions for the existence of subgame perfect equilibrium within stochastic game with Borel state space and compact metric action sets when the law of motion is continuous in some actions by the total variation norm and the payoff functions are bounded, Borel measurable function of the sequence of states and are continuous. Flesch et al. [13] provide if every payoff function u_j is bounded and lower semicontinuous, then an ϵ -SPE exists for each $\epsilon > 0$.

proximate the action sets by finite ones, then consider some subgame perfect equilibrium of the finite approximation, obtained by standard backward induction. At the limit when the finite approximation converges to the initial game, they get strategies that provide a subgame equilibrium, *up to small modifications of theses stragegies*. In other words, they use approximation of action sets by countable ones.

2.2 Discontinuous games

EXAMPLE 1: Consider a game with two players, denoted 1 and 2. Player 1 plays $x \in X = [0, 1]$, then player 2 plays $y \in Y = [0, 1]$. the payoffs are:

$$\begin{cases} u_1(x, y) = u_2(x, y) = y & \text{if} \quad x = 1 \\ u_1(x, y) = u_2(x, y) = -\frac{x}{2} & \text{if} \quad x \neq 1 \end{cases}$$

The unique subgame perfect equilibrium of the game: Player 1 plays $x = 1$, player 2 plays $y = 0$ if $x \in [0, 1[$ and $y = 1$ if $x = 1$.
This gives an equilibrium path $(1, 1)$.
This games does not have continuous payoffs with respect to the paths. But the equilibrium path is very predictable, because if player 1 plays $x = 1$, player 2 should play $y = 1$, and even in case of small mistakes of player 2, the final payoff is close to 1.
By opposition, if player 1 plays $x < 1$, he's sure to get less than 0.
Thus, despite the discontinuities of this game, there is no doubts that this unique subgame perfect equilibrium will be played.
A question is to find a method to detect this subgame perfect equilibrium in such games. The method of Reny, Robson and Hellwig works for continuous games, and may fail here. Indeed, consider the discretization $X^n = \{0, \dfrac{1}{n}, ..., \dfrac{n-1}{n}\}$ and $Y^n = \{0, \dfrac{1}{n}, ..., \dfrac{n-1}{n}\}$. This is a "*good*" approximation of $X = Y = [0, 1]$ in the sense of Reny, Robson and Hellwig (because X^n and Y^n converges to X and Y for the Hausdorff distance), but there is no hope

In the same context Purves and Sudderth [23] demonstrate if every payoff function u_j is bounded and upper semicontinuous, then a ϵ -SPE exists for each $\epsilon > 0$. When the set of actions is infinite, a SPE needs not exist, but one could follow the approach in Flesch and Predtetchinski [14] to obtain existence of a pure ϵ -SPE, for all $\epsilon > 0$. They prove that continuity ϵ -SPE exists for each $\epsilon > 0$ if the payoff functions are bounded and lower semicontinuous. Flesch and Predtetchinski [14] examine a re?nement of ϵ -SPE, which requires that, in every subgame, the payoff functions are continuous at the induced play.

to get the unique subgame perfect equilibrium from the descretization, since in the approximated game, the (unique) subgame perfect equilibrium path is $(0,0)$.

One of the aim of this paper is to define a class of discontinuous extensive-form game, we call path secure, which incorporates similar examples as this above, for which there always exists a subgame perfect equilibrium. The idea is to approximate the game not be a *given* finite approximation (which raises the question of its choice) but by *any* finite approximation: this will improve the quality of the existence result.

3. The general model, and the main existence result

3.1 The model

We consider an extensive-form game with $N \geq 2$ players[4] $i = 1, ..., N$. Each player i has an action set X_i, assumed to be a topological space[5] and a (possibly discontinuous) payoff function $u_i : X_1 \times ... \times X_N \to \mathbf{R}$.

The game can be formalized by the following directed tree Γ: the root is denoted x_0, and represents the moment just before player 1's action; the set of nodes is $V = \cup_{i=0}^{N} H_i$, where for every $i = 0, ..., N$,

$$H_i = \; X_0 \times X_1 \times X_2 \times ... \times X_i$$

where $X_0 = \{x_0\}$. For every $i = 1, ..., N-1$, H_i is called the set of *histories for player* $i+1$, $i = 0, ..., N-1$. In particular, the set of terminal nodes is H_N.

There is an edge from a node $v \in V$ to a node $v' \in V$ if there exists $i \in \{0, 1, ..., N-1\}$ and $(x_0, ..., x_i, x_{i+1})$ in H_{i+1} such that $v = (x_0, ..., x_i)$ and $v' = (x_0, ..., x_i, x_{i+1})$.

Assumptions

(A1) The payoff functions $u_i : X_1 \times ... \times X_N \to \mathbf{R}$ are assumed to be *bounded.*

[4]For simplicity of notations, we will also denote N the set of players.

