Efficient Ex Post Implementable Auctions and English Auctions for Bidders with Non-Quasilinear Preferences

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Abstract

We study efficient auction design for a single indivisible object when bidders have interdependent values and non-quasilinear preferences. Instead of quasilinearity, we assume only that bidders have positive wealth effects. Our setting nests cases where bidders are ex ante asymmetric, face financial constraints, are risk averse, and/or face ensuing risk. We give necessary and sufficient conditions for when there exists an ex post implementable and (ex post Pareto) efficient mechanism. These conditions differ between the cases where the object is a good and when the object is a bad.

In the good setting, there is an efficient ex post implementable mechanism if there is an efficient ex post implementable mechanism in a corresponding quasilinear setting. This result extends established results on efficient ex post equilibria of English auctions with quasilinearity to our non-quasilinear setting. Yet, in the bad setting (i.e. a procurement auction) there is no mechanism that has an ex post efficient equilibrium if the level of interdependence between bidders is sufficiently strong. This result holds even if bidder costs satisfy standard crossing conditions that are sufficient for efficient ex post implementation in the quasilinear setting.

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1 Introduction

Efficient auction design is a central question in mechanism design. In the private value single unit quasilinear benchmark case, the English auction has an efficient dominant strategy equilibrium. More recent research gives necessary and sufficient conditions for the English auction to have an efficient ex post equilibrium when bidders have interdependent values. Thus, there are well-understood settings where the English auction's efficient equilibrium is robust to asymmetries across bidders' beliefs and higher order beliefs.

While these notable results on English auctions show that it is robust to asymmetries in bidder beliefs, these results require strong restrictions on bidder preferences — namely, quasilinearity. Yet in many auction settings, bidders do not have quasilinear preferences, and violations of quasilinearity are frequently cited as being salient to bidders. For example, Maskin (2000) argues that financial market imperfections may result in liquidity-constrained bidders. Salant (1997) draws on his personal consulting experience to argue that financial constraints are a salient feature of how bidders determine their bids. In addition to access to credit, risk aversion and wealth effects are important features of auctions for larger items like houses.[1]

In this paper, we study the efficient auction design problem for a single indivisible unit when bidders have interdependent values. We remove the quasilinearity restriction on bidder preferences, and assume only that bidders have weakly positive wealth effects. We do not place functional form restrictions on bidder preferences, and our setting is sufficiently general to allow for asymmetric bidders who are risk averse, have financing constraints, have budgets, or face ensuing risk. Like much of the related literature on efficient auction design, we study auctions that are ex post implementable. Thus, our predictions are robust to asymmetries in bidders beliefs and higher order beliefs. Our main contribution is to provide conditions under which existing results from the literature on efficient auction design with quasilinearity can be extended to study efficient design on a more general preference domain. Interestingly, we show that the necessary and sufficient conditions for the existence of an efficient and ex post implementable auction differ between the case where the object being sold is a good or a bad (e.g. a procurement task).

Removing quasilinearity complicates the efficient auction design problem. With quasilinearity, an auction outcome is Pareto efficient if and only if the bidder with the highest value for the object wins. Hence, the space of efficient allocations is independent of bidder transfers. Yet, in our non-quasilinear setting, the presence of wealth effects imply that a losing bidder's willingness to pay for the object and the winning bidder's willingness to sell the object both

[1] Homes are often sold via auction. For example, in Melbourne, Australia an estimated 25-50% of homes are sold via auction (see Mayer (1998)).

depend on the amount they paid (or were paid) in the auction. Thus, the space of (ex post Pareto) efficient outcomes depends on both the allocation of the object and bidder transfers. While the space of efficient outcomes is qualitatively different without quasilinearity, our first result (Theorem 1) shows that we are able to extend results on the efficient and ex post implementable auctions for bidders with quasilinear preferences to the non-quasilinear setting when the object being sold is a good. The theorem shows that there is an auction with an efficient ex post equilibrium in our normal good setting if there is an auction with an efficient ex post equilibrium in a corresponding quasilinear setting. A bidder's valuation in the quasilinear setting is set equal to her willingness to pays in the non-quasilinear setting.

The existence of an efficient and ex post implementable auction in a corresponding quasilinear setting is a sufficient condition for the existence of such a mechanism in a normal good setting because positive wealth effects amplify the efficiency of the mechanism relative to the corresponding quasilinear setting. In our setting, the winning bidder feels wealthier when she wins the good and pays a price that is below her willingness to pay. The increase in the winner's perceived wealth increases her willingness to sell the good relative to her willingness to pay because she has positive wealth effects. Thus, the winner is relatively less inclined to trade with her rivals. We use this observation to show that an auction outcome that is efficient in the corresponding quasilinear setting is also efficient in our non-quasilinear. A corollary of Theorem 1 is that the English auction has an efficient ex post equilibrium in our normal good setting if bidder willingness to pays satisfy the crossing conditions established by Maskin (1992), Krishna (2003), or Birulin and Izmalkov (2011).

The implications of Theorem 1 do not extend beyond the normal good setting though. We illustrate this point by studying a procurement setting where bidders with positive wealth effects compete to win a procurement task (i.e. a normal bad). While positive wealth effects make the winning bidder less inclined to trade in the good setting, they make the winning bidder more inclined to trade in the procurement setting. In the good setting, the winning bidder feels wealthier after she wins the auction, and this increases her willingness to sell the good. The increase in the winning bidder's willingness to sell the good makes her less inclined to trade with her rival bidders after the auction. In the procurement auction setting the effect is reversed. When the winning bidder wins the auction, she is paid an amount that exceeds her reservation cost of completing the task. This makes the winning bidder feel wealthier. The increase in the winning bidder's wealth means that she is willing to offer a rival a relatively higher amount (relative to the winning bidder's reservation cost) to complete the task on her behalf, due to the positive wealth effects. This makes the winning bidder more inclined to pay one of her rivals to complete the procurement task on her behalf. In fact, we show that there are cases where there the English auction has an ex post equilibrium where the lowest cost bidder is always assigned the procurement task, but the auction outcome is inefficient. Theorem 2 formalizes this result. The theorem shows that there is no procurement auction that retains the English auction's desired incentive and efficiency properties — ex post implementability and Pareto efficiency — when the degree of interdependence in bidder preferences is sufficiently strong.

Related Literature

Prior work on efficient auction design without quasilinearity has primarily focused on private value settings. Much of this work studies either revenue maximizing auctions (see Maskin and Riley (1984), Laffont and Robert (1996), Pai and Vohra (2014), and Baisa (2017)), or bid behavior in standard auctions when bidders have private values and non-quasilinear preferences (see, for example, Matthews (1983, 1987), Che and Gale (1996, 1998)).

More toward the focus of this paper, there is also a literature that studies efficient auction design when bidders have non-quasilinear preferences. Within this literature, there are a number of papers that study auction design when bidders have a particular violation of quasilinearity — a hard budget constraint. In particular, Maskin (2000); Dobzinski, Lavi, and Nisan (2012); and Pai and Vohra (2014) all study efficient auction design when bidders have hard budgets. There is also a literature that studies efficient auction design on a general preference domain, like the one studied in this paper. Saitoh and Serizawa (2008), Morimoto and Serizawa (2015), and Baisa (2017) all provide necessary and sufficient conditions for efficient auction design in non-quasilinear settings where we do not make function form restrictions on bidder utility functions. In addition, like this paper, these three aforementioned papers study the design of auctions that are robust to asymmetries in bidder higher order beliefs and auctions that implement an ex post Pareto efficient allocation of resources.

Saitoh and Serizawa (2008) shows that the Vickrey rule is the unique mechanism that satisfies desirable incentive and efficiency properties when bidders have private values and non-quasilinear preferences. Our results show that Saitoh and Serizawa's positive implementation result does not extend to interdependent value settings, even if bidder demands satisfy single crossing conditions that ensure the existence of efficient ex post equilibria in quasilinear settings.

There are fewer papers that study auctions with the interdependent value and non-quasilinear preferences. The exceptions are Kotowski (2017), Fang and Parreiras (2002, 2003), and Hu, Matthews, and Zou (2015). Kotowski studies bid behavior in first price auctions where bidders have interdependent values and budgets. The Fang and Parreiras papers study revenue implications of information disclosure in English auctions when bidders have budgets and interdependent values. The focus of Hu, Matthews, and Zou (2015) is closer to our paper.

They study the efficiency properties of the English auctions when bidders are risk averse and face ensuing risk. In Section 5 we show that ensuing risk is nested as a particular case of our normal good setting.

The remainder of the paper proceeds as follows. After presenting a motivating example in Section 2, Section 3 introduces our formal model. Section 4 shows the relationship between ex post implementation without quasilinearity and ex post implementation in a corresponding quasilinear setting. Section 5 gives results on the normal good setting, and Section 6 then studies the procurement setting without quasilinearity.

2 Motivating Examples

We start by presenting two motivating examples that illustrate the main insights of the paper. Both examples are of English auctions. In the first example, bidders with positive wealth effects bid on a good; and in the second example, bidders with positive wealth effects bid in a procurement auction. In both cases there are ex post equilibria in which the bidder with the highest willingness to pay (or lowest cost) wins the auction. However, it is only in the first case that the equilibrium outcome is always Pareto efficient ex post. Wealth effects explain the difference between the two cases.

2.1 An English Auction for a Normal Good

Consider an English auction for a single indivisible good. There are two bidders. Bidder i has a private signal $s_i \in [0, 1]$, initial wealth of 1, and log-utility for money. Her utility is

$$s_i + \alpha s_j + ln(1 - p)$$

if she wins the good and pays p. Bidder i gets utility $ln(1 - p)$ if she does not win the good and pays price p. Note that α measures the degree of interdependence in bidders' preferences. We assume that $\alpha \in [0, 1]$. A simple calculation shows that bidder i is willing to pay $\hat{d}(s_i, s_j) = 1 - e^{-(s_i + \alpha s_j)}$ to win the good if her signal is s_i and her rival's signal is s_j.

The English auction has an ex post equilibrium where bidder i drops out when the price reaches $\hat{d}(s_i, s_i)$, which is analogous to an ex post equilibrium in a quasilinear setting, such as the one in Milgrom and Weber (1982), in which the bidder drops out at her expected valuation conditional on her opponent receiving the same signal.

