A Model of a Multilateral Proxy War with Spillovers

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Abstract

Motivated by the civil war in Syria, this paper models a proxy war with three sponsors and three combatants as a dynamic game. Sponsors are leaders that provide resources to combatants. Sponsors 1 and 2 have strong aversion to sponsor 3's proxy, but not against each other. This is modeled as a spillover effect between 1 and 2. We identify and characterize three pure strategy equilibria. It is shown that the comparative statics of the spillover effect varies from one equilibrium to another. Two mixed strategy equilibria are also studied. In the first, sponsor 3 spends less than others, and his participation probability is positively related to the cost of sponsorship. In the second, sponsor 3 spends the same amount as others, but his participation is negatively related to this cost. Finally, we explain why tacit coordination between sponsors 1 and 2 is better for them than forming an alliance.

1 Introduction

This study's motivation is the creation, growth, military advance, territorial conquests, and spectacular terrorist attacks of ISIS (The Guardian, 7 March, 2016).[1] ISIS is arguably the most powerful Islamic terrorist group that the world has ever seen.[2] ISIS emerged from the Syrian civil war and spread to Iraq. It rapidly grew into a powerful army of combatants, massacring enemies and amassing billions of dollars selling oil from its conquering territories. Its human and financial resources are now being used to launch terrorist attacks in Europe (e.g., in France and Belgium) and the U.S. (The Guardian, 2016).[3] The actual and potential costs of ISIS are considerable.[4] Understanding the conflict and milieu that generated ISIS is crucial to formulating a winning strategy to fight it and to prevent similar recurrences in the future.[5] We hope that the insights of this paper will be useful to countries such as the United States as they deliberate on the best ways to intervene in the Syrian conflict. These considerations are particularly important at this moment because of the recent change in the U.S. administration.

The Syrian civil war is not confined to two opposing sides – pro and anti-government. Syria's conflict involves multiple agents, separated by religious, ethnic, and political differences (and at a deeper level it is related to family and patronage loyalties). There are: three major confessional groups – two Islamic (Sunni, who comprise 74% of the population; Shia – and minor sects such as Alawite, Druze, etc. – who comprise 16% of the population), plus Christians (10% of the population); different ethnic groups – Arab (90%) and non-Arab (10%)[6]; and three broad political groups – pro-establishment dictatorship of the Baath party, democrats, and Islamic fundamentalists. Each of these combatant groups has one or more sponsors: the U.S. and western allies, supporting political opposition to the Syrian Assad regime (who they assume think like western liberals); Russia and Iran, supporting the Assad regime (which is thought to represent Syrian Shia, Allawite, Druze, and Christians); and finally a loose army of international Muslim Sunni volunteers (an important part composed of Muslim Europeans), supporting and fighting for ISIS.

[1] On terrorism spectaculars see Arce (2010), Hoffman (2006) and Enders and Sandler (2012).

[2] On the growth of Islamic terrorism see Enders and Sandler (2000) and Barros and Proenca (2005).

[3] On ISIS see Wood (2015).

[4] For an estimate of the economic impact of terrorism and conflicts on income per capita growth, see Gaibulloev and Sandler (2009). For macroeconomic impacts of terrorism see Blomberg et al. (2004) and Tavares (2004).

[5] Phares (2005) discusses the information war on Jihadism in academia and media, where Jihadists and their supporters actively participate, attempting to divert it, camouflage it, and move it in different directions.

[6] On conflict in dual population lands, see Levy and Faria (2007).

Syria is clearly plagued by communal cleavages. In its history Syria was never an independent country with a uniform ethnic-religious-political composition. After its independence from France in 1946 it struggled to find an identity. The Baath party gave Syria a much needed identity, with an ideology that is a mixture of Arab nationalism and socialism. It has a Bolshevik type of organization that permeates all regions, cities, villages, institutions, and groups of Syrian society. The Baath party and the army are the only institutions that allowed a significant share of Syrian society the opportunity for social mobility, in particular for religious minorities such as the Allawites that ended up forming the Army's elite.

General Hafiz al-Assad's 30-year dictatorship (1970-2000) was able to concentrate power because he had the political influence inside the Baath party and the control of the Army (Hinnebusch 2011). The only major group of Syrian society that was not under the influence of the government was the poor Sunni majority, profoundly influenced by the Islamic fundamentalism of the Muslim Brotherhood (Zollner 2009) and at odds with the secular and socialist outlook of the regime. Hafiz al-Assad fought and prevailed over the Islamic fundamentalist uprising during the period from 1978 to 1982. In spite of timid reforms, the regime of his son, Bashar al-Assad, was not able to accommodate either liberal or Islamic fundamentalist opposition. Tensions escalated until the Civil war broke out in 2011.

With the spread of the Syrian conflict to Iraq, it is important to recognize that Iraq is also a divided country with different religious confessions – Sunni and Shia – and significant non-Arab minority populations (e.g., Kurds). The sponsors in Iraq are Iran (for the Shia) and Saudi Arabia (Arab Sunni). Kurds and Christian minorities apparently have only the goodwill and wishful thinking of western powers. Therefore, the conflict that created and allowed ISIS to become a powerful combatant group and, at the same time, a terrorist group with lethal attacking capacity, involves multiple combatants and sponsors.

The objective of this paper is to formulate and analyze a dynamic game with multiple combatants and sponsors. With the real case of the conflict in Syria as motivation, the model has three combatants and three sponsors. The game is a proxy war, in which the combatants fight on behalf of their sponsors. Given the prominence of ISIS, we assume that sponsors 1 and 2 have a strong aversion to sponsor 3's proxy. Because of this strong aversion, we assume the existence of positive spillovers between sponsors 1 and 2.

The paper is organized as follows. In the next section we analyze the case of Lebanon, which is a closely related historical case to Syria and Islamic terrorism. We then review relevant literature in Section 3. We present a two-stage dynamic game in Section 4. The backward solution of the model for pure strategies appears in Sections 5 and 6. Section 7 presents the mixed strategies equilibrium. Concluding remarks appear in Section 8.

2 Historical Evidence: Proxy Civil War in Lebanon

History can guide us in formulating the model because, curiously and coincidentally, the closest example of what is presently happening in Syria – a conflict involving multiple ethnic-religious-political combatants and foreign sponsors – happened in Lebanon (Syria's neighbor and a closely related state) during the last quarter of the past century. In the Lebanese Civil war (1975-1990) there were multiple players (see, for example, Mansfield, 1990). On the combatants' sides, there were more than a dozen or so factions with heavily armed militias (Held, 2006). In terms of religion, the combatants were Christian, Sunni, Shia, and Druze. Ethnically they were Arab and non-Arab. Politically they were initially left and right wing, divided along the cold war lines, with fascists, Arab nationalists, Muslim Shia fundamentalists, and socialists. The conflict also had a massive participation of foreigners, such as Palestinian terrorists of the PLO.

Active sponsors in the Lebanese civil war were Israel (Christian militias), Iran (Shia), Saudi Arabia (Sunni), and Syria (defending its own interests and keeping alliances with different groups at different times). Some of these sponsors such as Syria and Israel intervened militarily in the conflict. A specific target was to limit and fight the terrorists of the PLO.

It is important to stress the point made by Goldschmidt and Davidson (2010) that in these Middle-eastern conflicts people may fight for reasons other than religion, nationality, class interest, or ideology. The loyalties are also related to family, patronage, habit, and neighborhood of habitation (Goldschmidt and Davidson, 2010, p. 356). When the political equilibrium broke down in Lebanon, territory was divided into a mosaic of mini-states, each with its clan group leader (Held, 2006, p.305). Moreover, group alliances were quite volatile. For instance, in Lebanon, after defeating the Palestinians, the Shiite Muslims fought among themselves. Hizbollah, backed by Iran, fought the Syrian-backed Amal Movement. The Christian Maronites also fought among themselves, with General Awn being supported by Saddam Hussein's Iraq and Harawi factions supported by Syria (Goldschmidt and Davidson, 2010, p. 409).[7]

The civil war in Lebanon "officially ended" with the Taif Agreement, and some peace and normalcy was achieved. The Taif Agreement (see Salem, 2011) ended the political accord of the 1926 constitution in which the Presidency, reserved to the Maronite Christians, had too much power. It created a polity with shared governance among the President (Christian), the Prime Minister (Sunni) and the President of the Chamber of Deputies (Shia).

As regards terrorism, while the PLO authority and its operatives were expelled from Lebanon, the conflict created and nurtured Hezbollah, a Shia militia linked to Iran and supported by the Syrian regime (see, for example, Norton, 2007). Among Hezbollah's main feats, the group was responsible for the terrorist attack on an American barracks in 1983, which killed 241 American service members and triggered the withdrawal of American troops from Lebanon (Shultz, 1993). After the Taif Agreement, Hezbollah became a force to be reckoned with, evolved into an army in itself, occupied territory in southern Lebanon, and was the main reason of the 2006 war with Israel (Burleigh, 2009, 348-349).

The irony of the history of the Lebanese conflict is that one of the main motivations for international participation in it was to fight and defeat Palestinian terrorists. They obtained a pyrrhic victory with the expulsion of the PLO from Lebanon. But in the end terrorism prevailed because it did not destroy the PLO and in Lebanon a new and different form of terrorism (based on Islamic fundamentalism) was born with Hezbollah.

Lebanon's proxy war example leads us to ask whether or not external sponsors should get involved with these types of multilateral religious-politico and ethnic conflicts to get rid of one of the major combatants, which happens to be a terrorist organization. The increasing involvement of the U.S. and Russia in the Syrian conflict was trigged by the growth and advances of ISIS. If foreign powers get involved as sponsors in the Syria conflict to destroy ISIS, what are the possible strategies available to them? What outcomes are possible? What are the chances of victory over ISIS, if U.S. and Russian actions have a positive impact on each other? Should Russia and western powers form an alliance – or perhaps coordinate their activities – to defeat the common enemy? Can terrorism be defeated?