[5]We do not require X_i to be Hausdorf.

(A2) For every $j = 1, ..., N$, X_j is a compact set.

These assumptions are assumed in all previous existence proof of SPE in extensive form games (e.g., [?], [?]). But in these previous papers, it is also assumed that the payoff functions are continuous (an assumption that will be replaced by a strictly weaker one), and also that the multivalued functions $X_j(x_1, ..., x_{j-1})$ are lower-semicontinuous (an assumption that we will remove, thanks to our existence proof).

Throughout this paper, $X_1 \times ... \times X_N$ will be endowed with the product topology, and each $H_i \subset X_1 \times ... \times X_i$ is endowed with the induced topology. In particular, under assuption $(A2)$, each H_i is compact.

Now, it is convenient to associate to the extensive-form game above the following normal form game denoted G:

- The players of G are $i = 1, ..., N$.
- A strategy s_i of player $i \in N$ is a mapping from $H_{i-1} \to X_i$ which associates to every possible history $h_{i-1} \in H_{i-1}$ for player i some action $s_i(h_{i-1}) \in X_i$. In particular, a strategy s_i for player $i = 1$ can also be equivalently described as an element $x_1 \in X_1$ (an equivalent description that we will used in this paper). For every $i \in N$, let S_i be the set of strategies of player i. To define the payoffs of G, fix a strategy profile $s = (s_1, ..., s_N) \in S := S_1 \times ... \times S_N$. This generates a *path* in the tree Γ, denoted $p(s)$, which is a sequence of decisions, $(x_1, ..., x_N)$ defined inductively[6] by $x_1 = s_1, x_2 = s_2(x_1), ...x_N = s_N(x_1, ..., x_{N-1})$.
- Last, the payoff of player $i \in \{1, ..., N\}$ in G is defined by

  $$U_i(s_1, ..., s_N) := u_i(p(s)).$$

  Each set S_i can be identified to $X_i^{H_{i-1}}$, and it is endowed with the product topology (and thus is compact, since X_i is compact). The set $S = S_1 \times ... \times S_N$ of strategy profiles is also endowed with product topology (and thus is compact).

[6] We should denote $x_1 = s_1(x_0), x_2 = s_2(x_0, x_1), ...x_N = s_N(x_0, x_1, ..., x_{N-1})$, etc. But for simplicity of notations, x_0 will be removed throughout this paper, without loss of generality.

Similarly, for every $i \in N$ and for every history $h_{i-1} = (x_1, ..., x_{i-1}) \in H_{i-1}$, a normal form game *played after* h_{i-1}, denoted $G(h_{i-1})$, can be defined as follows: the players' set is $\{i, i+1, ..., N\}$. A strategy s_j of player $j \in N$ is, again, a mapping from $H_{j-1} \to X_j$ (but in fact, only the values of $s_j(h_{i-1}, .)$ will be relevant for the definition of the payoffs in $G(h_{i-1})$). Every strategy profile $s = (s_1, ..., s_N)$ defines a unique path beginning at node $h_{i-1} = (x_1, ..., x_{i-1})$ and finishing at a terminal node, denoted $p_{|h_{i-1}}(s) = (x_i, ..., x_N)$, as follows: $x_i = s_i(x_1, ..., x_{i-1}), x_{i+1} = s_{i+1}(x_1, ..., x_{i-1}, x_i), ..., x_N = s_N(x_1, ..., x_{N-1})$. Remark that only the information about $s_i, .., s_N$ in s is relevant to be able to define $p_{|h_{i-1}}(s)$, thus we could note also $p_{|h_{i-1}}(s) = p_{|h_{i-1}}(s_i, s_{i+1}, ..., s_N)$. We shall use both notations (for simplication), depending on the context.

Also, for every strategy profile $s = (s_1, ..., s_N)$ and given a node $h_{i-1} = (x_1, ..., x_{i-1})$ and some action $\bar{x}_k$ at some $k \in \{i, i+1, ..., N\}$, we will denote by $p_{|h_{i-1}}(\bar{x}_k, s_{-k}) = (x_i, ..., x_N)$ the path constructed as $p_{|h_{i-1}}(s)$ except for stage k where $\bar{x}_k$ (some "deviation") is played: that is, formally,

$x_i = s_i(x_1, ..., x_{i-1}), ..., x_{k-1} = s_{k-1}(x_1, ..., x_{k-2})$, $x_k = \bar{x}_k$,

$x_{k+1} = s_i(x_1, ..., x_{k-1}, \bar{x}_k), ..., x_N = s_N(x_1, ...x_{k-1}, \bar{x}_k, x_{k+1}, ..., x_N)$.