The ex post equilibrium of the English auction assigns the good to the bidder with the highest willingness to pay. However, it is not immediate that the outcome is Pareto efficient. With wealth effects, the winning bidder's willingness to sell the good depends on the price she paid for the good. In this example the English auction is (ex post Pareto) efficient because there are no ex post Pareto improving trades among bidders. To see this, suppose that $s_1 > s_2$. In that case, bidder 1 wins the good and pays $\hat{d}(s_2, s_2)$ to win. A simple calculation shows that bidder 1's willingness to sell the good conditional on winning and paying $\hat{d}(s_2, s_2)$ is $e^{s_1-s_2} - e^{s_2+\alpha s_2}$.[2] A second simple calculation shows that bidder 1's willingness to sell exceeds her willingness to pay for the good $\hat{d}(s_1, s_2)$. Thus, the winning bidder's willingness sell is larger than her willingness to pay for the good, because the winning bidder has positive wealth effects.

If the winning bidder wins the good and pays exactly her willingness to pay, then the bidder's willingness to sell the good would exactly equal her willingness to pay. Yet, in the English auction, the winning bidder pays weakly less than her willingness to pay to win the good. Therefore, the auction outcome is as though the winning bidder wins the good, pays her willingness to pay for the good, and is then given a partial refund. The refund increases the bidder's willingness to sell the good, relative to her willingness to pay, because she has positive wealth effects. Since she has the highest willingness to pay before receiving the good, her willingness to sell after winning must exceed her rival's willingness to pay and there are no Pareto improving trades. This logic suggests a close connection between ex post efficiency results in quasilinear settings and ex post Pareto efficiency in a non-quasilinear setting in which bidders have positive wealth effects. Theorem 1 generalizes this intuition.

If bidders have negative wealth effects or the auctioned item is instead a bad, as in procurement, the results change. In the procurement case, there exist ex post equilibria in which the item is assigned to the bidder with the lowest cost but produce outcomes that are Pareto inefficient. The result is suggested by reversing some of the logic in the previous paragraph as we argue next.

2.2 An English Auction in a Procurement Setting with Positive Wealth Effects

Consider a similar setting where two bidders compete to win a procurement task in an English auction. Bidder i has a private signal $s_i \in [0, 1]$, initial wealth of 1, and log-utility for money. Her utility is

$$-(s_i + \alpha s_j) + ln(1 + p)$$

[2]Bidder 1 gets utility $s_1 + \alpha s_2 + ln(1 - \hat{d}(s_2, s_2))$ when she wins. Straightforward algebra shows that a bidder's willingness to sell is given by $h = e^{s_1-s_2} - e^{s_2+\alpha s_2}$, where h solves

$$s_1 + \alpha s_2 + ln(1 - \hat{d}(s_2, s_2)) = ln(1 - \hat{d}(s_2, s_2) + h).$$

if she is assigned the task and is paid p to complete the task. She gets utility $ln(1+p)$ if the task is not assigned to her but she is paid p. A simple calculation shows that bidder i's reservation cost to complete the procurement task is $c(s_i, s_j) = e^{(s_i+\alpha s_j)} - 1$ if her signal is s_i and her rival's signal is s_j.

The English auction has an ex post equilibrium in this setting. Bidder i drops out when the price falls to $c(s_i, s_i)$ and thus the bidder with the lowest cost is always assigned the good. While the English auction has an ex post equilibrium where the lowest cost bidder always wins, the auction does not necessarily satisfy ex post Pareto efficiency because the presence of wealth effects changes the winning bidder's incentive to trade with her rivals. To see this, suppose that $s_1 < s_2$. Then, bidder 1 has the lower reservation cost and is assigned the procurement task. Bidder 1 is paid $c(s_2, s_2)$ to complete the procurement task and a straightforward calculation shows that bidder 1 is then willing to offer bidder 2 up to $h = e^{(1+\alpha)s_2} - e^{s_2 - s_1}$ to complete the task on her behalf. There is an ex post Pareto improving trade between the two bidders if bidder 1's willingness to offer exceeds bidder 2's reservation cost. This occurs if

$$h = e^{(1+\alpha)s_2} - e^{s_2-s_1} > e^{s_2+\alpha s_1} - 1 = c(s_2, s_1).$$

or equivalently, if

$$e^{(1+\alpha)s_2} - e^{s_2+\alpha s_1} > e^{s_2-s_1} - 1.$$

The above condition holds for all $s_2 > s_1$ when α is sufficiently large. Thus, there are ex post Pareto improving trades where the lowest cost bidder sells the procurement task to her rival after the conclusion of the auction if the level of interdependence is sufficiently strong.

The inefficiency of the English auction is caused by the presence of wealth effects. This is most easily seen in the pure common value case where $\alpha = 1$. In that case, the above inequality holds for all $s_2 > s_1$. Thus, every outcome of the English auction is a Pareto dominated outcome. Note that in the pure common value case both bidders have the same reservation cost and the winning bidder is the bidder with the lower signal. The winning bidder is paid an amount that exceeds her reservation cost. This makes the winning bidder feel wealthier, and thus it increases the amount the winning bidder is willing to pay another bidder to complete the procurement task on her behalf. Thus, the winning bidder is willing to pay some amount that is greater than her reservation cost to have her rival complete the procurement task instead. In other words, positive wealth effects cause the winning bidder to be more inclined to trade with her rival. There is a Pareto improving trade because both bidders have the same reservation cost. Theorem 2 generalizes the intuition described in the above example to a general procurement setting where bidders have strictly positive wealth effects. Specifically, we show that when bidders have sufficiently strong interdependence, then there is no auction that satisfies individual rationality, ex post incentive compatibility, ex post Pareto efficiency, and no subsidies.

3 Model

3.1 Preferences

A seller has a single indivisible object and there are n bidders. Bidder i receives a private signal $s_i \in S_i \subset \mathbb{R}^k$.[3] For simplicity we let $S^n := \times_{i=1}^n S_i$. We assume bidders are utility maximizing and bidder i's preferences are described by utility functions u_i^x where

$$u_i^x : \mathbb{R} \times S^n \to \mathbb{R},$$

and $x \in \{0,1\}$ indicates whether the bidder wins the object or not. Bidder i gets utility $u_i^x(t,s)$ when she wins $x \in \{0,1\}$ units, receives transfer $t \in \mathbb{R}$, and bidders $1,\ldots,n$ have signals $s := (s_1,\ldots,s_n) \in S^n$.[4] We assume that a bidder's utility is strictly increasing, continuous, and differentiable in money. That is, $u_i^x(\cdot,s)$ is strictly increasing, continuous, and differentiable $\forall x \in \{0,1\},\ s \in S^n$. In addition, we assume that $u_i^x(t,\cdot)$ is continuous for all $x \in \{0,1\},\ t \in \mathbb{R}$.

The object being sold is a good if a bidder's utility increases from owning the good, holding all else equal,

$$u_i^1(t,s) > u_i^0(t,s)\ \forall i \in \{1,\ldots,n\},\ t \in \mathbb{R},\ s \in S^n.$$

When studying the good setting, we assume that bidders have finite demand for the unit. Thus, there exists a $\rho > 0$ such that

$$u_i^0(0,s) > u_i^1(-\rho,s)\ \forall i \in \{1,\ldots,n\},\ s \in S^n.$$

Similarly, we say the object being sold is a bad, as in a procurement setting, if

$$u_i^0(t,s) > u_i^1(t,s)\ \forall i = \{1,\ldots,n\},\ t \in \mathbb{R};\ s \in S^n.$$

[3] We will often study the case where a bidder's signal space is one-dimensional $S_i \subset \mathbb{R}$. Indeed, there are many impossibility results already in the quasilinear setting on efficient implementation with multi-dimensional types (see Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), Jehiel et al. (2006)). This is discussed in greater detail at the end of Section 4.

[4] Note, that it is without loss of generality to assume that a bidder begins with initial wealth normalized to zero. If a bidder has preferences $\hat{u}_i$ and initial wealth w_i, then we can define u_i has being $u_i^x(t,s) = \hat{u}_i^x(t+w_i,s)\ \forall x \in \{0,1\},\ t \in \mathbb{R},\ s \in S^n$. Alternatively, a bidder's initial wealth could be a dimension of her private information.

When studying the procurement setting, we assume that the bidder is willing to accept the bad (i.e. complete the task) for a finite amount of money. Thus, there exists a $\rho > 0$ such that

$$u_i^1(\rho, s) > u_i^0(0, s) \; \forall i \in \{1, \dots, n\}, \; s \in S^n.$$

Instead of quasilinearity, we assume that bidders have positive wealth effects. We define positive wealth effects for both the good and bad settings. In the good setting, our definition of positive wealth effects states that if a buyer is willing to buy the good for a price of p, then if we increase the bidder's initial wealth, she is still willing to buy the good for a price of p.

Definition 1. (Positive wealth effects for goods)

If the object being sold is a good, then bidders have weakly positive wealth effects if

$$u_i^1(t - p, s) \geq u_i^0(t, s) \implies u_i^1(t' - p, s) \geq u_i^0(t', s) \; \forall t' > t, \; i \in \{1, \dots, n\}, \; s \in S^n,$$

and strictly positive wealth effects if

$$u_i^1(t - p, s) \geq u_i^0(t, s) \implies u_i^1(t' - p, s) > u_i^0(t', s) \; \forall t' > t, \; i \in \{1, \dots, n\}, \; s \in S^n.$$

The definition implies that a bidder's demand for the good is weakly increasing as she becomes wealthier. We analogously define positive wealth effects in the procurement setting.

Definition 2. (Positive wealth effects in the procurement setting)

If the object being sold is a bad, then bidders have weakly positive wealth effects if

$$u_i^0(t - p, s) \geq u_i^1(t, s) \implies u_i^0(t' - p, s) \geq u_i^1(t', s) \; \forall t' > t, \; i \in \{1, \dots, n\}, \; s \in S^n,$$

and strictly positive wealth effects if

$$u_i^0(t - p, s) \geq u_i^1(t, s) \implies u_i^0(t' - p, s) > u_i^1(t', s) \; \forall t' > t, \; i \in \{1, \dots, n\}, \; s \in S^n.$$

To understand the above definition, suppose that bidder i is assigned to complete the procurement task. Bidder i is willing to offer a rival some amount of money to complete the procurement task on her behalf. Then an increase in bidder i's wealth leads to an increase in the amount of money that she is willing to offer another bidder to complete the procurement task.

