

Oligopolistic Equilibrium and Financial Constraints*

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Abstract

We model a dynamic duopoly in which firms can potentially drive their rivals from the market (bankrupt them). A consequence is that, for some range of parameters, the static Cournot equilibrium outcome cannot be sustained in an infinitely dynamic setting. In those cases, there is a Markov perfect equilibrium in mixed strategies in which one firm will eventually be driven from the market with probability one. We consider the consequences of potentially bankruptcy on the set of outcomes supportable via tacit collusion, showing the set can be different than in the absence of bankruptcy. We show that total payoff in the maximum collusive outcome is greater under bankruptcy consideration than in the absence of bankruptcy.

Key words: Financial Constraints, Bankruptcy, Firm Behavior, Dynamic Games.

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1. Introduction

There is ample evidence that financial constraints play an important role in the behavior of firms (Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997). We begin with the observation that the punishment for violation of a financial constraint must be severe or otherwise firms would default all the time. Suppose that the punishment is so severe that firms violating financial constraints lose the capacity to compete and disappear (Bolton and Scharfstein, 1990).¹ Firms might then have incentives to take actions that would make it impossible for competitors to fulfill financial constraints in the hope of getting rid of them.

In this paper, we model a duopoly in which firms take into account their and their rivals' financial limits, specifically that firms go bankrupt (exit) if they earn negative profits in a period. We introduce the concept of bankruptcy-free outputs (BF hereafter). These are output profiles (pairs) in which each firm's profit is non-negative (so no firm goes bankrupt) and in which no firm could, by changing its output, bankrupt another without bankrupting itself. A critical insight of our analysis is that static Cournot equilibrium outputs can fail to be BF when firms' cost functions differ. In particular, if firms have constant, but differing, average costs, then the Cournot outcome is not BF: a lower-cost firm can bankrupt a higher-cost rival without bankrupting itself by increasing its output to the point that price falls between their average costs. In an infinitely dynamic game, such a move could be profitable if firms are patient because it gets rid of a competitor.

In our dynamic game, the unique Markov Perfect Equilibrium (MPE) in pure strategies, if it exists, is the Cournot equilibrium. But if the Cournot outcome is not BF and firms have incentives to predate for some discounts, MPE must entail mixed strategies. *Inter alia*, this suggests that the commonly used constant-marginal-cost Cournot model could be misleading if firms have different marginal costs and are financially constrained.

We show that the mixed MPE exists. Assuming constant average costs and concave profit functions we characterize the equilibrium. The support of each firm's mixed strategy contains exactly one interval. The support of the inferior firm has a mass point in the upper extreme of the interval which coincides with the best reply in the static game to the superior firm's mixed strategy. For the superior firm the support also contains an isolated mass point which coincides with the best

¹Even though firms can be reorganized after bankruptcy and continue business, the survival rate of firms after bankruptcy is typically low, 18% US, 20% in UK and 6% in France, see Couwenberg (2001).

reply in the static game to the inferior firm's mixed strategy. The mass point lies strictly below the interval support. Since the superior firm would not produce a larger output than the best reply unless bankruptcy occurs, outputs in the interval support reflect the predatory activities of the superior firm. The inferior firm becomes bankrupt with positive probability in each period. The introduction of financial constraints implies that monopolization (by the efficient firm) will occur with positive probability in each period and thus, it will occur almost surely in the long run. The probability of predation increases with the discount factor. Moreover, any of the outputs chosen by the superior firm are larger than the Cournot output and the outputs of the inferior firm are smaller than its Cournot output.

We consider the consequences of potentially bankruptcy on the set of outcomes supportable via tacit collusion. We show that total payoff in the maximum collusive outcome is greater under bankruptcy consideration than in the absence of bankruptcy. Thus, in this case, bankruptcy considerations help collusion.

Lastly we examine the validity of the celebrated "folk theorem" under decreasing returns to scale. It turns out that in this case, mixed strategies can be disposed of: We show that the folk theorem remains to be valid in our model, once the set of feasible and individually rational profits are appropriately modified. In the original folk theorem (without bankruptcy consideration), each firm's profit becomes individually rational if it exceeds a minimax value, i.e., the minimum profit that a firm can guarantee itself even though the opponent takes the severest output, and this becomes zero in our duopoly model. Therefore, any combination of feasible and positive profits can be sustained by an equilibrium. The modification of feasibility is straightforward. Instead of considering all possible profits, we should focus only on BF outputs, since at least one firm has an incentive to ruin the other if the output is not BF. The modification of the individual rationality condition comes from the fact that under financial constraints, firms that take severe punishments may become bankrupt as a consequence of their own action. To avoid such scenario, we define a new concept called the minimax BF value where minimization and maximization are taken over only BF outputs.² Then, we establish the modified folk theorem that reflects bankruptcy consideration. Namely, we show that any BF output profile that gives profits greater than the minimax BF value can be supported as an SPNE and that profits less than the minimax BF value cannot be sustained

²For technical reasons we assume here that average costs are increasing.

in any SPNE for a discount factor close to 1.

We end this introduction with a preliminary discussion of the literature (see more on this in the final section). Although a number of papers demonstrate that the financial structure does affect market outcomes in an oligopoly, most previous studies adopt either static or two-stage models. There are at least two exceptions, Spagnolo (2000) and Kawakami and Yoshida (1997). Both papers make use of games with infinite time like ours. The former examines the role of stock options in repeated Cournot games. In his model, unlike standard repeated games, firms do not necessarily maximize average discounted profits because stock options affect managers' incentives. Taking this effect into consideration, Spagnolo (2000) shows that collusion becomes easier to achieve. In our model, by contrast, collusion becomes more difficult to support, at least when firms adopt the grim-trigger strategies. The latter incorporates a simple exit constraint into the repeated prisoners' dilemma. In their model, each firm must exit from the market no matter how it plays if the rival deviates over certain number of periods, and hence no output profile can be bankruptcy free. They show that predations inevitably occur when bankruptcy constraints are asymmetric and firms are long-sighted.

Finally, our approach might provide support to the notion that firms may engage in predatory activities when pursuing profit maximization. Standard explanations of this behavior are based on incomplete information (Milgrom and Roberts, 1982), the learning curve (Cabral and Riordan, 1994) or firms playing an attrition game (Roth, 1996). In our model, firms have complete information, the technology is fixed and firms play standard quantity-setting games. Nevertheless, we obtain predation as a competitive equilibrium in mixed strategies. More importantly, both predation and tacit collusion can be derived (as different equilibria) in a single model, which is a completely new result to the best of our knowledge..

2. The model

Two firms compete in an infinite number of periods. In each period firms simultaneously choose quantities. Firms produce an homogeneous product. In order to focus in the strategic decisions regarding outputs we assume that firms cannot accumulate profits. Firms become bankrupt if they suffer losses in a period. A bankrupt firm exits the market (i.e., 'produces' zero every period thereafter). When making its quantity decision in a period, each firm knows what any firm has

produced in all previous periods and which firms became bankrupt. The equilibrium concept that we use is Subgame Perfect Nash Equilibrium (SPNE). The formal definitions are given in Section 4. In the rest of this section we present the elements of the game that is played in each period. For simplicity, the time dimension is not considered yet.

We refer to one of the firms as the superior (S) and the other as the inferior (I). Let $j \in \{S, I\}$ denote a firm and $x_j \in \mathbb{R}_+$ the output of firm j . Let $C_i(x_i)$ denote the cost function, where $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an everywhere twice differentiable function and $C_i(0) = 0$. Assume that for all output, x , $AC_S(x) \leq AC_I(x)$, where $AC_j(\cdot)$ is firm j 's average-cost schedule. Assume that average cost is nondecreasing and twice differentiable. Let $\mathbf{x} = (x_S, x_I)$ denote an output profile, and let $X = x_S + x_I$ be the aggregate output. Let $p(X)$ be the inverse demand function assumed to be strictly decreasing in X for any positive price and twice differentiable. Derivatives are denoted by primes; e.g., $p'(X)$ is the slope of the inverse demand at X , etc. Profits for firm i are $\pi_i \equiv p(X)x_i - C_i(x_i)$, and written as $\pi_i(\mathbf{x})$ or as $\pi_i(x_i, x_j)$. We assume the classical conditions that guarantee existence and uniqueness of a Cournot equilibrium namely, for all $\mathbf{x} = (x_S, x_I)$,

$$p''(X)x_i + p'(X) < 0, \text{ for all } i \in \{S, I\}, \quad (2.1)$$

$$p'(X) - C_i''(x_i) < 0, \text{ for all } i \in \{S, I\}. \quad (2.2)$$

These conditions are satisfied if, for example, demand is linear and cost functions are quadratic.

We denote by $\mathbf{x}^C = (x_S^C, x_I^C)$ the Cournot output profile and by π_i^C firm i 's profit at the Cournot output profile.

Central to the analysis of our dynamic set up is the concept of bankruptcy-free (BF) output profiles.

As defined in the Introduction, bankruptcy-free (BF) output profiles are those in which no firm makes negative profit and there is no deviation such that either firm can drive the other into bankruptcy without bankrupting itself. Formally,

Definition 1. An output profile $\hat{\mathbf{x}} = (\hat{x}_S, \hat{x}_I)$ is bankruptcy-free (BF) if

- a) $\pi_i(\hat{\mathbf{x}}) \geq 0$ for both $i \in \{S, I\}$; and
- b) $\pi_i(\hat{x}_i, x) \geq 0$ for any x such that $\pi_j(x, \hat{x}_j) \geq 0$ ($i \neq j$).

Note that if firm i is required to make some profit v_i (it could be either positive or negative) to avoid bankruptcy, we can define a new profit function as $\tilde{\pi}_i(\mathbf{x}) \equiv \pi_i(\mathbf{x}) - v_i$ and redefine BF with

respect to this new profit function.

3. Properties of the BF set

In this section we characterize the BF output profiles. The characterizations will become important for the analysis of the dynamic game.

Lemma 1. *An output profile $\mathbf{x} = (x_S, x_I)$ is BF if and only if $\pi_j(\mathbf{x}) \geq 0$ for both j and*

$$AC_j(x_j) \leq AC_k(D(AC_j(x_j)) - x_j) \text{ if } x_j > 0, \quad (3.1)$$

where $D(\cdot)$ is aggregate demand and $k \neq j$.

Proof. The definition of BF entails $\pi_j(\mathbf{x}) \geq 0$ for both j . If a firm does not produce, it cannot be driven to bankruptcy; hence, assume firm j 's output is positive and consider whether firm k can bankrupt it. Define

$$x(x_j) = \inf\{x \in \mathbb{R}_+ \mid \pi_j(x_j, x) < 0\}. \quad (3.2)$$

By continuity, $\pi_j(x_j, x(x_j)) = 0$. It follows that

$$p(x_j + x(x_j)) = AC_j(x_j); \text{ hence, } x(x_j) = D(AC_j(x_j)) - x_j. \quad (3.3)$$

Demand slopes down, so price and, thus, a rival's profit decrease with output. Those facts and the assumption that average cost is non-decreasing mean firm k can bankrupt j without bankrupting itself if and only if

$$0 \leq p(x_j + \hat{x}_k) - AC_k(\hat{x}_k) < p(x_j + x(x_j)) - AC_k(x(x_j)) = AC_j(x_j) - AC_k(D(AC_j(x_j)) - x_j) \quad (3.4)$$

for some $\hat{x}_k > x(x_j)$. But (3.4) can hold if and only if (3.1) does not. ■

Corollary 1. *If the average cost is constant for both firms, $AC_j(x) = c_j$, $j \in \{S, I\}$, and $c_S < c_I$ no output profile with both firms active is BF. In a BF output profile only the superior firm is producing.*

Proposition 1. *Let \mathbf{x} be BF and let \mathbf{x}' be a smaller profile (i.e., $\mathbf{x}' < \mathbf{x}$ in the usual vector order). Then, \mathbf{x}' is also BF.*

Proof. We have

$$0 \leq AC_k(D(AC_j(x_j)) - x_j) - AC_j(x_j) \leq AC_k(D(AC_j(x'_j)) - x'_j) - AC_j(x'_j), \quad (3.5)$$

where the first inequality follows from Lemma 1 given the assumption \mathbf{x} is BF and the second because demand curves slope down and average cost is nondecreasing. Invoking Lemma 1, this chain implies \mathbf{x}' is BF. ■

A useful characterization of the BF set can be provided under the following additional assumption.

Assumption 1. Both firms have increasing average cost and there exists x_S^0 such that $AC_I(0) = AC_S(x_S^0) < p(x_S^0)$.

Observe Assumption 1 holds provided the average cost curve of the superior firm rises above the minimum of the inferior firm's average cost curve and such that the break even output of the superior firm yields a price above the minimum of the inferior firm's average cost. Assumption 1 always holds if for example demand is linear and $C_i(x_i) = \gamma_i x_i^2$ with $\gamma_i > 0$.

In the next Lemma we show a consequence of Assumption 1 that is useful to characterize the BF set under Assumption 1.