We can define the same thing for more than one deviation: for every strategy profile $s = (s_1, ..., s_N)$ and given a node $h_{i-1} = (x_1, ..., x_{i-1})$ and some actions $\bar{x}_{k_1}, ..., \bar{x}_{k_l}$ at some $k_1 < k_2 < ... < k_l \in \{i, i+1, ..., N\}$, we will denote by $p_{|h_{i-1}}(x_{k_1}, ..., x_{k_l}, s_{-k_1 - k_2 ... - k_l}) = (x_i, ..., x_N)$ the path constructed as prescribed $p_{|h_{i-1}}(s)$ except for stage $k_1, ..., k_l$ where $\bar{x}_{k_1}, ..., \bar{x}_{k_1}$ are played: that is, formally, $x_i = s_i(x_1, ..., x_{i-1}), ..., x_{k_1-1} = s_{k-1}(x_1, ..., x_{k_1-2})$, $x_{k_1} = \bar{x}_{k_1}$, $x_{k_1+1} = s_i(x_1, ..., x_{k_1-1}, \bar{x}_{k_1})$, ...,$x_{k_2-1} = s_i(x_1, ..., x_{k_1-1}, \bar{x}_{k_1}, x_{k_1-1}, ..., x_{k_2-1}, \bar{x}_{k_2} - 2)$, $x_{k_2} = \bar{x}_{k_2}$, ... etc

When $h_{i-1} = s_0$, $p_{|s_0}(s)$ will be also simply be denoted $p(s)$, and $p_{|s_0}(x_k, s_{-k})$ will be also simply be denoted $p(x_k, s_{-k})$, and similarly if there are more than one deviation with respect to s.

Finally, from $s = (s_1, ..., s_N)$ we can define a payoff in $G(h_{i-1})$ for each player $j = i, ..., N$, denoted $U_{j|h_{i-1}}$, by $U_{j|h_{i-1}}(s_i, ..., s_N) = u_j(h_{i-1}, p_{|h_{i-1}}(s))$. The game $G(h_{i-1})$ is called the subgame of G beginning at h_{i-1} (in particular $G(\{x_0\}) = G$).

Throughout this paper, we shall denote $\Gamma = ((X_i)_{i \in N}, (u_i)_{i \in N})$ the pair of action sets and payoff functions, defining the extensive-form game. The associated normal game will be denoted $G = ((S_i)_{i \in N}, (U_i)_{i \in N})$.

We now define the basic equilibrium notion associated to such game G.

Definition 3.1.— *A strategy profile $s = (s_1, ..., s_N)$ is a subgame-perfect equilibrium (SPE) of $\Gamma = ((X_i)_{i\in N}, (u_i)_{i\in N})$ if for every $i \in N$ and every history $h = (x_0, ..., x_{i-1})$, $(s_i, s_{i+1}, ..., s_N)$ is a Nash equilibrium of $G(h)$.*

Equivalently, $s = (s_1, ..., s_N)$ is a subgame-perfect equilibrium (SPE) of Γ if for every $i = 1, ..., N$, for every history $h_{i-1} = (x_0, ..., x_{i-1}) \in H_{i-1}$ and every deviation d_i of every player i,

$$u_i(h_{i-1}, d_i, p_{|(h_{i-1},d_i)}(s_{i+1}, .., s_N)) \leq u_i(h_{i-1}, p_{|h_{i-1}}(s_i, .., s_N)).$$

3.2 path-secure games

Let $\Gamma = ((X_i)_{i\in N}, (u_i)_{i\in N})$ be an extensive-form game. For every history h_k, denote $G^{finite}_{|h_k}$ the set of finite subgames of G beginning at h_k.
The following notion plays a central role in the paper.

Definition 3.2.— *Player $i \in N$ has a strictly profitable path-secure deviation finite set $D_k \subset X_k$ at some history $h_{k-1} \in \mathcal{H}_{k-1}$ given $\bar{s} \in S$ if there exists some open neighborhoods $V'(d_k)$ of $(h_{k-1}, d_k, p_{|(h_{k-1},d_k)}(\bar{s}))$ for every $d_k \in D_k$ and some open neighborhood V of $(h_{k-1}, p_{|h_{k-1}}(\bar{s}))$, such that for every finite game G^{finite} and every SPE s of G^{finite}, $p(s) \in V$ and $p(d_k, s_{-k}) \in V'(d_k)$ for every $d_k \in D_k$ implies that*

$$\max_{d_k\in D_k} u_k(p(d_k, s_{-k})) > u_k(p(s))$$

Definition 3.3.— *An extensive-form game $G = (X, u)$ is path-secure if for every strategy profile s^* which is not a SPE, some player k has a strictly profitable path-robust deviation set D_k in some subgame h_{k-1}.*

Theorem 3.1.— *Under Assumptions (A1) and (A2), for every path secure game G, there exists some SPE.*

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