Nash Equilibrium in Games with Quasi-Monotonic Best-Responses*

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Abstract

This paper develops a new existence result for pure-strategy Nash equilibrium. For a two-player game with scalar action sets, existence entails that one reaction curve be increasing and continuous and the other quasi-increasing (i.e, not have any downward jumps). The latter property amounts to strategic quasi-complementarities. The paper provides a number of ancillary results of independent interest, including sufficient conditions for a quasi-increasing argmax (or non-monotone comparative statics), and new sufficient conditions for uniqueness of fixed points. For maximal accessibility of the results, the main results are presented in a Euclidean setting. We argue that all these results have broad and elementary applicability by providing simple illustrations with commonly used models in economic dynamics and industrial organization.

JEL codes: C72, D43, L13.

Key words and phrases: Existence of Nash equilibrium, uniqueness of of Nash equilibrium, quasi-monotone functions, non-monotone comparative statics, supermodularity, Tarski's Theorem.

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1 Introduction

In the course of developing new game-theoretic models of economic behavior, the existence of Nash equilibrium often emerges as the first critical test to discriminate between alternative candidate models. In most economic settings, a long-standing preference for pure-strategy Nash equilibrium (henceforth, PSNE) still constitutes the dominant norm. The primary requirement of existence applies to any investigation, whenever the analysis covers general functional forms. Existence of PSNE is then virtually always obtained via the application of a fixed point theorem, following a long-standing practice going back to Nash (1951). While Nash considered mixed-strategy equilibrium for finite games, Rosen (1965) extended his basic insight to the case of PSNE and Euclidean action spaces. In this traditional approach, existence follows from Brouwer's (or Kakutani's) fixed point theorem, and is thus predicated on the best response functions being continuous on compact convex action spaces. Stepping back to payoff functions, the relevant properties are joint continuity in the strategies and quasi-concavity in own action. We refer to this method as the topological approach.

A new approach to the existence of PSNE, which relies on the best response mapping being increasing and on the action spaces being complete lattices, was proposed by Topkis (1978, 1979). Based instead on Tarski's fixed point theorem for monotone functions (Tarski, 1955), this approach of an order-theoretic nature has given rise to the class of supermodular games. In addition to the existence of PSNE, this approach has also proven useful for the characterization of equilibrium learning and comparative statics properties (Vives, 1990, and Milgrom and Roberts, 1990).

The purpose of this paper is to develop a new class of games that possess PSNEs, which is not covered by either of the two paradigms. The result pertains to two-player games with scalar action sets. This result imposes different requirements on the two players' reaction curves. For one player, this curve must be both continuous and increasing, while for the other player all that is needed is that his reaction curve not possess any downward jump discontinuities (see Fig. 1).

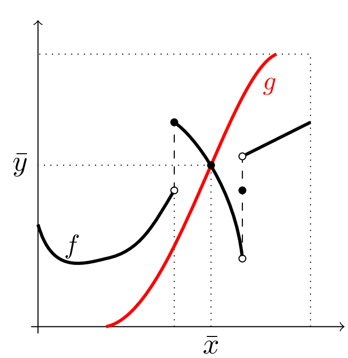

Fig. 1: Illustration of our fixed point theorem: $(\bar{x}, \bar{y}) = (g(\bar{y}), f(\bar{x}))$.

We follow a lesser known part of Tarski's (1955) classical paper, and call quasi-increasing functions that cannot have downward jumps.[1] Tarski proved an intersection point theorem between a quasi-increasing function and a quasi-decreasing function whenever these have the same domains and ranges (both complete chains) and the former starts above the latter and ends below it.[2] An important special case obtains when one takes the quasi-decreasing function to be the identity function, in which case the result boils down to a fixed point theorem for quasi-increasing functions. Interestingly, variants of this fixed point theorem have been rediscovered in the economics literature and applied a number of times to establish existence of symmetric PSNE in symmetric oligopoly settings. As a result, the existence of a symmetric PSNE then follows at once from the existence of a fixed point for the common reaction curve of all the players.[3]

While the economic applications of this result have so far all shared the critical property that the underlying game is symmetric, the starting point for the present paper is the idea that the technique may be used to establish existence of PSNE for asymmetric two-player games. To this end, it suffices to apply this fixed point theorem to the composition of the two reaction curves, one of which is a quasi-increasing function and the other an increasing and continuous function. The key observation is that the composition of two such functions is itself necessarily quasi-increasing. While this structure makes it clear that the underlying class of games overlaps with the two existing classes of games that are known to possess PSNEs, as described above, it is easy to see that it is not nested with either of them.

It is instructive to provide some basic insight into the restrictiveness of the needed assumptions for this new existence result by comparing them to those used for supermodular games. The comparison yields a mixed outcome. On the one hand, for one player, the present framework requires continuity in addition to upward monotonicity of the reaction curve. On the other hand, for the other player, the requirement of upward monotonicity has been relaxed to that of just quasi-monotonicity. Amending the standard terms to refer to these properties in the economics literature in a suitable way, one may designate such games as being characterized by continuous strategic complementarity for one player and limited strategic substitutability for the other player, or what

[1] Tarski's definition of quasi-increasing is given in purely lattice-theoretic terms for functions mapping chains into chains. The simpler version for real functions, the focus of this paper, follows Milgrom and Roberts (1994).

[2] A quasi-decreasing function is defined by the dual property of not possessing any upward jump discontinuities. See Section 2 for a more detailed explanation and illustration of Tarski's intersection point theorem.

[3] See MacManus (1962, 1964), Roberts and Sonnenschein (1976), Reinganum (1982), Vives (1990), Milgrom and Roberts (1994), Amir (1996), Amir and Lambson (2000), and Hoernig (2003), among others. For a textbook treatment with an overview of this literature, see Vives (1999).

we may call *strategic quasi-complementarity.* While the former property may be seen as combining the requirements from each of two known approaches to existence of PSNE so far, i.e., continuity and monotonicity, no such connection holds for the property of quasi-increasing reaction curves. Recall that the latter rules out only downward jump discontinuities.

Despite the partial link with the topological approach just alluded to, we stress at the outset that the present approach is lattice-theoretic. One consequence of this fact is that the result admits an order dual for games where the reaction function of one player is continuous and decreasing and that of the other player is quasi-decreasing. While the two dual results are mathematically equivalent, they tend to apply to quite different classes of economic models (as will be seen in the applications section below). In delineating the proper scope for the new result at hand, it is important to point out two basic shortcomings. The first is that the players' strategy spaces must be a totally ordered set (or chain), a limitation that stems directly from the use of Tarski's intersection point theorem, which is not valid in partially ordered sets. The second is that this basic existence result does not extend at the same level of generality to games with any number of players.

While the existence results constitute the main goal of the paper, the underlying analysis requires a number of ancillary results that are of independent interest. Some of these are crucially needed as building blocks to construct the basic machinery for the existence result, while others are useful supplementary results. The first of the building blocks is to derive some simple lemmas that capture the essential features of quasi-monotonic functions, provide a basic calculus for useful operations involving them, as well as sufficient conditions that isolate a useful subclass of quasi-monotonic functions (the so-called upper or lower Liptschitz functions). The second block consists of sufficient conditions on a parametric optimization problem to yield a quasi-monotonic argmax correspondence. To this end, we follow some existing work (in particular Granot and Veinott, 1985 and Curtat, 1996) and introduce parameter-dependent changes of variable that allow the desired conclusion to obtain in much the same way as the usual conclusion of monotonicity of argmax's.