[7] For a theoretical model of faction behavior and competition or cooperation, see Siqueira (2005).

3 Related Literature

In this paper we present and examine a dynamic contest game with multiple agents to answer the questions posed above.[8] The definition of a contest is a game where the players exert effort in order to increase their probability of winning a reward. Our game is a dynamic two-stage game with three sponsors (principals) that provide resources to support three combatants (proxies) who exert fighting effort. The combatants fight each other in the second stage of the game. As in the Syrian conflict, in which the U.S. and Russia both have aversion to ISIS, we assume that sponsors 1 and 2 have strong aversion to sponsor 3's proxy but not against each other; on the contrary, they have positive spillovers between them.

War is often modelled as a bargaining problem (e.g., Anderton and Carter, 2009, and Baliga and Sjostrom, 2013), following Schelling's (1966) observation that military power is a special case of bargaining power. According to Brito and Intrilligator (1985) conflicts happen because of incomplete information, which prevents rational agents from negotiating transfer payments. Jackson and Morelli (2007) show that there are cases in which political bias leads to war, no matter the amount of transfer payments. Corchon and Yildizparlak (2013) study a war game in which war can occur with little asymmetric information. In our context, if one considers ISIS as an aggressor, then there is no doubt ISIS is not bluffing – so appeasement is not an option, but neither are transfer payments. Every player knows the nature and type of combatant that ISIS is. War is unavoidable. Thus, our model is one of complete information, in which the distribution of resources is heterogeneous.[9]

The contest success function (CSF) (see, for example, Hwang, 2012) is an essential tool for analyzing conflicts. We consider a CSF of Tullock (1980) ratio-form[10] in which the winning probability is a function of the aggregate level of effective efforts in the contest – that is, it depends on combatants' effort and sponsors' resources. As a consequence, it is not a simple lottery-CSF, but is in-line with the generalization of a CSF (e.g., Dixit, 1987; Hirshleifer, 1989, Skaperdas, 1996) in which the CSF is weighted (e.g., Dahm and Porteiro, 2008, Brown, 2011) by sponsors' resources.

The game is a dynamic contest (see Konrad, 2012), albeit a simple one played in only two periods. Generally, two-period dynamic contest games consider 2-period contests where the first

[8]For surveys on contest games see Corchon (2007) and Corchon and Serena (2016).

[9]Bevia and Corchon (2010) examine a complete information war game in which the initial distribution of resources is heterogeneous.

[10]See Choudhury and Sheremeta (2011).

period has a positive effect on the probability of winning in the second period (see Sela, 2011, Moller, 2012 and Beviá and Corchón, 2013). In our setup the sequential contest is played by different players in periods 1 and 2. In the first period it is played by the leaders (i.e., the sponsors) and in the second period by the followers (i.e., the combatants). As a result, combatants exert effort as a function of the sponsors' resources. In our game sponsors and combatants form a team in which there is no free-rider problem (e.g., Olson, 1985) on the part of the combatants, because they know that if they free-ride their survival is at stake.[11]

In our model, because sponsors 1 and 2 have strong aversion to sponsor 3's proxy, we assume that they have positive spillovers between them. If the proxy of sponsor 1 loses, it would prefer combatant 2 to win rather than 3. Similarly, if sponsor 2's proxy loses, it would prefer combatant 1 to win rather than 3. Externalities have recently been a focus of the conflict literature. Faria et al. (2016) analyzes two types of externalities – temporal and spatial – in a dynamic game between two national governments that fight a common terrorist organization. They show that when government takes into account terrorists' reactions both domestically and abroad, both terrorism and (costly) counterterror policies are reduced, irrespective of the nature of the policy externality. Oliveira et al. (2016) examine the link between coalition formation, counterterrorism (CT) and spillovers. In a symmetric model, CT and terrorism decrease with the size of the externality regardless of the degree of cooperation between nations. In the asymmetric model, as the externality of the "smaller" nation increases, the "larger" nations reduce their efforts, and the smaller nation reacts by increasing its own efforts.

In our model the probability of winning and the sponsors' payoffs vary as a function of the size of the spillover effect. However, there is no unique way of characterizing the comparative statics of the spillover effect and the answer varies from one equilibrium to another.

The contest prize in our model is victory, which is a unique prize. In the literature the prize in a winner-take-all contest generally leads players to display the highest possible level of performance, while a multiple-prize contest induces a general increase of activity (e.g., Moldovanu and Sela, 2001). In our set up the value of winning, whether it is large or small, is crucial for determining which kind of equilibrium holds.

[11] Katz et al. (1990) analyze group contest where groups vary in number of members. They find that when all members are identical, all groups exert the same aggregate effort regardless of asymmetries in group size.

Our model has multiple equilibria and participation costs are crucial to characterize and differentiate mixed strategy equilibria. Choudhury and Sheremeta (2011) show that the uniqueness of equilibrium in Tullock contests depends crucially on the specification of the cost and spillover parameters in the payoff function. They show that multiple equilibria may exist in symmetric Tullock games, even with a standard lottery CSF.

4 Model

Consider a contest in which there are three sponsors si and three combatants ci, for $i = 1, 2, 3$. Sponsor si supports combatant ci. There are two stages in the game. In period 1, si provides resource $r_i \geq 0$ to ci. Examples of such resources include arms and training. The total cost of providing resources is given by

$$tr_i^2;\ t > 0.$$

In period 2, each combatant exerts effort x_i in the contest. The *effective effort* that combatant i exerts is $r_i x_i$. Notice that for a combatant, the resource provided by the sponsor is a complementary input because it changes the productivity of the combatant. If si chooses $r_i = 0$, then the effective effort of ci will be 0 for any chosen value of $x_i \geq 0$.

Let p_i be the probability that si and ci together win the contest. This is defined as follows:

$$p_i = \frac{r_i x_i}{\theta}$$

where

$$\theta = \sum_{i=1}^{3} r_i x_i \tag{1}$$

is the aggregate effective effort in the contest. Notice that p_i is not defined if for all i either $r_i = 0$ or $x_i = 0$. In such a case, the probability that each sponsor wins is assumed to be $\frac{1}{3}$. The payoff of ci is given by the following function:

$$U_i = p_i - x_i. \tag{2}$$

In this contest, each sponsor prefers to win. However, the interesting aspect of this scenario is that sponsors do not have the same degree of aversion to the proxy agents of each rival. In particular, $s1$ and $s2$ do not dislike each other's proxy as much as they dislike $c3$. Therefore, if $s1$'s proxy loses, it would prefer $c2$ to win rather than $c3$. Similarly, if $s2$'s proxy loses, it would prefer $c1$ to win rather than $c3$. Sponsor $s3$ however only cares for whether its proxy wins or loses.

This structure mimics the case of the Syrian civil war. Combatants $c1$, $c2$ and $c3$ are respectively similar to the Assad regime, the opposition known as the Syrian National Coalition, and ISIS (Zorthian 2015). The Assad regime is supported by Russia and Iran, who are therefore similar to $s1$. The Syrian National Coalition is supported by the U.S. and other western nations, who are therefore similar to $s2$. Regarding ISIS, they are partly self-financed (Gause 2014) and partly supported by private donations (Rogin 2014).[12] Therefore $s3$ is similar to the private donors.

Among the belligerents, ISIS is the most hated and feared group. According to Gause (2014), ISIS "...has the unique ability to unite most of the players in the new Middle East cold war against it. Iran and Iran's allies detest it because of its fiercely anti-Shia ideology. The Saudis fear it as a potential domestic threat, turning Salafism into a revolutionary political ideology rather than the pro-regime bulwark it has usually been in Saudi Arabia. Turkey, the Kurds, the United States, the EU and Russia all stand to lose if ISIS wins." In order to capture this feature, we assume that the payoff of sponsor si is given by the following function:

$$W_i = (p_i + \beta p_j)\, V - tr_i^2 \text{ if } i = 1, 2 \tag{3}$$

and by

$$W_3 = p_3 V - tr_3^2 \tag{4}$$

for sponsor $s3$, with $\beta \in [0, 1]$. Note that $s1$ and $s2$ earn their lowest payoff of 0 when $s3$ wins. The parameter V is the *ex post* value of winning of the sponsor. Finally, the parameter β captures the positive externality – or spillover effect – that si enjoys if sj wins the contest, for $i, j = 1, 2$, $i \neq j$. The spillover effect implies that si $(i = 1, 2)$ can earn a positive payoff even if it does not participate in the contest. Thus, one impact of the spillover effect is that it reduces the incentives for $s1$ and $s2$ to participate in the contest.

[12] It has been alleged that ISIS also has state sponsors, but to the best of our knowledge that has not been clearly established. We therefore refrain from this issue.

Notation	Description
si	Sponsor i
ci	Combatant i
r_i	Amount of resources provided by si to ci
t	Scale parameter that determines the cost to a sponsor of providing resources
x_i	Effort of ci in the contest
p_i	Probability that si and ci together win the contest
θ	Aggregate effective effort in the contest. This captures the intensity of combat
V	*Ex post* value of winning of the sponsor
U_i	Payoff of ci
W_i	Payoff of si
β	Positive spillover to sj $(j = 1, 2; j \neq i)$ if si $(i = 1, 2)$ wins
$E_{x,r}$	Elasticity of combat effort with respect to the amount of resources
Δ	$1 + \sqrt{1 + 8(1 - \beta)}$
α_i	Probability that si participates in the contest
$p_i(1, 2, 3)$	Probability that si wins the contest when all three sponsors participate
$p_i(i, j)$	Probability that si wins the contest when si and sj participate while sk abstains
τ	Payoff of $s1$ or $s2$ (resp., $s3$) if it loses to $s3$ (resp., $s1$ or $s2$)
A	Alliance between $s1$ and $s2$

Table 1: List of Notations

We solve this game using backward induction, solving for period 2 first. The notations used in the paper are summarized in Table 1.