It is useful to denote a bidder's willingness to pay and willingness to sell a unit of the good.[5] We let $d_i(t,s)$ be bidder i's willingness to pay for a unit if bidders $1,\dots,n$ have signals $s=(s_1,\dots,s_n)\in S^n$ and bidder i received transfers $t\in\mathbb{R}$ prior to her purchasing decision. More formally $d_i(t,s)$ is implicitly defined as

$$d_i(t,s)=d_i \text{ s.t. } u_i^1(t-d_i,s)=u_i^0(t,s).$$

We will often be interested in a bidder's willingness to pay prior to receiving any payment from the auctioneer $d_i(0,s)$. For simplicity we write this as $\hat{d}_i(s)$.

Similarly, bidder i's willingness to sell the unit is $h_i(t,s)$ where

$$h_i(t,s)=h_i \text{ s.t. } u_i^1(t,s)=u_i^0(t+h_i,s).$$

Our assumptions imply that:

1. $d_i(\cdot,s)$ and $h_i(\cdot,s)$ are continuous and differentiable.
2. In the good setting, $d_i(t,s),h_i(t,s)>0\ \forall s\in S^n,\ t\in\mathbb{R}$.
3. In the procurement setting, $0>d_i(t,s),h_i(t,s)\ \forall s\in S^n,\ t\in\mathbb{R}$.
4. If bidders have weakly positive wealth effects $|d_i(\cdot,s)|$ and $|h_i(\cdot,s)|$ are weakly increasing $\forall s\in S^n$.
5. If bidders have strictly positive wealth effects $|d_i(\cdot,s)|$ and $|h_i(\cdot,s)|$ are strictly increasing $\forall s\in S^n$.

3.2 Mechanisms

By the revelation principle, it is without loss of generality to consider direct revelation mechanisms. A direct revelation mechanism maps bidder signals to an outcome. An outcome specifies a feasible assignment of the unit and transfers. An assignment of a unit is feasible if $q\in\{0,1\}^n$ and $\sum_{i=1}^n q_i\le 1$. We let Q be the set of all feasible assignments and $\hat{Q}$ be the set of all feasible assignments where the unit is allocated to a bidder $\sum_{i=1}^n q_i=1$. A (deterministic) assignment rule q maps the profile of reported signals to a feasible assignment $q:S^n\to Q$.[6] The transfer rule t maps the profile of reported signals to transfers $t:S^n\to\mathbb{R}^n$. A direct revelation mechanism Γ consists of an assignment rule and a transfer rule.

[5] With quasilinearity, both quantities equal a bidder's valuation.

[6] We limit attention to auctions that have deterministic outcomes. This assumption is partly motivated by practical concerns — randomization is rarely used in practice. Moreover, from a theoretical perspective, prior research shows that there is no mechanism that has desirable incentive and efficiency properties when we allow for randomization. When we allow for stochastic mechanisms, efficient auction design for a single unit when bidders are non-quasilinear is equivalent to efficient auction design in a multi-unit setting. Bidders

We study mechanisms that satisfy ex post incentive compatibility (hereafter, EPIC) and the taxation principle (see Rochét (1985)) states that a bidder's payment depends on (1) whether or not she wins, and (2) her rivals' reported types if a mechanism satisfies EPIC. Thus, a bidder's payment is constant conditional on the number of units that she wins and her rivals' reports. Let $\tau_i^0(s_{-i})$ be the amount that bidder i pays when she wins no units and her rivals report types $s_{-i} \in \times_{j \neq i} S_j$. Similarly, let $\tau_i^1(s_{-i})$ be the marginal price bidder i pays to win a unit when her rivals report types s_{-i}. If the mechanism is such that bidder i always wins a unit given her rivals' reports (i.e. given $s_{-i} \in \times_{j \neq i} S_j$, $q_i(s_i, s_{-i}) = 1\ \forall s_i \in S_i$), then let $\tau_i^1(s_{-i}) = 0$. Similarly, if the mechanism is such that bidder i never wins a unit given her rivals' reports (i.e. given $s_{-i} \in \times_{j \neq i} S_j$, $q_i(s_i, s_{-i}) = 0\ \forall s_i \in S_i$), then we let $\tau_i^1(s_{-i}) = \infty$. Hence, the transfer rule is such that

$$t_i(s_i, s_{-i}) = \begin{cases} -\tau_i^0(s_{-i}) & \text{if } q_i(s_i, s_{-i}) = 0, \\ -\tau_i^0(s_{-i}) - \tau_i^1(s_{-i}) & \text{if } q_i(s_i, s_{-i}) = 1. \end{cases}$$

Mechanism Γ satisfies EPIC if truthful reporting is always a Nash equilibrium of the game where the signal realization $(s_1, \ldots, s_n)$ is common knowledge.

Definition 3. (Ex post incentive compatibility)

A mechanism Γ is ex post incentive compatible (EPIC) if for $\forall s \in S^n$, $i \in \{1, \ldots, n\}$, and $\forall s' \in S^n$ such that $s'_{-i} = s_{-i}$,

$$u_i^{q_i(s)}(t_i(s), s) \geq u_i^{q_i(s')}(t_i(s'), s).$$

A mechanism satisfies ex post individual rationality if a bidder is made no worse off by participating.

Definition 4. (Ex post individual rationality)

A mechanism Γ is ex post individually rational (IR) if $\forall s \in S^n$ and $i \in \{1, \ldots, n\}$,

$$u_i^{q_i(s)}(t_i(s), s) \geq u_i^0(0, s).$$

With quasilinearity an outcome is efficient if the bidder with the highest value gets that good. Thus, there is a generically unique efficient assignment, and the efficient auction design

have downward sloping demand for additional probability units (see Baisa (2017a)). Moreover, Baisa (2017b) shows that efficient auction design in a multi-unit setting is generally impossible even with private values. Hence, the restriction to deterministic mechanisms is a natural second-best analysis. Furthermore, related literature on efficient auctions without quasilinearity similarly restricts attention to deterministic allocation rules (see Saitoh and Serizawa (2008), Morimoto and Serizawa (2015), and Hu, Matthews, and Zou (2015)).

problem is solved by finding a transfer rule that implements the efficient assignment rule. Without quasilinearity, a bidder's demand for the object varies with her payment. Thus, the space of efficient outcomes depends on both the assignment and the transfers. Like much of the literature on efficient auction design without quasilinearity, we consider mechanisms that implement ex post Pareto efficient outcomes as defined by Holmstrom and Myerson (1983). An outcome is ex post Pareto efficient if there are no ex post Pareto improving trades among bidders. This is the same efficiency notion Saitoh and Serizawa (2008); Dobzinski, Lavi, and Nisan (2012); Hu, Matthews, and Zou (2015); and Morimoto and Serizawa (2015), among others, use to study efficient auction design without quasilinearity.

Definition 5. (Ex post Pareto Efficient)

Fix $s = (s_1, \dots, s_n)$. An outcome $(q, t) \in Q \times \mathbb{R}^n$ is ex post Pareto efficient if $q \in \hat{Q}$ and for any $(\tilde{q}, \tilde{t}) \in \hat{Q} \times \mathbb{R}^n$ such that

$$u_i^{\tilde{q}_i}(\tilde{t}_i, s) > u_i^{q_i}(t_i, s)$$

then either

$$u_j^{\tilde{q}_j}(\tilde{t}_j, s) < u_j^{q_j}(t_j, s)$$

for some $j \neq i$ or

$$\sum_{i=1}^{n} t_i > \sum_{i=1}^{n} \tilde{t}_i.$$

The definition says that an outcome is ex post Pareto efficient if any other outcome that makes one bidder strictly bidder off necessarily decreases revenue or makes another bidder strictly worse off. In the good setting it is without loss of generality to restrict attention to feasible allocations where the good is always assigned. Any outcome where the good is unassigned is Pareto dominated by an outcome where the good is assigned. Yet, in the procurement setting, we assume that efficiency requires that some bidder be assigned the object (task). That is, the auctioneer requires that the task be completed. If we do not impose this restriction in the procurement setting, we can trivialize the problem by never assigning the task. We say that a mechanism satisfies ex post Pareto efficiency (hereafter, efficiency) if the outcome $(q(s), t(s))$ is an ex post Pareto efficient outcome for all type realizations $s \in S^n$.

Note that without quasilinearity ex post Pareto efficiency is different from simply requiring that the auction assign the object to the bidder with the highest willingness to pay. Later in the paper, we show that assigning the good to the bidder with the highest willingness to pay is neither necessary nor sufficient for an auction to be ex post Pareto efficient (see Example 3 and Theorem 2 on procurement auctions).

4 A Corresponding Quasilinear Setting

In this section, we define a corresponding quasilinear setting for a direct revelation mechanism Γ. From our definition, it is simple to show that a mechanism Γ satisfies IR and EPIC in our non-quasilinear setting if and only if the mechanism Γ satisfies IR and EPIC in the corresponding quasilinear setting.

Recall that $\tau_i^0(s_{-i})$ and $\tau_i^1(s_{-i})$ represent the price that bidder i pays for her zeroth and first units, respectively, given her rivals reported types. Thus, bidder i's contingent demand for her first unit, given that she participates in the auction is $d_i(-\tau_i^0(s_{-i}), s)$ where $s = (s_i, s_{-i})$. In the corresponding quasilinear setting, bidder i's valuation for the object is set equal to her contingent demand for her first unit.

Definition 6. A mechanism Γ has a corresponding quasilinear setting $QL(\Gamma)$ where bidders have quasilinear preferences with payoffs determined by the value function

$$v_i(s) = d_i(-\tau_i^0(s_{-i}), s) \ \forall s = (s_i, s_{-i}) \in S^n.$$

If bidder i is assigned the item, her ex post payoff is $v_i(s) - \tau_i^1(s_{-i}) - \tau_i^0(s_{-i})$. Otherwise, it is $-\tau_i^0(s_{-i})$.

Note that the bidders' valuations in the corresponding quasilinear setting vary with Γ because a bidder's willingness to pay for the object varies with how much she is paid to participate $\tau_i^0(s_{-i})$.

Remark 1. Bidder i has valuation $v_i(s) = \hat{d}_i(s) \ \forall s \in S^n$ in the corresponding quasilinear setting of any mechanism that satisfies IR and no subsidies.

Lemma 1 below shows that a mechanism Γ satisfies EPIC and IR if and only if mechanism Γ satisfies EPIC and IR in the corresponding quasilinear setting $QL(\Gamma)$.