In line with the theory of supermodular games, our basic existence results do not address uniqueness of PSNE. Nonetheless, some key ancillary results provide novel sufficient conditions for the uniqueness of PSNE. While the underlying conditions are related in one form or another to a contraction property in the reaction curves, the latter is required to hold only in a local sense. As such, the results are more reminiscent of the uniqueness results in equilibrium theory that are based on degree theory (e.g., Dierker, 1972). In addition to applying to some settings covered by the existence results given here, these uniqueness results could also apply more broadly to other classes of games.

In particular, the uniqueness results apply to symmetric PSNE of symmetric games, for which we also state the basic existence result that forms the general version of those that have appeared in specific oligopoly contexts (e.g., Roberts and Sonnenschein, 1976).

Last but not least, as with any advance in abstract theory, a crucial test is: How broad is the scope of its applicability, and how accessible is the overall tool kit developed to facilitate its use? In order to make a compelling case that both questions can be answered along very positive lines, we provide four detailed examples of well known economic models for which new results are obtained via the direct use of our results. Furthermore, as we provide all the concomitant details of the various steps needed to apply the results for the different models, the reader can easily appreciate the practical value of our results. While a large variety of applications may be given, it suffices to develop the following selection of models in some detail: A model of growth with increasing returns, a hybrid duopoly model of price-quantity competition with differentiated products, and a model of Bertrand competition with increasing returns for one firm. In addition, the Bertrand model is also used as a vehicle to illustrate some of the ancillary results of the paper.

The organization of the paper is as follows. We begin in Section 2 with a full exposition of the definitions of the new notions, their main properties, and a derivation of the basic background results for real action sets. In Section 3, the new results are stated in the form of existence results for PSNE in games, along with the associated uniqueness results. Section 4 contains a detailed discussion of some economic applications of the new theory. Section 6 provides a brief conclusion.

2 Quasi-monotone functions on $\mathbb{R}$

In the framework of real parameter and decision spaces, this section lays out all the fundamental notions and basic results that are needed as preliminaries for our new approach to the existence of pure-strategy Nash equilibrium (henceforth PSNE) for games with scalar real action spaces. The present theory is based on the properties of quasi-increasing and quasi-decreasing functions, introduced by Tarski (1955) for general totally ordered lattices (chains). We begin in Section 2.1 with the basic definitions and properties of quasi-monotone functions in one-dimensional Euclidean space along with some basic practical tests for this property. We provide useful sufficient conditions for quasi-increasingness in section 2.2, before moving to the analysis of parametric optimization problems that yield quasi-monotonic functions as optimal solutions in Section 2.3. Then Section 2.4 describes our fixed point results, which are used in section 3 for equilibrium existence in games.

2.1 Definition and basic properties

In the same article that contains his well known fixed point theorem for increasing maps on a complete lattice, Tarski (1955) also proved an intersection point theorem (his Theorem 3) on chains. To present this much less known result for the special case of real-valued functions on a real domain, the main new concepts needed are now presented.

Definition 1 *Let $X, Y \subset \mathbb{R}$. A function $f : X \to Y$ is quasi-increasing if for every $x \in X$,*

$$\limsup_{y \uparrow x} f(y) \leqslant f(x) \leqslant \liminf_{y \downarrow x} f(y). \tag{1}$$

Likewise f is quasi-decreasing if $-f$ is quasi-increasing, or $\liminf_{y\uparrow x} f(y) \geqslant f(x) \geqslant \limsup_{y\downarrow x} f(y)$. A function is quasi-monotone if it is either quasi-increasing or quasi-decreasing.

Milgrom and Roberts (1994) refer instead to functions satisfying (1) as "continuous but for upward jumps." Indeed, one crucial implication of (1) is that any jump discontinuity for a quasi-increasing function must be upward (downward for quasi-decreasing functions).

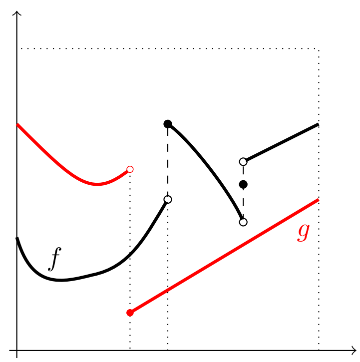

Fig. 2: f is quasi-increasing; g is quasi-decreasing.

From the above definitions, the following facts follow easily.[4] Since properties related to quasi-decreasing functions can be obtained directly from the analogs for quasi-increasing functions via standard duality arguments, we shall limit most of our discussion to the latter.

Proposition 2 *Let $\lambda \geqslant 0$, $X \subset \mathbb{R}$ and $f, g : X \to \mathbb{R}$ be two functions. The following hold:*
(i) f is continuous if and only if it is both quasi-increasing and quasi-decreasing.
(ii) If f is increasing, then f is quasi-increasing.
(iii) If f is quasi-increasing and g is continuous and increasing, $g \circ f$ and $f \circ g$ are quasi-increasing.
(iv) If $f, g : X \to \mathbb{R}$ are quasi-increasing functions, then $\lambda f + g$ is quasi-increasing.

[4]"Increasing" means weakly increasing. The same holds for all order statements. Else "strict" is added.

In Proposition 2(iii) , it is not sufficient that g be just increasing (indeed, counter-examples to this effect are easily contructed, and thus left to the interested reader).

Using these concepts of quasi-increasing and quasi-decreasing functions, Tarski (1955) proved an order-theoretic theorem (Tarski's intersection point theorem) whose real-valued version follows.

Theorem 3 *If $f : [a, b] \to \mathbb{R}$ is quasi-increasing, $g : [a, b] \to \mathbb{R}$ is quasi-decreasing, $f(a) \geqslant g(a)$ and $f(b) \leqslant g(b)$, then the set $\{x \in [a, b] : f(x) = g(x)\}$ is non-empty, and has as largest element $\vee\{x \in [a, b] : f(x) \geqslant g(x)\}$ and as smallest element $\wedge\{x \in [a, b] : f(x) \leqslant g(x)\}$.*

Theorem 3 is illustrated in Fig. 3. The conditions $f(a) \geqslant g(a)$ and $f(b) \leqslant g(b)$ are indispensable. As Theorem 3 pertains to two functions with the same domains and ranges, it is more aptly called an intersection point theorem (between two curves). We shall refer to it as such, motivated also by the need to distinguish it from the well known fixed point theorem (Tarski, 1955).

Theorem 3 may also be viewed as a more general version of the classical Intermediate-Value Theorem for the function $f - g$, where continuity is replaced by quasi-increasingness.

Milgrom and Roberts (1994) proved a fixed-point result for quasi-increasing self maps on a compact interval, which can be obtained as a special case of Theorem 3 by taking f to be the mapping of interest and g to be the 45° line. Then the extra conditions $f(a) \geqslant g(a)$ and $f(b) \leqslant g(b)$ are automatically satisfied (see Corollary 11). This result has a remarkable history in the economics literature in that special cases were independently discovered by McManus (1964) and Roberts and Sonnenschein (1976). These two studies used this fact as an intermediate result with the principal aim of establishing existence of a symmetric equilibrium for symmetric Cournot oligopoly. The key property of the underlying reaction curve (of any one firm) in symmetric Cournot oligopoly with convex costs is that all its slopes are bounded below by -1, which is a sufficient condition for a function to be quasi-increasing. Vives (2000) offers a detailed overview of this literature. For an extension to the case of symmetric firms with non-convex costs, using Topkis's monotonicity result for the first time in this literature, see Amir and Lambson (2000) and Amir (1996).

While the property of quasi-monotonicity might at first sight appear quite esoteric as far as its relevance to economic modeling is concerned, we shall derive functional and convenient sufficient conditions for quasi-increasingness that arise in quite natural ways in economics. The next subsections discuss some of these conditions and how they naturally arise.