5 Period 2: Combat Stage

In this period, the combatants determine their effort levels. In this stage, ci solves the following problem:

$$\max_{x_i} \quad U_i.$$

There are two kinds of equilibria possible in this stage, depending upon r_1, r_2 and r_3.

5.1 All three combatants actively compete

An equilibrium in which all three combatants actively compete is described by the following proposition.

Proposition 1 *Suppose r_1, r_2 and r_3 satisfy the following inequalities:*

$$r_i \geq \frac{r_j r_k}{r_j + r_k};\ i, j, k = 1, 2, 3;\ i \neq j \text{ or } k. \tag{5}$$

In equilibrium, the effort levels of the combatants in period 2 are

$$x_i^* \left(r_{1,} r_2\right) = \frac{\left(r_i - \theta^*\right) \theta^*}{r_i^2}, \tag{6}$$

and the aggregate effective effort is

$$\theta^* = \frac{2 r_i r_j r_k}{r_i r_j + r_j r_k + r_i r_k}. \tag{7}$$

Proof. See the Appendix. ■

In this equilibrium, (5) guarantees that $x_i^* \left(r_{1,} r_2\right) \geq 0$ for $i = 1, 2, 3$. Notice that (5) can be re-written as follows:

$$\frac{1}{r_j} + \frac{1}{r_k} \geq \frac{1}{r_i}. \tag{8}$$

This means that given r_j and r_k, the value of r_i cannot be too small for it to be the case that all three combatants actively compete.

Let us consider how the aggregate effective effort θ^* changes when there is an increase in r_i. It follows from (7) that

$$\frac{\partial \theta^*}{\partial r_i} = \frac{1}{2 r_i^2} \theta^{*2} > 0. \tag{9}$$

That is, everything else remaining constant, an increase in r_i increases the aggregate effective effort. Notice from Proposition 1 that the probability of winning of ci ($i = 1, 2$) is

$$p_i^* = 1 - \frac{\theta^*}{r_i}. \tag{10}$$

Hence, it can also be shown that

$$\frac{\partial p_i^*}{\partial r_i} = -\frac{\partial}{\partial r_i} \left(\frac{\theta^*}{r_i} \right) = \frac{1}{r_i} \left(\frac{r_j + r_k}{r_i r_j + r_j r_k + r_i r_k} \right) \theta^* > 0. \tag{11}$$

We now examine the effect of a change in r_i on the effort levels of the combatants and their winning probabilities. In order to do so, let us define the elasticity of combat effort with respect to the amount of resources as follows:

$$E_{x,r} = \frac{\partial x_i^*}{\partial r_i} \frac{r_i}{x_i^*}.$$

We now have the following result:

Corollary 1 *Consider the subgame in which all contestants actively compete. In equilibrium, the elasticity of combat effort with respect to resources is given by the following:*

$$E_{x,r} = (1 - 2p_i^*) .$$

Proof. See the Appendix. ■

An important point to note from Corollary 1 is that an increase in r_i does not always increase ci's effort level x_i^*. When sponsor si spends more resources, then ci responds with more effort when its probability of winning the contest is small enough. However, ci chooses to cut back on its effort x_i^* when its probability of winning is high enough.

5.2 One combatant drops out (and the other two remain)

In this subsection, we consider equilibria of the subgame in which combatant k drops out while i and j remain. Such equilibria are described by the following proposition.

Proposition 2 *Suppose r_i, r_j and r_k satisfy the following inequalities:*

$$r_k < \frac{r_i r_j}{r_i + r_j}. \tag{12}$$

In equilibrium, the effort levels of the combatants in period 2 are:

$$x_i^{**} = x_j^{**} = \frac{r_i r_j}{(r_i + r_j)^2} \text{ for } i = 1, 2,$$

and

$$x_k^{**} = 0,$$

and the aggregate effective effort is

$$\theta^{**} = \frac{r_i r_j}{r_i + r_j}.$$

Proof. See the Appendix. ■

Notice from Proposition 1 that the probability of winning for ci $(i = 1, 2, 3)$ is

$$p_i^{**} = 1 - \frac{\theta^{**}}{r_i} = \frac{r_i}{r_i + r_j}. \tag{13}$$

Therefore, everything else remaining constant, p_i^{**} increases in r_i.

Also notice that (12) can be re-written as follows:

$$\frac{1}{r_i} + \frac{1}{r_j} < \frac{1}{r_k}.$$

This inequality will be satisfied if r_k is sufficiently small. Hence, if sk provides a relatively small amount of resources, then ck prefers to not fight at all.

6 Period 1: Resource Allocation Stage

Having fully characterized the subgame equilibrium choices of effort levels by the combatants, in this section we determine the equilibrium values of r_i, for $i = 1, 2, 3$. In particular, we are interested in identifying subgame perfect Nash equilibria in which each sponsor selects a pure strategy in the first period. We show that there are three types of equilibria in pure strategies. In the spirit of subgame perfection, each equilibrium depends on the outcome in period 2. The major results in this section are summarized in Table 2.

6.1 Type 1 Equilibrium: All three combatants actively compete in period 2

The payoff function of sponsor si $(i = 1, 2)$ is given by (3) and of $s3$ is given by (4). The equilibrium levels of resource allocation are described below in the proposition below, with the following notation: $\Delta = 1 + \sqrt{1 + 8(1 - \beta)}$.

Proposition 3 *Consider an equilibrium in which all three combatants actively compete in period*

2. *In period* 1 *of such an equilibrium, si* $(i = 1, 2)$ *provides resources of*

$$r_1^* = r_2^* = r^* = \frac{V}{t}\frac{2\Delta^2}{(8+\Delta)^2} \tag{14}$$

*and s*3 *provides resources of*

$$r_3^* = \frac{V}{t}\frac{8\Delta}{(8+\Delta)^2}. \tag{15}$$

The subsequent effort levels of the combatants in period 2 *are given by Proposition 1.*

Proof. See the Appendix. ■

In this equilibrium, the winning probabilities are given by

$$p_i^* = \frac{\Delta}{8+\Delta} \text{ for } i = 1, 2,$$

and

$$p_3^* = \frac{8-\Delta}{8+\Delta} = p_i^*\left(1 - p_i^*\right); \ i = 1, 2.$$

When $\beta = 0$, then $\Delta = 4$, in which case $p_1^* = p_2^* = p_3^* = \frac{1}{3}$.

Further, since $\frac{\partial\Delta}{\partial\beta} < 0$, it follows that

$$\frac{\partial p_i^*}{\partial\beta} < 0 \text{ for } i = 1, 2 \text{ and } \frac{\partial p_3^*}{\partial\beta} > 0. \tag{16}$$

Therefore, the larger the spillover effect, the smaller are the winning probabilities of sponsors 1 and 2, and the larger is the winning probability of $s3$. When $\beta = 1$, then $\Delta = 2$, in which case $p_1^* = p_2^* = 0.2$ and $p_3^* = 0.6$.

Let us examine conditions under which this equilibrium is valid. In the type 1 equilibrium, all three combatants actively compete in period 2. This requires (5) to hold, which in turn requires $\Delta \leq 8$. Since the maximum possible value of Δ is 4, this condition is satisfied. Therefore, if the combatants are provided the equilibrium level of resources, then each of them prefers to participate in the contest. However, that alone does not guarantee that the type 1 equilibrium occurs, because it is possible that a sponsor might find it preferable to not participate in the contest. Below, we examine the equilibrium payoff of each sponsor and find conditions under which each of them prefers to participate in the contest.

6.1.1 Payoff of Sponsors

The equilibrium payoffs of sponsors - derived using (3), (4), and Proposition 3 - are presented in the corollary below:

Corollary 2 *In the type* 1 *equilibrium, the payoffs of* $s1$, $s2$ *and* $s3$ *are:*

$$W_i^* = p_i^* \left\{ 1 + \beta - 4p_i^{*3} \frac{V}{t} \right\} V \text{ for } i = 1, 2$$

and

$$W_3^* = \left\{ 1 - 2p_i^* - p_i^{*2} \left(1 - p_i^*\right)^2 \frac{V}{t} \right\} V.$$

Let us first consider the participation decision of si ($i = 1, 2$). If this sponsor does not participate in the contest (and the others do), then its expected payoff will be

$$\frac{p_j^*}{p_j^* + p_3^*} \beta V; \; j = 1, 2, \; j \neq i. \tag{17}$$

In the above expression, $\frac{p_j^*}{p_j^* + p_3^*}$ is the conditional probability that sj will win the contest if si pulls out and in this case si will enjoy the spillover effect.[13] Now, using the facts that $p_j^* + p_3^* = 1 - p_i^*$ and $p_j^* = p_i^*$, we can re-write the outside option of si as $\frac{p_i^*}{1 - p_i^*} \beta V$ for $i = 1, 2$. The participation constraint of si is therefore

$$W_i^* \geq \frac{p_i^*}{1 - p_i^*} \beta V \tag{18}$$

Note that the value of the outside option for $s1$ and $s2$ increases in V and β. The participation constraint of $s3$ is simply $W_3^* \geq 0$. The conditions under which these participation constraints are satisfied is shown in the left panel of Figure 1. Notice that the participation constraint is satisfied when the *ex post* value of winning V is relatively small or if the spillover effect β is relatively large.

Another question is the relationship between the equilibrium payoffs of the sponsors and the spillover effect β. It is interesting to note that this relationship need not be monotonic. To see this, consider the right panel of Figure 1, which illustrates the equilibrium payoffs of the sponsors when

[13] For details, see Extended Appendix.