Lemma 2. *Assumption 1 implies that there exist a unique $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I) \neq (0, 0)$ such that $\pi_i(\bar{\mathbf{x}}) = \mathbf{0}$ for all $i \in \{S, I\}$.*

Proof. Since average costs are increasing, the functions $AC_I(\cdot)$ and $AC_S(\cdot)$ are invertible, so for any level $\lambda \in [AC_S(x_S^0), \infty)$ there exist $x_I(\lambda)$ and $x_S(\lambda)$, continuous in λ , such that

$$\lambda = AC_I(x_I(\lambda)) = AC_S(x_S(\lambda)). \quad (3.6)$$

By Assumption 1, if $\lambda = AC_S(x_S^0)$, then

$$p(x_I(\lambda) + x_S(\lambda)) > AC_I(x_I(\lambda)) = AC_S(x_S(\lambda)). \quad (3.7)$$

But as $\lambda \rightarrow \infty$ the left term goes to 0, while the middle and right terms stay above 0; by continuity and given that demand slopes down and average costs are increasing, there must exist a unique λ^* such that

$$p(x_I(\lambda^*) + x_S(\lambda^*)) = AC_I(x_I(\lambda^*)) = AC_S(x_S(\lambda^*)). \quad (3.8)$$

Let $\bar{\mathbf{x}} = (x_I(\lambda^*) + x_S(\lambda^*))$. By construction, $\pi_i(\bar{\mathbf{x}}) = \mathbf{0}$ for all $i \in \{S, I\}$. ■

Lemma 3. *Suppose Assumption 1 holds. Let $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I)$ be as described in Lemma 2. Then the set of BF output profiles is:*

$$BF = \{(x_S, x_I) \mid 0 \leq x_i \leq \bar{x}_i \text{ for } i \in \{S, I\}\}. \quad (3.9)$$

Proof. Trivially, $\bar{\mathbf{x}}$ is BF. By Proposition 1, it follows that all $\mathbf{x} < \bar{\mathbf{x}}$ are BF. To complete the proof, we need to show that no \mathbf{x} can be BF when $x_j > \bar{x}_j$ for one or more j . Because demand slopes down and average costs are increasing, if $\mathbf{x} > \bar{\mathbf{x}}$, then $\pi_i(\mathbf{x}) < \pi_i(\bar{\mathbf{x}}) = 0$ for all $i \in \{S, I\}$, so \mathbf{x} is not BF. Suppose then, $x_j > \bar{x}_j$, but $x_i < \bar{x}_i$. Observe $x_j < \bar{x}_j + \bar{x}_i$; if not, the chain

$$p(x_i + x_j) - AC_j(x_j) \leq p(\bar{x}_i + \bar{x}_j) - AC_j(x_j) < p(\bar{x}_i + \bar{x}_j) - AC_j(\bar{x}_j) = 0 \quad (3.10)$$

shows \mathbf{x} is not BF. Given $x_j < \bar{x}_j + \bar{x}_i$, there exists $\hat{x}_i > 0$ such that $\hat{x}_i + x_j = \bar{x}_j + \bar{x}_i$. Let

$$p^* = p(\hat{x}_i + x_j) = p(\bar{x}_j + \bar{x}_i). \quad (3.11)$$

Average cost is increasing; hence,

$$p^* - AC_i(\hat{x}_i) > p^* - AC_i(\bar{x}_i) = 0 = p^* - AC_j(\bar{x}_j) > p^* - AC_j(x_j). \quad (3.12)$$

Expression (3.12) entails that firm i can bankrupt firm j without bankrupting itself; hence, \mathbf{x} is not BF. ■

To close this section, we work out in an example the conditions under which joint profit maximum and Cournot's output profiles are BF.

Example 1. *Let demand be linear, $p(x_S + x_I) = A - x_S - x_I$, and cost be quadratic, $C_S(x) = \gamma_S x^2$, $C_I(x) = \gamma_I x^2$ with $0 < \gamma_S \leq \gamma_I$. By Lemma 3 the BF set is completely characterized by the output profile $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I)$ such that $AC_I(\bar{x}_I) = AC_S(\bar{x}_S) = p(\bar{x}_S + \bar{x}_I)$, thus, $\gamma_S \bar{x}_S = \gamma_I \bar{x}_I = A - \bar{x}_S - \bar{x}_I$.*

$$\bar{x}_S = \frac{\gamma_I A}{(1 + \gamma_S)(1 + \gamma_I) - 1}; \quad \bar{x}_I = \frac{\gamma_S A}{(1 + \gamma_S)(1 + \gamma_I) - 1}. \quad (3.13)$$

The Cournot equilibrium is given by

$$x_S^C = \frac{(1 + 2\gamma_I)A}{4(1 + \gamma_I)(1 + \gamma_S) - 1}; \quad (3.14)$$

$$x_I^C = \frac{(1 + 2\gamma_S)A}{4(1 + \gamma_I)(1 + \gamma_S) - 1} \quad (3.15)$$

For the superior firm is always the case that $x_S^C \leq \bar{x}_S$. For the inferior firm, $x_I^C \leq \bar{x}_I$ if and only if

$$\gamma_S \geq \frac{\gamma_I}{2(1 + \gamma_I)}. \quad (3.16)$$

Thus, the Cournot equilibrium is BF if and only if condition (3.16) holds. The right hand side of (3.16) is increasing in γ_I , which is the parameter defining the marginal cost of the inferior firm. Thus the larger γ_I is, i.e. the more inefficient the inferior firm is, the easier is for the superior firm to bankrupt the inferior firm. Conversely, the larger γ_S the more difficult is for the superior firm to bankrupt the inferior firm.

Finally, in this example, the joint-profit-maximizing output, $\mathbf{x}^J = (x_S^J, x_I^J)$, is always BF because the marginal cost of both firms is the same, $\gamma_S x_S^J = \gamma_I x_I^J$ (production efficiency) and profits are non-negative for both firms. Furthermore, since $\gamma_S \bar{x}_S = \gamma_I \bar{x}_I$, trivially $x_j^J \leq \bar{x}_j$ for all $j \in \{S, I\}$. By the same reason, the perfectly competitive equilibrium, which is also the efficient allocation is also BF.

4. Dynamic Competition with Bankruptcy

In this section we focus on the dynamic model.

In each period t each firm $i \in \{S, I\}$ chooses an output denoted by x_i^t . Let $\mathbf{x}^t = (x_I^t, x_S^t)$ be a profile of outputs in period t . The profits obtained by firm i in period t are $\pi_i(\mathbf{x}^t)$ $t = 0, 1, \dots, \tau, \dots$. We define a state at t as the set of firms that did not fall into bankruptcy in previous periods called *active* firms. Let $\delta \in (0, 1)$ be the common discount factor. Discounted profits for firm i are $\Pi_i = \sum_{t=0}^{\infty} \delta^t \pi_i(\mathbf{x}^t)$. The continuation profit in period t is given by $\Pi_i^t = \sum_{r=0}^{\infty} \delta^r \pi_i(\mathbf{x}^{t+r})$. At period 0 the game begins with the null history h^0 . For $t \geq 1$, a history, denoted by $h^t = (\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^{t-1})$, is a list all outputs at all periods before t . A strategy for firm i , σ_i , (pure or mixed) is a sequence of maps, one for each period t , mapping all possible period t histories into a probability distribution in outputs. Let $\sigma = (\sigma_I, \sigma_S)$ denote a strategy profile (pure or mixed). A Markovian strategy for firm i is a mapping from the set of active firms into a probability distribution on outputs. A Subgame Perfect Nash Equilibrium (SPNE) is a collection of strategies which are a NE in every possible subgame. A Markov perfect equilibrium (MPE) is a SPNE in which firms use Markovian strategies only. To ease notation whenever no confusion can arise, we drop the time superindex.

In infinitely repeated games without bankruptcy considerations there is only one state, and hence the Markovian strategy exactly coincides with the Cournot output. Under bankruptcy con-

siderations, when the Cournot output profile is BF, the unique MPE is that both firms produce the Cournot outcome when both are active and, when only one firm is active, this firm produces the monopoly outcome. In equilibrium both firms are active in every period. A different analysis has to be made when the Cournot outcome profile is not BF because it could be the case that for some discounts factors one firm may have incentives to bankrupt the other firm.³ In the following Lemma we provide the range of the discount factor that is needed in order to prevent such a deviation.

Lemma 4. *Suppose that (x_S^C, x_I^C) is not BF. Then, there exists $\underline{\delta} < 1$ such that the Markovian strategy $x_i = x_i^C, i \in \{S, I\}$ in the states with all firms active, and $x_i = x_i^M$ in states where only firm i is active constitutes a MPE if and only if $\delta \leq \underline{\delta}$.*

Proof. Deviations in states at which only one firm is active are not profitable because the active firm is producing the monopoly outcome and the other firm is out of the market. Thus, only deviations at states with both firms active are possible. Note that if both firms are active and produce Cournot outputs, profits for firm i are $\pi_i^C/(1-\delta)$. Given that the Cournot output profile is not BF, a potential profitable deviation is such that one firm drives the other to bankruptcy without bankrupting itself. The discounted profits for this move are $\pi_i^D + \delta\pi_i^M/(1-\delta)$, where π_i^M are monopoly profits and π_i^D are profits in the deviation for firm i . Firm i drives firm j to bankruptcy by producing an outcome $\hat{x}_i > x(x_j^C)$, where $x(x_j^C)$ is such that $\pi_j(x_j^C, x(x_j^C)) = 0$. Given that $\pi_j(x_j^C, x_i^C) \geq 0$, $x(x_j^C) \geq x_i^C$, and therefore, for all $\hat{x}_i > x(x_j^C)$, $\pi_i(\hat{x}_i, x_j^C) < \pi_i(x(x_j^C), x_j^C)$. Thus, driving firm j to bankruptcy is not a profitable deviation for firm i if and only if

$$\pi_i^C \geq (1-\delta)\pi_i(x(x_j^C), x_j^C) + \delta\pi_i^M. \quad (4.1)$$

For $\delta \simeq 0$ the right hand side of 4.1 is approximately $\pi_i(x(x_j^C), x_j^C)$, and then the inequality holds because $\pi_i^C \geq \pi_i(x(x_j^C), x_j^C)$. For $\delta \simeq 1$ the right hand side of 4.1 is approximately π_i^M , and the inequality does not hold because $\pi_i^C < \pi_i^M$. Since the right hand side of 4.1 is decreasing in δ , by the intermediate value theorem there is $\underline{\delta}_i$ such that $\pi_i^C \geq (1-\delta)\pi_i(x(x_j^C), x_j^C) + \delta\pi_i^M$ if and only if $\delta \leq \underline{\delta}_i$. In conclusion, by taking $\underline{\delta} = \min\{\underline{\delta}_i\}_{i \in \{S, I\}}$, we get the result. ■

When $\delta > \underline{\delta}$, the Markovian strategy $x_i = x_i^C, i \in \{S, I\}$ in the state with all firms active, and $x_i = x_i^M$ in states where only firm i is active is not a MPE because firm S has incentives to

³Recall that if firms have constant marginal costs, the Cournot outcome is BF if (i) costs are identical or (ii) they are sufficiently different so that only one firm produces in equilibrium.

bankrupt firm I . Thus, a MPE may involve mixed strategies. We start by a general observation that will be useful later on.

Lemma 5. *For any pure strategy SPNE, no firm goes bankrupt.*

Proof. Suppose that firm i goes bankrupt in some period t , which happens only if its profit in t is negative. Since the profits after bankruptcy are always zero, the i 's continuation profit at t is zero. However, producing nothing at t and at any following periods, assures zero profits, so firm i can profitably deviate by choosing $x_i^t = 0$ at t . Thus we derive contradiction. ■

Note that Lemma 5 holds even when strategies are not constrained to be Markovian. The following lemma shows that when the repeated Cournot outcome cannot be a MPE equilibrium, no MPE equilibrium in pure strategies exists when δ is large.

Lemma 6. *For any $\delta > \underline{\delta}$, there is no MPE in pure strategies.*

Proof. Given no bankruptcy occurs in an equilibrium in pure strategies (Lemma 5), the repeated Cournot outcome when both firms are active is a unique mutual best reply in pure strategies that are Markovian. However, it cannot be an equilibrium for $\delta > \underline{\delta}$ by Lemma 4. ■

In light of Lemma 6, we study equilibria in mixed strategies when the infinite repetition of the Cournot output cannot be an equilibrium, i.e., $\delta > \underline{\delta}$. The existence of a MPE is guaranteed by an extension of a theorem proved in Dasgupta and Maskin (1986) which we leave in the Appendix.

Proposition 2. *For any δ , there exists at least one MPE, possibly, in mixed strategies.*

The characterization of the mixed strategy equilibrium is not an easy task. To characterize the equilibrium support, we impose two assumptions.

Assumption AC The average cost for each firm is constant.

Assumption EC Each firm i 's expected per-period profit function (given other firm's mixed strategy) is strictly concave in x_i for any $x_i \geq 0$.

Let $E[\pi_i(x_i, x_j) \mid \sigma_j]$ be the expected per-period profit of firm i given that firm j is using the mixed strategy σ_j .