Lemma 1. *The direct revelation mechanism Γ satisfies EPIC and IR in the non-quasilinear setting if and only if Γ satisfies EPIC and IR in setting $QL(\Gamma)$.*

The proof of Lemma 1 is straightforward because the corresponding quasilinear setting is constructed to be such that the non-quasilinear bidder demands a unit for price $\tau_i^1(s_{-i})$ if and only if the corresponding quasilinear bidder demands a unit for price $\tau_i^1(s_{-i})$. Thus, ex post incentive compatibility in one setting implies ex post incentive compatibility in the other.

Note that we will often study the case where a bidder's signal space is one-dimensional, $S_i \subset \mathbb{R}$. Indeed, there are many impossibility results in the quasilinear setting for implementation problems with multi-dimensional types (see Dasgupta and Maskin (2000), Jehiel and

Moldovanu (2001), and Jehiel et al. (2006)). In addition, all of our examples also impose that a bidder's willingness to pay for the object is monotone in her signal. Yet our general results do not require that we put structure on the signal space. Instead, our main results will show that we can study a non-quasilinear setting by considering a corresponding quasilinear setting. For example, if we study a setting where signal space is multi-dimensional and ex post implementation of nontrivial social choice functions is impossible in a mechanism's corresponding quasilinear setting, then Lemma 1 similarly shows that the social choice function is not ex post implementable in our non-quasilinear setting.

Our contribution is to show when the existing tools on efficient auction design from the quasilinear setting can be used in our non-quasilinear setting. This problem is simplified by considering ex post equilibrium, because we are able to restrict attention to settings where bidders have complete information over the rivals' private information. However, the efficient auction design problem is complicated by the presence of wealth effects. In what follows, we show that when the object being sold is a normal good, results from the literature on efficient auction design with quasilinear bidders can be extended to our non-quasilinear setting (Theorem 1). However, this positive result does not extend to settings where bidders compete to win an inferior good, or procurement settings.

5 The Normal Good Setting

In this section, we study efficient and ex post implementable mechanisms in the good setting where bidders have positive wealth effects (i.e., a normal good setting). We show that a mechanism is efficient and ex post implementable if the mechanism is efficient and ex post implementable in the corresponding quasilinear setting (Theorem 1). We then use Theorem 1 and recent results in the literature on efficient ex post equilibria of English auctions in the quasilinear setting to obtain analogous sufficient conditions for the existence of efficient ex post equilibria of English auctions in the non-quasilinear setting (Corollary 1).

In the motivating example given in Section 2.1, bidders have additively separable preferences for the good and wealth. It is analogous to the financially constrained bidder case in Che and Gale (1998).[7] Positive wealth effects may also arise in situations where there is

[7] In Che and Gale (1998), bidders have a (private) valuation for the good and are risk neutral, but bidders also borrow money to finance their auction bids, and the cost of borrowing is increasing. A special case of the financial constrained bidders case is the case where bidders have hard budgets. In this case, we assume that a bidder has a hard budget that is less than her valuation. This case is studied by Che and Gale (1996, 2000), Maskin (2000), and Pai and Vohra (2014), among others. In the hard budget case f_i is such that

$$f_i(t) = \begin{cases} t & \text{if } t \geq -b_i, \\ -\infty & \text{if } t < -b_i, \end{cases}$$

some residual uncertainty about the value of the good and bidders are risk averse, which the following example illustrates.

Example 1. (Ensuing risk) Bidder i maximizes her expected utility from money and she has decreasing absolute risk aversion. If bidder i wins the good, then she is paid a dividend of $v_i(s) + z_i$. The term z_i is a type-independent random variable. This case is studied by Hu, Matthews, and Zou (2015). The risky asset is a normal good because declining absolute risk aversion implies that a bidder becomes less risk averse as her wealth increases. Thus, she is willing to pay more for the risky asset when her wealth increases.

While the space of efficient outcomes in the quasilinear setting is distinct from the space of efficient outcomes in the normal good setting, Theorem 1 shows that efficient auction design in the quasilinear setting is related to efficient design in the normal good setting. More precisely, if Γ satisfies (1) EPIC, (2) IR, and (3) efficiency in setting $QL(\Gamma)$, then Γ satisfies (1)–(3) in the normal good setting.

Theorem 1. *A direct revelation mechanism Γ satisfies (1) IR, (2) EPIC and (3) efficiency in the non-quasilinear setting if Γ satisfies EPIC, IR, and efficiency in the corresponding quasilinear setting.*

Before discussing the intuition behind the proof, note that Lemma 1 shows that if Γ satisfies (1) EPIC and (2) IR in setting $QL(\Gamma)$, then Γ satisfies (1) and (2) in the normal good setting. Thus, the proof of Theorem 1 shows that if Γ satisfies (1) EPIC, (2) IR, and (3) efficiency in setting $QL(\Gamma)$, then Γ satisfies efficiency in the normal good setting.

The intuition for the result is the same as that given in our motivating example in Section 2.1. Consider a mechanism that satisfies the no subsidy condition.[8] Therefore, a losing bidder makes zero payment $\tau_i^0(s_{-i}) = 0 \; \forall s_{-i}$. If mechanism Γ is efficient in the corresponding quasilinear setting, then the mechanism assigns the good to the bidder with the highest willingness to pay. Thus the allocation rule is such that $q_i(s) = 1 \implies \hat{d}_i(s) \geq \hat{d}_j(s) \; \forall j \neq i$. If a bidder wins the good and pays her willingness to pay for the good, then her willingness to sell the good after the auction equals her willingness to pay, because $\hat{d}_i(s) = h_i(-\hat{d}_i(s), s)$. Thus after buying at her willingness to pay $\hat{d}_i(s)$, the bidder is indifferent between selling and keeping the good at this price. Yet, the winning bidder pays a price p where p is less than her willingness to pay to win the good. Since bidders have positive wealth effects,

$$p \leq \hat{d}_i(s) \implies \hat{d}_i(s) = h_i(-\hat{d}_i(s), s) \leq h_i(-p, s),$$

and b_i is bidder i's hard budget.

[8] The formal proof for a general class of mechanisms is in the appendix.

where the first equality holds by the construction of d_i and h_i and the final inequality holds because of positive wealth effects. Therefore, the winning bidder's willingness to sell the object after winning and paying p exceeds her willingness to pay for the good, and since she has the highest willingness to pay of all bidders, there are no ex post Pareto improving trades of the good.

Theorem 1 is a sufficient — but not necessary — condition to ensure a mechanism satisfies Properties (1)–(3). This is illustrated via Example 2 in Section 5.2. In the example, there are two ex ante symmetric bidders, and bidder 1 always wins the good for free. While bidder 1 does not necessarily have the highest willingness to pay, bidder 1's willingness to sell the good is sufficiently large (relative to her rival's willingness to pay for the good) because she wins the good for a low price (free). Thus, in our example, there are no ex post Pareto improving trades, even though the bidder with the highest willingness to pay may not win the good.

5.1 Application: English Auctions

Theorem 1 extends results on the efficiency of English auctions with quasilinearity to the non-quasilinear setting. Note that the English auction satisfies the no subsidy condition by construction. Thus, we have that $\tau_i^0(s_{-i}) = 0 \ \forall s_{-i}$. Therefore, Theorem 1 states that the English auction has an efficient ex post equilibrium if there is an efficient ex post equilibrium of the corresponding quasilinear setting where bidder i has valuation $v_i(s) = \hat{d}_i(s) \ \forall s \in S^n$.

Corollary 1. *The English auction in the normal good setting has an efficient ex post equilibrium if there is an efficient ex post equilibrium of the English auction in the quasilinear setting where bidder i has value function*

$$v_i(s) = \hat{d}_i(s) \ \forall s \in S^n.$$

Thus, we can say that an English has an efficient ex post equilibrium if $\hat{d}_i(s)$ satisfies Krishna's (2003) average or weighted crossing conditions or the generalized single crossing condition of Birulin and Izmalkov (2011). Bidder strategies in the efficient ex post equilibrium in the normal good setting are identical to bidder strategies in the corresponding quasilinear setting, because an efficient ex post equilibrium in the latter implies the existence of an efficient ex post equilibrium in the former.

5.2 The Bounds of Theorem 1

Theorem 1 provides a sufficient condition for efficient auction design. The theorem shows that a mechanism Γ is efficient and ex post implementable if the mechanism Γ is efficient and ex post implementable in the corresponding quasilinear setting $QL(\Gamma)$. If mechanism Γ does not provide bidders with an upfront subsidy (i.e. $\tau_i^0(s_{-i}) = 0,\ \forall s_{-i}, i$) then Theorem 1 implies the mechanism Γ is efficient if the mechanism assigns the good to the bidder with the highest willingness to pay for the good.

In this subsection, we show that assigning the good to the bidder with the highest willingness to pay is not a necessary condition for efficient auction design without quasilinearity (Example 2). In addition, we show that this condition is not sufficient for efficient auction design if we remove the normal good assumption. Thus, there are cases where we can construct an auction with an ex post equilibrium where the bidder with the highest willingness to pay for the good is always assigned the good, and yet the auction is inefficient (Example 3).

Example 2. (An efficient auction that does not assign the good to the bidder with the highest willingness to pay)

Suppose there are two bidders and that $S_1 = S_2 = [1, 2]$, where

$$u_i^x(m, s) = \left(s_i + \frac{1}{2}s_j\right) x + ln(m + 1).$$

Then,

$$\hat{d}_i(s) = 1 - e^{-(s_i + \frac{1}{2}s_j)},$$

and

$$h_i(t, s) = (1 + t)(e^{s_i + \frac{1}{2}s_j} - 1).$$

Let Γ be a mechanism where

$$q_1(s) = 1 \text{ and } q_2(s) = t_1(s) = t_2(s) = 0\ \forall s \in S^2.$$

The proposed mechanism always assigns the good to bidder 1 for free. The construction of the mechanism immediately implies that it satisfies (1) IR, (2) no subsidies, and (3) efficiency. There are no Pareto improving trades between bidders because

$$h_1(0, s) \geq \hat{d}_2(s)\ \forall s \in S^2.$$

In Example 2, both bidders are ex ante symmetric, but the good is always assigned to bidder 1 for free. Bidder 1's willingness to sell the good after winning $h_1(0, s)$ exceeds her willingness to pay for the good $\hat{d}_1(s)$ she wins the good for free. This is because winning the good for free makes bidder 1 feel wealthier, and increases her willingness to sell the good relative to her willingness to pay, because bidder 1 has positive wealth effects.

In the next example, we study an English auction where two bidders compete to win an inferior good (i.e. bidders have negative wealth effects). We show that the English auction has an ex post equilibrium where the bidder with the highest willingness to pay for the good always wins the good. However, the auction is inefficient.