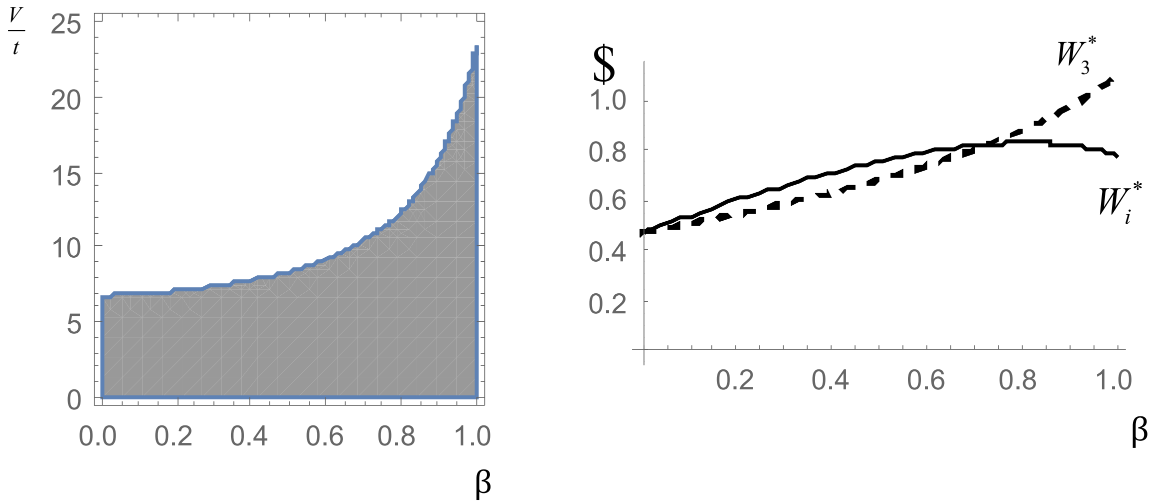

Figure 1: (a) The shaded region in the left panel shows the combinations of β and $\frac{V}{t}$ for which the type 1 equilibrium exists. In this case, the participation constraint of each sponsor is satisfied. (b) The right panel shows the payoffs of sponsor i ($i = 1, 2$) and sponsor 3 in the type 1 equilibrium for $\frac{V}{t} = 2$. The payoff function of sponsor i is non-monotonic in β.

$\frac{V}{t} = 2$. Notice that W_i^* first increases with β and then decreases. The intuition is as follows. When β increases, then si gains from the spillover effect. However the problem is that sj also gains from the spillover effect, which induces sj to reduce its expenditure. Such a reduction now hurts si. As long as the former effect is the stronger one, then an increase in β increases W_i^*. However, when the latter effect is the stronger one, then an increase in β decreases W_i^*.

6.1.2 Payoff of Combatants

The equilibrium payoffs of the combatants can be derived by plugging in the values of r_i from Proposition 3 into the payoff function of the combatants, given by (2). It can be shown that in the type 1 equilibrium, the payoff of the combatants are as follows:

$$U_i = \begin{cases} p_i^* \left[1 - p_i^{*2} \left(1 - p_i^*\right) \frac{V}{t}\right] & \text{if} \quad i = 1, 2, \\ p_3^{*2} & \text{if} \quad i = 3. \end{cases}$$

If a combatant does not participate in the contest, then its payoff is 0. Therefore, in equilibrium, every combatant must earn at least 0. To check if a combatant earns at least 0 in equilibrium, consider the payoff function of ci given in (2). Notice that $U_i > 0$ as long as $p_i^* > x_i^*$. Since

$p_i^* = \frac{r_i^* x_i^*}{\theta^*}$, this ultimately reduces to a requirement that (5) is satisfied (which we have argued is always true, since $\Delta \leq 4$).

6.2 Type 2 Equilibrium: Either $c1$ or $c2$ drops out (and others stay) in period 2

Let us now consider the equilibrium in period 1 that induces only ci ($i = 1$ or 2) to drop out in period 2. This equilibrium is presented in the proposition below.

Proposition 4 *Consider the equilibrium in which only ci ($i = 1, 2$) drops out in period 2. In period 1 of such an equilibrium, sj ($j = 1, 2$; $j \neq i$) and s3 provide resources of:*

$$r_j^{**} = r_3^{**} = \sqrt{\frac{1}{8}\frac{V}{t}} \quad \textit{for } j = 1, 2,\ j \neq i.$$

The subsequent effort levels of the combatants in period 2 are given by Proposition 2.

Proof. See the Appendix. ■

Using (12), it can be shown that this equilibrium exists only if the following condition holds:

$$\frac{V}{t} \geq 32. \tag{19}$$

This implies that when the *ex post* value of winning is sufficiently large, then $s1$ and $s2$ coordinate their actions such that only one of them actively fights. However, such coordination is not possible when the *ex post* value of winning is small.

6.2.1 Payoff of Sponsors

The equilibrium payoffs of sponsors in the type 2 equilibrium are presented in the following corollary.

Corollary 3 *Suppose in the type 2 equilibrium, si drops out and sj participates in the contest ($i, j = 1, 2$; $i \neq j$). In that case, the payoffs of the sponsors are:*

$$W_i^{**} = \frac{1}{2}\beta V$$

and

$$W_j^{**} = W_3^{**} = \frac{3}{8}V.$$

In this equilibrium, the payoff of the participants is independent of β, while the payoff of the non-participant increases in β. Further, we have the curious result that $W_i^{**} > W_j^{**}$ for $\beta > \frac{3}{4}$ – this reveals that there are strong incentives to stay out of the contest when β is sufficiently high, since si realizes a larger payoff from dropping out than sj realizes from participating.

6.2.2 Payoff of Combatants

The equilibrium payoffs of the combatants can be derived by plugging in the results from Proposition 4 into the payoff function of the combatants, given by (2). Suppose sponsor si decides not to participate in the contest. In the absence of resources from si, the corresponding combatant ci will also not participate in the contest. It can be shown that in the type 2 equilibrium, the payoffs of the combatants are:

$$U_k = \begin{cases} \frac{1}{4} & \text{if} \quad k = j, 3, \\ 0 & \text{if} \quad k = i. \end{cases}$$

Notice that theses payoffs of the combatants are independent of β in the type 2 equilibrium.

6.3 Type 3 Equilibrium: $c3$ drops out (and others stay) in period 2

Let us now consider the equilibrium in period 1 that induces only $c3$ to drop out in period 2. This equilibrium is presented in the proposition below.

Proposition 5 *Consider the equilibrium in which only* $c3$ *drops out in period* 2*. In period* 1 *of such an equilibrium,* si *(*$i = 1, 2$*) provides resources of:*

$$r_i^{***} = r^{***} = \sqrt{\frac{(1-\beta)}{8}\frac{V}{t}} \text{ for } i = 1, 2.$$

The subsequent effort levels of the combatants in period 2 *are given by Proposition 2.*

Proof. See the Extended Appendix. ■

When $\beta = 0$, the resource spent by each participant in this equilibrium is $\sqrt{\frac{1}{8}\frac{V}{t}}$, which is the same as the expenditure in the type 2 equilibrium. Consequently, when $\beta = 0$ the type 2 equilibrium and the type 3 equilibrium are the same. However, there are meaningful differences between them in the presence of spillover effects.

Let us first check the validity of this equilibrium. It was discussed above in Proposition 2 that $c3$ drops out if the following condition holds: $r_1 + r_2 \leq r_1 r_2$. Using the equilibrium values of r_1 and r_2 from Proposition 5, it follows that the following condition must hold for the Type 3 Equilibrium to exist:

$$\frac{V}{t}(1-\beta) \geq 32. \tag{20}$$

Note that (20) is a more restrictive condition than (19). When β is sufficiently large, then (20) is violated. In such a case, the Type 3 Equilibrium cannot exist.

6.3.1 Payoff of Sponsors

Below, we present the payoff of the sponsors in the type 3 equilibrium.

Corollary 4 *In the type 3 equilibrium, the payoffs of $s1$, $s2$ and $s3$ are as follows:*

$$W_1^{***} = W_2^{***} = \left(\frac{3+5\beta}{8}\right) V$$

and

$$W_3^{***} = 0.$$

We can also compare the aggregate payoffs of $s1$ and $s2$ when both type 2 and type 3 equilibria exist. Notice that $\left(\frac{3+5\beta}{8}\right) V > \max\left\{\frac{1}{2}\beta V, \frac{3}{8}V\right\}$. Hence, both $s1$ and $s2$ are better off in the type 3 equilibrium compared to the type 2 equilibrium. In contrast, $s3$ is worse off in the type 3 equilibrium.

6.3.2 Payoff of Combatants

The equilibrium payoffs of the combatants can be derived by plugging in the results from Proposition 5 into the payoff function of the combatants, given by (2). It can be shown that in the

		Equilibrium		
		Type 1	Type 2	Type 3
Quantity of	r_i $(i=1,2)$	$\frac{V}{t}\frac{2\Delta^2}{(8+\Delta)^2}$	$\sqrt{\frac{1}{8}\frac{V}{t}}$ or 0	$\sqrt{\frac{(1-\beta)}{8}\frac{V}{t}}$
Resources	r_3	$\frac{V}{t}\frac{8\Delta}{(8+\Delta)^2}$	$\sqrt{\frac{1}{8}\frac{V}{t}}$	0
Winning	p_i $(i=1,2)$	$\frac{\Delta}{8+\Delta}$	$\frac{1}{2}$ or 0	$\frac{1}{2}$
Probability	p_3	$\frac{8-\Delta}{8+\Delta}$	$\frac{1}{2}$	0
Payoff	W_i $(i=1,2)$	$p_i^*\left\{1+\beta-4p_i^{*3}\frac{V}{t}\right\}V$	$\frac{3}{8}V$ or $\frac{1}{2}\beta V$	$\left(\frac{3+5\beta}{8}\right)V$
of Sponsors	W_3	$\left\{1-2p_i^*-p_i^{*2}\left(1-p_i^*\right)^2\frac{V}{t}\right\}V$	$\frac{3}{8}V$	0
Payoff	U_i $(i=1,2)$	$p_i^*\left[1-p_i^{*2}\left(1-p_i^*\right)\frac{V}{t}\right]$	$\frac{1}{4}$ or 0	$\frac{1}{4}$
of Combatants	U_3	p_3^{*2}	$\frac{1}{4}$	0
Aggregate Effort	θ	$\frac{256\Delta}{32+4\Delta}\frac{V}{t}$	$\frac{1}{2}\sqrt{\frac{1}{8}\frac{V}{t}}$	$\frac{1}{2}\sqrt{\frac{(1-\beta)}{8}\frac{V}{t}}$

Table 2: Summary of all Pure Strategy Equilibria

type 3 equilibrium, the payoff of the combatants are:

$$U_i = \begin{cases} \frac{1}{4} & \text{if} \quad i=1,2, \\ 0 & \text{if} \quad i=3. \end{cases}$$

Notice that the payoff of $c1$ and $c2$ are independent of β in the type 3 equilibrium.