2.2 On some subclasses of quasi-monotone functions

With the exception of the key implication of ruling out downward jumps, the general definition of quasi-increasing hardly imposes any useful structure on functions that would make them amenable to practical analysis from the perspective of economic applications. For instance, quasi-monotonic functions may fail to possess left and right limits at points of their domains, or to possess any smoothness properties. In this section, we derive sufficient conditions for quasi-monotonicity that impart crucial structure to the associated subclass of functions, akin to that enjoyed by monotonic functions. Importantly, these sufficient conditions correspond precisely to properties that are naturally satisfied when quasi-monotonic functions arise as argmax's of parametric optimization problems whose objective functions satisfy some quasi-complementarity conditions to be identified below.

An important sub-class of quasi-increasing functions that arise naturally in economics is the class of lower-Liptschitz functions, defined as follows. A function $f : X \to \mathbb{R}$ is K-lower-Liptschitz if for some $K \in \mathbb{R}$, we have $f(x) - f(y) \geqslant K(x - y)$ for all $x, y \in X$ such that $x \geqslant y$.[5] A function $f : X \to \mathbb{R}$ is K-upper-Liptschitz if $-f$ is K-lower-Liptschitz, with corresponding dual properties.

Lemma 4 *Let $X \subset \mathbb{R}$ and a function $f : X \to \mathbb{R}$ be K-lower-Liptschitz. Then*
(a) f is quasi-increasing,[6] *and*
(b) f is a function of bounded variation.

Proof. (a) The property of lower-Liptschitz can be rewritten as $x \geqslant y \Longrightarrow f(x) - Kx \geqslant f(y) - Ky$, that is, the function $\widehat{f}(x) = f(x) - Kx$ is increasing. Therefore, it is quasi-increasing by Proposition 2(ii). If we add to it the function $x \mapsto Kx$, which is continuous and therefore also quasi-increasing, the sum $x \mapsto f(x)$ is quasi-increasing by Proposition 2(v).
(b) From part (a), we have $f(x) = \widehat{f}(x) + Kx$. Hence, when $K \geqslant 0$, f is increasing and when $K < 0$, f is the difference between two increasing functions. Either way, f is of bounded variation. ∎

A useful consequence of this result is that lower-Liptschitz functions inherit all the useful properties of functions of bounded variation, such as the existence of a left and right limit at every point of their domain, a countable number of jump discontinuities, and differentiability almost everywhere.

We now characterize another useful subclass of quasi-monotone functions for use below.

[5] Recall that f is K-Liptschitz if for some $K \geqslant 0$, we have $|f(x) - f(y)| \leqslant K|x - y|$ for all $x, y \in X$.
[6] If $K \geqslant 0$, f is necessarily increasing and hence quasi-increasing. Thus this lemma is useful only for $K < 0$.

Lemma 5 *Let $r : Y \to X$, $\beta : X \times Y \to \mathbb{R}$ be two functions. Define $f : Y \to \mathbb{R}$ as $f(y) \equiv \beta(r(y), y)$. Assume that r is increasing and $x \mapsto \beta(x, y)$ is increasing for all y. The following then holds:*
(a) If β is jointly continuous, f is quasi-increasing.
(b) If $y \mapsto \beta(x, y)$ is K-lower-Liptschitz for all x, f is K-lower-Liptschitz.
(c) If X, Y are compact intervals and β is continuously differentiable in y for each x, then f is K-lower-Liptschitz.

One noteworthy aspect of the above result is that the function β is not required to be monotonic in its second argument y. In general, f as defined in this Lemma may well fail to be increasing.

We are now ready for an investigation of the emergence of the aforementioned subclasses of quasi-monotone functions as solutions to parametric optimization problems.

2.3 Sufficient conditions for quasi-increasing argmax

In what follows, we shall be primarily interested in quasi-monotone functions that arise as selections of players' reaction correspondences in a game. To this end, we must therefore investigate how such functions arise as solutions to parametric optimization problems. Thus we shall consider a selection $r : Y \to X$ of the correspondence $R : Y \rightrightarrows X$, with $X, Y \subset \mathbb{R}$ defined by

$$r(y) \in R(y) \equiv \arg\max_{x \in X} M(x, y), \tag{2}$$

for some objective function $M : X \times Y \to \mathbb{R}$. We wish to provide conditions on M such that some or all of the selections of the correspondence r are quasi-increasing in the parameter y.

It is useful to recall that the extremal selections of r will be increasing in y if M satisfies a single-crossing property (or SCP) with respect to $(x; y)$, i.e., if for any $(x'; y') \geqslant (x; y)$, we have[7]

$$M(x', y) \geqslant (>) M(x, y) \Longrightarrow M(x', y') \geqslant (>) M(x, y'). \tag{3}$$

For an argmax to be quasi-increasing in the parameter y, we need instead the notion of *shifted single-crossing*, which we introduce next via a judicious (non-separable) change of decision variable.

Let there be given a continuous function $\alpha : X \times Y \to Z$ that is strictly increasing in x for fixed s and increasing in y for fixed x. Here, Z is the range of α and it is given by $Z \equiv [\alpha(\underline{x}, \underline{y}), \alpha(\overline{x}, \overline{y})]$, where $\underline{x} = \inf X, \overline{x} = \sup X, \underline{y} = \inf Y$, and $\overline{y} = \sup Y$.

[7]Throughout, while we use the notions of increasing differences and single-crossing as sufficient conditions on an objective function for an increasing argmax, we could also use the interval dominance order (Quah and Strulovici, 2009) for the same purpose. As increasing differences is the only one that survives addition without restrictions (see Shannon, 1995 and Quah and Strulovici, 2012 for more on this), we rely on this property in the applications section.

If one defines a new variable $z = \alpha(x, y)$, then since α is continuous and strictly increasing in x, there exists a (parametrized inverse) function $\beta : Z \times Y \longrightarrow X$ such that $x = \beta(z, y)$. In other words, α has an inverse in its first argument, that is, there exists a function β satisfying $\alpha(\beta(z, y), y) = z$ and $\beta(\alpha(x, y), y) = x$. It is not difficult to see that β must be increasing in its first argument. Further, assume that β is continuous.

In view of the monotonicity properties of α, and of the fact that $X = [\underline{x}, \overline{x}]$ is independent of y, the set of feasible values of z for a fixed value of y is $Z(y) \equiv [\alpha(\underline{x}, y), \alpha(\overline{x}, y)]$. This set is clearly ascending in y (since α is increasing in y). We have the following:

Definition 6 *Let $\alpha : X \times Y \to Z$ be continuous, strictly increasing in x and increasing in y, and $\beta(z, y)$ be the continuous inverse of α with respect to the first variable. A function $M : X \times Y \longrightarrow \mathbb{R}$ satisfies a β-shifted SCP with respect to $(x; y)$ if $\widetilde{M}(z, y) \equiv M(\beta(z, y), y)$ has the SCP (3) with respect to $(z; y) \in Z \times Y$, i.e., for any $(z'; y') \geqslant (z; y)$,*

$$\widetilde{M}(z', y) \geqslant (>) \widetilde{M}(z, y) \Longrightarrow \widetilde{M}(z', y') \geqslant (>) \widetilde{M}(z, y'). \tag{4}$$

Moreover, M satisfies a β-shifted strict SCP with respect to $(x; y)$ if $\widetilde{M}(z, y) \equiv M(\beta(z, y), y)$ has the strict SCP with respect to $(z; y) \in Z \times Y$, that is, for any $(z'; y') \geqslant (z; y)$, $(z'; y') \neq (z; y)$,

$$\widetilde{M}(z', y) \geqslant \widetilde{M}(z, y) \Longrightarrow \widetilde{M}(z', y') > \widetilde{M}(z, y'). \tag{5}$$