Proposition 3. *Under assumptions AC and EC, and $\delta > \underline{\delta}$, a mixed strategy MPE must satisfy the following conditions:*

- (i) Firm S randomizes over $x_S^0 \cup (\underline{x}_S^*, \overline{x}_S^*]$ where $x_S^0 < \underline{x}_S^*$.
- (ii) Firm I randomizes over $(\underline{x}_I^*, \overline{x}_I^*) \cup \overline{x}_I^*$.
- (iii) $\overline{x}_I^* = \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$.
- (iv) $x_S^0 = \arg \max_{x_S} E[\pi_S(x_I, x_S) \mid \sigma_I]$.
- (v) $\pi_I(\underline{x}_I^*, \overline{x}_S^*) = 0$.
- (vi) $\pi_I(\overline{x}_I^*, \underline{x}_S^*) = 0$.
- (vii) $\pi_I(\overline{x}_I^*, x_S^0) > 0$.

The proof of Proposition 3 is long and involved and it is left to the Appendix. We highlight here some of the properties. i) The support of each firm's mixed strategy contains exactly one interval. And the superior firm there is a unique mass point which coincides with the best reply in the static game to the inferior firm's mixed strategy. ii) The inferior firm becomes bankrupt with positive probability in each period (so monopolization occurs with positive probability), thus, the inferior firm eventually goes bankrupt with probability one. As output increases, the probability of bankruptcy increases but is never one. This reflects that the possibility of bankruptcy makes the inferior firm prudent when choosing outputs. iii) For the superior firm, the mass point is isolated and lies strictly below the interval support. Since the superior firm would not produce a larger output than the best reply unless bankruptcy occurs, outputs in the interval support reflect the predatory activities of the superior firm.

In the following proposition we give the characterization of the probability distribution on the support for each firm under the assumption that demand is linear.

Proposition 4. *Suppose that AC and EC holds and demand is linear, $p(X) = a - X$. Let c_I and c_S the marginal cost of I and S respectively, let $a_I = a - c_I$ and $a_S = a - c_S$. For any $\delta > \underline{\delta}$, a mixed strategy MPE is such that:*

- (i) Firm S randomizes according to a mixture cdf $p_S + (1 - p_S)F_S(x_S)$ with support $x_S^0 \cup (\underline{x}_S^*, \overline{x}_S^*]$

where $F_S(x_S)$ is a cdf on $(\underline{x}_S^*, \overline{x}_S^*]$ and p_S is the probability of producing x_S^0 with

$$F'_S(x_S) = \frac{2(x_S - \underline{x}_S^*)}{(\overline{x}_S^* - \underline{x}_S^*)^2}, \text{ and} \quad (4.2)$$

$$p_S = 1 - \frac{(1 - \delta)(\overline{x}_S^* - \underline{x}_S^*)^2}{\delta(\overline{x}_S^* - 2\underline{x}_S^* + a_I)(a_I - \overline{x}_S^*)} \quad (4.3)$$

(ii) Firm I randomizes according to a mixture cdf $p_I + (1 - p_I)G_I(x_I)$ with support $(\underline{x}_I^*, \overline{x}_I^*) \cup \overline{x}_I^*$ where $G_I(x_I)$ is a cdf on $(\underline{x}_I^*, \overline{x}_I^*)$ and p_I is the probability of producing \overline{x}_I^* with

$$G'_I(x_I) = \frac{2(x_S^0 - (a_I - x_I))}{(\overline{x}_I^* - \underline{x}_I^*) \left(\overline{x}_I^* + \underline{x}_I^* - 2a_I + 2x_S^0 \right)}, \text{ and} \quad (4.4)$$

$$p_I = 1 - \frac{(\overline{x}_I^* - \underline{x}_I^*) \left(\overline{x}_I^* + \underline{x}_I^* - 2a_I + 2x_S^0 \right)}{\delta \left[(2x_S^0 - a_I + \underline{x}_I^*)(a_I - \underline{x}_I^*) - \pi_S^M \right]}. \quad (4.5)$$

where $\pi_S^M = \left(\frac{a_S}{2}\right)^2$

(iii) Finally, $x_S^0, \underline{x}_S^*, \overline{x}_S^*, \underline{x}_I^*, \overline{x}_I^*$ are the solutions of the following system:

$$2\underline{x}_S^* - a_I = p_S x_S^0 + (1 - p_S) \int_{\underline{x}_S^*}^{\overline{x}_S^*} x_S F'_S(x_S) dx_S \quad (4.6)$$

$$a_S - 2x_S^0 = p_I \overline{x}_I^* + (1 - p_I) \int_{\underline{x}_I^*}^{\overline{x}_I^*} x_I G'_I(x_I) dx_I \quad (4.7)$$

$$\frac{1}{1 - \delta} (x_S^0)^2 = (2x_S^0 - \overline{x}_S^*) \overline{x}_S^* + \delta \frac{\pi_S^M}{1 - \delta} \quad (4.8)$$

$$\overline{x}_I^* = a_I - \underline{x}_S^* \quad (4.9)$$

$$\underline{x}_I^* = a_I - \overline{x}_S^* \quad (4.10)$$

The proof of Proposition 4 is left to the Appendix. We comment here the content of each of the equations that define the support of the mixed strategy for each firm. Equations (4.9) and (4.10) follows from (v) and (vi) in Proposition 3. The right hand side of equation (4.6) is the expected value of the outputs in the support of firm S 's mixed strategy. This expected value has to be equal to $2\underline{x}_S^* - a_I$ because of two reasons: first, by (iii) in Proposition 3, $\overline{x}_I^* = \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$, since demand is linear, \overline{x}_I^* is the solution of $a_I - 2\overline{x}_I^* - E x_S = 0$. Secondly, by (vi) in Proposition 3, $\overline{x}_I^* = a_I - \underline{x}_S^*$. Thus, $E x_S = 2\underline{x}_S^* - a_I$. The right hand side of equation (4.7) is the expected value of the outputs in the support of firm I 's mixed strategy. By (iv) in Proposition 3, $x_S^0 = \arg \max_{x_S} E[\pi_S(x_I, q_S) \mid \sigma_I]$, since demand is linear, x_S^0 is the solution of $a_S - 2x_S^0 - E x_I = 0$.

Thus, $Ex_I = a_S - 2x_S^0$. The left hand side of equation (4.8) is the expected payoff of firm S at (x_S^0, σ_I) . By (vii) in Proposition 3, $\pi_I(\bar{x}_I^*, x_S^0) > 0$, thus, when firm S produces x_S^0 , the probability of bankruptcy for firm I is zero, and consequently the payoff of firm S is $(a_S - x_S^0 - Ex_I)x_S^0/(1 - \delta) = (x_S^0)^2/(1 - \delta)$. The right hand side of equation (4.8) is the expected payoff of firm S at (\bar{x}_S^*, σ_I) . Since by (v) in Proposition 3, $\pi_I(x_I^*, \bar{x}_S^*) = 0$, the probability of bankruptcy for firm I is 1. Thus, the payoff at (\bar{x}_S^*, σ_I) is $\pi_S(\bar{x}_S^*, \sigma_I) + \delta\pi_S^M/(1 - \delta)$. Finally, since both x_S^0 and \bar{x}_S^* are in the support of S 's mixed strategy, the payoffs are equal.

Finally, we highlight that when $\delta- > \underline{\delta}$, $p_I = 1$, $p_S = 1$, $x_S^0 = x_S^C$, $\bar{x}_I^* = x_I^C$, $\underline{x}_I^* = \bar{x}_I^*$, $\underline{x}_S^* = \bar{x}_S^* = a_I - x_I^C$, is a solution of the system.

Example 2. We cannot say much in general given the complicated expressions that define the equilibrium. But in the following examples we can show how the system evolve with δ , and we can also see the consequences on the best collusive outcome that can be supported using the MPE strategy as trigger.

Example 3. In this example we compute the MPE for $\delta \in (\underline{\delta}, 1)$ and highlight some of its properties. Demand and costs are linear $a_I = 100$, and $a_S = 125$. In this case, $\underline{\delta} \approx 0.31$. The table below shows how the support of the mixed strategy changes with δ , for $\delta > \underline{\delta}$.

| δ | \underline{x}_I^* | \bar{x}_I^* | x_S^0 | \underline{x}_S^* | \bar{x}_S^* |
|----------|---------------------|---------------|---------|---------------------|---------------|
| 0.4 | 19.48 | 23.83 | 50.89 | 76.17 | 80.52 |
| 0.5 | 13.66 | 21.36 | 52.63 | 78.64 | 86.34 |
| 0.6 | 8.24 | 17.98 | 54.68 | 82.02 | 91.76 |
| 0.7 | 3.71 | 13.22 | 56.95 | 86.78 | 96.29 |
| 0.8 | 1.06 | 7.43 | 59.27 | 92.57 | 98.94 |
| 0.9 | 0.17 | 2.87 | 61.16 | 97.13 | 99.83 |
| 0.99 | 0.0012 | 0.23 | 62.39 | 99.77 | 99.9988 |

We see that the length of both supports becomes smaller when δ increases, so, in a way, the behavior of both firms becomes much more predictable when future counts heavily. As expected, the support for firm S increases with δ and the support for firm I decreases with δ . This reflects the strategic superiority of firm S as predation is more profitable when the future counts. Finally, when $\delta- > 1$, the firm S becomes totally dominant and set $x_S^0 = x_S^M = 62.50$.

In the following table we show how p_I , the probability of producing $\overline{x_I^*}$, and p_S , the probability of producing x_S^0 changes with δ .

| δ | p_I | p_S |
|----------|-------|-------|
| 0.4 | 0.73 | 0.95 |
| 0.5 | 0.60 | 0.85 |
| 0.6 | 0.54 | 0.72 |
| 0.7 | 0.57 | 0.54 |
| 0.8 | 0.70 | 0.31 |
| 0.9 | 0.87 | 0.14 |
| 0.999 | 0.99 | 0.04 |

We see that the probability to produce x_S^0 decreases with δ showing how the predation outputs in $[x_S^*, \overline{x_S^*}]$ become dominant when the future counts heavily. In contrast, the probability of producing $\overline{x_I^*}$ is not monotonic on δ .

In the following table we show how the expected output changes with δ .

| δ | Ex_I | Ex_S |
|----------|--------|--------|
| 0.4 | 23.22 | 52.34 |
| 0.5 | 19.74 | 57.28 |
| 0.6 | 15.64 | 64.04 |
| 0.7 | 11.1 | 73.56 |
| 0.8 | 6.46 | 85.14 |
| 0.9 | 2.68 | 94.26 |
| 0.99 | 0.22 | 99.54 |

We see that the fear to be predated makes firm I less and less important when predation becomes more and more profitable (large δ) which, in turn, makes room for firm S to be more and more prominent.

Next we compute the probability that both firms survive, p_1 .

| | |
|----------|-------|
| δ | p_1 |
| 0.4 | 0.95 |
| 0.5 | 0.87 |
| 0.6 | 0.77 |
| 0.7 | 0.61 |
| 0.8 | 0.38 |
| 0.9 | 0.18 |
| 0.99 | 0.04 |

Again we see that despite the shrinking of firm I this firm is bankrupt with a larger and larger probability when δ increases reflecting the increased profitability of predation. This is reflected in the next table which shows payoffs in the Markov equilibrium.

| | | | |
|----------|---------------------------|---------------------------|--------|
| δ | $V_I(\sigma_I, \sigma_S)$ | $V_S(\sigma_S, \sigma_I)$ | |
| 0.4 | 914.911 | 4316.3 | |
| 0.5 | 793.919 | 5539.83 | |
| 0.6 | 571.032 | 7474.57 | |
| 0.7 | 281.094 | 10810.5 | (4.11) |
| 0.8 | 73.14 | 17564.3 | |
| 0.9 | 9.469 | 37401.5 | |
| 0.99 | 0.055 | 389197 | |

We continue by focusing on the consequences of bankruptcy on the maximum collusive outcome (the one that maximizes joint profits $\pi_S(x_S, x_I) + \pi_I(x_I, x_S)$) when grim trigger strategies are used and $\delta > \underline{\delta}$.