We also compare the payoffs of $c1$ and $c2$ in the type 2 and type 3 equilibria. To make a meaningful comparison, we restrict ourselves to the case in which (20) holds. The payoff of ci $(i=1,2)$ is $\frac{1}{4}$ in the type 3 equilibrium and is either $\frac{1}{4}$ or 0 in the type 2 equilibrium. Hence, if (19) holds, then $c1$ and $c2$ both weakly prefer the type 3 equilibrium over the type 2 equilibrium. It is easy to see that $c3$ prefers the type 2 equilibrium over the type 3 equilibrium.

7 Mixed Strategy Equilibrium

Until now, we have considered only pure strategy equilibria. However, there are also several mixed strategy equilibria in this game, which we now consider. Recognize that from the analysis thus far, there are some parameter values for which no equilibrium in pure strategies exists. More precisely, for an arbitrarily fixed value of β there is no pure strategy equilibrium when the value of $\frac{V}{t}$ is in the "intermediate range" for which both (18) does not hold (i.e., the unshaded region in Figure 1) and (19) does not hold. If we restrict attention to only pure strategy equilibria, then we will not have any predicted outcome for such cases.

Suppose sponsor i provides resources $r_i > 0$ with probability α_i and $r_i = 0$ with probability $(1-\alpha_i)$; $i = 1, 2, 3$. Let us describe these actions as "Participate" and "Abstain." Let $p_i(1,2,3)$ denote the probability that si wins the contest when all three sponsors participate. From (7) and (10) we have $p_i(1,2,3) = \frac{r_i(r_j+r_k)-r_j r_k}{r_i(r_j+r_k)+r_j r_k}$. Let $p_i(i,j)$ denote the probability that si wins the contest when si and sj participate but sk abstains. From (13) we have $p_i(i,j) = \frac{r_i}{r_i+r_j}$.

The expected payoff of $s1$ conditional on participating is

$$\begin{aligned}&\alpha_2\alpha_3\{p_1(1,2,3)+\beta p_2(1,2,3)\}V+\alpha_2(1-\alpha_3)\{p_1(1,2)+\beta p_2(1,2)\}V\\&+\alpha_3(1-\alpha_2)p_1(1,3)V+(1-\alpha_2)(1-\alpha_3)V-tr_1^2,\end{aligned} \tag{21}$$

whereas the expected payoff of $s1$ conditional on abstaining is

$$\alpha_2\alpha_3\beta p_2(2,3)V+\alpha_2(1-\alpha_3)\beta V+(1-\alpha_2)(1-\alpha_3)\frac{1+\beta}{3}V. \tag{22}$$

The payoff function of $s2$ can be derived analogously. Finally consider $s3$. The expected payoff of $s3$ conditional on participating is

$$\begin{aligned}&\alpha_1\alpha_2 p_3(1,2,3)V+\alpha_1(1-\alpha_2)p_3(1,3)V+(1-\alpha_1)\alpha_2 p_3(2,3)V\\&+(1-\alpha_1)(1-\alpha_2)V-tr_3^2,\end{aligned} \tag{23}$$

and conditional on abstaining is

$$(1-\alpha_1)(1-\alpha_2)\frac{1}{3}V. \tag{24}$$

Sponsor $s1$ prefers to participate (resp., abstain) if (21) is greater (resp., less) than (22). Moreover, $s1$ randomizes between participating and abstaining if (21) is equal to (22). Similarly, $s3$ randomizes if (23) is equal to (24). There are three conditions (one for each sponsor) and six variables- three α's and three r's. Therefore, this system is overdetermined and there are many possible solutions. Below, we discuss two solutions.

7.1 Type 4 Equilibrium: Mixed strategy equilibrium with $r_1 = r_2 = r$ and $r_3 = \frac{1}{2}r$

Suppose $s1$ and $s2$ choose the same amount of resource r conditional on participating while $s3$ chooses half of that. Notice that (8) is satisfied and therefore every combatant will exert a positive effort in the second period if the corresponding sponsor provides a positive amount of resources in period 1. In this case, there is a mixed strategy equilibrium as follows:

Proposition 6 *Consider a mixed strategy equilibrium in which the amount of resource that si $(i = 1, 2)$ expends conditional on participating is $r_1 = r_2 = r$ while the amount of resource that $s3$ expends conditional on participating is $r_3 = \frac{1}{2}r$. In this case, si $(i = 1, 2)$ participates with probability*

$$\alpha_i^* = 1 - \frac{3}{8}\frac{t}{V}r^2 \tag{25}$$

and $s3$ participates with probability

$$\alpha_3^* = \frac{1}{2\beta}\left\{\left(\frac{5-3\beta}{8}\right)\frac{t}{V}r^2 - 3\left(1-\beta\right)\right\}. \tag{26}$$

Proof. See the Appendix. ■

An increase in r increases the cost of participation. When the cost of participation increases, then si tries to free ride on sj $(i, j = 1, 2;\ i \neq j)$ and therefore reduces the probability of participation. Notice that $s3$ is different from $s1$ and $s2$ on two counts: *(i)* $s3$ cannot free-ride on others, and *(ii)* in equilibrium, its cost of participation is lower than others. The latter means that a unit increase in r increases the cost of participation of $s3$ by a relatively small amount. Hence, when the cost of participation increases, then $s3$'s participation probability α_3^* increases in order to take advantage of the lower participation probabilities of the others.

Next, consider the effect of β on the participation probabilities. It follows from the above expressions that β does not affect the participation probability of sponsors 1 and 2. However the participation probability of $s3$ decreases in β. The intuition is as follows. When β increases then, everything else remaining constant, sponsors 1 and 2 have a stronger incentive to not participate in the contest because they would like to free ride on each other. Therefore, in order to make them indifferent between participating and not participating, $s3$ decreases its probability of participation.

The following corollary specifies conditions under which the type 4 equilibrium exists.

Corollary 5 *The mixed strategy equilibrium described in Proposition 6 exists if the following conditions are satisfied:*

$$24\left(\frac{1-\beta}{5-3\beta}\right)\frac{V}{t} \leq r^2 \leq 8\min\left\{\frac{1}{3},\left(\frac{3-\beta}{5-3\beta}\right)\right\}\frac{V}{t}. \tag{27}$$

The range of r specified above is non-empty if

$$\beta \geq \frac{2}{3}. \tag{28}$$

Proof. See the Appendix. ■

From this corollary we see that this mixed strategy equilibrium exists when the spillover effect is sufficiently large. Recall, from our previous discussion, for an arbitrarily fixed value of β there is no pure strategy equilibrium when the value of $\frac{V}{t}$ is in the "intermediate range" for which both (18) does not hold (i.e., the unshaded region in Figure 1) and (19) does not hold. We find here that if $\beta \geq \frac{2}{3}$, then there is always an equilibrium in mixed strategies for all values of $\frac{V}{t}$. For some parameter values, the only equilibria are in mixed strategies, while for other parameter values there are both mixed strategy and pure strategy equilibria.

It is interesting to compare the winning probabilities of this mixed strategy equilibrium with those of the pure strategy equilibria discussed above. Suppose all three sponsors enter the contest (that is, spend a positive amount of resources instead of 0) in this mixed strategy equilibrium. This situation is comparable to the type 1 pure strategy equilibrium. Therefore, let us compare the winning probabilities in these two cases. In this mixed strategy equilibrium, $p_i(1,2,3) = \frac{1}{2}$ for $i = 1, 2$ and $p_3(1,2,3) = 0$. Therefore, $s1$ and $s2$ each has a 50% chance of winning and $s3$ has no chance of winning conditional on the event that all three sponsors choose to spend a positive amount of resources. Also, these winning probabilities do not depend upon β. In contrast, in the type 1 pure strategy equilibrium, the winning probabilities of all three players vary depending upon β (see (16)).

It can be shown that the winning probabilities do not depend on β in the other situations also. Consider the case in which sj ($j = 1, 2$) chooses not to spend any resources while the other two spend a positive amount. In this case, the winning probabilities in this mixed strategy equilibrium are $p_i(i,3) = \frac{2}{3}$ for $i = 1,2$ and $p_3(i,3) = \frac{1}{3}$. Finally, consider the case in which $s3$ chooses not to spend any resources while the other two spend a positive amount. In this case, the winning probabilities are $p_i(1,2) = \frac{1}{2}$ for $i = 1,2$ and $p_3(1,2) = 0$. It follows from the discussion above that in this mixed strategy equilibrium, the winning probabilities are always independent of β. This is however not the case for the pure strategy equilibria.

Next, we consider the equilibrium payoffs of the three sponsors. In the mixed strategy equilibrium, the payoff of each sponsor is simply equal to its payoff if it chooses to abstain (since it is indifferent between participating and abstaining). It follows that the equilibrium payoff of $s3$ is $W_3 = \frac{3}{64}\frac{t^2}{V}r^2$, which is independent of β. It can also be shown that in this case $\frac{\partial W_i}{\partial \beta} > 0$ for $i = 1,2$. In the pure strategy equilibrium of type 1, we found that these payoff functions vary depending on β and, in particular, W_i $(i = 1,2)$ has a non-monotonic relationship with β.