Example 3. Suppose there are two bidders and $S_1 = S_2 = [0, \frac{1}{2}]$.

$$u_i^x(t, s) = (s_1 + (1 - \epsilon)s_2)\mathbb{I}_{x=1} + e^t,$$

where $\epsilon > 0$ is small. Bidder i's willingness to pay for the good is

$$\hat{d}_i(s) = -ln(1 - (s_i + (1 - \epsilon)s_j)).$$

The English auction has an ex post equilibrium where the bidder with the highest type always wins the good. Bidder i remains in the auction until the price reaches $b(s_i)$ where

$$b(s_i) = \hat{d}_i(s_i, s_i) = -ln(1 - (2 - \epsilon)s_i).$$

The auction outcome is inefficient. To see this suppose that bidder 1 wins, and hence $s_1 > s_2$. Then bidder 1 wins and pays $b(s_2)$. A straightforward calculation shows that bidder 1 is willing to sell the good for a price of at least

$$ln(1 + s_1 - s_2) - ln(1 - (2 - \epsilon)s_2). \tag{1}$$

Bidder 2 is willing to buy the good for

$$-ln(1 - (s_2 + (1 - \epsilon)s_1)). \tag{2}$$

It is straightforward to show that for any $s_1 > s_2$, expression (1) exceeds expression (2) when ϵ is sufficiently small. Thus, the auction in inefficient because there is an ex post Pareto improving trade where the winning bidder sells the good to the losing bidder for a price above the winning bidder's willingness to sell and below the losing bidder's willingness to pay.

In the above example, the winning bidder pays a price that is below her willingness to pay. This makes the winning bidder feel wealthier, and because bidders have negative wealth effects, this lowers the price at which the winning bidder is willing to sell the good relative to her willingness to pay for the good. At the same time, the losing bidder's willingness to pay approximately equals to the winning bidder's willingness to pay, because this is an almost common value setting. Hence, there are ex post Pareto improving trades between the two bidders.

In the next section, we show that the presence of positive wealth effects similarly leads to inefficiencies in procurement auctions, even when the auction assigns the good to the lowest cost bidder.

6 Efficient Procurement Auctions

In this section, we study a procurement setting where bidders have positive wealth effects and we obtain results that are qualitatively different from our results in the normal good setting. Theorem 1 shows that an auction is efficient and ex post implementable if the auction has an ex post equilibrium where the good is assigned to the bidder with the highest willingness to pay. Thus, assigning the good to the bidder with the highest willingness to pay is a sufficient condition for ex post Pareto efficiency. However, in the procurement setting, we show that assigning the procurement task to the bidder with the lowest cost is necessary, but not sufficient, for ex post Pareto efficiency in the procurement setting. In fact, we show that there is no auction with an efficient ex post equilibrium if there is sufficiently strong interdependence in bidder preferences. The result holds even in the case where there is an auction with an ex post equilibrium that assigns the task to the lowest cost bidder.

We limit attention to mechanisms that provide no subsides for losing bidders.

Definition 7. (No subsidies) A mechanism Γ satisfies no subsidies if a losing bidder is never paid to participate. That is,

$$q_i(s) = 0 \implies t_i(s) \leq 0.$$

All commonly used auction formats satisfy the no subsidy condition. Morimoto and Serizawa (2015) also note that the no subsidy restriction is necessary to prevent entry of bidders who are not interested in the item being auction, and are instead interested in "leaching" surplus from the auctioneer. In addition, without quasilinearity subsidies can be used to change bidder preferences endogenously. Baisa (2017b) shows that the space of implementable social choice rules is expanded if we allow the mechanism designer to change bidder preferences by providing bidders with ex ante subsidies.

A bidder's reservation cost in the procurement setting is the absolute value of the bidder's willingness to pay for the task,

$$c_i(s) = |\hat{d}_i(s)|.$$

A bidder who has received no upfront payment would accept a take-it-or-leave-it offer to complete the task if and only if she is offered an amount that exceeds her reservation cost.

We similarly define $a_i(m, s)$ as the analog to a bidder's willingness to sell — we call a_i bidder i's willingness to offer. It is the most amount of money that bidder i would be willing to offer someone to complete the procurement task for them, conditional on having been assigned the task and receiving m from the auctioneer,

$$a_i(m, s) = |h_i(m, s)|.$$

Thus, a bidder who is responsible for completing the procurement task, and has received m in compensation, would prefer to pay another bidder p to complete the task if and only if $p \leq a_i(m, s)$. Note that positive wealth effects imply that a_i is increasing in the first argument. Moreover, a_i is strictly increasing in the first argument if bidders have strictly positive wealth effects.

We provided one example of such a setting in Section 2.2. The next example shows that bidders can have positive wealth effects that are induced by limited liability constraints and some residual uncertainty over the value of the good. The example derives from the model in Burguet, Ganuza, and Hauk (2012).

Example 4. (Procurement with limited liability — Burguet, Ganuza, and Hauk (2012)) Consider an auction to procure a government contract. There are 2 firms that compete to win the contract. The monetary cost of completing the project, $\kappa \in [0, 1]$, is unknown at the time the auction is conducted. Each firm has a privately known asset level, $s_i \in [0, 1]$ and has limited liability. If the project is assigned to firm i for p and $s_i + p < \kappa$, the project is unfinished and limited liability prevents the government from pursuing damages.

A bidder's utility is normalized to zero if she does not win and makes no transfers. In addition, the expected utility of bidder i after winning for a price of p is

$$u_i^1(p, s_i) = s_i + p - E[\kappa|\kappa \leq s_i + p].$$

Recall that $a_i(m, s)$ is defined as solving

$$u_i^1(m, s) = u_i^0(m - a_i, s),$$

or equivalently, $a_i(m, s)$

$$a_i(m, s) = \mathbb{E}[\kappa|\kappa \leq s_i + m] - s_i.$$

Since the expectation term is increasing in m, bidder i's willingness to offer is also increasing in m, indicating that bidder i has positive wealth effects.

The example presented is a pure private value environment, which is nested in our setting.

We could generalize the example to include interdependence by assuming that bidder beliefs on κ are affiliated with their signals.

6.1 Necessary and Sufficient Conditions

Lemma 2 below shows that if a mechanism Γ satisfies (1) IR, (2) no subsidies, (3) EPIC, and (4) efficiency, then the bidder with the lowest cost wins the task. Or equivalently, if Γ satisfies Properties (1)–(4) in the non-quasilinear setting then Γ satisfies Properties (1)–(4) in a corresponding quasilinear $QL(\Gamma)$ setting where bidder i has reservation cost $c_i(s)$. Note that Lemma 2 gives a necessary, but not sufficient condition for efficient implementation.

Lemma 2. *If Γ satisfies (1) IR, (2) no subsidies, (3) EPIC, and (4) efficiency, then*

$$q_i(s) = 1 \implies c_i(s) \le \min_{j \neq i} c_j(s).$$

The proof is by contradiction. Suppose a mechanism satisfies Properties (1)–(4) and assigns the task bidder i and bidder i does not have the lowest reservation cost. If bidder i wins the project and is paid an amount exactly equal to her reservation cost, then she would be willing to offer another bidder up to her reservation cost to avoid having to complete the project. That is,

$$c_i(s) = a_i(c_i(s), s),$$

where the above expression follows directly from the definitions of c_i and a_i. Yet, the mechanism satisfies IR, so the winning bidder is paid (weakly) more than her reservation cost to complete the task. Or equivalently, bidder i is paid her reservation cost and then given an additional non-negative refund. The refund makes bidder i wealthier. Since bidder i has strictly positive wealth effects, then the increase in her wealth weakly increases the amount she is willing to offer another bidder to complete the procurement task. Thus, she is willing to offer another bidder an amount that is at least her reservation cost $c_i(s)$ to complete the procurement task for her. Yet if the winning bidder does not have the lowest reservation cost, then there is another bidder j who is willing to complete the task for an amount that is below bidder i's reservation cost. Hence, there is a Pareto improving trade where bidder i pays bidder j to complete the procurement task.

Lemma 2 illustrates differences between the good setting and the procurement setting. First, note that in the good setting, Example 2 shows that there is an auction that satisfies Properties (1)-(4) stated above, but the auction does not assign the good to the bidder with the highest willingness to pay. In addition, Theorem 1 shows that in the good setting, an ex post implementable mechanism that assigns the good to the highest willingness to pay bidder is efficient. In contrast to both points, Lemma 2 shows that in the procurement setting, assigning the good to the lowest cost bidder is a necessary, but not sufficient, condition for efficiency.

We show that the question of whether there exists a mechanism satisfying (1) IR, (2) no subsidies, (3) EPIC and (4) efficiency can be reduced to the question of whether there is a mechanism within a particular class of candidate mechanisms that satisfies (1)–(4). In other words, there exists a mechanism satisfying (1)–(4) if and only if a there is a candidate Γ^* mechanism does, where the class of Γ^* mechanisms is defined below (Lemma 3). We use this necessary and sufficient condition to establish Theorem 2 which shows that there is no mechanism that satisfies (1)–(4) when the level of interdependence among bidders is sufficiently strong.

Definition 8. All Γ^* mechanisms have the following properties. They always assigns the task to a bidder with the lowest cost, i.e.,

$$q_i(s) = 1 \implies c_i(s) \le \max_{j \neq i} c_j(s), \text{ and } \sum_{i=1}^{n} q_i(s) = 1 \; \forall s = (s_1, \ldots, s_n) \in S^n.$$

For all $s \in S^n$ and all i, $\tau_i^0(s_{-i}) = 0$. If s is such that $q_i(s) = 1$, $\tau_i^1(s) = -\omega_i^*(s_{-i})$ where

$$\omega_i^*(s_{-i}) = \sup_{\tilde{s}_i \in S_i} c_i(\tilde{s}_i, s_{-i}) \text{ s.t. } q_i(\tilde{s}_i, s_{-i}) = 1.$$

Note that $\omega_i^*(s_{-i})$ is the highest possible reservation cost that bidder i could have reported, under the constraint that bidder i is still selected as the winner.