A sufficient condition for the β-shifted SCP is β-shifted increasing differences, defined by $\widetilde{M}(z, y) \equiv M(\beta(z, y), y)$ having increasing differences in (z, y), when β is continuous in (z, y) and strictly increasing in z. If both M and β are C^2, this is equivalent to[8,9]

$$\widetilde{M}_{12}(z, y) = [M_{11}(\beta(z, y), y)\beta_2(z, y) + M_{12}(\beta(z, y), y)]\, \beta_1(z, y) + M_1(\beta(z, y), y)\beta_{21}(z, y) \geqslant 0$$

Defining the set-valued functions $R : Y \rightrightarrows X$ and $Z^* : Y \rightrightarrows Z$ by

$$R(y) \equiv \arg\max_{x \in X} M(x, y) \tag{6}$$

and

$$Z^*(y) \equiv \arg\max_{z \in Z(y)} M(\beta(z, y), y), \tag{7}$$

we have a one-to-one mapping between selections r of R and selections z^* of Z^*: given $z^*(\cdot) \in Z^*(\cdot)$, we have $r(y) = \beta(z^*(y), y) \in R(y)$ and given $r(\cdot) \in R(\cdot)$, we have $z^*(y) = \alpha(r(y), y) \in Z^*(y)$.

[8] In this paper, we use subscripts on functions to denote their corresponding partial derivatives.

[9] This equivalence is a well-known result about increasing differences (Topkis, 1978).

Proposition 7 *Let $M : X \times Y \to \mathbb{R}$ satisfies a β-shifted SCP with respect to $(x; y)$, with β jointly continuous and strictly increasing in its first argument. Assume $R(y) \neq \varnothing$ for all $y \in Y$. Then*
(a) The maximal and minimal selections of R, $\overline{r}$ and $\underline{r}$, are both quasi-increasing in y.
(b) If in addition β is continuously differentiable, $\overline{r}$ and $\underline{r}$ are K-lower Liptschitz in y for some K.
(c) If M satisfies the β-shifted strict SCP, then all selections of R are quasi-increasing in y.

Proof. (a) Since $M(\beta(z, y), y)$ has the single-crossing property with respect to $(z; y)$, and the feasible set $Z(y) \equiv [\alpha(\underline{x}, y), \alpha(\overline{x}, y)]$ is ascending in y (since α is increasing in y), we know from the Topkis-Milgrom-Shannon monotonicity theorem that the extremal selections of $S(y)$, $\overline{s}^*$ and $\underline{s}^*$ are increasing functions of y. Since β is increasing in its first coordinate and continuous by assumption, the assumptions of Lemma 5(a) are satisfied by β and $\overline{s}^*$ and $\underline{s}^*$ and we conclude that $\overline{r}(y) = \beta(\overline{s}^*(y), y)$ and $\underline{r}(y) = \beta(\underline{s}^*(y), y)$ are both quasi-increasing in y.

(b) This follows directly from Lemma 5(c).

(c) It is well known that the strict single-crossing property implies that all selections are monotonic. Thus, the proof of (c) is similar to the proof of (a) and therefore omitted. ∎

The following economic application illustrates the change of variable used above in the context of a familiar model. In particular, this provides some guidance as to how a suitable choice of the function α is arrived at. Several more relevant examples are given in the applications section.

Example 8 *Consider a Bertrand duopoly with differentiated substitute products wherein firms 1 and 2 choose prices x, y (in some given price set $[0, \overline{p}]$) and face a demand system $(D, \widehat{D})$ for their products, respectively. Assume linear cost functions with marginal costs c and $\widehat{c}$. In what follows, we focus only on firm 1 (say). Its profit function is*

$$F(x, y) = (x - c)D(x, y). \tag{8}$$

Its demand D is continuously differentiable and satisfies $D_1 < 0$ (the law of demand) and $D_2 > 0$ (products are substitutes in demand). Our aim here is to show that any selection $f(y)$ of Firm 1's reaction correspondence $\Gamma(y) = \arg\max_{x \in [0,\overline{p}]}(x - c)D(x, y)$ is quasi-decreasing in y, under the assumption that

$$D_2 D_1^2 - D\,[D_1 D_{12} - D_2 D_{11}] > 0 \ \text{ for all } (x, y). \tag{9}$$

To see this, let firm 1 respond by choosing its own output z instead of its price x, i.e. let $z = D(x, y)$. Since $D_1 < 0$ and D's domain and range are compact, parametric inversion will yield a jointly continuous function h such that

$$x = h(z, y) \iff z = D(x, y).$$

In fact, h is differentiable and it is easy to check that the partials of h and D are then related by

$$h_1 = \frac{1}{D_1}, h_2 = -\frac{D_2}{D_1}, \text{ and } h_{12} = \frac{1}{D_1^3}(D_2 D_{11} - D_1 D_{12}). \tag{10}$$

Given y, the best response problem of firm 1 may be equivalently viewed as

$$\max \left\{ \widetilde{F}(z,y) \triangleq z[h(z,y) - c] : z \in [D(\overline{p}, y), D(0,y)] \right\}. \tag{11}$$

The first step is to derive conditions under which the argmax $z^*(y)$ is increasing in y. Since the feasible set $[D(\overline{p}, y), D(0,y)]$ is ascending in y (as $D_2 > 0$), by Topkis's Theorem, all the selections of $z^*(y)$ are increasing in y if $\widetilde{F}$ has strictly increasing differences in (z,y). For this, it suffices that $\widetilde{F}_{12}(z,y) = h_2(z,y) + zh_{12}(z,y) > 0$, for all (z,y). Using (10), the latter is equivalent to

$$-\frac{D_2}{D_1} + D\frac{1}{D_1^3}(D_1 D_{12} - D_2 D_{11}) > 0,$$

which is the same as (9).

Since $f(y) = h(z^*(y), y)$, $z^*(\cdot)$ is increasing, and h is decreasing in its first argument and jointly continuous, the dual of Lemma 5(a) implies that $f(y)$ is quasi-decreasing in y.

The interpretation is that when firm 2 raises its price y, firm 1 may optimally react by decreasing or increasing its price x, but, in the latter case, never by so much that firm 1's output would end up decreasing. In other words, while strategic complementarity in pricing decisions is allowed to any extent, a limited form of strategic substitutability can also be accommodated by condition (9). More precisely, condition (9) accommodates what we named strategic quasi-complementarity*.*

We shall return to this economic application below to illustrate some other results.

The following remark introduces a particular change of variable with a separable structure that will prove useful in some of the economic applications presented in section 4.