7.2 Type 5 Equilibrium: Mixed strategy equilibrium with $r_1 = r_2 = r_3 = r$

Suppose that, conditional upon participating, all three sponsors choose the same amount of resource r. Recognize that (8) is satisfied, so that every combatant will exert a positive effort in the second period if the corresponding sponsor provides a positive amount of resources in period 1. In this case, there is a mixed strategy equilibrium as follows:

Proposition 7 *Consider a mixed strategy equilibrium in which the amount of resource that each sponsor expends conditional on participating is r. In this case, si $(i = 1,2)$ participates with probability*

$$\alpha_i^{**} = 2 - 3\frac{t}{V}r^2$$

and $s3$ participates with probability

$$\alpha_3^{**} = 2 - 3\left(\frac{1-\beta}{1-2\beta}\right)\frac{t}{V}r^2.$$

The proof of this result is omitted, since the argument mimics that of the proof of Proposition 6. In this equilibrium, the probability of participation decreases in the cost of participation (i.e., t) for every sponsor. In contrast, in the type 4 mixed strategy equilibrium we saw that α_3^* was increasing in the cost of participation. Shifting attention to changes in r, for this equilibrium the participation probabilities of every sponsor decrease in response to an increase in r, whereas in the type 4 equilibrium $s3$'s participation probability is decreasing in r. This is due to the fact that in the type 4 equilibrium (for which $s3$ would choose a relatively low level of r_3) the cost of participation is relatively low; therefore, $s3$ chooses a greater probability of participation. In contrast, in the type 5 equilibrium (for which $s3$ would choose a relatively higher level of r_3), $s3$'s cost of participation is relatively higher; therefore it prefers to save cost by reducing its own participation probability. It can also be shown that α_3^{**} decreases in β and this is similar to our finding for the type 4 equilibrium.

Below, we describe the conditions under which this equilibrium exists.

Corollary 6 *The mixed strategy equilibrium described in Proposition 7 exists if the following conditions are satisfied:*

$$\frac{1}{3}\left(\frac{1-2\beta}{1-\beta}\right)\frac{V}{t} \leq r^2 \leq \frac{2}{3}\left(\frac{1-2\beta}{1-\beta}\right)\frac{V}{t}.$$

The range of r specified above is non-empty if

$$\beta < \frac{1}{2}.$$

It follows from the above corollary that this mixed strategy equilibrium exists when the spillover effect is sufficiently small. It can also be argued that if $\beta < \frac{1}{2}$, then there is always an equilibrium in mixed strategies for all values of $\frac{V}{t}$. Again, for some parameter values, the only equilibria are in mixed strategies, while for other parameter values there are both mixed strategy and pure strategy equilibria.

Next, let us consider the winning probabilities in this mixed strategy equilibrium. It can be shown that $p_i(1,2,3) = \frac{1}{3}$ for $i = 1,2,3$ and $p_i(i,j) = \frac{1}{2}$ for $i,j = 1,2,3$, $i \neq j$. Notice that these winning probabilities do not depend upon β. Regarding the equilibrium payoffs, it can be shown that W_3 is independent of β while $\frac{\partial W_i}{\partial \beta} > 0$ for $i = 1,2$. All of these results on winning probabilities and payoffs are similar to our findings for the type 4 equilibrium.

As we mention above, there are other mixed strategy equilibria also. However, they do not admit an analytical solution and we exclude them from the discussion.

7.3 Discussion on mixed strategy equilibria

Below, we discuss the major takeaways from our analysis of mixed strategy equilibria. First, considering mixed strategies is particularly appropriate since for some parameter values no pure strategy equilibria exist (i.e., when thinking of an equilibrium as a predicted outcome, for these parameter values we would not have a predicted outcome if we restricted attention to only pure strategy equilibria). Moreover, for many of the parameter values for which no pure strategy equilibria exists, we have been able to explicitly identify and provide a closed form solution for a class of mixed strategy equilibria, thereby offering a predicted outcome in such cases.

Second, given a set of parameter values, a mixed strategy equilibrium predicts a distribution of outcomes rather than one specific outcome. This makes the solution set much richer when we consider mixed strategies. For example, a mixed strategy equilibrium can be used to predict that the number of sponsors that ultimately participate in a contest can range from 0 to 3. If the number of sponsors is 0, then it means that war does not occur at all. Similarly, if only one sponsor enters the contest, then this sponsor wins without a fight. This is an example of a peaceful conquest. In this model, these two outcomes can never emerge in a pure strategy equilibrium. However, each of these outcomes can occur with positive probability in a mixed strategy equilibrium. In this sense, the mixed strategy equilibria strengthens our analysis.

In addition, the mixed strategy equilibria provides a different characterization of the role of β. Regarding the winning probabilities, we find that they often depend on β when the equilibrium is in pure strategies. However, when the equilibrium is in mixed strategies, then these probabilities no longer depend on β. Similarly, the payoff function of $s3$ (who is disliked the most by $s1$ and $s2$) is also independent of β when the equilibrium is in mixed strategies, which was not the case in pure strategies. Judging from these aspects, it seems that β has less influence on the endogenous variables when the equilibrium is in mixed strategies. However, that does not mean that β is irrelevant in the case of mixed strategies. Its importance lies in specifying the domain of a solution. For example, we know that the type 4 equilibrium cannot hold if β is sufficiently small, whereas the type 5 equilibrium cannot hold if β is sufficiently large.

8 Extensions

8.1 Negative outside option

In this paper, we assumed that si's $(i = 1, 2)$ payoff is 0 when $s3$ wins and conversely $s3$'s payoff is also 0 when si wins. In other words, the outside option of a sponsor is assumed to be 0.[14] In some circumstances, it is plausible that this outside option might take a negative value. For example, if ISIS wins the Syrian Civil War, it is quite likely that it would be in a better position to plan attacks against the U.S. or Russia and therefore their payoff might be negative. To capture these kinds of situations, we assume that the payoff of si $(i = 1, 2)$ is $-\tau$ when $s3$ wins; $\tau > 0$. To keep the model symmetric, we assume that the payoff of $s3$ is also $-\tau$ when si $(i = 1, 2)$ wins.

The impact of this change depends on the equilibrium in consideration. For all pure strategy equilibria, it increases the incentive of the players to participate in the contest. Consider the case of the Type 1 equilibrium. In this case, it follows from the discussion below (17) that the payoff of si $(i = 1, 2)$ from abstaining is $\frac{p_j^* \beta V - \tau p_3^*}{p_j^* + p_3^*}$ for $j = 1, 2, j \neq i$. Since $p_j^* = p_i^*$, from this expression it follows that the participation constraint of si $(i = 1, 2)$ is given by $W_i^* \geq \frac{p_i^* \beta V - \tau p_3^*}{1 - p_i^*}$. Notice that the right hand side of this expression is decreasing in τ. Hence, with the introduction of a negative outside option, the participation constraint for si is satisfied for a larger range of parameter values. The participation constraint for $s3$ is $W_3^* \geq -\tau$. Thus, the participation constraint for $s3$ is also satisfied for a wider range of parameter values in the case of a negative outside option.

For the mixed strategy equilibria, the effect of a negative outside option is not so obvious. First consider the payoff function of $s3$. The expected payoff of $s3$ conditional on participating is

$$\begin{aligned} &(23) - \alpha_1 \alpha_2 p_1 (1,2,3) \tau - \alpha_1 \alpha_2 p_2 (1,2,3) \tau \\ &- \alpha_1 (1 - \alpha_2) p_1 (1,3) \tau - (1 - \alpha_1) \alpha_2 p_2 (2,3) \tau \end{aligned} \tag{29}$$

and its payoff conditional on abstaining is

$$(24) - \left[1 - (1 - \alpha_1)(1 - \alpha_2) \frac{1}{3} \right] \tau. \tag{30}$$

[14] We still allow for spillover affects between $s1$ and $s2$. Therefore, the term "outside option" refers to the payoff of a sponsor when the worst adversary wins.

Therefore a negative outside option reduces the expected payoff of $s3$ both when it participates and when it abstains. Without more specific information about the probabilities (e.g., $p_3(1,2,3)$ and $p_3(1,3)$) it is not possible to say which of these terms is reduced by more.

Let us consider the case of Type 5 Equilibrium, for which (29) can be written as

$$\begin{aligned} & (23) - \frac{2}{3}\alpha_i^2\tau - \alpha_i(1-\alpha_i)\tau \\ = & (23) - \left[\alpha_i - \frac{1}{3}\alpha_i^2\right]\tau, \end{aligned} \tag{31}$$

and (30) can be written as

$$(24) - \left[1 - \frac{1}{3}(1-\alpha_i)^2\right]\tau. \tag{32}$$

In the Type 5 mixed strategy equilibrium, si ($i = 1, 2$) participates with probability $\alpha_i = 2 - \frac{3tr^2}{V+\tau}$, for which $\frac{\partial \alpha_i}{\partial \tau} > 0$. That is, any worsening of the outside option of $s3$ (captured by an increase in τ) results in an increase in the participation probability of si. This result is similar to what we obtained for the case of pure strategy equilibria, but the intuition is different. In the mixed strategy equilibrium, si keeps $s3$ indifferent between participating and abstaining. When τ increases, participating becomes relatively more attractive for $s3$, and in order to restore indifference, si increases α_i.

For a reasonably large value of α_i, an increase in τ makes abstention more attractive to si. Therefore, in order to keep si ($i = 1, 2$) indifferent between participating and abstaining, $s3$ decreases α_3. This is an interesting result because it shows that with a negative outside option, there is a possibility for ISIS to reduce its probability of participation in the conflict.

8.2 Alliance Formation

In this paper, we have allowed for implicit coordination between $s1$ and $s2$ (in the Type 2 equilibrium), but have not considered a formal alliance between them. Below, we discuss the impact of such an alliance. We will use the results of the Type 2 equilibrium as a benchmark.