Let $T_S(x_S, x_I)$ be the set of outputs that fulfill the grim trigger strategy constraint for firm S namely

$$T_S(x_S, x_I) = \{(x_S, x_I) \in \mathbb{R}_+^2 \mid \pi_S(x_S, x_I) \geq (1 - \delta)\pi_S^D(x_I) + \delta\pi_S^R\}$$

where $\pi_S^D(x_I)$ is the maximum profit that can be obtained when the inferior firm is producing x_I and π_S^R is the profits in the reversion. This reversion is the profit corresponding to the Cournot equilibrium output when $\delta \leq \underline{\delta}$ and for larger δ is the discounted profit corresponding to the mixed

strategy equilibrium, that is $(1 - \delta)V_S(\sigma_I, \sigma_S)$. The set $T_I(x_I, x_S)$ is defined similarly, namely

$$T_I(x_I, x_S) = \{(x_S, x_I) \in \mathbb{R}_+^2 \mid \pi_I(x_I, x_S) \geq (1 - \delta)\pi_I^D(x_S) + \delta\pi_I^R\}$$

where $\pi_I^D(x_S)$ is the maximum profit that can be obtained when the superior firm is producing x_S and π_I^R is the profit corresponding to the Cournot equilibrium output when $\delta \leq \underline{\delta}$ and for larger δ is the discounted profit corresponding to the mixed strategy equilibrium, that is $(1 - \delta)V_I(\sigma_I, \sigma_S)$

Let $N(x_S, x_I)$ be the set of outputs that do not encourage predation, namely

$$N(x_S, x_I) = \{(x_S, x_I) \in \mathbb{R}_+^2 \mid \pi_S(x_S, x_I) \geq (1 - \delta)\pi_S^K(x_I) + \delta\pi_S^M\},$$

where $\pi_S^K(x_I)$ is the supremum of profits when the inferior firm is bankrupted. Let

$$A(x_S, x_I) = (T_S(x_S, x_I) \cap T_I(x_I, x_S) \cap N(x_S, x_I))$$

The set $A(x_S, x_I)$ encapsulates all the relevant constraints for joint profit maximization defined as

$$\max \pi_S(x_S, x_I) + \pi_I(x_I, x_S) \text{ subject to } (x_S, x_I) \in A(x_S, x_I). \quad (4.12)$$

Let (x_S^J, x_I^J) be the solution to the above problem.⁴ Finally, let ∂B be the boundary of an arbitrary set B . Now we assume the following

Assumption G. a) $T_S(x_S, x_I) \cap N(x_S, x_I)$ is convex. b) $\pi_S(x_S, x_I) + \pi_I(x_I, x_S)$ is strictly concave.

Assumption G is standard in mathematical programming. It is fulfilled when demand and costs are linear.

Proposition 5. *Let us assume G and constant return to scale. Then $(x_S^J, x_I^J) \in \partial T_I(x_I, x_S)$.*

Proof. Suppose (x_S^J, x_I^J) belongs to the interior of $T_I(x_I, x_S)$. Consider a linear convex combination of (x_S^J, x_I^J) and $(x_S^M, 0)$, $(\alpha x_S^J + (1 - \alpha)x_S^M, \alpha x_I^J)$ $\alpha \in (0, 1)$. By our assumption of constant returns to scale, $(x_S^M, 0)$ maximizes joint profits without constraints and by definition, $(x_S^M, 0) \in T_S(x_S, x_I) \cap N(x_S, x_I)$. Thus $(\alpha x_S^J + (1 - \alpha)x_S^M, \alpha x_I^J) \in T_S(x_S, x_I) \cap N(x_S, x_I)$ for any $\alpha \in (0, 1)$ because $T_S(x_S, x_I) \cap N(x_S, x_I)$ is convex. And choosing α small enough

⁴Note that despite the use of mixed strategies in the reversion, we do not use it in the objective function. This is because this function is strictly concave so any mixed strategy is payoff dominated by a pure strategy which is the combination of the strategies in the support of the mixed strategy.

$(\alpha x_S^J + (1-\alpha)x_S^M, \alpha x_S^J) \in T_I(x_I, x_S)$. Thus $(\alpha x_S^J + (1-\alpha)x_S^M, \alpha x_S^J) \in A(x_S, x_I)$ for some $\alpha \in (0, 1)$. By strict concavity of joint profits $(\alpha x_S^J + (1-\alpha)x_S^M, \alpha x_S^J)$ yields larger profits than (x_S^J, x_I^J) . Contradiction.

This result may help us to compute the maximum joint profit that solves (4.12). For a moment, let us forget constraints other than $T_I(x_I, x_S)$ and solve the following maximization program

$$\max \pi_S(x_S, x_I) + \pi_I(x_I, x_S) \text{ subject to } \pi_I(x_I, x_S) = (1-\delta)\pi_I^D(x_S) + \delta\pi_I^R. \quad (4.13)$$

To check that this solution solves (4.12), it suffices to plug the solutions into the constraints corresponding to no predation and the trigger strategy of the superior firm. If both constraints are satisfied, we have found the solution to the constrained joint profit maximization. In this case, the possibility of bankruptcy only affects achievable joint profits throughout π_I^R .

Note that the above result can be applied also to the standard case (Friedman (1971)) considering the feasible set $A(x_S, x_I) = T_S(x_S, x_I) \cap T_I(x_I, x_S)$ and requiring just the convexity of $T_S(x_S, x_I)$.

Going back to Example 3 we can compare the total payoff in the maximum collusive outcome in both cases, the standard one and our case.

Example 4. *In the following table we show the maximum joint profits under bankruptcy considerations (left part of the table, named B), and without bankruptcy considerations (right part of the table, named NB).*

| | δ | $\pi_I + \pi_S$ | δ | $\pi_I + \pi_S$ | |
|----------|----------|-----------------|----------|-----------------|-----------|
| | 0.4 | 3437.29 | 0.4 | 3412.59 | |
| | 0.5 | 3549.8 | 0.5 | 3437.5 | |
| <i>B</i> | 0.6 | 3667.24 | 0.6 | 3457.6 | <i>NB</i> |
| | 0.7 | 3775.84 | 0.7 | 3474.68 | |
| | 0.8 | 3845.61 | 0.8 | 3489.47 | |
| | 0.9 | 3881.07 | 0.9 | 3502.45 | |
| | 0.99 | 3903.92 | 0.99 | 3512.88 | |

We see that in our case joint profits supported by grim trigger strategies are larger than those in the standard case. In order to understand it, we construct the profit possibility frontier, Note that

from the definitions of profits of both firms we obtain that

$$\pi_S = (a_S - x_I - (a_I - x_I - \frac{\pi_I}{x_I}))(a_I - x_I - \frac{\pi_I}{x_I}) = (25 + \frac{\pi_I}{x_I})(100 - x_I - \frac{\pi_I}{x_I}).$$

Maximizing joint profits for given π_I we obtain

$$x_I = \sqrt[3]{\frac{1}{25}\pi_I^2 + \sqrt{\pi_I^3 + \frac{1}{625}\pi_I^4}} - \frac{\pi_I}{\sqrt[3]{\frac{1}{25}\pi_I^2 + \sqrt{\pi_I^3 + \frac{1}{625}\pi_I^4}}}.$$

Plugging this into the previous expression we obtain the maximum profit possibility frontier, which is the solid line in Figure 1

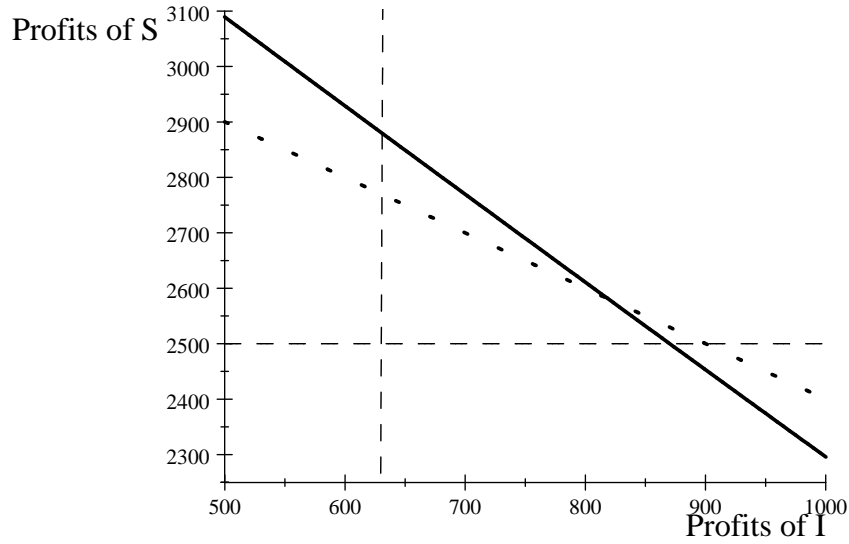


Figure 1

Cournot equilibrium profits are (2500, 625). These are the dotted lines in Figure 1. Iso joint profits are pictured in dots. We can see that the maximum joint profit supportable by trigger strategies would be at 3514 which correspond at $\pi_I = 625$. In the standard infinitely repeated game, this would be achievable as a SPNE when $\delta \approx 1$. For $\delta = 0$, only the Cournot equilibrium profits are achievable. For intermediate values of δ , we can imagine a set that is getting larger with δ and that allows larger and larger joint profits because higher iso joint profits are achievable. Given $\delta > \underline{\delta}$, let $(\bar{\pi}_I, \bar{\pi}_S)$ be the profits associated to the maximum collusive outcome for the standard case, and let $(\hat{\pi}_I, \hat{\pi}_S)$ the ones obtained in our case. Using table 4.11 and a simple computation,

we can see that $\pi_I^C > (1 - \delta)V_I$. By Proposition 5,

$$\begin{aligned}\bar{\pi}_I &= (1 - \delta)\pi_I^D(x_S) + \delta\pi_I^C, \text{ and} \\ \hat{\pi}_I &= (1 - \delta)\pi_I^D(x_S) + \delta(1 - \delta)V_I(\sigma_I, \sigma_S),\end{aligned}$$

therefore $\bar{\pi}_I > \hat{\pi}_I$. Given the profit possibility frontier in Figure 1 and the iso joint profits, $\hat{\pi}_I + \hat{\pi}_S > \bar{\pi}_I + \bar{\pi}_S$.

5. Equilibrium with Increasing Average Cost and Patient Firms

The folk theorem of repeated games states that when firms are sufficiently patient, arbitrary feasible profits larger than the minimax can be obtained as the average profit of an SPNE of the repeated game. Thus a natural question is to ask what kinds of profits can be supported as an SPNE in our model for sufficiently patient firms. This section is devoted to this task under Assumption 1 (see Section 3). We concentrate here on pure-strategy equilibria.

We first see that, for sufficiently patient firms (δ close to one), any SPNE of the dynamic game yields *BF* action profiles in each period. This result is independent on both demand and costs conditions. Denoting monopoly profits for firm i as π_i^M , we have the following:

Proposition 6. *Consider a sequence of output profiles that constitute a SPNE such that each firm's profit in every period is strictly bounded away from monopoly profits (i.e., there exists a $\varepsilon > 0$ such that $\varepsilon \leq \pi_i^M - \pi_i(\mathbf{x}^t)$ for both i and all t). Then, there exist a large discount factor such that each period's output profile is *BF*.*

Proof. Suppose that in period t , (x_S^t, x_I^t) is not *BF*. Thus, one firm can bankrupt the other. Suppose, without loss of generality, that the superior firm can bankrupt the inferior one. Consider the following strategy for firm S . In period t , firm S produces an output \tilde{x}_S that drives firm I into bankruptcy, and produces the monopoly output thereafter. The continuation profit for firm S is

$$(1 - \delta)(\pi_S(\tilde{x}_S, x_I^t) + \delta\pi_S^M + \delta^2\pi_S^M + \dots). \quad (5.1)$$

The continuation profit at t for the sequence $((x_S^1, x_I^1), \dots, (x_S^t, x_I^t), \dots)$ is:

$$(1 - \delta)(\pi_S(\mathbf{x}^t) + \delta\pi_S(\mathbf{x}^{t+1}) + \delta^2\pi_S(\mathbf{x}^{t+2}) + \dots). \quad (5.2)$$

By the definition of an SPNE,

$$\pi_S(\mathbf{x}^t) + \delta\pi_S(\mathbf{x}^{t+1}) + \delta^2\pi_S(\mathbf{x}^{t+2}) + \dots \geq \pi_S(\tilde{x}_S, x_I^t) + \delta\pi_S^M + \delta^2\pi_S^M + \dots \quad (5.3)$$

or

$$\pi_S(\mathbf{x}^t) - \pi_S(\tilde{x}_S, x_I^t) \geq \delta(\pi_S^M - \pi_S(\mathbf{x}^{t+1})) + \delta^2(\pi_S^M - \pi_S(\mathbf{x}^{t+2})) + \dots \geq \delta\varepsilon + \delta^2\varepsilon + \dots = \delta\frac{\varepsilon}{1-\delta}. \quad (5.4)$$

Clearly, when $\delta \rightarrow 1$, the above inequality is impossible, contradicting that we were in an SPNE. ■

The condition that $\pi_i(\mathbf{x}^t) + \varepsilon \leq \pi_i^M$ is satisfied, for instance, for stationary sequences. This result shows that when firms are sufficiently patient, incentives for predation are high so firms only choose BF allocations in equilibrium.⁵

In what follows we characterize the profit that can be supported as a SPNE of the dynamic game for sufficiently patient firms. For this purpose, we adapt the standard definition of a minimax profit to the case in which outputs are constrained to be BF.