Remark 9 *For many problems, a simple change of variable is as follows. Let $z = \alpha(x,y) = x + k(y)$ for some strictly increasing function k. Then for $Z^*(\cdot)$ to be increasing in y, it is sufficient that $\widetilde{M}(z,y) \equiv M(z - k(y), y)$ has the single-crossing property with respect to $(z;y)$. A sufficient condition that is easy to check is that $\widetilde{M}(z,y)$ has increasing differences with respect to (z,y). When $\widetilde{M}$ and k are both twice continuously differentiable, this is equivalent to*

$$\widetilde{M}_{12}(z,y) = -M_{11}(z - k(y), y)k'(y) + M_{12}(z - k(y), y) \geqslant 0. \tag{12}$$

The idea of a change of variable to perform comparative statics of a non-monotonic sort has appeared repeatedly in the literature on supermodular games. In a setting with scalar decision and parameter variables, Granot and Veinott (1985) define the notion of doubly-increasing differences for an objective function as being the conjunction of the properties of increasing differences and of β-shifted increasing differences with an additively separable function (as in the previous remark). Curtat (1996) extends their result to multi-dimensional decision and parameter sets. Similar ideas have also appeared in the context of oligopoly applications with quasi-increasing reaction curves, e.g., in Amir (1996) and Amir and Lambson (2000), where the relevant change of variable is $z = x + y$.[10]

The modern theory of monotone comparative statics is often qualified as being of a qualitative nature. Indeed, it aims to predict the directions of change of endogenous variables in response to changes in exogenous parameters, but usually not the associated magnitudes of these changes. In contrast, the conclusion that an argmax is quasi-decreasing in a parameter can be viewed as a comparative statics result of a non-monotonic and quantitative sort. As an illustration, consider the Bertrand duopoly example above. The derived conclusion may be re-stated as $f'(y) \leqslant -D_2(f(y), y)/D_1(f(y), y)$, for almost all y (w.r.t. Lebesgue measure),[11] which provides a lower bound on the rate of decrease of $f(y)$ as y changes. If one adds the reasonable further assumption on demand that $D_2(f(y), y)/\left|D_1(f(y), y)\right| < 1$ for all y, then one can conclude the firm 1 never lowers its price by as much as the increase in its rival's price, a conclusion of a clearly quantitative nature.

Observing that a similar (dual) argument can handle the derivation of upper bounds on the rate of the change of argmax's, as will be illustrated in the last section below, this method can easily be used to provide sufficient conditions on the players' reaction curves in a game to constitute contraction mappings in a global sense, thus ensuring uniqueness of PSNE. For instance, Amir (1996) uses such arguments to establish uniqueness of Cournot equilibrium.

2.4 A fixed point result

This subsection states the simplest form of our basic fixed point theorem, and relates it to Tarski's intersection point theorem. To fix ideas, the functions may be thought of as selections from players' best response correspondences in a strategic game. In the next section, we provide sufficient conditions on the payoff functions that yield the given properties on players' best responses.

[10] More specifically, in Cournot duopoly with convex costs, although the reaction curve $x^*(y)$ need not be monotonic (up or down), the aggregate reaction $z^*(y) = x^*(y) + y$ is increasing, in rival's output y. This means that the reaction curve $x^*(y)$ is 1-lower Liptchitz (thus quasi-increasing). An unusual specification of demand and cost functions for which these properties are nicely illustrated is given in [Novshek, 1985, Example 2 pp. 88-89].

[11] This is justified since f is a function of bounded variation (Lemma 4).

Theorem 10 *Let $f : [a,b] \to [c,d]$ be quasi-increasing, $g : [c,d] \to [a,b]$ be continuous and increasing, and $h(x,y) = (g(y), f(x))$. Then there exists $(\bar{x}, \bar{y}) \in [a,b] \times [c,d]$ such that $h(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$.*

Proof. By Proposition 2(iii), the composition $f \circ g : X \longrightarrow X$ is quasi-increasing. Let $\iota : X \to X$ be the identity. As ι is continuous, it is quasi-decreasing by Proposition 2(i). If $\inf X = \underline{x}$ and $\sup X = \bar{x}$, we have $f \circ g(\underline{x}) \geqslant \underline{x} = \iota(\underline{x})$ and $f \circ g(\bar{x}_1) \leqslant \bar{x} = \iota(\bar{x})$. Hence the assumptions of Tarski's Intersection Point Theorem 3 hold for $f \circ g$ and ι and $\bar{X}_1 \equiv \{x \in X : f(g(x)) = x\}$ is nonempty. If $\bar{x} \in \{x \in [a,b] : g \circ f(x) = x\}$ and we let $\bar{y} = f(\bar{x})$, then $(\bar{x}, \bar{y})$ is a fixed point of h. ■

From this proof, it is clear that the key idea here is to translate Tarksi's result from an intersection point result to a fixed point theorem for an important subclass of bivariate maps, namely those that are formed by the conjunction of two one-dimensional functions, as is the best response mapping for a two-player game. Put differently, the idea is to translate an intersection point result between two functions with the same domains and the same ranges to an intersection point result between two functions with interchanged domains and ranges, in line with the usual graphical depiction of intersecting reaction curves in economics as a simple way of representing PSNE.

A well-known, interesting, and immediate corollary is the following.

Corollary 11 *(a) Let $f : [a,b] \longrightarrow [a,b]$ be quasi-increasing. Then f has a fixed-point.*
(b) Let $f : [a,b] \longrightarrow [a,b]$ be such that $\frac{f(x')-f(x)}{x'-x} \geqslant -k$, for some $k \geqslant 0$ and any $x', x \in [a,b], x \neq x'$. Then f has a fixed-point.

Proof. (a) Simply apply Theorem 10 to f and $g(x) = x$ (the identity).
(b) The slope condition means that f is k-lower-Liptschitz, hence quasi-increasing (Lemma 4). ■

As noted earlier, the result in part (b) with $k = 1$ was proved and used by MacManus (1964) and Roberts and Sonnenschein (1976) to establish existence of symmetric Cournot equilibrium in symmetric Cournot oligopoly with convex costs.[12] The latter property alone ensures that each firm's reaction curve has all its slopes above -1 (though it may be discontinuous), so that each firm always reacts to rivals' output in a way that increases total output. Existence then follows from this property alone, even though the game is neither of strategic substitutes nor of strategic complements. Amir and Lambson (2000) extends this result to oligopolies with some level of non-convex costs. Milgrom and Roberts (1994) prove part (a) independently and use it to conduct comparative statics of equilibrium points.

[12]The proof of this result by Mac Manus is not fully rigorous.

The following is the order-dual of Theorem 10. The economic models to which it applies may be substantially different from those of Theorem 10, as confirmed in some examples below.

Theorem 12 *Let $f : [a, b] \to [c, d]$ be quasi-decreasing, $g : [c, d] \to [a, b]$ be continuous and decreasing and $h(x, y) = (g(y), f(x))$. There exists $(\bar{x}, \bar{y}) \in [a, b] \times [c, d]$ such that $h(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$.*

The fixed point theorems presented in this section will be translated into equilibrium existence results for games in section 3. The next subsection deals with the issue of uniqueness of fixed points of quasi-monotone functions and will be useful to establish uniqueness of PSNE.

2.5 Uniqueness of fixed points

In some economic applications, beyond the issue of existence, the uniqueness of fixed points is often a highly desirable property. In fact, most studies in applied microeconomics postulate game-theoretic models with a unique PSNE, whether specific functional forms are adopted or not. The standard methods used to establish uniqueness of fixed points or PSNEs typically rely on dominant diagonal conditions on payoff functions or, equivalently, on contraction arguments for best response mappings (Rosen, 1965; Milgrom and Roberts, 1990). Both of these conditions are generally postulated to hold in a global sense. In this section, we present two results that establish the uniqueness of fixed points in the present setting without requiring global contraction arguments. Our first result of this form allows the function to be quasi-decreasing (instead of continuous) and satisfy a local contraction property along the diagonal, that is, only at candidate fixed points.[13]

Proposition 13 *(a) Let $f : [a, b] \to [a, b]$ be a quasi-decreasing function, with the property that*

$$\text{at every } x \in [a, b] \text{ such that } x = f(x), \ \limsup_{y \to x} \frac{f(y) - f(x)}{y - x} < 1, \tag{13}$$

then f has at most one fixed point.
(b) If f is a correspondence such that every one of its selections satisfies the assumptions in (a), then f has at most one fixed point, i.e. there is a unique $\overline{x} \in [a, b]$ with $\overline{x} \in f(\overline{x})$.