Let W_A denote the aggregate payoff of the alliance. This is given by

$$W_A = p_A(1+\beta)V - tr_A^2,$$

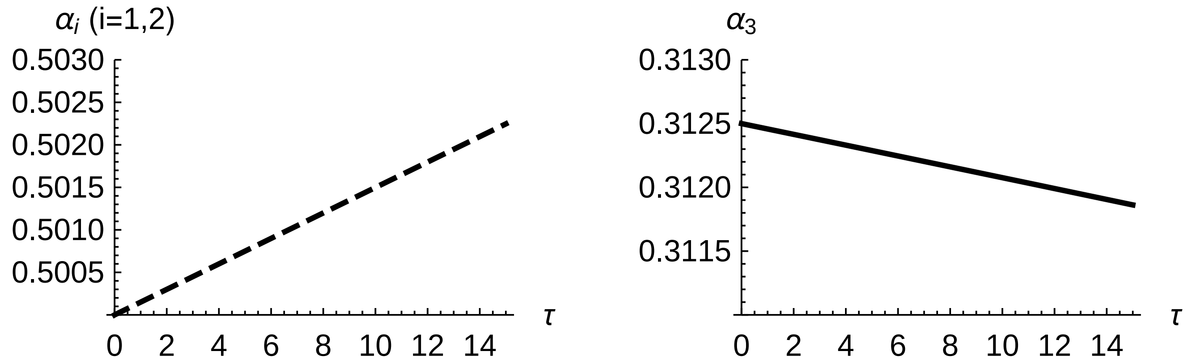

Figure 2: The left panel is a plot of α_i and the right panel is a plot of α_3 for the Type 5 Equilbrium. The parameter values are as follows: $V = 10,000$, $t = 50$, $r = 10$ and $\beta = 0.1$. Also, τ is assumed to lie between 0 and 15.

where p_A is the probability of the alliance winning the contest and r_A is the total amount of resources that the sponsors provide. It follows from (13) that $p_A = \frac{r_A}{r_A+r_3}$.

The alliance maximizes W_A with respect to r_A. In equilibrium, the probabilities of winning for the alliance and $s3$ respectively are $\hat{p}_A = \frac{1}{1+\sqrt{1+\beta}}$ and $\hat{p}_3 = \frac{\sqrt{1+\beta}}{1+\sqrt{1+\beta}}$. Note that $\hat{p}_A \leq 0.5$ and $\hat{p}_3 \geq 0.5$.

When $s1$ and $s2$ coordinate their actions with one of them sitting out of the contest, then the probability that $s3$ wins is 0.5. However when $s1$ and $s2$ form an alliance, then $s3$'s winning probability goes up. Under coordination, the participating sponsor does not internalize the positive externality and hence is unlikely to be aggressive. Therefore, $s3$ also spends less resources in the contest. In contrast, an alliance internalizes the externality and hence has a stronger incentive to be aggressive. This in turn induces $s3$ to spend more resources and consequently the winning probability of the alliance is lower under an alliance than under coordination. This result implies that an alliance of U.S. and Russia, as apparently being considered by the Trump administration, may lead to an increase in the winning chances of ISIS.

Finally, it can be shown that the equilibrium payoff of the alliance is

$$\hat{W}_A = \left[\hat{p}_A - 0.5\,(1+\beta)\,\hat{p}_3^2\right](1+\beta)\,V.$$

It follows from Corollary 3 that under implicit coordination, the aggregate payoff of $s1$ and $s2$ is $W_1^{**} + W_2^{**} = \frac{3+4\beta}{8}V$. From here it can be shown that $\hat{W}_A \leq W_1^{**} + W_2^{**}$, holding with equality only for $\beta = 0$. Thus, forming an alliance is a losing proposition and $s1$ and $s2$ will be better off by just coordinating their actions.

9 Concluding Remarks

This model shows that a civil war might evolve in several different ways depending upon the context of the war and the strategies followed by the belligerents. The context is captured in this model by parameters such as V (the *ex post* value of winning of the sponsor), t (a scale parameter of the cost of sponsoring a proxy), and β (the magnitude of a spillover effect). This model illustrates how the likely outcome of a proxy war can depend critically upon the values of all of these different parameters.

Equilibrium strategies can be pure or mixed. We identify conditions under which each type of equilibrium arises. There are several striking results that we find. First, we show that an increase in the spillover effect can affect the payoff of the sponsors non-monotonically. Second, we show that there are situations in which a sponsor who stays away from the contest is better off than a sponsor who participates. Third, using mixed strategies, we show that there are situations in which sponsor 3's participation probability increases with an an increase in the cost of participation. Fourth, a worsening of the outside option of a sponsor can decrease the probability of participation under some circumstances.

The analysis adds to our understanding of proxy wars by characterizing the different kinds of equilibria. It shows that such wars can evolve in many non-obvious ways. These predictions can be tested, an exercise which we leave for future research.

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Appendix

A Proof of Proposition 1

Consider $c1$ and $c2$. In the combat subgame, each of them solve the following problem:

$$\max_{x_i} \quad U_i \quad = \quad \tfrac{r_i x_i}{\theta} - x_i$$

for $i = 1, 2$. It follows from the above that

$$\frac{\partial U_i}{\partial x_i} = \frac{k r_i - r_i x_i \frac{\partial \theta}{\partial x_i}}{\theta^2} - 1 \text{ for } i = 1, 2.$$

Since

$$\frac{\partial \theta}{\partial x_i} = r_i \text{ for } i = 1, 2,$$

therefore,

$$\frac{\partial U_i}{\partial x_i} = \frac{r_i \sum_{j \neq i} r_j x_j}{\theta^2} - 1.$$

Also,

$$\frac{\partial^2 U_i}{\partial x_i^2} = -2 \frac{r_i^2 \sum_{j \neq i} r_j x_j}{\theta^3} < 0,$$

that is, U_i is a concave function of x_i. At the optimum, $\frac{\partial U_i}{\partial x_i} = 0$. Hence, at the optimum,

$$r_i \sum_{j \neq i} r_j x_j = \theta^2.$$

For convenience, we re-write the first order condition as follows:

$$r_i \left(\theta - r_i x_i\right) = \theta^2. \tag{33}$$

It follows from (33) that

$$r_i x_i = \theta \left(1 - \frac{\theta}{r_i}\right). \tag{34}$$

By substituting (34) into (1), we obtain the following:

$$\begin{aligned}\theta &= \sum_{i=1}^{3} r_i x_i \\ &= \theta \sum_{j \neq i} \left(1 - \frac{\theta}{r_i}\right).\end{aligned}$$

Hence,

$$1 = \sum_{j \neq i} \left(1 - \frac{\theta}{r_i}\right) = 3 - \theta \sum_{i=1}^{3} \frac{1}{r_i},$$

that is,

$$\theta = \frac{2}{\sum_{i=1}^{3} \frac{1}{r_i}}. \tag{35}$$

The above expression yields the equilibrium value of θ given in (7). The equilibrium value is denoted by θ^*.

The equilibrium effort level of x_i is therefore given by (34) and (35).

B Proof of Corollary 1

(a) We first determine the value of $\frac{\partial}{\partial r_i}(r_i x_i^*)$. It follows from (34) that

$$r_i^2 x_i^* = (r_i - \theta^*)\,\theta^*.$$

Differentiating both sides with respect to r_i and using (9), we obtain the following:

$$\begin{aligned}2r_i x_i + r_i^2 \frac{\partial x_i^*}{\partial r_i} &= \left(1 - \frac{\partial \theta^*}{\partial r_i}\right)\theta^* + (r_i - \theta^*)\frac{\partial \theta^*}{\partial r_i} \\ &= \theta^* + (r_i - 2\theta^*)\frac{\partial \theta^*}{\partial r_i}. \\ &= \theta^* + \frac{1}{2r_i^2}(r_i - 2\theta^*)\,\theta^{*2}.\end{aligned}$$

Now substitute the expression for x_i from (6) in the left hand side of the above expression to obtain the following:

$$2\frac{(r_i - \theta^*)\,\theta^*}{r_i} + r_i^2 \frac{\partial x_i^*}{\partial r_i} = \theta^* + \frac{1}{2r_i^2}(r_i - 2\theta^*)\,\theta^{*2}.$$

Finally, we substitute (10) in the above expression to obtain the following:

$$\frac{r_i^2}{\theta^*}\frac{\partial x_i^*}{\partial r_i} = p_i^* \left(1 - 2p_i^*\right).$$

The left hand side of the above expression can be written in the following form:

$$p_i^* E_{x,r}.$$

Hence, we have the following result:

$$E_{x,r} = \left(1 - 2p_i^*\right).$$

C Proof of Proposition 2

First consider combatants ci and cj. Following the same steps as in the proof of Proposition 1, the first order conditions can be shown to be the following:

$$\frac{\partial U_i}{\partial x_i} = \frac{r_i r_j x_j}{\theta^2} - 1 \ = 0 \tag{36}$$

and

$$\frac{\partial U_j}{\partial x_j} = \frac{r_i r_j x_i}{\theta^2} - 1 \ = 0.$$

Therefore, in equilibrium,

$$x_i = x_j = x.$$

Hence, the aggregate effective effort is given by

$$\theta = r_i x_i + r_j x_j = \left(r_i + r_j\right) x. \tag{37}$$

It follows from (36) and (37) that

$$r_i r_j x = \left(r_i + r_j\right)^2 x^2,$$

that is,

$$x_i^{**} = x_j^{**} = \frac{r_i r_j}{(r_i + r_j)^2}$$

and

$$\theta^{**} = \frac{r_i r_j}{r_i + r_j}.$$

We also need to check that

$$\frac{\partial U_k}{\partial x_k}|_{x_k=0} < 0.$$

It can be shown that when $x_i = x_i^{**}$ and $x_j = x_j^{**}$, then

$$\frac{\partial U_k}{\partial x_k}|_{x_k=0} = \frac{r_k}{\theta^{**}} - 1.$$

Hence,

$$\frac{\partial U_k}{\partial x_k}|_{x_k=0} < 0 \text{ if } r_k < \theta^{**} = \frac{r_i r_j}{r_i + r_j}.$$

Hence, the result follows.