Assumption 1 guarantees that the set of BF output profiles is not empty and, as we have shown in Lemma 3, is characterized as $BF = \{(x_S, x_I) \mid 0 \leq x_i \leq \bar{x}_i \text{ for } i \in \{S, I\}\}$, where $\bar{\mathbf{x}} = (\bar{x}_S, \bar{x}_I)$ with $\bar{x}_i \neq 0$ for all $i \in \{S, I\}$ is such that $\pi_i(\bar{\mathbf{x}}) = 0$ for all $i \in \{S, I\}$. **The minimax BF profit for firm i** is defined as:

$$\pi_{im} = \min_{x_j \in [0, \bar{x}_j]} \max_{x_i \in [0, \bar{x}_i]} \pi_i(x_i, x_j) = \max_{x_i \in [0, \bar{x}_i]} \pi_i(x_i, \bar{x}_j). \quad (5.5)$$

Note that, since $\pi_i(\bar{\mathbf{x}}) = 0$, $\pi_{im} > 0$ because at (\bar{x}_i, \bar{x}_j) firm i by reducing his output gets positive profits. The standard minimax, when applied to our model, yields a minimax profit of zero because firm $j \neq i$ could produce an output, call it x_j , such that the best reply of i is to produce zero. But x_j might not fulfill the definition of minimax BF profits because it might drive firm j to bankruptcy.

In the standard approach, when firms minimax each other there might experience huge losses. In our setting these losses kill firms and so the possibilities of punishment are reduced with respect to the standard framework. Thus the set of outputs yielding more profits than the minimax BF is smaller than the set of outputs yielding non-negative outputs.

The next proposition shows that, for a sufficiently large δ , no SPNE of the dynamic game can give any firm a profit lower than its minimax BF profit.

⁵This result can not be extended to n firms. The difficulty is that, after a firm is bankrupted, the strategies of the surviving firms can be anything. An example of this is obtainable under request from the authors.

Proposition 7. *Under Assumption 1, $\delta' \in (0, 1)$ exists such that for all $\delta \in (\delta', 1)$, $\pi_i < \pi_{im}$ cannot be supported in any SPNE.*

Proof. For each $i \in \{S, I\}$, let $\delta^i \in (0, 1)$ be such that $\delta^i \pi_i^M = \pi_{im}$ where π_i^M is the monopoly profits for firm i and π_{im} is the minimax BF profit. Since $\pi_i^M > \pi_{im}$, δ^i exists. Let $\delta' = \max_{i \in N} \delta^i$ and let $\delta \in (\delta', 1)$. Suppose that $\pi_i < \pi_{im}$ is supported as an SPNE for $\delta \in (\delta', 1)$. If $x_j^t \in [0, \bar{x}_j]$ for all t on and off the equilibrium path, firm i could have achieved at least π_{im} irrespective of δ by choosing an output $x_i^t \in [0, \bar{x}_i]$ (the standard argument in repeated games can be applied here because in this case the output profile at each t is in the BF set). Therefore, if $\pi_i < \pi_{im}$ happens in equilibrium, $x_j^t > \bar{x}_j$ must hold for some t either on or off the equilibrium path. We show that if this is the case, the continuation profit for i at t in equilibrium, Π_i^t , must be such that $\Pi_i^t \geq \delta \pi_i^M$. Suppose that $\Pi_i^t < \delta \pi_i^M$; since $x_j^t > \bar{x}_j$, firm i can make firm j bankrupt retaining non-negative profits, and can achieve a monopoly profit in every period from $t + 1$. Although the bankruptcy of firm j has a cost at period t , the continuation profit for firm i if it deviates from equilibrium will be at least $\delta \pi_i^M$. However, if $\delta \pi_i^M > \Pi_i^t$ such a deviation would be profitable for firm i and would contradict the notion that we are in equilibrium. Therefore, $\Pi_i^t \geq \delta \pi_i^M$. Since $\delta \in (\delta', 1)$, $\Pi_i^t > \pi_{im}$. Thus, π_i must exceed π_{im} which concludes the proof. ■

Note that when δ is very small, firms may have little incentives to engage in predatory activities and allocations which are not BF might be supported as an SPNE. For instance, if $\delta = 0$ only the profits corresponding to the Cournot equilibrium can be supported as an SPNE, but Cournot equilibrium outputs may be not BF (see Figure 2).

We are now ready to prove a folk theorem regarding BF allocations. We say that π_i is an **individually rational BF profit** if $\pi_i > \pi_{im}$. An individually rational BF vector profit $(\pi_i)_{i \in \{S, I\}}$ is **feasible** if a BF output profile (x_i, x_j) exists such that $\pi_i = \pi_i(x_i, x_j)$ for all $i, j \in \{S, I\}$, $i \neq j$. In Figure 2 the BF output profiles that give an individually rational BF profit are the ones in the area limited by the minimax BF isoprofits.

Proposition 8. *Suppose Assumption 1 holds. Let $\pi = (\pi_i)_{i \in \{S, I\}}$ be a feasible and individually rational BF profit vector. Then, $\delta' \in (0, 1)$ exists such that for all $\delta \in (\delta', 1)$, π is the average profits in some SPNE.*

Proof. The proof is given by constructing an equilibrium which is originally proposed by Fudenberg and Maskin (1986). Let $(\pi_i)_{i \in \{S, I\}}$ be feasible and individually rational BF profit vector.

By the definition of feasibility, there is a BF output profile (x_i, x_j) such that $\pi_i = \pi_i(x_i, x_j)$ for $i, j \in \{S, I\}$, $i \neq j$.

Suppose each firm $i \in \{S, I\}$ produces output x_i in each period if no deviation has occurred, but both $i \in \{S, I\}$ produce \bar{x}_i , for T periods once one of them unilaterally deviates from the equilibrium path. If no one deviates during these T periods, then firms go back to the original path. Otherwise, if one of them deviates, then firms restart this phase for T more periods. We prove that this strategy constitutes an SPNE.

First consider a deviation from the equilibrium path. Suppose firm i produces $x'_i \neq x_i$ in some period, say period t . By the one-stage-deviation principle (e.g. Fudenberg and Tirole, 1991, p.110), a deviation is profitable if and only if firm i could profit by deviating from the original strategy in period t only and conforming thereafter. Therefore, firm i can benefit by deviation if and only if x'_i exists such that

$$(1 - \delta)\pi_i(x'_i, x_j) + (1 - \delta)(\delta + \dots + \delta^T)\pi_i(\bar{x}_i, \bar{x}_j) + \delta^{T+1}\pi_i > \pi_i, \quad (5.6)$$

or equivalently,

$$(1 - \delta)\pi_i(x'_i, x_j) + \delta^{T+1}\pi_i > (1 - \delta)(1 + \delta + \dots + \delta^T)\pi_i + \delta^{T+1}\pi_i, \quad (5.7)$$

which it holds whenever:

$$(1 - \delta)\{\pi_i(x'_i, x_j) - \pi_i\} - (\delta + \dots + \delta^T)\pi_i > 0. \quad (5.8)$$

Let $\Delta_i = \max_{x'_i} \pi_i(x'_i, x_j) - \pi_i$ and choose T such that $\Delta_i < T\pi_i$. Note that the left hand side of (5.8) is weakly less than $(1 - \delta)\{\Delta_i - (\delta + \dots + \delta^T)\pi_i\}$. This term is non-positive when δ is close to 1. Therefore, (5.8) cannot be satisfied for such T .

By the same argument as above, firm i can benefit by deviating from the mutual minmax phase if and only if x''_i exists such that

$$\begin{aligned} & (1 - \delta)\pi_i(x''_i, \bar{x}_j) + (1 - \delta)(\delta + \dots + \delta^T)\pi_i(\bar{x}_i, \bar{x}_j) + \delta^{T+1}\pi_i \\ & > (1 - \delta)(1 + \delta + \dots + \delta^{T-1})\pi_i(\bar{x}_i, \bar{x}_j) + \delta^T\pi_i, \end{aligned} \quad (5.9)$$

which can be written as:

$$\pi_i(x''_i, \bar{x}_j) > \delta^T\pi_i. \quad (5.10)$$

Note that $\pi_i(x_i'', \bar{x}_j) \leq \max_{x_i \in [0, \bar{x}_i]} \pi_i(x_i, \bar{x}_j) = \pi_{im}$. Since $\pi_i > \pi_{im}$ by assumption, (5.10) never holds when δ is close to 1.

Thus there is no profitable deviation when δ is sufficiently close to 1. Since π is an arbitrary feasible and individually rational BF profit vector, the proof is complete. ■

6. Final Remarks

In this paper we have developed a theory of dynamic competition in which firms may bankrupt other firms. We have shown that this theory provides new insights into the theory of dynamic games. Cournot may not constitute a Markovian equilibrium. When this is the case, collusive outcomes supported as an SPNE by grim-trigger strategies without bankruptcy considerations may not be supported now, because in those outcomes, the superior firm have incentives to predate. For sufficiently high δ the Markov equilibrium involves mixed strategies and predation occurs with positive probability. Finally, we have shown limit results, a folk theorem kind of result. Collusion is more difficult to sustain than in standard supergames and, in particular, not every individually rational profit can be supported by a SPNE.

Our results are obtained at the cost of making several simplifying assumptions to make the model tractable. Here we discuss some of the issues arising from these simplifications.

No accumulation

In this paper we focused on outputs that make other firms bankrupt, but we did not consider the funds that might support or deter aggressive strategies (the "deep pocket" argument). Our research strategy is to analyze the incentives to prey in the simplest possible case where no funds can be accumulated. A full fledged model of accumulation and predation is, no doubt about it, preferable but it is beyond the scope of our paper. In other cases, accumulation of profits might play an important role in shaping the SPNE set as in the model of Rosenthal and Rubinstein (1984).⁶

Credit

If credit is given on the basis of past performance, the redefinition of the BF set can be applied here and credits can be incorporated into the model. However, if credit is given on the basis of

⁶They characterize a subset of the Nash equilibria in the repeated game with no discounting (i.e., $\delta = 1$) where each player regards ruin of the other player as the best possible outcome and his own ruin as the worst possible outcome.

future performance, future performance also depends on credit (via the BF constraints), which makes this problem extremely complex. This points to a deep conceptual problem about credit in oligopolistic markets where firms might be made bankrupt. This topic should be the subject of future research.

Entry

In this paper we assumed that the disappearance of a firm does not bring a new one into the market. Of course this should not be taken literally. What we mean is that if entry does not quickly follow, it makes sense, as a first approximation, to analyze the model with a given number of firms. For instance it can be shown that when firms are very patient and costs and demand are linear, ruining a firm is a good investment even if monopoly lasts for one period. In other cases, though, the nature of equilibria will be altered if, for instance, entry immediately follows the ruin of a competitor as in the model of Rosenthal and Spady (1989).⁷

Buying Competitors

In our model, there is no option to buy a firm. Sometimes it is argued that buying an opponent may be a cheaper and safer strategy than ruining it. We do not deny that buying competitors plays an important role in business practices. However, we do not agree that under the option of buying, ruining a competitor is irrational. First, buying competitors may be forbidden by a regulatory body because of anticompetitive effects. Second, when the owner of a firm sells it to competitors, this does not stop her from creating a new firm and financing it with the money received from selling the old one. In other words, selling a firm is not equivalent to a contract in which the owner commits not to enter into a market again. Thus, bankruptcy may be the only credible way of getting rid of a competitor. Finally, buying and ruining competitors may complement each other because the acquisition value may depend on the aggressiveness of the buyer in the past; see Burns (1986) for some evidence in the American tobacco industry. Thus, it seems that a better understanding of the mechanism of ruin might help to further enhancement of our understanding of how the buying mechanism works in this case.

Summing up, the model presented in this paper sheds some light on certain aspects of the equilibrium in oligopolistic markets in which firms may make each other bankrupt. We hope that

⁷They consider a prisoner's dilemma in continuous time in a market with room for two firms only. When a firm goes bankrupt, this firm is immediately replaced by a new entrant. They show that some kind of predatory behavior can arise in equilibrium.

the insights obtained here can be used in further research in this area.

7. APPENDIX

Proof of Proposition 2. By Lemma 4, the existence of MPE is established for $\delta \leq \underline{\delta}$. So, in the following proof, we only consider cases in which $\delta > \underline{\delta}$.

There are 4 possible states in our dynamic game. Namely, the set of active firms is (1) S and I, (2) S (3) I, and (4) empty.⁸ Since there is a single active firm (monopoly) in states (2) and (3), that firm simply chooses the monopoly output. In (4), no strategic decision can be made. In this way, for any MPE, Markovian strategies in states (2), (3) and (4) are uniquely pinned down, and continuation profits in those states are derived accordingly. Note also that state transition is irreversible. If a state changes from (1) to any other one among (2) to (4), it is impossible for players to move this back to (1). Moreover, the continuing game in MPE must stay there forever; no further transition can occur.

