On Algorithms That Approach Correlated Equilibrium

Ayan Bhattacharya*

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Short Abstract

The set of correlated equilibria is a closed convex set. Using this fact, I show that one may characterize a correlated equilibrium in strategic-form games in terms of a weak notion of approachability. While Blackwell (1956) defined approachability at the level of individual play, to characterize correlated equilibrium, I use a related notion of approachability at the level of collective play. Approachability also lends itself to an algorithmic interpretation, and one can define step-by-step procedures that shrink the space of play to the approachable set. I use the topological notion of a retraction to obtain properties of such algorithms. This allows me to explore the generality of the link between approachability algorithms and correlated equilibrium.

*Baruch College, The City University of New York, New York, NY 10010. Email: Ayan.Bhattacharya@baruch.cuny.edu

Extended Abstract

Given the large fraction of economic interactions intermediated by algorithms today, it has become important for economists to understand the "collective play" properties of such procedures. Important work by several researchers (see Fudenberg and Levine (1998), Cesa-Bianchi and Lugosi (2006), Shoham and Leyton-Brown (2010), Hart and Mas-Colell (2013), and the references therein) has uncovered a number of algorithmic procedures that converge to a variety of economic equilibrium. In particular, for many algorithmic procedures that feature the correlated equilibrium, a common attribute seems to be their reducibility, in some sense, to the Blackwell approachability algorithm (Blackwell (1956)). In this paper, I explore the generality of this link. For this, I first show that empirical play convergence to a correlated equilibrium is equivalent to imposing a weak approachability condition on the algorithms at play. Next, I obtain a characterization of *all* algorithms that satisfy Blackwell approachability. Finally, using this characterization, I establish some general properties of algorithms whose empirical play converge to a correlated equilibrium.

An important result in Blackwell's influential paper (Blackwell (1956)) concerns the characterization of the value of a two player zero-sum game in terms of approachability. Blackwell demonstrates that the highest payoff that is approachable for one player, while not made excludable by the other player, is the value of the game. I show that an approachability based characterization is viable for correlated equilibrium, too, in general strategic-form games, if one weakens the notion of approachability suitably. Instead of defining approachability for individual player strategies, one needs a notion of approachability at the level of collective play. Like the target set in Blackwell approachability, the set of correlated equilibria is convex and closed—thus if one works within the probability simplex, many of the assumptions from the Blackwell setup continue to hold. This allows me to reformulate the notion of correlated equilibrium in terms of the approachability of

sets.

The characterization of algorithms that satisfy approachability comes from recognizing the topological properties of the approach process. Broadly, an approachability algorithm shrinks the average payoff space (and consequently, the space of play) over time to a target convex set. Topologically, the process bears a strong resemblance to a retraction. Let X be a space and $A \subseteq X$ be closed. One says that A is a retract of X provided that there is a continuous function $r : X \to A$ such that r restricted to A is an identity, in which case r is called a retraction. Using the fact that the target set is closed convex, I use tools from the theory of retracts (Borsuk (1966)) to establish a Cauchy convergence like criterion that all algorithms satisfying approachability must fulfill. The Blackwell algorithm is one example of this class, but there could be others (see Hart and Mas-Colell (2001) for some generalizations), all satisfying this characterization.

Finally, using the above characterization, I explore some general properties of algorithms whose empirical distribution of play converge to a correlated equilibrium. Well known algorithms like regret-matching or calibrated learning can be shown to have these properties.

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