Lemma 3. *There exists a mechanism that satisfies (1) IR, (2) EPIC, (3) no subsidies, and (4) efficiency if and only if there is a Γ^* that mechanism satisfies (1)–(4).*

The proof can be understood intuitively. If there is a mechanism $\tilde{\Gamma}$ that satisfies the four properties, then we show that a Γ^* mechanism satisfies the four properties as well. The key step of the proof is to show that the outcome of some Γ^* mechanism is efficient if the outcome of $\tilde{\Gamma}$ is efficient. We show this in two steps. First, Lemma 2 implies that $\tilde{\Gamma}$ must assign the task to the lowest cost bidder, just as any Γ^* mechanism does. Let $\tilde{\Gamma^*}$ be the mechanism with the same allocation rule as $\tilde{\Gamma}$ but the transfer rule of a Γ^* mechanism. Second, we note that any mechanism that satisfies the four properties must pay bidder i at least $\omega_i^*(s_{-i})$ to complete the task. If bidder i is assigned the task and paid an amount p that is less than $\omega_i^*(s_{-i})$, then there exists a $\tilde{s}_i$ such that bidder i has the lowest costs, yet bidder i is paid an amount less then her reservation cost $p < c_i(\tilde{s}_i, s_{-i})$ and this would violate IR. Therefore, if a $\tilde{\Gamma}$ satisfies the four properties, it must pay the winning bidder a weakly greater amount than the winning bidder is paid in $\tilde{\Gamma^*}$. In other words, the winning bidder is weakly richer under the outcome of $\tilde{\Gamma}$, and positive wealth effects imply that the winning bidder is more inclined to pay one of her rival's to complete the task under the outcome where she is paid more. Thus, if there are no ex post Pareto improving trades in the outcome implemented by $\tilde{\Gamma}$, then there are no Pareto improving trades under $\tilde{\Gamma^*}$, because the winning bidder is weakly poorer, and hence less inclined to trade under the outcome in $\tilde{\Gamma^*}$.

In the next subsection, we study the efficiency of Γ^* mechanisms in a symmetric setting where the outcome of a Γ^* mechanism is equivalent to the outcome of an English auction.

6.2 A Symmetric Procurement Environment

In this section, we show that there is no mechanism that satisfies (1) IR, (2) no subsidy, (3) EPIC, and (4) efficiency when there is sufficiently strong interdependence in bidder demands. We study a symmetric environment that is parameterized by α, which measures the level interdependence of bidder preferences. We show that the necessary and sufficient condition for efficient design from Lemma 3 is satisfied if and only if the level of interdependence α is sufficiently small. In particular, we show that all Γ^* mechanisms are inefficient when the level of interdependence is sufficiently large.

We assume bidders have single dimensional types and $S_i = [0, 1] \subset \mathbb{R}\ \forall i \in \{1, \dots, n\}$ and we normalize a bidder's utility to be zero if she does not win and does not make any payment,

$$u_i^0(0, s) = 0\ \forall s \in S^n.$$

In addition, there is function u such that

$$u_i^1(m, s) = u(m, s_i + \alpha \sum_{j \neq i} s_j),\ \forall i \in \{1, \dots, n\},$$

where u is continuous and strictly increasing in both arguments. We assume bidders have strictly positive wealth effects. If $\alpha = 0$, this is a pure private value setting and if $\alpha = 1$ this is a pure common value setting. For notational simplicity, we write bidder i's reservation cost $c_i(s)$ as $c(s_i + \alpha \sum_{j \neq i} s_j)$. Note that $c_i(\cdot)$ is strictly decreasing.

Next, we derive conditions under which a Γ^* mechanism satisfies Properties (1)–(4). Note that in this setting, a Γ^* mechanism satisfies (1) IR, (2) no subsides, and (3) EPIC by construction. Thus, a Γ^* mechanism has an ex post equilibrium in which the lowest cost bidder always wins the good. However, the candidate mechanism violates efficiency when the level of interdependence among bidders, α, is sufficiently large. Hence, Lemma 3 implies that there is a mechanism that satisfies Properties (1)–(4) if and only if α is sufficiently small.

This is stated in Theorem 2 below.

Theorem 2. *There exists an $\alpha^* \in [0, 1)$ such that a mechanism satisfying (1) IR, (2) no subsidies, (3) EPIC, and (4) efficiency exists if and only if $\alpha \leq \alpha^*$.*

The intuition for the result can be understood by studying the extreme cases of pure common values ($\alpha = 1$) and pure private values ($\alpha = 0$) with two bidders.

In the pure common value setting, suppose that bidder 1 wins the procurement contract under a Γ^* mechanism. Because the mechanism is individually rational bidder 1 is paid an amount that exceeds her reservation cost.[9] In addition, since we are in a pure common value setting, bidder 2 has the same reservation cost as bidder 1. If bidder 1 were paid exactly her reservation cost, then she would be willing to offer bidder 2 an amount up to her reservation cost to complete the task. This is because bidder 1 is indifferent between completing the task and not completing the task when she is paid her reservation cost. Yet, bidder 1 is typically paid more than her reservation cost because she is compensated for her information rents in a mechanism that satisfies EPIC. Thus, it is as though bidder 1 was paid her reservation cost, and then given some additional money. When bidder 1 is given additional money, she is willing to offer bidder 2 more money to complete the task, because bidder 1 has strictly positive wealth effects. Thus, bidder 1 is willing to pay bidder 2 an amount that strictly exceeds her reservation cost to complete the project. In addition, bidder 2 is willing to complete the task for an amount that equals her reservation cost, $c(s_1 + s_2)$. Therefore, bidder 1 is willing to offer an amount that strictly exceeds bidder 2's reservation cost in order to have bidder 2 complete the project. Thus, there is an ex post Pareto improving trade where bidder 1 pays bidder 2 to complete the task, and any Γ^* mechanism is inefficient. Lemma 3 then implies that there is mechanism that satisfies the four desired properties in this setting.

In contrast, when bidder's have pure private values ($\alpha = 0$), all Γ^* mechanisms are efficient (and equivalent to a second price auction). We can see the intuition by again considering a two-bidder setting where bidder 1 wins the procurement contract under a Γ^* mechanism. Thus, $s_1 > s_2$. When bidder 1 wins, she is paid her rival's reservation cost $c(s_2)$, where $c(s_2) > c(s_1)$. Therefore, bidder 1 is paid an amount that exceeds her reservation cost, and her utility increases.[10] If bidder 1 wanted to pay bidder 2 to complete the task, then bidder 1 must offer bidder 2 at least $c(s_2)$. This is because $c(s_2)$ is bidder $2's$ reservation cost. If bidder 1 pays bidder 2 an amount $p \geq c(s_2)$ to complete the task, she does not complete the task and her wealth decreases (weakly) relative to her wealth prior to the auction. Bidder 1's wealth decreases because she pays bidder 2 an amount p that exceeds the amount the

[9] More formally, bidder 1 is paid $w_1^*(s_2) = c(2s_2) > c(s_1 + s_2)$ if $s_1 > s_2$.

[10] $c(s_2) > c(s_1) \implies u^1(c(s_2), s_1) > u^1(c(s_1), s_1) = u^0(0, s_1) = 0$ where the first equality follows from the definition of $c(\cdot)$.

government paid her upon winning $c(s_2)$. Thus, if bidder 1 pays bidder 2 an amount that she would be willing to accept, then she (bidder 1) is made worse off. Hence, there are no Pareto improving trades between the two bidders and any Γ^* mechanism is efficient.

Note that in our symmetric setting, a Γ^* mechanism is implemented by an English auction with a particular tie-breaking rule. Thus, we can say that there is an efficient auction in our symmetric procurement if and only if the English auction is efficient.

Corollary 2. *An English auction has a symmetric ex post equilibrium that is equivalent to the outcome of a* Γ^* *mechanism.*

Corollary 2 follows from Theorem 10 in Milgrom and Weber (1982). Milgrom and Weber (1982) show that in a symmetric quasilinear setting, the English auction has an ex post equilibrium where the high value bidder (here lowest cost) wins the object. Lemma 1 then implies that the English auction satisfies IR, no subsidies, and EPIC in non-quasilinear setting because it satisfies the three conditions in the corresponding quasilinear setting. Thus, we see that the English auction has an ex post equilibrium where the bidder with the lowest cost wins. However, when the degree of interdependence among bidders is sufficiently strong, the outcome of the English auction is inefficient.

The intuition for the impossibility result given above is different from intuition used to prove the impossibility results in other interdependent value auction settings where bidders have quasilinear preferences. The impossibility theorems given by Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), and Jehiel et al. (2006) are all related to interdependent value settings where bidders have quasilinear preferences and multi-dimensional private information. The efficient social choice function is a social choice function that assigns the good to the lowest cost (or highest willingness to pay) bidder. Those papers show that the efficient social choice function is not ex post implementable. Bergemann and Morris (2009) also present an impossibility result related to implementation in a single unit auction setting where bidders are quasilinear and have interdependent and additive values (Section 7). They show that robust implementation is possible if and only if the interdependence in bidder preferences is not too strong. Bergemann and Morris' notion of robust implementation is stronger than EPIC, and implementation fails because truthful reporting is not the unique solution to iterative elimination of never best replies when bidder interdependence is sufficient strong. In all of these papers, efficient implementation fails because there is no mechanism that implements an assignment where the bidder with the highest valuation (lowest cost) wins.

In contrast, we study a setting where there is an ex post equilibrium where the lowest cost bidder wins the good. This is both necessary and sufficient for efficient implementation with quasilinearity. However, without quasilinearity the space of efficient outcomes is different and assigning the good to the lowest cost bidder is necessary, but not sufficient, for efficient auction design. Instead, we show positive wealth effects make the winning bidder more inclined to pay a rival to complete the task, and this leads to Pareto improving trades when there is strong interdependence.

7 Conclusion

We study efficient and ex post implementable mechanisms when bidders have non-quasilinear preferences with positive wealth effects. Our setting nests well-studied cases such as auctions with risk averse bidders, auctions with financially constrained bidders, and auctions for risky assets. We study the incentive and efficiency properties of mechanisms in the non-quasilinear setting by considering the mechanism's corresponding quasilinear setting.

We show that when the object being allocated is a good, a mechanism is efficient in the non-quasilinear setting if the mechanism is efficient in the corresponding quasilinear setting. This implies that conditions guaranteeing the efficiency of the English auction in a quasilinear setting translate to the non-quasilinear setting when bidders have positive wealth effects. Moreover, this gives us a simple method for computing equilibrium bid behavior in English auctions without quasilinearity. We get distinct results in procurement settings. We show that there are cases where there is no mechanism that has desirable incentive and efficiency properties, even if there is an ex post implementable mechanism that assigns the task to the lowest cost bidder.