The proof (in Appendix) essentially shows that the four Dini derivates satisfy (13) at any candidate fixed point. Part (b) is the useful form of this result, since our change of variable approach delivers via Topkis's Theorem optimal argmax correspondences with all selections being strictly quasi-decreasing functions whenever the objective satisfies a strict dual SCP property. The

[13] To underscore the novelty of this result, an application to Bertrand competition is presented in Section 4.

key property that makes Part (b) an easy extension of Part (a) is that such argmax's are multi-valued at countably many points, thus being essentially like functions as far as uniqueness is concerned.

Relying on first order conditions under smoothness assumptions, another convenient test for the uniqueness of a fixed point that is non-global in character can be given in the following form.

Proposition 14 *Let $X = [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$. Assume that $M : X^n \to \mathbb{R}$ is differentiable in its first coordinate and satisfies the following:*

$$x, x' \in X, x' > x \text{ and } M_1(x, x, ..., x) \leqslant 0 \text{ imply } M_1(x', x', ..., x') < 0. \tag{14}$$

Then a function $r : X \to X$ satisfying $r(x) \in \arg\max_{y \in X} M(y, x, ..., x)$has at most one fixed point.

Proof. The proof rules out multiple interior fixed points, and then multiple corner fixed points. Since $r(x) \in \arg\max_{y \in X} M(y, x, ..., x)$ and M is differentiable in its first variable, we must have $M_1(r(x), x, ..., x) = 0$ if x is an interior point of X. Assume that x and x' are interior fixed points of r, with $x' > x$. Then, $M_1(x, x, ..., x) = 0$ and $M_1(x', x', ..., x') = 0$, but this contradicts (14). Assume now that the endpoint a is a fixed point of r. In this case, we must have $M_1(a, ..., a) \leqslant 0$. By (14), $M_1(x', ..., x') < 0$ for all $x' > a$, which shows that there is no other fixed point of r above a. Similarly, if the endpoint b is a fixed point of r, then $M_1(b, ..., b) \geqslant 0$ and (14) implies that $M_1(x, ..., x) > 0$ for all $x < b$. Therefore, there are no other fixed point of r below b. ■

Interestingly, (14) may be seen as a strict dual single-crossing condition for the partial $M_1(x, ..., x)$, viewed as a function of one variable. As such, it becomes transparent that a sufficient condition for (14) is that $M_1(a, a, ..., a)$ is strictly decreasing in a, which (if M is smooth) is in turn implied by

$$M_{11}(a, ..., a) + \sum_{j \neq 1} M_{1j}(a, a, ..., a) < 0.$$

This is a dominant diagonal condition: every row sum of the Hessian matrix of M is negative. Nevertheless, this condition is significantly less restrictive than the typical conditions in the literature, in that it is not required to hold globally, but rather only along the diagonal of the domain X^n.

3 Pure-strategy Nash equilibrium in games

This section contains our results about the existence and uniqueness of pure-strategy Nash equilibrium (henceforth PSNE) in games. Its main objective is to translate our fixed points results into conclusions about the existence of PSNE. In the process, the relevant sufficient conditions shall be placed on the primitives of the game. We begin by describing results for two players games in subsection 3.1. Then n-player symmetric games are the object of subsection 3.2.

3.1 Two-player games

Consider a two-player strategic game with action spaces X and Y and payoff functions $F, G : X \times Y \longrightarrow \mathbb{R}$. We first translate Theorem 10's assumptions onto conditions on the game payoffs.

Theorem 15 *Assume that X and Y are compact intervals in $\mathbb{R}$, F and G are upper semi-continuous in own action, and that:*
(a) F satisfies a β-shifted SCP in $(x; y)$ for some β that is continuous and increasing, and
(b) G is strictly quasi-concave in y for each fixed x, and satisfies the SCP with respect to $(y; x)$.

Then the set E of PSNE is non-empty.

Proof. Due to the upper semi-continuity assumption, the best-reply correspondences have non-empty values. By Proposition 7, the maximal and minimal selections of the best-reply correspondence for the first player, $\bar{x}(\cdot)$ and $\underline{x}(\cdot)$, are quasi-increasing.

From the assumption that G is strictly quasi-concave and upper semi-continuous in y, we know that the best-reply correspondence is a single-valued continuous function denoted $\bar{y}$. From the single-crossing property, $\bar{y}(\cdot)$ is increasing in x. By Theorem 10, there exists (x^*, y^*) such that $\bar{x}(y^*) = x^*$ and $\bar{y}(x^*) = y^*$, that is, (x^*, y^*) is a PSNE of the game and E is non-empty. ■

It is instructive to contrast this result with its counterpart from the theory of supermodular games for the present setting. While the latter relies on Tarski's well kwown fixed point theorem for increasing maps, the present result is based on a reinterpretation of Tarski's intersection point theorem as a fixed point theorem for bivariate maps that arise as best-response maps of two-player games. In terms of scope, the two approaches are not nested. On the one hand, for player 1, the present result imposes less structure since his reaction curve is only quasi-decreasing and not necessarily increasing. On the other hand, for player 2, the present result requires continuity of the reaction curve in addition to monotonicity while the latter property is all that is needed for supermodular games. Put differently, the present result relaxes strategic complementarity to strategic quasi-complementarity (i.e., allows a limited form of strategic substitutability) for one player, but imposes continuity as an extra condition on the reaction curve of the other player.

As the result relies on a mix of continuity and quasi-monotonicity conditions, it may be regarded as a synthesis of the two existing methodologies for establishing existence of PSNE in general games: the classical (topological) approach via the Brouwer/Kakutani fixed point theorem (Nash, 1951 and Rosen, 1965) and the algebraic approach via Tarski's fixed point theorem (Topkis, 1979).

Theorem 15 admits an order-dual, which is as follows.

Theorem 16 *Assume that X and Y are compact intervals in $\mathbb{R}$, and F and G are upper semi-continuous in own action. Let α be a continuous function on $X \times Y$ that is strictly decreasing in x and decreasing in y, and $\beta(\cdot, y)$ be the inverse of α with respect to the first variable. Assume that*
(a) $\widehat{F}(z, y) \equiv F(\beta(z, y), y)$ satisfies the dual SCP with respect to $(x; y)$, and
(b) G is strictly quasi-concave in y for each fixed x, and satisfies the dual SCP with respect to $(y; x)$.
Then the set of PSNE of this game is non-empty.

Proof. If the order on (say) player 2's action set is reversed, the assumptions of this theorem turn into those of Theorem 15. Hence, the conclusion follows from the latter result. ∎

While equivalent to Theorem 15 from a mathematical point of view, in terms of economics applications, this order-dual will apply to models that are quite distinct from those of Theorem 15. In fact, this version will be directly invoked in some of the applications later on.

Theorem 16 can be extended to uniqueness of PSNE by using the insight from Proposition 13.

Proposition 17 *In addition to the assumptions of Theorem 16, assume that the composition of the two reaction curves $f \circ g$ satisfies (13) at all of its fixed points. Then there exists a unique PSNE.*

Proof. Since the composition $f \circ g$ is quasi-decreasing, the uniqueness conclusion follows directly from Proposition 13, when one takes into account that the set of PSNE coincides with the set of fixed points of the composition $f \circ g$. ∎

Theorem 15 and its order-dual (Theorem 16) are a priori valid only for two-player asymmetric games. Nevertheless, for the case of symmetric games, the N-player case is just as easily handled.

3.2 N-player symmetric games

Consider a n-player game, where all players have the same action space X, a compact interval in $\mathbb{R}$, and the same payoff function $F : X \times X^{n-1} \to \mathbb{R}$, where the first entry is the player's own action. With an abuse of notation, we write a joint action vector $x \in X^n$ as (x_i, x_{-i}) for any $i \in N$.