Since equilibrium Markovian strategies in (2), (3) and (4) are derived above, we focus on Markovian strategies in the remaining state (1) where both firms are active. Let σ_S and σ_I be a (mixed) strategy in (1) for S and I , respectively. Note that, depending on realized pure actions of σ_S and σ_I , state transition, i.e., bankruptcy, may occur. Let $p_s(\sigma_S, \sigma_I)$, $s = 1, 2, 3, 4$, be a probability such that state (s) would realize when firms play σ_S and σ_I . Then, continuation profit in state (1) for each firm, denoted by V_i , $i = S, I$, is expressed as follows.

$$V_S(\sigma_S, \sigma_I) = \pi_S(\sigma_S, \sigma_I) + \delta \left[p_1(\sigma_S, \sigma_I)V_S(\sigma_S, \sigma_I) + p_2(\sigma_S, \sigma_I)\frac{\pi_S^M}{1-\delta} \right] \quad (7.1)$$

$$V_I(\sigma_S, \sigma_I) = \pi_I(\sigma_S, \sigma_I) + \delta \left[p_1(\sigma_S, \sigma_I)V_I(\sigma_S, \sigma_I) + p_3(\sigma_S, \sigma_I)\frac{\pi_I^M}{1-\delta} \right] \quad (7.2)$$

where π_i^M is a monopoly profit of firm $i = S, I$. Solving each equation, we obtain the following.

$$V_S(\sigma_S, \sigma_I) = \frac{1}{1-\delta p_1(\sigma_S, \sigma_I)} \left[\pi_S(\sigma_S, \sigma_I) + \delta p_2(\sigma_S, \sigma_I)\frac{\pi_S^M}{1-\delta} \right] \quad (7.3)$$

$$V_I(\sigma_S, \sigma_I) = \frac{1}{1-\delta p_1(\sigma_S, \sigma_I)} \left[\pi_I(\sigma_S, \sigma_I) + \delta p_3(\sigma_S, \sigma_I)\frac{\pi_I^M}{1-\delta} \right] \quad (7.4)$$

⁸Given our assumption that firm S goes bankrupt only if firm I goes bankrupt, state (3) never realizes. Such property, to some extent, simplifies the following proof, but is not needed to establish the existence result. To clarify this point, we do not impose this assumption in what follows.

Note that $V_i, i = S, I$ is discontinuous in pure actions (x_S, x_I) , since the probability of state realization, $p_s, s = 1, 2, 3, 4$, is discontinuous in (x_S, x_I) . To establish the existence of MPE, it is necessary and sufficient to show that a Markovian strategy profile (σ_S, σ_I) constitutes an SPNE among all Markovian strategies.⁹ In our dynamic game, deviation in states (2) and (3) cannot be profitable for a remaining active firm, and no deviation is possible in (4). Therefore, without loss of generality, we can focus on deviation in state (1) alone. This implies that an MPE of our dynamic game is identical to a Nash equilibrium of a static game in which the payoff function of each firm is set equal to $V_i(\sigma_S, \sigma_I), i = S, I$. We shall denote this (modified) static game by G .

Fortunately, Theorem 5b in Dasgupta and Maskin (1986) (D&M hereinafter) can be invoked to show the existence of a mixed strategy Nash equilibrium of G . Roughly speaking, the existence is guaranteed when utility functions are bounded and continuous except in a set of measure zero in the (joint) pure strategy space. More precisely, the theorem requires that (a) discontinuities occur in a set whose dimension is strictly lower than the dimension of the strategy space, (b) strategy sets are intervals, and (c) when we approach a discontinuity if a firm profit falls, another rises. Our game clearly satisfies (b). Since profit function, $\pi_i, i = S, I$, is continuous in (x_S, x_I) , discontinuity possibly occurs only at a point in which $p_s, s = 1, 2, 3, 4$, is discontinuous, or a state transition occurs. Our bankruptcy conditions assure that such state transitions occur in a set whose dimension is strictly lower than the dimension of the strategy space, and hence (a) is also satisfied. Therefore, we only need to check (c).

Given the irreversibility of state transition mentioned above, there are only 4 possible cases of discontinuities of p_s (or V_i): (i) $p_1 = 1 \rightarrow p_2 = 1$, (ii) $p_1 = 1 \rightarrow p_3 = 1$, (iii) $p_2 = 1 \rightarrow p_4 = 1$, and (iv) $p_3 = 1 \rightarrow p_4 = 1$. In (iii), firm I is not active in both before and after the state transition from (2) to (4) occurs. That is, $V_I = 0$ in both (2) and (4), and hence the condition (c) is trivially satisfied. By the symmetric argument, (c) is also satisfied in (iv). In (i) and (ii), when the profit of one firm falls (because this firm goes bankrupt), the profit of the other firm must rise (because the remaining firm becomes a monopolist). So, (c) is satisfied in cases (i) and (ii) as well.

We have shown that G satisfies all the conditions of D&M, and hence the existence of a mixed strategy Nash equilibrium of G is guaranteed. Since this equilibrium is essentially identical to the

⁹Necessity is obvious. To understand sufficiency, note that, if the opponent uses Markovian strategy, the player always has a best reply that is Markovian as well. See, for example, pp.501 of Fudenberg and Tirole (1991) for more detailed discussion.

MPE of our dynamic game, the existence of an MPE is also guaranteed. ■

Characterization of the support of the Markovian mixed strategy equilibrium.

The Proof of Proposition 3 follows from the following lemmas.

Lemma 7. *For any mixed strategy of the other firm, the optimal output that maximizes a firm's expected profit in a period (i.e. $E[\pi_i(x_i, x_j) \mid \sigma_j]$) is always unique.*

Proof. From Assumption EC, $\arg \max_{x_i > 0} E[\pi_i(x_i, x_j) \mid \sigma_j]$ is unique (if it exists), and i 's optimal output is either $\arg \max_{x_i > 0} E[\pi_i(x_i, x_j) \mid \sigma_j]$ or 0. Assumption AC implies that the expected profit of the former is always positive, so it cannot be the case that both become optimal. ■

Lemma 8. *Given $\delta > \underline{\delta}$, for any equilibrium in mixed strategies,*

- (i) *at least one firm goes bankrupt with strictly positive probability,*
- (ii) *both firm use totally mixed strategies.*

Proof. (i) Since $\delta > \underline{\delta}$, at least one firm is using a totally mixed strategy. Suppose without loss of generality that is firm j and suppose that no firm goes bankrupt. Pick any two outputs x, x' from the support of the mixed strategy of firm j . Then, the firm must be indifferent between choosing x and x' . However, given that no bankruptcy occurs, the firm's optimal output (to the other firm's equilibrium strategy) is always unique by Lemma 7. Thus we get a contradiction.

(ii) Suppose on the contrary that firm i uses pure strategy x_i (and j uses mixed strategy σ_j). Then, firm j does not go bankrupt in equilibrium, since choosing such an output in the support of j is clearly suboptimal. Let $x < x'$ be two different outputs in the support of σ_j . Lemma 7 implies that, in order for j to be indifferent between x and x' , bankruptcy must occur in either output. Since π_i is decreasing in x_j , i must go bankrupt under x' but not under x . Moreover, the support of σ_j cannot contain a point other than x and x' (since choosing such an output cannot yield the same profit as x and x' do). However, given that j randomizes only over the two points x and x' , x must be best reply to x_i in the one period game, since bankruptcy does not occur in such case. Then, j has incentive to set x' as small as possible to make i going bankrupt. But since the set of outputs that make i bankrupt is open, such a minimum output does not exist. In this way, (x_i, σ_j) cannot be a mutual best reply. ■

Let us denote the inferior and the superior outputs in the support of the equilibrium (mixed) strategy for firm i by \underline{x}_i and \overline{x}_i , respectively. By Lemma 8 (ii), we have $\underline{x}_i < \overline{x}_i$ for each i .

Lemma 9. *In equilibrium, the following condition must hold for every firm i :*

$$\pi_i(\overline{x}_i, \underline{x}_j) > 0.$$

Proof. Suppose $\pi_i(\overline{x}_i, \underline{x}_j) \leq 0$. By choosing $x_i = \overline{x}_i$, firm i always receives non-positive profit, and strictly negative profit when $x_j > \underline{x}_j$ (note $\pi_i(x_i, x_j)$ is decreasing in x_j). Since the latter case occurs with positive probability, i would always become strictly better off by choosing $x_i = 0$. ■

Lemma 10. *In equilibrium, the following condition must hold for at least one firm:*

$$\pi_i(\underline{x}_i, \overline{x}_j) > 0.$$

Proof. Suppose on contrary that $\pi_i(\underline{x}_i, \overline{x}_j) \leq 0$ and $\pi_j(\underline{x}_j, \overline{x}_i) \leq 0$. Combining with Lemma 9, the following inequalities must hold.

$$\pi_i(\overline{x}_i, \underline{x}_j) > 0 \text{ and } \pi_j(\underline{x}_j, \overline{x}_i) \leq 0 \Rightarrow AC_i(\overline{x}_i) < p(\overline{x}_i + \underline{x}_j) \leq AC_j(\underline{x}_j), \quad (7.5)$$

$$\pi_j(\overline{x}_j, \underline{x}_i) > 0 \text{ and } \pi_i(\underline{x}_i, \overline{x}_j) \leq 0 \Rightarrow AC_j(\overline{x}_j) < p(\overline{x}_j + \underline{x}_i) \leq AC_i(\underline{x}_i). \quad (7.6)$$

Since average costs are constant, the above conditions imply

$$p(\overline{x}_j + \underline{x}_i) \leq AC_i(\underline{x}_i) = AC_i(\overline{x}_i) < p(\overline{x}_i + \underline{x}_j), \quad (7.7)$$

$$p(\overline{x}_i + \underline{x}_j) \leq AC_j(\underline{x}_j) = AC_j(\overline{x}_j) < p(\overline{x}_j + \underline{x}_i), \quad (7.8)$$

which is an obvious contradiction. ■

Lemma 11. *If $\pi_i(\underline{x}_i, \overline{x}_j) > 0$ holds, then*

(i) \underline{x}_i maximizes the expected per period profit given j 's mixed strategy, $E[\pi_i(x_i, x_j) \mid \sigma_j]$,

(ii) \underline{x}_i must be isolated from other part of the support of i 's equilibrium strategy.

Proof. Since $\pi_j(\overline{x}_j, \underline{x}_i) > 0$ by Lemma 9, j 's profit $\pi_j(x_j, \underline{x}_i)$ is strictly positive for any $x_j \in [0, \overline{x}_j]$. This implies that probability such that j goes bankrupt is 0 when i chooses output sufficiently close to \underline{x}_i . Therefore, (i) \underline{x}_i must be optimal given that no bankruptcy occurs, and (ii) no output close to \underline{x}_i can be contained in i 's equilibrium support. ■

Lemma 12. *In equilibrium, firm I never produces strictly higher output than the one which maximizes its per-period profit given firm S 's equilibrium mixed strategy. That is,*

$$\bar{x}_I \leq \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S].$$

Proof. Note first that I can make S bankrupt only if I itself goes bankrupt. Hence, firm I can never be better off by bankrupting firm S . Choosing $x_I > \bar{x}_I$ weakly increases the risk of bankruptcy and strictly reduces π_I in that period. Therefore, it must be suboptimal. ■

Lemma 13. *The following conditions must hold:*

(i) $\pi_I(\underline{x}_I, \bar{x}_S) \leq 0$.

(ii) $\pi_S(\underline{x}_S, \bar{x}_I) > 0$.

Proof. We first verify (i). Suppose on the contrary that $\pi_I(\underline{x}_I, \bar{x}_S) > 0$ holds. Then, by Lemma 11, I must choose a strictly larger output than $\arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$, which contradicts Lemma 12. Given that (i) holds, (ii) must be satisfied by Lemma 10. ■

Lemma 14. *Let $x_S^0 = \arg \max_{x_S} E[\pi_S(x_S, x_I) \mid \sigma_I]$. The equilibrium support of firm S 's mixed strategy is such that $\underline{x}_S = x_S^0$.*

Proof. By Lemma 13, $\pi_S(\underline{x}_S, \bar{x}_I) > 0$. Thus, by Lemma 11, \underline{x}_S maximizes S 's expected profit (per period) given I 's mixed strategy and it is an isolated point. Therefore, $\underline{x}_S = x_S^0$.