The methodology used in this paper could also be useful in further research on other mechanism design settings without quasilinearity. More precisely, we would be interested in studying how we could construct corresponding quasilinear settings in other mechanism design problems where agents have non-quasilinear preferences. We think that this could help us to understand the limits of efficient and ex-post implementation without quasilinearity in multi-unit auctions, combinatorial assignment problems, or double auction models.

8 Proofs

Proof of Lemma 1

Using the definition of $d_i(t,s)$, we have that $v_i(s) = d_i(-\tau_i^0(s_{-i}), s) \gtreqless \tau_i^1(s_{-i})$ as $u^1(-\tau_i^0(s_{-i}) - \tau_i^1(s_{-i}), s) \gtreqless u_i^0(-\tau_i^0(s_{-i}), s)$ for all $s \in S^n$. Note that these inequalities continue to be satisfied in the cases where the mechanism never assigns the good to bidder i or always assigns the good to bidder i for some s_{-i}, because in these cases $\tau_i^1(s_{-i}) = \infty$ or $\tau_i^1(s_{-i}) = 0$ and we assumed that the bidder's utility would increase if assigned the good for free. It follows that Γ is EPIC in the quasilinear setting if and only if it is EPIC in the non-quasilinear setting. In either setting, given s_{-i} if there is a s_i such that $q_i(s) = 0$, then the mechanism is IR if and only if $\tau_i^0(s_{-i}) \leq 0$. When $q_i(s) = 1$ for all s_i given some s_{-i}, bidder i's payoff in the quasilinear (non-quasilinear) setting is $d_i(-\tau_i^0(s_{-i}), s)$ $(u_i^1(-\tau_i^0(s_{-i}), s))$ which is greater than 0 $(u_i^0(-\tau_i^0(s_{-i}), s))$ if and only if $\tau_i^0(s_{-i}) \leq 0$.

Proof of Theorem 1

Proof. If Γ satisfies EPIC and IR in setting $QL(\Gamma)$, then Lemma 1 shows that Γ satisfies EPIC and IR in the non-quasilinear setting. Thus, we need to show that if Γ satisfies EPIC, IR, and efficiency in setting $QL(\Gamma)$, then Γ satisfies efficiency in the non-quasilinear setting. We assume Γ is efficient in setting $QL(\Gamma)$. Thus,

$$v_i(s) \geq v_j(s) \text{ if } q_i(s) = 1 \text{ and } j \neq i.$$

By construction of v_i, then

$$d_i(-\tau_i^0(s_{-i}), s) = v_i(s) \geq v_j(s) = d_j(-\tau_j^0(s_{-j}), s).$$

In addition, EPIC implies that

$$d_i(-\tau_i^0(s_{-i}), s) \geq \tau_i^1(s_{-i}) \text{ if } q_i(s) = 1.$$

Thus, if we let $d^* = d_i(-\tau_i^0(s_{-i}), s)$, then

$$h_i(-\tau_i^0(s_{-i}) - d^*, s) = d_i(-\tau_i^0(s_{-i}), s) \implies h_i(-\tau_i^0(s_{-i}) - \tau_i^1(s_{-i}), s) \geq h_i(-\tau_i^0(s_{-i}) - d^*, s)$$

where the inequality holds because positive wealth effects imply that h_i is increasing in the second argument. Thus,

$$h_i(-\tau_i^0(s_{-i}) - \tau_i^1(s_{-i}), s) \geq d_i(-\tau_i^0(s_{-i}), s) \geq d_j(-\tau_j^0(s_{-j}), s) \text{ if } q_i(s) = 1.$$

Therefore, there are no ex post Pareto improving trades between the winning bidder and her rivals, and thus Γ is efficient. □

Proof of Lemma 2

Proof. The proof is by contradiction. Suppose Γ satisfies (1)–(4) yet there is a $s \in S^n$ where $q_i(s) = 1$ and a bidder $j \neq i$ such that $c_j(s) < c_i(s)$. Since Γ satisfies IR, then we know that if bidder i wins the procurement auction, she is paid some amount p where $p \geq c_i(s)$. Moreover positive wealth effects imply that $a_i(p, s)$ is increasing in p, and the construction of a_i and c_i imply that

$$a_i(c_i(s), s) = c_i(s).$$

Thus,

$$p \geq c_i(s) \implies a_i(p, s) \geq c_i(s).$$

Thus, bidder i is weakly better off by paying another bidder $c_i(s)$ to complete the task. Bidder j is made strictly better off completing the task and being paid $c_i(s)$, because her reservation cost $c_j(s)$ is strictly smaller than $c_i(s)$ by assumption. Hence there is a Pareto improving trade where bidder i pays bidder j the amount $c_i(s)$ to complete the task. This contradicts efficiency. □

Proof of Lemma 3

Proof. The if direction is obvious, so we focus on the only if direction. Suppose that $\tilde{\Gamma}$ satisfies Properties (1)–(4) and has the allocation rule $\tilde{q}$. By Lemma 2 $\tilde{\Gamma}$ must allocation to the lowest cost bidder. Therefore, there is a Γ^* mechanism which we call $\tilde{\Gamma^*}$ that has the same allocation rule as $\tilde{\Gamma}$ but the transfer rule defined for Γ^* mechanisms. We use $\tilde{q}$ for both mechanisms' allocation rules, $\tilde{\tau}$ for the transfer rule of $\tilde{\Gamma}$ and τ^* for the transfer rule of $\tilde{\Gamma^*}$. We show that $\tilde{\Gamma^*}$ satisfies Properties (1)–(4). Its construction guarantees that $\tilde{\Gamma^*}$ satisfies IR and no subsidies. To argue that $\tilde{\Gamma^*}$ satisfies EPIC, we define $\mathcal{S}_i(s_{-i}) = \{s_i | \tilde{q}_i(s_i, s_{-i}) = 1\}$ and suppose for a contradiction that $\tilde{\Gamma^*}$ violates EPIC. Lemma 1 implies that there is either a $s_i \in \mathcal{S}_i(s_{-i})$ such that

$$-\tau_i^{1*}(s_{-i}) = \omega_i^*(s_{-i}) < c_i(s_i, s_{-i}) = -\hat{d}_i(s_i, s_{-i}),$$

or there is a $s_i \notin \mathcal{S}_i(s_{-i})$ such that

$$-\hat{d}_i(s_i, s_{-i}) = c_i(s_i, s_{-i}) < \omega_i^*(s_{-i}) = -\tau_i^{1*}(s_{-i}).$$

Yet, the first can not hold because the construction of $\omega_i^*(s_{-i})$ implies that

$$\omega_i^*(s_{-i}) \geq c_i(s_i, s_{-i}) \ \forall s_i \in \mathcal{S}_i(s_{-i}).$$

Thus, if $\tilde{\Gamma^*}$ violates EPIC, then there exists a $s_i \notin \mathcal{S}_i(s_{-i})$ such that $c_i(s_i, s_{-i}) < \omega_i^*(s_{-i})$. Using Lemma 1 and the fact that $\tilde{\Gamma}$ satisfies EPIC,

$$s_i \notin \mathcal{S}_i(s_{-i}) \implies -\tilde{\tau_i^1}(s_{-i}) \leq c_i(s_i, s_{-i}).$$

But this implies that $-\tilde{\tau_i^1}(s_{-i}) < \omega_i^*(s_{-i})$, which implies there exists a $s_i \in \mathcal{S}_i(s_{-i})$ such that

$$-\tilde{\tau_i^1}(s_{-i}) < c_i(s_i, s_{-i})$$

because by definition $\omega_i^*(s_{-i}) = \sup_{\tilde{s}_i \in \mathcal{S}_i(s_{-i})} c_i(\tilde{s}_i, s_{-i})$. Hence, $\tilde{\Gamma}$ violates EPIC, a contradiction. Therefore, if $\tilde{\Gamma}$ satisfies Properties (1)–(4), $\tilde{\Gamma^*}$ satisfies EPIC.

Finally, we show that $\tilde{\Gamma^*}$ is efficient. The above argument implies that

$$\omega_i^*(s_{-i}) \leq -\tilde{\tau}_i^1(s_{-i}) \ \forall s_{-i}.$$

In addition, since $\tilde{\Gamma}$ is efficient, there are no ex post Pareto improving trades among bidders,

$$s_i \in \mathcal{S}_i(s_{-i}) \implies \max_{j \neq i} c_j(s_i, s_{-i}) \geq a_i(-\tilde{\tau}_i^1(s_{-i}), (s_i, s_{-i})).$$

And positive wealth effects imply that $a_i(-\tilde{\tau}_i^1(s_{-i}), s) \geq a_i(\omega_i^*(s_{-i}), s)$ since $\omega_i^*(s_{-i}) \leq -\tilde{\tau}_i^1(s_{-i}) \ \forall s_{-i}$. Thus,

$$s_i \in \mathcal{S}_i(s_{-i}) \implies \max_{j \neq i} c_j(s_i, s_{-i}) \geq a_i(-\tilde{\tau}_i^1(s_{-i}), (s_i, s_{-i})) \geq a_i(\omega_i^*(s_{-i}), (s_i, s_{-i})),$$

and there are no ex post Pareto improving trades in mechanism $\tilde{\Gamma^*}$, making it efficient. □

Proof of Theorem 2

Proof. Let Γ be a Γ^* mechanism. By construction Γ satisfies Properties IR and no subsides. In addition, Γ satisfies EPIC because Lemma 1 states that a mechanism satisfies EPIC if and only if it satisfies EPIC in the corresponding quasilinear environment. The corresponding quasilinear setting is a subset of the preferences studied by Milgrom and Weber (1982), and the outcome of Γ is equivalent to an (ex post) implementable English auction.

When $\alpha = 0$, Γ satisfies Properties (1)–(4) because in the private value setting Γ is equivalent to the Vickrey allocation rule studied by Saitoh and Serizawa (2008). Saitoh and Serizawa show that the Vickrey allocation rule is efficient in the single unit setting, hence any Γ satisfies efficiency when $\alpha = 0$.