We now state a basic existence result for symmetric games, special versions of which have surfaced in the economics literature a number of times in specific contexts (e.g., Amir and Lambson, 2000 and Milgrom and Roberts, 1994). This result provides conditions on the payoff function that lead to the (common) reaction correspondence satisfying Corollary 11(a).

Theorem 18 *Assume that*
(a) F is upper semi-continuous in own action (or first argument);

(b) for each $a \in X, \widehat{F}(x,a) \equiv F(x,a,...,a)$ satisfies a β-shifted single-crossing property with respect to $(x;a)$ for some $\beta : X \times X \to X$ that is continuous and increasing.

Then the set of symmetric PSNE is non-empty.

Proof. The assumption that F is upper semi-continuous in its first entry guarantees that the best-reply correspondence is non-empty. By Proposition 7, the maximal best-reply restricted to symmetric actions $\bar{x} : X \to X$ is quasi-increasing. By Corollary 11, the set of fixed points of $\bar{x}$ is a non-empty chain. Since a fixed point of $\bar{x}$ is an equilibrium, the set of equilibria is non-empty. ■

For the next uniqueness result, existence of PSNE may be guaranteed either by the continuity of the (common) reaction curve or by the fact that it is quasi-increasing.

Theorem 19 *Assume that F is C^2 and*

(a) for each $a \in X, \widehat{F}(x,a) \equiv F(x,a,...,a)$ is either strictly quasi-concave in x or satisfies a β-shifted SCP with respect to $(x;a)$ for some $\beta : X \times X \to X$ that is continuous and increasing.

(b) for each $a \in X$, $\widehat{F}$ satisfies $\widehat{F}_{11}(a,a) + \widehat{F}_{12}(a,a) < 0$.

Then, there exists a unique symmetric PSNE of this game.

Proof. The existence of a symmetric PSNE follows from either of the two assumptions in (a), the quasi-concavity in x or the β-shifted single-crossing property, see Theorem 18. Since F is twice continuously differentiable, we know from (b) that for each $\overline{a}$, $\widehat{F}_{11}(x,\overline{a}) + \widehat{F}_{12}(x,\overline{a}) < 0$ for all (x,a) in some neighborhood of $(\overline{a},\overline{a})$. Hence, with the change of variable $z = x+a$, $\widehat{F}(z-a,a)$ has strongly increasing differences in (z,a), i.e., $\partial\widehat{F}/\partial a$ is strictly increasing in z; see Amir (1996b), Edlin and Shannon (1998) or [Topkis, 1998, p. 79]. It follows that all the selections of $z^*(a) \equiv \widehat{F}(z-a,a)$ are strictly increasing in a. In other words, $x^*(a)$ has all its slopes strictly less than 1 in a neighborhood of $\overline{a}$. The uniqueness of symmetric PSNE then follows directly from Proposition 13. ■

Again, the main novelty in the underlying argument is that the assumption in part (b) generates a local contraction property for the reaction curve along the diagonal, which is not required to hold in a global sense. Due to the latter point, multiple asymmetric PSNEs are not ruled out.

Before considering an extension to a class of n-player games, we provide a new uniqueness result for symmetric pure-strategy Nash equilibrium for symmetric normal-form games, which is of independent interest for many potential economic applications.

Theorem 20 *Assume that:*

(a) F is differentiable in its first variable (own action) and

(b) For any $x', x \in X$ with $x' > x$, $F_1(x, x, ..., x) \leqslant 0 \Longrightarrow F_1(x', x', ..., x') < 0$.

Then, there exists at most one symmetric equilibrium of this game.

Proof. This follows directly from Proposition 14. ■

Observe here that the given assumptions do not preclude the existence of other PSNEs as long as they are asymmetric.

A sufficient condition for the assumption in Theorem 20(b) is that $\partial_1 F_i(x, ..., x)$ is strictly decreasing in x (see comments after Proposition 14).

We now provide an illustration of Proposition 13 in a familiar setting.

Example 21 *Consider the symmetric version of the Bertrand model of Example 8 with demand function $D(x, y)$. We shall show that there can be at most one symmetric Bertrand equilibrium provided one assumes that demand satisfies (9) and*

$$D_2(x, x) < |D_1(x, x)| \text{ for all } x. \tag{15}$$

Recall that Example 8 showed that every selection of the best-response correspondence $f(y)$ is quasi-decreasing in y under Assumption (9), i.e., that $z^*(y) = D(f(y), y)$ is increasing in y.

Consider any candidate fixed point $y_0 = f(y_0)$. In the Appendix, we show that (13) holds at every such y_0 (and for every selection of f).

Hence, there is a unique symmetric PSNE by Proposition 13.

The reaction curve is required to be contractive only locally, around fixed points; it need not be a global contraction. For the latter property to prevail, one would need to strengthen Assumption (A3) to hold at all pairs (x, y). This would be a much more restrictive assumption, which is not satisfied by many known (non-linear) demand systems (see Vives, 1999 for details).

4 Some selected applications

This section presents a selection of well known models in applied microeconomics for which the results of the present paper apply quite naturally and in a straightforward manner to yield novel results. Despite the fact that some of these models have extensive literature dealing specifically with the existence of PSNEs, the results proposed below constitute either significant generalizations of their counterparts in the literature or new versions that are not nested with existing ones. Since the results presented below are actually new to the separate literatures dealing with each model, we present the results in the form of formal propositions with concise proofs.

The first model is an illustration of our ancillary results and the role that quasi-monotonic functions can play in economic models, in particular nonmonotone comparative statics, rather than the existence of PSNE. The next two applications serve to illustrate how our results can be used to tackle the issue of PSNE existence for classes of games that cannot be handled via the existing approaches, thus giving some economic insight as to how the key asymmetry between the two players' reaction curves can arise naturally. The third illustrates how the present approach can lead to alternative sufficient conditions for the existence of PSNE for a well known model, relative to the existing approach.

4.1 Optimal growth without decreasing returns

We consider a modified version of Solow's growth model allowing increasing returns, and show that the optimal policy is 1-upper Lipschitz (all slopes ≤ 1), but not necessarily increasing, under very general conditions. In addition, an instructive two-period example is solved in closed-form.[14]

With a utility function u and a production function f, the problem writes (with $0 < \delta < 1$)

$$\max \sum_{t=0}^{\infty} \delta^t u(c_t) \text{ subject to } s_{t+1} = f(s_t - c_t)$$

where s_t is the stock of capital/labor ratio and $c_t \in [0, s_t]$ is the consumption at period $t = 0, 1, ...$

Assume that u and f are both twice continuously differentiable and satify $u' > 0, u'' < 0$ and $f' > 0$. However, unlike the standard treatment, we do not impose the usual assumption that $f'' < 0$, so as to allow for increasing returns to scale in production, either locally or globally.

The Bellman equation is then

$$V(s) = \max_{0 \leq c \leq s} \{u(c) + \delta V[f(s-c)]\}$$

In the standard treatment, one assumes that $f'' < 0$, in which case V is concave and the maximand has increasing differences in (c, s). Since the constraint $[0, s]$ is ascending in s, the optimal consumption $c^*(s)$ is increasing in the stock s (this is a well known property of the standard model).

However, under the present assumptions, the maximand need not have increasing differences (nor the SCP), and $c^*(s)$ need not increase in s (see example below). Consider the change of variable

[14] For a generalization to optimal growth in the multi-dimensional case, see Amir (1996b).