Lemma 15. *In equilibrium, firm I goes bankrupt with positive probability.*

Proof. By Lemma 8 at least one firm goes bankrupt. If I does not go bankrupt then firm S is bankrupt, but this is impossible because whenever firm S is bankrupt firm I is also bankrupt. ■

Lemma 16. *For all x_I in I 's mixed strategy support, $\pi_I(x_I, \bar{x}_S) \leq 0$.*

Proof. This follows immediately from Lemma 13. ■

Lemma 17. *The probability of bankruptcy at the infimum of the support of I 's mixed strategy, \underline{x}_I , is zero. There is no other output in the support with zero probability of bankruptcy.*

Proof. Suppose on the contrary that the probability that I goes bankrupt at \underline{x}_I is positive. This probability is the probability that the superior firm produces $x_S \in (\tilde{x}_S, \overline{x}_S]$, for \tilde{x}_S such that $p(\underline{x}_I + \tilde{x}_S) = c_I$. By Lemma 9 we know that $\pi_I(\overline{x}_I, \underline{x}_S) > 0$, consequently, $\pi_I(\underline{x}_I, \underline{x}_S) > 0$. Thus, $\tilde{x}_S \in (\underline{x}_S, \overline{x}_S]$. If \tilde{x}_S is in the support of S 's mixed strategy, firm S , by concentrating all the mass placed at $[\tilde{x}_S + \varepsilon, \overline{x}_S]$ in $\tilde{x}_S + \varepsilon$, will not change the probability of bankruptcy for firm I and will increase the per period profit of firm S . If \tilde{x}_S is not in the support of S 's mixed strategy, let $\hat{x}_S > \tilde{x}_S$ be the closest point to \tilde{x}_S in the support of S 's mixed strategy. Again, firm S , by placing all the mass placed at $[\hat{x}_S, \overline{x}_S]$ in \hat{x}_S , will not change the probability of bankruptcy for firm I and will increase its per period profit. Finally, by Lemma 12, $\overline{x}_I \leq \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$. If there is another output in the support different from \overline{x}_I with zero probability of bankruptcy, \tilde{x}_I , should be such that $\underline{x}_I < \tilde{x}_I \leq \overline{x}_I$. But then, since firm i 's expected profit function (given other firm's mixed strategy) is strictly concave in x_i , it could not be that the profit of firm I is the same at \underline{x}_I and at \tilde{x}_I . ■

Lemma 18. *The support of I 's mixed strategy contains at least an interval and the infimum of the interval is \underline{x}_I .*

Proof. Let us see first that \underline{x}_I can not be an isolated mass point. If it were, \underline{x}_I would be the minimum of the support of I 's mixed strategy. By Lemma 17 the probability of bankruptcy at \underline{x}_I is zero. Thus, $\underline{x}_I = \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$, but this contradicts Lemma 12. Thus, \underline{x}_I is not an isolated point and, for the same argument as before, \underline{x}_I can not be in the support of I 's mixed strategy. ■

Lemma 19. *The equilibrium support of firm S 's mixed strategy contains at least one interval.*

Proof. By Lemma 18 the support of I 's mixed strategy contains at least one interval, and since $\overline{x}_I \leq \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$, I 's per period profit is strictly increasing and also continuous at any x_I in the interval. Thus, the probability of bankruptcy must be increasing with x_I (in order for I to be indifferent in the interval). This is possible only when the support of S 's mixed strategy also contains an interval where the distribution of x_S does not make any jump. ■

Lemma 20. *The support of I 's mixed strategy contains a unique interval and no isolated points.*

Proof. Suppose on the contrary that it contains two intervals $(\underline{x}_I^1, \bar{x}_I^1)$ and $(\underline{x}_I^2, \bar{x}_I^2)$ with $\underline{x}_I^1 < \bar{x}_I^1 < \underline{x}_I^2 < \bar{x}_I^2$, and that there is no isolated outputs between the intervals. Let $x_S(\bar{x}_I^1)$ and $x_S(\bar{x}_I^2)$ be such that $\pi_I(\bar{x}_I^1, x_S(\bar{x}_I^1)) = \pi_I(\bar{x}_I^2, x_S(\bar{x}_I^2)) = 0$. Note first that it is not optimal for firm S to have in the support of its mixed strategy outputs in $(x_S(\bar{x}_I^2), x_S(\bar{x}_I^1))$, because otherwise firm S , by concentrating all the mass of the interval $(x_S(\bar{x}_I^2), x_S(\bar{x}_I^1))$ in $x_S(\bar{x}_I^2)$, will not change the probability of bankruptcy of firm I and will increase its per period profit. If there are isolated points between the two intervals, we can replicate the same argument by considering the biggest isolated point and the greatest interval. Without loss of generality we can consider just the two intervals without isolated points in the middle. Let $x_I = \bar{x}_I^1 + \varepsilon$ with ε sufficiently small. Since there is no mass in $(x_S(\bar{x}_I^2), x_S(\bar{x}_I^1))$, the probability of bankruptcy at $\bar{x}_I^1 + \varepsilon$ is the same that at \bar{x}_I^1 but the per period payoff is bigger. So it can not be optimal for firm I to produce \bar{x}_I^1 . Thus, there is only one interval in the support of I 's mixed strategy. The same argument can be applied in order to discard the isolated points in the support of I 's mixed strategy ■

Lemma 21. *At equilibrium the following should hold:*

- (i) $\pi_I(\underline{x}_I, \bar{x}_S) = 0$;
- (ii) $\pi_I(\bar{x}_I, \underline{x}_S^1) = 0$, where \underline{x}_S^1 is the infimum of the first interval in the support of S 's mixed strategy.

Proof. (i) Lemma 17 implies $\pi_I(\underline{x}_I, \bar{x}_S) \geq 0$. By Lemma 13 $\pi_I(\underline{x}_I, \bar{x}_S) \leq 0$. Thus, $\pi_I(\underline{x}_I, \bar{x}_S) = 0$.

(ii) Suppose that $\pi_I(\bar{x}_I, \underline{x}_S^1) < 0$. By Lemmas 9 and 14, $\pi_I(\bar{x}_I, x_S^0) > 0$. Then, $x_S(\bar{x}_I)$ defined by $\pi_I(\bar{x}_I, x_S(\bar{x}_I)) = 0$ is such that $x_S^0 < x_S(\bar{x}_I) < \underline{x}_S^1$. This implies that for ε sufficiently small, the probability of bankruptcy at any $x_I \in (\bar{x}_I - \varepsilon, \bar{x}_I)$ does not change. But then, I can not be indifferent between any two outputs in the interval $(\bar{x}_I - \varepsilon, \bar{x}_I)$. Thus, $\pi_I(\bar{x}_I, \underline{x}_S^1) \geq 0$. Suppose that $\pi_I(\bar{x}_I, \underline{x}_S^1) > 0$, then, at any $x_S \in (\underline{x}_S^1, x_S(\bar{x}_I))$, the probability of bankruptcy for I is the same, but then, firm S cannot be indifferent in the interval $(\underline{x}_S^1, \underline{x}_S^1 + \varepsilon)$. Thus, $\pi_I(\bar{x}_I, \underline{x}_S^1) = 0$. ■

Lemma 22. *At equilibrium, $\bar{x}_I = \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$.*

Proof. Suppose that $\bar{x}_I < \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$. By Lemmas 9 and 14, $\pi_I(\bar{x}_I, x_S^0) > 0$ and by Lemma 21, $\pi_I(\bar{x}_I, \underline{x}_S^1) = 0$. At an output $x_I = \bar{x}_I + \varepsilon$ with ε sufficiently small such that $\bar{x}_I + \varepsilon < \arg \max_{x_I} E[\pi_I(x_I, x_S) \mid \sigma_S]$, the probability of bankruptcy is the same than at \bar{x}_I , but the

per period payoff is greater. So \bar{x}_I can not be best reply to σ_S . Thus, $\bar{x}_I = \arg \max_{x_I} E[\pi_I(x_I, x_S) | \sigma_S]$. ■

Lemma 23. *The support of S 's mixed strategy contains a unique interval and one isolated point.*

Proof. By Lemma 14 $\underline{x}_S = x_S^0$ and is isolated. By Lemma 21, more than one interval in the support of S 's mixed strategy would imply that some outputs in the support of I 's mixed strategy produce the same probability of bankruptcy for firm I . But, if this is the case, firm I can not be indifferent among those outputs. The same kind of argument can be applied to discard any isolated point bigger than the supremum of the interval. Finally, given that $\pi_I(\bar{x}_I, \underline{x}_S^1) = 0$, if there are two isolated points smaller than the infimum of the interval then none of those outputs can bankrupt firm I , but then firm S can not be indifferent among them. ■

Lemma 24. *The infimum of the interval in the support of firm S 's mixed strategy is not in the support.*

Proof. Let \underline{x}_S^* the infimum of the interval in the support. Since by Lemma 21 $\pi_I(\bar{x}_I, \underline{x}_S^*) = 0$, by producing \underline{x}_S^* the probability of bankruptcy for firm I is zero. But this probability is also zero by producing x_S^0 (the isolated point in S 's support which is the minimum output in the support). But then, firm S cannot be indifferent between x_S^0 and \underline{x}_S^* . ■

Those Lemmas complete the proof of Proposition 3.

Proof of Proposition 4

We start by showing that if demand is linear, (v) and (vi) in Proposition 3 implies that $\bar{x}_I^* = a_I - \underline{x}_S^*$ and $\underline{x}_I^* = a_I - \bar{x}_S^*$. If \underline{x}_I^* is greater than zero, the implication is direct. Suppose that $\underline{x}_I^* = 0$ and $a_I \neq \bar{x}_S^*$. Suppose that $\bar{x}_S^* > a_I$. At any x_S in a neighborhood of \bar{x}_S^* , the probability of bankruptcy for firm I is 1, but then firm S can not be indifferent among all the x_S in that neighborhood. Suppose that $\bar{x}_S^* < a_I$, and let $\tilde{x}_I \in (0, \bar{x}_I^*)$ be such that $\tilde{x}_I = a_I - \bar{x}_S^*$, since $\pi_I(\bar{x}_I^*, \underline{x}_S^*) = 0$, \tilde{x}_I exists and is greater than zero. At any $x_I \in (0, \tilde{x}_I)$ the probability that firm I is bankrupt is zero, but then firm I can not be indifferent among all $x_I \in (0, \tilde{x}_I)$.

Let (σ_I, σ_S) a mixed strategy profile. The payoffs for each firm given (σ_I, σ_S) are:

$$\begin{aligned}
V_I(\sigma_I, \sigma_S) &= \frac{1}{1 - \delta p_1} \pi_I(\sigma_I, \sigma_S); \\
V_S(\sigma_I, \sigma_S) &= \frac{1}{1 - \delta p_1} \left[\pi_S(\sigma_I, \sigma_S) + \delta p_2 \frac{\pi_S^M}{1 - \delta} \right]
\end{aligned}$$

where p_1 is the probability that both firms survive given that strategy and p_2 is the probability that only S survive.

Let us first prove (i). For firm I , every x_I in the support of σ_I has to be a best reply to σ_S . Thus for each x_I in the support,

$$\frac{\partial V_I(x_I, \sigma_S)}{\partial x_I} = 0,$$

or equivalently

$$\frac{\delta \frac{\partial p_1(x_I, \sigma_S)}{\partial x_I}}{(1 - \delta p_1)^2} \pi_I(x_I, \sigma_S) + \frac{1}{1 - \delta p_1} \frac{\partial \pi(x_I, \sigma_S)}{\partial x_I} = 0 \quad (7.9)$$

Given (x_I, σ_S) , let us compute first the probability that both firms survive, $p_1(x_I, \sigma_S)$. By the characterization of the support in Proposition 3, $\pi_I(\underline{x}_I^*, \overline{x}_S^*) = 0$ and $\pi_I(\overline{x}_I^*, \underline{x}_S^*) = 0$. This implies that for each x_I in the support of I , the infimum output that bankrupt firm I , $(a_I - x_I)$, is in the support of firm S mixed strategy. Thus,

$$\begin{aligned}
p_1(x_I, \sigma_S) &= P(x_S \leq a_I - x_I) = p_S + (1 - p_S) \int_{\underline{x}_S^*}^{a_I - x_I} F'_S(x_S) dx_S, \text{ and} \\
\frac{\partial p_1(x_I, \sigma_S)}{\partial x_I} &= -(1 - p_S) F'_S(a_I - x_I).
\end{aligned}$$

Furthermore,

$$\frac{\partial \pi(x_I, \sigma_S)}{\partial x_I} = (a_I - 2x_I - Ex_S)$$

where Ex_S is the expected value of the outputs in σ_S support. Let $F(x_S) = (1 - p_S)F_S(x_S)$. Thus, expression (7.9) can be written as

$$\frac{-\delta F'(a_I - x_I)}{(1 - \delta p_1)^2} \pi(x_I, \sigma_S) + \frac{1}{1 - \delta p_1} \frac{\partial \pi(x_I, \sigma_S)}{\partial x_I} = 0,$$

or equivalently

$$\frac{-\delta F'(a_I - x_I)}{1 - \delta p_1} (a_I - x_I - Ex_S)x_I + (a_I - 2x_I - Ex_S) = 0 \quad (7.10)$$

Given that for each x_I , $a_I - x_I \in (\underline{x}_S^*, \overline{x}_S^*)$, and for each $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$, there is $x_I \in (\underline{x}_I^*, \overline{x}_I^*)$

such that $a_I - x_I = x_S$, (7.10) can be expressed in terms of $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$,

$$\frac{-\delta F'(x_S)}{1 - \delta p_S - \delta F(x_S)}(x_S - Ex_S)(a_I - x_S) + (2x_S - a_I - Ex_S) = 0 \quad (7.11)$$