Then, we show that Γ is inefficient when $\alpha = 1$. Hence $c_i(s) = c_j(s)\ \forall s \in S^n$. Let $\mathcal{S}_i(s_{-i}) := \{s_i | q_i(s_i, s_{-i}) = 1\}$. Since $\sum_{i=1}^n q_i(s) = 1\ \forall s \in S^n$, then there exists a bidder i such that $s_i^h, s_i^\ell \in \mathcal{S}_i(s_{-i})$, where $s_i^h > s_i^\ell$. Then $\omega_i^*(s_{-i}) \geq c_i(s_i^\ell, s_{-i}) > c_i(s_i^h, s_{-i})$. Positive wealth effects imply that if $s^h = (s_i^h, s_{-i})$, then

$$c_j(s^h) = c_i(s^h) = a_i(c_i(s^h), s^h) < a_i(\omega_i^*(s_{-i}), s^h),$$

where the final inequality follows because bidder i has strictly positive wealth effects and $\omega_i^*(s_{-i}) > c_i(s^h)$. Thus, there is a Pareto improving trade between bidder i and bidder j. Hence, Γ is inefficient. Thus, Lemma 3 implies that there is no mechanism that satisfies Properties (1)-(4) when $\alpha = 1$.

Finally, we show if Γ satisfies efficiency in an environment with level of interdependence $\alpha^h < 1$, then there is a Γ^* mechanism that satisfies efficiency in any environment where the level of interdependence is $0 \leq \alpha^\ell < \alpha^h$. Fix $(s_1, \ldots, s_n) \in S^n$ and suppose without loss of generality that $s_1 \geq s_2 \geq s_j\ \forall j \neq 1, 2$. Let

$$\begin{aligned} \tilde{s}_1 &= s_1 + s_2(\alpha^\ell - \alpha^h \tfrac{1+\alpha^\ell}{1+\alpha^h}), \\ \tilde{s}_2 &= \tfrac{1+\alpha^\ell}{1+\alpha^h} s_2, \\ \tilde{s}_j &= \tfrac{\alpha^\ell}{\alpha^h} s_j. \end{aligned}$$

The construction of $(\tilde{s}_1, \ldots, \tilde{s}_n)$ is such that $\tilde{s}_j \in S$ and $\tilde{s}_j \leq s_j\ \forall j = 1, \ldots, n$. Both follow immediately from the construction of $\tilde{s}_j$ when $j = 2, \ldots, n$. When $j = 1$, note that $-1 < \alpha^\ell - \alpha^h \frac{1+\alpha^\ell}{1+\alpha^h} < 0$ because $\alpha^\ell < \alpha^h$. Hence, $s_2 < s_1 \implies 0 \leq \tilde{s}_1 \leq s_1$. Thus, $\tilde{s}_1 \in S$ and $\tilde{s}_1 \leq s_1$.

Moreover, $\tilde{s}_1 \geq \tilde{s}_2 \geq \tilde{s}_j$, where $j \neq 1, 2$. To prove the first inequality, note that if $s_1 = s_2$, then $\tilde{s}_1 = \tilde{s}_2$. In addition, $\tilde{s}_1$ is increasing in s_1 and $\tilde{s}_2$ is unchanged in s_1. Hence, $s_1 \geq s_2 \implies \tilde{s}_1 \geq \tilde{s}_2$. To prove the second inequality note that $1 \geq \alpha^h > \alpha^\ell \geq 0$. Hence $\frac{1+\alpha^\ell}{1+\alpha^h} \geq \frac{\alpha^\ell}{1+\alpha^h}$, and thus

$$s_2 \geq s_j \forall j \neq 1, 2 \implies \tilde{s}_2 = \frac{1+\alpha^\ell}{1+\alpha^h} s_2 \geq \frac{\alpha^\ell}{\alpha^h} s_j = \tilde{s}_j\ \forall j \neq 1, 2.$$

Since $(\tilde{s}_1, \ldots, \tilde{s}_n) \in S^n$, then when the level of interdependence is α^h and $q_1(\tilde{s}) = 1$, we have that

$$c_2\left(\tilde{s}_2 + \alpha^h \sum_{j\neq 2} \tilde{s}_j\right) - a\left(c\left(\tilde{s}_2 + \alpha^h \sum_{j\neq 1} \tilde{s}_j\right), \tilde{s}_1 + \alpha^h \sum_{j\neq 1} \tilde{s}_j\right) \geq 0.$$

That is, the reservation cost of the lowest cost bidder of bidder 1's rivals exceeds bidder 1's willingness to offer another bidder to complete the task. This inequality holds because we have assume the candidate mechanism is efficient when the level of interdependence is α^h. Then, by our definition of $(\tilde{s}_1, \ldots, \tilde{s}^n)$,

$$\begin{aligned} s_2 + \alpha^\ell \textstyle\sum_{j\neq 2} s_j &= \tilde{s}_2 + \alpha^h \textstyle\sum_{j\neq 2} \tilde{s}_j, \\ s_1 + \alpha^\ell \textstyle\sum_{j\neq 1} s_j &= \tilde{s}_1 + \alpha^h \textstyle\sum_{j\neq 1} \tilde{s}_j, \\ s_2 + \alpha^\ell \textstyle\sum_{j\neq 1} s_j &= \tilde{s}_2 + \alpha^h \textstyle\sum_{j\neq 1} \tilde{s}_j. \end{aligned}$$

Hence

$$c_2\left(s_2 + \alpha^\ell \sum_{j\neq 2} s_j\right) - a\left(c\left(s_2 + \alpha^\ell \sum_{j\neq 1} s_j\right), s_1 + \alpha^\ell \sum_{j\neq 1} s_j\right) \geq 0.$$

Since this holds for arbitrary $(s_1, \ldots, s_n)$ such that $s_1 \geq s_2 \geq s_j \forall j \neq 1, 2$, the reservation cost of the lowest cost losing bidder exceeds the willingness to offer of the winning bidder. Hence, a Γ^* mechanism (with the same allocation rule) is efficient when the level of interdependence is $\alpha^\ell < \alpha^h$.

Thus, there is an $\alpha^* \in [0, 1)$ such that a Γ^* mechanism is efficient if and only if $\alpha \leq \alpha^*$. $\square$

References

Baisa, B. (2017a). Auction design without quasilinear preferences. *Theoretical Economics 12*(1), 53–78.

Baisa, B. (2017b). Efficient multi-unit auctions for normal goods. *Working paper*.

Bergemann, D. and S. Morris (2005, November). Robust mechanism design. *Econometrica 73*(6), 1771–1813.

Bergemann, D. and S. Morris (2009). Robust implementation in direct mechanisms. *Review of Economic Studies 76*, 1175–1206.

Birulin, O. and S. Izmalkov (2011, July). On efficiency of the english auction. *Journal of Economic Theory 146*(4), 1398–1417.

Che, Y.-K. and I. Gale (1996). Expected revenue of all-pay auctions and first-price sealed-bid auctions with budget constraints. *Economics Letters 50*, 373–379.

Che, Y.-K. and I. Gale (1998). Standard auctions with financially constrained bidders. *The Review of Economic Studies 51*(1), 1–21.

Che, Y.-K. and I. Gale (2000). The optimal mechanism for selling to a budget-constrained buyer. *Journal of Economic Theory 92*, 198–233.

Che, Y.-K. and I. Gale (2006). Revenue comparisons for auctions when bidders have arbitrary types. *Theoretical Economics*, 95–118.

Dasgupta, P. and E. Maskin (2000, May). Efficient auctions. *The Quarterly Journal of Economics 115*(2), 341–388.

Dobzinski, S., R. Lavi, and N. Nisan (2012). Multi-unit auctions with budget limits. *Games and Economic Behavior 74*(2), 486–503.

Fang, H. and S. Perreiras (2002). Equilibrium of affiliated value second price auctions with financially constrained bidders: The two-bidder case. *Games and Economic Behavior 39*, 215–236.

Fang, H. and S. Perreiras (2003). On the failure of the linkage principle with financially constrained bidders. *Journal of Economic Theory 110*, 374–392.

Holmstrom, B. and R. B. Myerson (1983). Efficient and durable decision rules with incomplete information. *Econometrica 51*(6), 1799–1819.

Hu, A., S. A. Matthews, and L. Zou (2015). English auctions with ensuing risks and heterogeneous bidders. *Penn Institute for Economic Research Working Paper*.

Jehiel, P. and B. Moldovanu (2001, September). Efficient design with interdependent valuations. *Econometrica 69*(5), 1237–1259.

Jehiel, P., M. M. ter Vehn, B. Moldovanu, and W. R. Zame (2006). The limits of ex post implementation. *Econometrica 74*(3), 585–610.

Kotowski, M. H. (2017, November). First-price auctions with budget constraints. *Working paper*. https://sites.hks.harvard.edu/fs/mkotows/1001-fpabc-wpWEB.pdf.

Krishna, V. (2003, October). Asymmetric english auctions. *Journal of Economic Theory 112*(2), 261–288.

Laffont, J.-J. and J. Robert (1996). Optimal auction with financially constrained buyers. *Economic Letters 52*, 181–186.

Maskin, E. (1992). Auctions and privatization. *in: H. Siebert (Ed.), Privatization, Institut fur Weltwirtschaften der Universitat Kiel, Kiel*, 115–136.

Maskin, E. (2000). Auctions, development and privatization: Efficient auctions with liquidity-constrained buyers. *European Economic Review 44*, 667–681.

Maskin, E. and J. Riley (1984, November). Optimal auctions with risk averse buyers. *Econometrica 52*(6), 1473–1518.

Matthews, S. (1987). Comparing auctions for risk averse buyers: A buyer's point of view. *Econometrica 55*(3), 633–646.

Matthews, S. A. (1983). Selling to risk averse buyers with unobservable tastes. *Journal of Economic Theory 30*, 370–400.

Mayer, C. J. (1998). Assessing the performance of real estate auctions. *Real Estate Economics 26*, 41–66.

Milgrom, P. and R. Weber (1982). A theory of auctions and competitive bidding. *Econometrica 50*, 1089–1122.

Morimoto, S. and S. Serizawa (2015). Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price walrasian rule. *Theoretical Economics*, 445–487.

Nöldeke, G. and L. Samuelson (2015). The implementation duality. *Cowles Foundation Discussion Paper*.

Pai, M. M. and R. Vohra (2014, March). Optimal auctions with financially constrained buyers. *Journal of Economic Theory*, 383–425.

Saitoh, H. and S. Serizawa (2008). Vickrey allocation rule with income effect. *Economic Theory 35*(2), 391–401.

Salant, D. J. (1997). Up in the air: GTE's experience in the mta auction for personal communication services licenses. *Journal of Economics and Management Strategy 6*(3), 549–572.

Wilson, R. (1987). Game-theoretic analyses of trading processes. *In Advances in Economic Theory: Fifth World Congress, ed. Truman Bewley*, 33–70.