D Proof of Proposition 3

First consider sponsors $s1$ and $s2$. The payoff function of these sponsors is given by (3). By differentiating this expression with respect to r_i, we obtain the following:

$$\frac{\partial W_i}{\partial r_i} = \left(\frac{\partial p_i^*}{\partial r_i} + \beta \frac{\partial p_j^*}{\partial r_i}\right) V - 2tr_i \text{ for } i = 1, 2;\ i \neq j.$$

In the above expression, the values of $\frac{\partial p_i^*}{\partial r_i}$ and $\frac{\partial p_j^*}{\partial r_i}$ are given by (11). Substituting these values into the expression for $\frac{\partial W_i}{\partial r_i}$, we obtain the following:

$$\begin{aligned} \frac{\partial W_i}{\partial r_i} &= \frac{r_j + (1-\beta) r_3}{r_i r_j + r_3 (r_i + r_j)} \theta^* V - 2tr_i \\ &= 2 \left\{ \frac{r_j + (1-\beta) r_3}{[r_i r_j + r_3 (r_i + r_j)]^2} r_j r_3 V - t \right\} r_i. \end{aligned}$$

In an interior solution, $\frac{\partial W_i}{\partial r_i} = 0$, that is,

$$\frac{r_j + (1-\beta) r_3}{[r_i r_j + r_3 (r_i + r_j)]^2} r_j r_3 V = t \text{ for } i = 1, 2;\ i \neq j. \tag{38}$$

For this to be a valid solution, the second order condition must hold. Let us therefore check the second order conditions. It can be shown that

$$\begin{aligned} \frac{\partial^2 W_i}{\partial r_i^2} &= -4 \frac{r_j + (1-\beta) r_3}{[r_i r_j + r_3 (r_i + r_j)]^3} r_i r_j r_3 (r_j + r_3) V \\ &\quad +2 \left\{ \frac{r_j + (1-\beta) r_3}{[r_i r_j + r_3 (r_i + r_j)]^2} r_j r_3 V - t \right\}. \end{aligned}$$

In an interior solution, the second term above must be 0. Hence in an interior solution,

$$\frac{\partial^2 W_i}{\partial r_i^2} = -4 \frac{r_j + (1-\beta) r_3}{[r_i r_j + r_3 (r_i + r_j)]^3} r_i r_j r_3 (r_j + r_3) V < 0$$

if

$$r_i > 0,\ r_j > 0 \text{ and } r_3 > 0.$$

Therefore, the second order condition is satisfied if all three sponsors spend a positive amount.

Sponsor $s3$'s payoff function is given by (4). By applying a similar argument, the first order condition in this case can be shown to be the following:

$$\frac{(r_i + r_j) r_i r_j}{[r_i r_j + r_3 (r_i + r_j)]^2} V = t. \tag{39}$$

Next, notice from (38) that

$$r_i = r_j = r.$$

Hence, the first order conditions (38) and (39) can be written as follows:

$$\frac{r + (1-\beta) r_3}{r^2 (r + 2r_3)^2} r r_3 V = t$$

and

$$\frac{2r^3}{r^2 (r + 2r_3)^2} V = t. \tag{40}$$

The above two conditions can be combined to imply the following:

$$2r^2 - r_3 r - (1-\beta)\, r_3^2 = 0.$$

Hence,

$$r(r_3) = \frac{1}{4} r_3 \left\{ 1 \pm \sqrt{1 + 8(1-\beta)} \right\}$$

are two candidate solutions. Among these two solutions,

$$\frac{1}{4} r_3 \left\{ 1 - \sqrt{1 + 8(1-\beta)} \right\}$$

is negative and is not a valid value for r.

Hence, in equilibrium,

$$\begin{aligned} r(r_3) &= \frac{1}{4} r_3 \left\{ 1 + \sqrt{1 + 8(1-\beta)} \right\} \\ &= \frac{1}{4} \Delta r_3. \end{aligned} \tag{41}$$

By substituting (41) into (40), we obtain (15). Finally, we obtain (14) from (41) and (15).

E Proof of Proposition 4

The payoff function of sj $(i = 1, 2;\ j \neq i)$ is given by

$$W_j = p_j^{**} V - t r_j^2$$

where it follows from (13) that

$$p_j^{**} = \frac{r_j}{r_j + r_3}.$$

Hence, sj solves the following first order condition:

$$\frac{\partial W_j}{\partial r_j} = \frac{\partial p_j^{**}}{\partial r_j} V - 2 t r_j = 0.$$

Since

$$\frac{\partial p_j^{**}}{\partial r_j} = \frac{r_3}{\left(r_j + r_3\right)^2},$$

therefore, the first order condition for sj can be re-written as follows:

$$r_j V = 2tr_3 \left(r_j + r_3\right)^2. \tag{42}$$

Similarly, it can be shown that $s3$ solves the following first order condition:

$$r_3 V = 2tr_j \left(r_j + r_3\right)^2.$$

It follows from the above two first order conditions that

$$r_3 = r_j = r. \tag{43}$$

By substituting (43) into (42), we obtain the following:

$$V = 8tr^2.$$

Hence the result follows.

F Proof of Proposition 6

In the mixed strategy equilibrium, $s3$ must be indifferent between participating and not participating. Hence, (23) must be equal to (24). Further, $r_1 = r_2 = r$ and $r_3 = \frac{1}{2}r$. We also look for an equilibrium in which $\alpha_1 = \alpha_2$. These conditions imply that the following equation must be satisfied in the mixed strategy equilibrium:

$$\alpha_1 \left(1 - \alpha_1\right) \frac{2}{3} V + \left(1 - \alpha_1\right)^2 V - t\frac{1}{4} r^2 = \left(1 - \alpha_1\right)^2 \frac{1}{3} V$$

which upon simplification reduces to (25).

Next, consider $s1$. Sponsor $s1$ must also be indifferent between participating and not participating. By equating (21) to (22) and simplifying, we obtain (26).

G Proof of Corollary 5

Since $\alpha_i^* \geq 0$, therefore the following condition must hold:

$$r^2 \leq \frac{8}{3}\frac{V}{t}.$$

Also, $\alpha_3^* \geq 0$ and this implies the following:

$$r^2 \geq 24\left(\frac{1-\beta}{5-3\beta}\right)\frac{V}{t}.$$

Finally, $\alpha_3^* \leq 1$ imposes the following restriction:

$$r^2 \leq 8\left(\frac{3-\beta}{5-3\beta}\right)\frac{V}{t}.$$

Summarizing the above restrictions, we obtain (27).

We also need to ensure that the upper bound of r exceeds its lower bound. Notice that $8\left(\frac{3-\beta}{5-3\beta}\right)\frac{V}{t}$ is always greater than $24\left(\frac{1-\beta}{5-3\beta}\right)\frac{V}{t}$. However,

$$\frac{8}{3}\frac{V}{t} \geq 24\left(\frac{1-\beta}{5-3\beta}\right)\frac{V}{t}$$

only if (28) holds.

Extended Appendix

A Proof of (17)

Suppose $s1$ chooses to deviate and not participate in the contest (and let the others play their equilibrium strategies). In that case, $s1$'s payoff will be βV if $s2$ wins the contest and will be 0 otherwise. The probability that $s2$ will win the contest can be derived by setting $r_1 = 0$ in the contest success function and is given by

$$\frac{r_2^* x_2^*}{r_2^* x_2^* + r_3^* x_3^*}.$$

Let us now divide both the numerator and the denominator of the above expression by

$$r_1^* x_1^* + r_2^* x_2^* + r_3^* x_3^*$$

to obtain the following:

$$\begin{aligned} & \frac{\frac{r_2^* x_2^*}{r_1^* x_1^* + r_2^* x_2^* + r_3^* x_3^*}}{\frac{r_2^* x_2^* + r_3^* x_3^*}{r_1^* x_1^* + r_2^* x_2^* + r_3^* x_3^*}} \\ = & \frac{p_2^*}{p_2^* + p_3^*}. \end{aligned}$$

Hence we obtain the result.

B Proof of Proposition 5

The payoff function of si ($i = 1, 2$) is given by

$$W_i = \left(p_i^{**} + \beta p_j^{**}\right) V - t r_i^2 \text{ for } i = 1, 2;\ i \neq j,$$

where p_i^{**} and p_j^{**} is given by (13). Hence, the first order condition is as follows:

$$\frac{\partial W_i}{\partial r_i} = \left(\frac{\partial p_i^{**}}{\partial r_i} + \beta \frac{\partial p_j^{**}}{\partial r_i} \right) V - 2tr_i.$$

Using (13) it follows that

$$\frac{\partial p_i^{**}}{\partial r_i} = \frac{r_j}{(r_i + r_j)^2} = -\frac{\partial p_j^{**}}{\partial r_i}\ .$$

Hence the first order condition can be re-written as follows:

$$\frac{\partial W_i}{\partial r_i} = \frac{(1-\beta)\, r_j}{(r_i + r_j)^2} V - 2tr_i \tag{44}$$

In an interior solution, $\frac{\partial W_i}{\partial r_i} = 0$ and $\frac{\partial W_j}{\partial r_j} = 0$. Therefore, in an interior solution, the following equalities must hold:

$$(1-\beta)\, r_j V = 2tr_i\, (r_i + r_j)^2$$

and

$$(1-\beta)\, r_i V = 2tr_j\, (r_i + r_j)^2\,.$$

It follows from the above two conditions that in an interior solution, we must have

$$r_1 = r_2 = r. \tag{45}$$

Using (44) and (45), it can be shown that

$$r^{**} = \sqrt{\frac{(1-\beta)}{8} \frac{V}{t}}.$$