Let $H(x_S) = 1 - \delta p_S - \delta F(x_S)$, and $f(x_S) = (x_S - Ex_S)(a_I - x_S)$. Note that $H'(x_S) = -\delta F'(x_S)$, and $f'(x_S) = a_I - x_S - x_S + Ex_S = -(2x_S - a_I - Ex_S)$. Using this transformation, (7.11) can be written as:

$$\frac{H'(x_S)}{H(x_S)}f(x_S) - f'(x_S) = 0$$

Thus, for $H(x_S) = Kf(x_S)$ the above equation will hold. Which will implied that

$$1 - \delta p_S - \delta F(x_S) = K(x_S - Ex_S)(a_I - x_S), \text{ and} \quad (7.12)$$

$$\delta F'(x_S) = K(2x_S - a_I - Ex_S) \quad (7.13)$$

By (iii) in Proposition 3, $\overline{x}_I^* = \arg \max_{x_I} \pi_I(x_I, \sigma_S)$. Therefore, \overline{x}_I^* has to be the solution of the first order condition

$$a_I - 2\overline{x}_I^* - Ex_S = 0.$$

By point (vi) in Proposition 3, $\pi_I(\overline{x}_I^*, \underline{x}_S^*) = 0$, which implies that $\overline{x}_I^* = a_I - \underline{x}_S^*$. Thus,

$$Ex_S = 2\underline{x}_S^* - a_I \quad (7.14)$$

Which implies that

$$\delta F'(x_S) = 2K(x_S - \underline{x}_S^*) \quad (7.15)$$

Furthermore,

$$1 - p_S = \int_{\underline{x}_S^*}^{\overline{x}_S^*} F'(x_S) dx_S.$$

Thus,

$$\frac{K}{\delta}(\overline{x}_S^* - \underline{x}_S^*)^2 = 1 - p_S. \quad (7.16)$$

By (7.15), (7.16) and given that $(1 - p_S)F'_S(x_S) = F'(x_S)$,

$$F'_S(x_S) = \frac{2(x_S - \underline{x}_S^*)}{(\overline{x}_S^* - \underline{x}_S^*)^2}, \quad (7.17)$$

which proves the first part of (i). For the second part, substituting p_S and Ex_S in (7.12) using (7.14) and (7.16) we get

$$\begin{aligned} 1 - \delta + K(\overline{x_S^*} - \underline{x_S^*})^2 - \delta F(x_S) &= K(x_S - 2\underline{x_S^*} + a_I)(a_I - x_S) \\ F(x_S) &= \frac{1}{\delta} - 1 + \frac{K}{\delta}(\overline{x_S^*} - \underline{x_S^*})^2 - \frac{K}{\delta}(x_S - 2\underline{x_S^*} + a_I)(a_I - x_S) \end{aligned}$$

Given that $F(\overline{x_S^*}) = \frac{K}{\delta}(\overline{x_S^*} - \underline{x_S^*})^2$,

$$\frac{1}{\delta} - 1 - \frac{K}{\delta}(\overline{x_S^*} - 2\underline{x_S^*} + a_I)(a_I - \overline{x_S^*}) = 0.$$

Which implies that K should be

$$K = \frac{1 - \delta}{(\overline{x_S^*} - 2\underline{x_S^*} + a_I)(a_I - \overline{x_S^*})}$$

Thus, by (7.16)

$$p_S = 1 - F(\overline{x_S^*}) = 1 - \frac{(1 - \delta)(\overline{x_S^*} - \underline{x_S^*})^2}{\delta(\overline{x_S^*} - 2\underline{x_S^*} + a_I)(a_I - \overline{x_S^*})} \quad (7.18)$$

Secondly, we prove (ii). For firm S , every x_S in the support of σ_S has to be a best reply to σ_I . In particular, for each $x_S \in (\underline{x_S^*}, \overline{x_S^*})$

$$\frac{\partial V_S(x_S, \sigma_I)}{\partial x_S} = 0,$$

or equivalently,

$$\frac{\delta \frac{\partial p_1(x_S, \sigma_I)}{\partial x_S}}{(1 - \delta p_1)} \left[\pi_S(x_S, \sigma_I) + \delta p_2 \frac{\pi_S^M}{1 - \delta} \right] + \left[\frac{\partial \pi(x_S, \sigma_I)}{\partial x_S} + \delta \frac{\pi_S^M}{1 - \delta} \frac{\partial p_2(x_S, \sigma_I)}{\partial x_S} \right] = 0 \quad (7.19)$$

By the (v) and (vi) in Proposition 3, $\pi_I(\underline{x_I^*}, \overline{x_S^*}) = 0$ and $\pi_I(\overline{x_I^*}, \underline{x_S^*}) = 0$. This implies that for each $x_S \in (\underline{x_S^*}, \overline{x_S^*})$ in the support of S , the infimum output for firm I that bankrupt firm I is $x_I = a_I - x_S$, and $a_I - x_S \in (\underline{x_I^*}, \overline{x_I^*})$. Thus, for $x_S \in (\underline{x_S^*}, \overline{x_S^*})$ the probability that both firms survive, $p_1(x_S, \sigma_I)$ is

$$p_1(x_S, \sigma_I) = P(x_I \leq a_I - x_S) = (1 - p_I) \int_{\underline{x_I^*}}^{a_I - x_S} G'_I(x_I) dx_I, \text{ and} \quad (7.20)$$

$$\frac{\partial p_1(x_S, \sigma_I)}{\partial x_S} = -(1 - p_I) G'_I(a_I - x_S). \quad (7.21)$$

Given $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$ the probability that only firm S survives (or equivalently, the probability that firms S bankrupt firm I) is

$$p_2(x_S, \sigma_I) = P(x_I > a_I - x_S) = p_I + (1 - p_I) \int_{a_I - x_S}^{\overline{x}_I^*} G'_I(x_I) dx_I \text{ and} \quad (7.22)$$

$$\frac{\partial p_2(x_S, \sigma_I)}{\partial x_S} = (1 - p_I) G'_I(a_I - x_S) \quad (7.23)$$

Let $G(x_I) = (1 - p_I)G_I(x_I)$. Plugging (7.20), (7.21), (7.22) and (7.23) into (7.19) we get

$$\frac{-\delta G'(a_I - x_S)}{1 - \delta G(a_I - x_S)} \left[\pi_S(x_S, \sigma_I) + \delta p_2 \frac{\pi_S^M}{1 - \delta} \right] + \left[\frac{\partial \pi(x_S, \sigma_I)}{\partial x_S} + \delta \frac{\pi_S^M}{1 - \delta} G'(a_I - x_S) \right] = 0, \quad (7.24)$$

For each $x_S \in (\underline{x}_S^*, \overline{x}_S^*)$,

$$\begin{aligned} \pi_S(x_S, \sigma_I) &= (a_S - x_S - Ex_I)x_S = (a_S - a_I + x_I - Ex_I)(a_I - x_I) \\ \frac{\partial \pi(x_S, \sigma_I)}{\partial x_S} &= (a_S - 2x_S - Ex_I) = (a_S - 2(a_I - x_I) - Ex_I), \end{aligned}$$

where Ex_I is the expected value of the outputs in σ_I support.

Given (??) and the transformation $G(x_S) = (1 - p_I)G_I(x_S)$, $p_2(x_S, \sigma_I) = 1 - G(a_I - x_S)$ and

$$\delta \frac{\pi_S^M}{1 - \delta} G'(a_I - x_S) \left[1 + \frac{-\delta + \delta G(a_I - x_S)}{1 - \delta G(a_I - x_S)} \right] = \frac{\delta \pi_S^M G'(a_I - x_S)}{1 - \delta G(a_I - x_S)}.$$

Thus, equation (7.24) can be expressed in terms of $x_I \in (\underline{x}_I^*, \overline{x}_I^*)$ as

$$\frac{-\delta G'(x_I)}{1 - \delta G(x_I)} \left[(a_S - a_I + x_I - Ex_I)(a_I - x_I) - \pi_S^M \right] + (a_S - 2(a_I - x_I) - Ex_I) = 0. \quad (7.25)$$

Let $J(x_I) = 1 - \delta G(x_I)$ and $g(x_I) = (a_S - a_I + x_I - Ex_I)(a_I - x_I) - \pi_S^M$. Note that $J'(x_I) = -\delta G'(x_I)$ and $g'(x_I) = -(a_S - 2(a_I - x_I) - Ex_I)$. Using this transformation, (7.25) can be written as:

$$\frac{J'(x_I)}{J(x_I)} g(x_I) - g'(x_I) = 0$$

Thus, for $J(x_I) = Bg(x_I)$ the above equation will hold. Which will implied that

$$\begin{aligned} 1 - \delta G(x_I) &= B \left[(a_S - a_I + x_I - Ex_I)(a_I - x_I) - \pi_S^M \right] \\ -\delta G'(x_I) &= B(2(a_I - x_I) - a_S + Ex_I) \end{aligned}$$

By (iv) in Proposition 3, $x_S^0 = \arg \max_{x_S} \pi_S(x_S, \sigma_I)$. Therefore, x_S^0 has to be the solution of the first order condition $a_S - 2x_S^0 - Ex_I = 0$. Thus,

$$Ex_I = a_S - 2x_S^0 \quad (7.26)$$

Which implies that

$$G(x_I) = \frac{1}{\delta} - \frac{B}{\delta} [(2x_S^0 - a_I + x_I)(a_I - x_I) - \pi_S^M], \text{ and} \quad (7.27)$$

$$\delta G'(x_I) = B2(x_S^0 - (a_I - x_I)) \quad (7.28)$$

Note that

$$1 - p_I = \int_{\underline{x}_I^*}^{\overline{x}_I^*} G'(x_I) dx_I = \frac{B}{\delta} [(a_I - \overline{x}_I^*)^2 - (a_I - \underline{x}_I^*)^2 + 2x_S^0(\overline{x}_I^* - \underline{x}_I^*)] = \quad (7.29)$$

$$= \frac{B}{\delta} (\overline{x}_I^* - \underline{x}_I^*) (\overline{x}_I^* + \underline{x}_I^* - 2a_I + 2x_S^0) \quad (7.30)$$

Since $G'_I(x_I) = G'(x_I)/(1 - p_I)$, then

$$G'_I(x_I) = \frac{2(x_S^0 - (a_I - x_I))}{(\overline{x}_I^* - \underline{x}_I^*) (\overline{x}_I^* + \underline{x}_I^* - 2a_I + 2x_S^0)} \quad (7.31)$$

which proves the first part of (ii). For the second part, since $G(\underline{x}_I^*) = 0$, and by (7.27) $G(\underline{x}_I^*) = \frac{1}{\delta} - \frac{B}{\delta} [(2x_S^0 - a_I + \underline{x}_I^*)(a_I - \underline{x}_I^*) - \pi_S^M]$,

$$B = \frac{1}{(2x_S^0 - a_I + \underline{x}_I^*)(a_I - \underline{x}_I^*) - \pi_S^M}$$

which implies that

$$p_I = 1 - \frac{(\overline{x}_I^* - \underline{x}_I^*) (\overline{x}_I^* + \underline{x}_I^* - 2a_I + 2x_S^0)}{\delta [(2x_S^0 - a_I + \underline{x}_I^*)(a_I - \underline{x}_I^*) - \pi_S^M]}. \quad (7.32)$$

Finally, to prove (iii), (4.6) follows from (7.14) (that is, $Ex_S = 2\underline{x}_S^* - a_I$); equation (4.7) follows from (7.26) (that is, $Ex_I = a_S - 2x_S^0$); equation (4.8) is obtained because if x_S^0 and \overline{x}_S^* are in the support of σ_S , $V_S(x_S^0, \sigma_I) = V_S(\overline{x}_S^*, \sigma_I)$. Since by (vii) in Proposition 3 $\pi_I(\overline{x}_I^*, x_S^0) > 0$, the probability that both firms survive, $p_1(x_S^0, \sigma_I) = 1$, and the probability that only firm S survive, $p_2(x_S^0, \sigma_I) = 0$. By (v) in Proposition 3, $\pi_I(\underline{x}_I^*, \overline{x}_S^*) = 0$. Thus, at $(\overline{x}_S^*, \sigma_I)$, $p_1(\overline{x}_S^*, \sigma_I) = 0$, and the probability that only firm S survive, $p_2(\overline{x}_S^*, \sigma_I) = 1$. Therefore,

$$V_S(x_S^0, \sigma_I) = \frac{1}{1 - \delta} \pi_S(x_S^0, \sigma_I);$$

$$V_S(\overline{x}_S^*, \sigma_I) = \pi_S(\overline{x}_S^*, \sigma_I) + \delta \frac{\pi_S^M}{1 - \delta}.$$

Since any output in the support of σ_S has to give the same payoff,

$$\begin{aligned} \frac{1}{1 - \delta} \pi_S(x_S^0, \sigma_I) &= \pi_S(\overline{x}_S^*, \sigma_I) + \delta \frac{\pi_S^M}{1 - \delta} \\ \frac{1}{1 - \delta} (a_S - x_S^0 - Ex_I)x_S^0 &= (a_S - \overline{x}_S^* - Ex_I)\overline{x}_S^* + \delta \frac{\pi_S^M}{1 - \delta} \end{aligned}$$

By (7.28), $Ex_I = a_S - 2x_S^0$. Thus

$$\frac{1}{1-\delta}(x_S^0)^2 = (2x_S^0 - \bar{x}_S^*)\bar{x}_S^* + \delta \frac{\pi_S^M}{1-\delta}.$$

Finally, (4.9) and (4.10) follows from (v) and (vi) in Proposition 3 and our remark at the beginning of this proof. ■

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