$y = s - c$ (i.e., use investment as the decision), and rewrite the Bellman equation equivalently as

$$V(s) = \max_{0 \le y \le s} \{u(s-y) + \delta V[f(y)]\}.$$

Then clearly the optimal investment $y^*(s)$ is increasing in s (irrespective of the properties of both f and V). Hence $c^*(s)$ is 1-upper Lipschitz (i.e., all its slopes are ≤ 1), and hence quasi-decreasing.[15]

As a specification of interest for a closed-form illustration, consider the two-period problem with $u(c) = \sqrt{c}$, $f(s) = s^3$ and $\delta = 1$. The RHS of the Bellman equation is then

$$\max_{0 \le c \le s} \{\sqrt{c} + \sqrt{(s-c)^3}\}$$

A long but straightforward computation yields the optimal policy correspondences

$$c^*(s) = \begin{cases} s, \text{ if } s \le \widehat{s} \\ \frac{1}{6}[3s + (9s^2 - 4)^{1/2}], \text{ if } s \ge \widehat{s} \end{cases} \quad \text{and} \quad y^*(s) = \begin{cases} 0, \text{ if } s \le \widehat{s} \\ \frac{1}{6}[3s + (9s^2 - 4)^{1/2}], \text{ if } s \ge \widehat{s} \end{cases}$$

where $\widehat{s}$ is the unique solution to

$$\sqrt{s} = (3s - (9s^2 - 4)^{1/2} + [(3s + (9s^2 - 4)]^{3/2}.$$

It is easy to see that $c^*(s)$ has a downward jump at $\widehat{s}$, and that all its slopes are ≤ 1. The corresponding optimal investment $y^*(s)$ has selections that are all increasing in s.

This application serves to illustrate how non-monotone comparative statics arises very naturally in economic dynamics and stochastic games. In addition, an explicit solution is also of interest here.

4.2 Price-quantity duopoly

In pioneering work on linear duopoly, Singh and Vives (1984) introduced a hybrid notion of duopoly wherein one firm is a price setter and the other firm a quantity setter. They analyzed only the case of linear demands and derived closed-form PSNEs, with the reaction curves being linear, downward-sloping for the price setter and upward-sloping for the quantity setter. Due to this two-way monotonicity, no lattice-theoretic argument can establish existence of PSNE in more general cases. Instead, one would therefore a priori resort back to a Brouwer-type fixed point argument to obtain PSNEs (and thus impose continuity of the reaction curves). Nevertheless, we invoke our main existence result to provide the very first general PSNE existence result for this hybrid duopoly.

[15] This property arises in dynamic economics models. Amir (1996c) uses this space of consumption functions as the set of Markov strategies in a stochastic game. Dechert and Nishimura (1983) use the strict version (slopes < 1) to study the asymptotic properties of the optimal stock for the growth model at hand.

Consider an asymmetric duopoly with differentiated substitute products and linear costs c_1 and c_2. Let the direct and inverse demand functions by $D^i(p_1, p_2)$ and $P^i(q_1, q_2)$, $i = 1, 2$, where (p_1, p_2) and (q_1, q_2) are the prices and the outputs for the two firms.[16] To introduce the new hybrid duopoly, we need to consider the related standard price (Bertrand) and quantity (Cournot) games. For the price (Bertrand) game, firm i's payoff function is

$$\Pi^i_B(p_1, p_2) = (p_i - c_i)D^i(p_1, p_2).$$

For the quantity (Cournot) game, firm i's payoff function is

$$\Pi^i_C(q_1, q_2) = q_i[P^i(q_1, q_2) - c_i].$$

To define the price-quantity duopoly, assume that D^i and P^i are smooth (for convenience only in establishing the relevant complementarities via Topkis's cross partial test), and that $D^i_i < 0$ and $P^i_i < 0$ (i.e., the Law of Demand). To avoid technicalities, we also assume that the firms' outputs and prices lie in compact sets.[17] For the products to be substitutes, we need to postulate that

$$\frac{\partial D^i(p_1, p_2)}{\partial p_j} > 0 \text{ and } \frac{\partial P^i(q_1, q_2)}{\partial q_j} < 0.$$

By taking parametrized inverse functions, there are functions H_1 and H_2 such that

$$p_1 = P^1(q_1, q_2) \iff q_1 = H^1(p_1, q_2)$$

and

$$q_2 = D^2(p_1, p_2) \iff p_2 = H^2(p_1, q_2).$$

The payoffs in the price-quantity duopoly (with firm 1 as the price setter) are then

$$F(p_1, q_2) = (p_1 - c_1)H^1(p_1, q_2)$$

and

$$G(p_1, q_2) = q_2[H^2(p_1, q_2) - c_2].$$

We can now state our main result for this section. A discussion and economic interpretation of the assumptions used here follows the proof below.

[16]The micro-economic foundation for this demand system is the customary one of a representative consumer maximizing a smooth utility function that is quasi-linear in a numeraire good and has a negative definite Hessian matrix in the two goods under consideration. Under these assumptions, the inverse demands (P_1, P_2) are well-defined as the gradient of the utility functions, and invertible to form the direct demands (D_1, D_2). We omit the details of this derivation for the sake of brevity and refer the reader to [Vives, 1999, Sections 3.1 and 2.5]. With this foundation in place, we may take the demand system as given and the four functions as primitives for our purposes here.

[17]This may be justified on economic grounds as firms may face capacity constraints and regulatory price ceilings.

Proposition 22 *Assume that the original demand system satisfies the following conditions*

$$P_2^1(q_1, q_2) + q_1 P_{12}^1(q_1, q_2) < 0, \text{ for all } q_1, q_2 \geq 0 \tag{16}$$

$$D_1^2(p_1, p_2) + (p_2 - c_2) D_{21}^2(p_1, p_2) > 0, \text{ for all } p_1, p_2 \geq 0 \tag{17}$$

and $\frac{1}{D^2(p_1,p_2)}$ *is strictly convex in* p_2*, or*

$$D^2(p_1, p_2) D_{22}^2(p_1, p_2) - 2 \left[D_2^2(p_1, p_2) \right]^2 < 0 \tag{18}$$

Then the price-quantity duopoly possesses a PSNE.

Proof. Denote the reaction correspondences in the price-quantity duopoly by

$$r_1(q_2) = \arg\max_{p_1} F(p_1, q_2) \text{ and } r_2(p_1) = \arg\max_{q_2} G(p_1, q_2).$$

Consider the change of variable

$$z_1 = H^1(p_1, q_2) \iff p_1 = h^1(z_1, q_2).$$

Then firm 1's payoff function can be rewritten as

$$\widetilde{F}(z_1, q_2) = z_1[h^1(z_1, q_2) - c_1)]$$

For firm 1, we aim to show that $z_1^* = \arg\max_{z_1} \widetilde{F}(z_1, q_2)$, or equivalently $H^1(r_1(q_2), q_2)$, is decreasing in q_2, so that $r_1(q_2)$ is quasi-increasing. To this end, a sufficient condition is that $\widetilde{F}(z_1, q_2)$ has decreasing differences in (z_1, q_2), or

$$h_2^1(z_1, q_2) + z_1 h_{12}^1(z_1, q_2) < 0 \tag{19}$$

Using the simple formulas that relate the partial derivatives of a function and its parametric inverse (10) a first time, it is easy to see that (19) is equivalent to

$$-\frac{H_2^1(p_1, q_2)}{H_1^1(p_1, q_2)} + \frac{H^1(p_1, q_2)}{\left[H_1^1(p_1, q_2)\right]^3} [H_2^1(p_1, q_2) H_1^1(p_1, q_2) - H_{12}^1(p_1, q_2) H_1^1(p_1, q_2)] < 0$$

which via (10) is in turn equivalent to (16) upon simplification.

As for firm 2, to show continuity of $r_2(p_1)$, (18) implies that $\Pi_B^2(p_1, p_2)$ is strictly quasi-concave in p_2 (Caplin and Nalebuff, 1991). Since $G(p_1, q_2)$ is obtained from $\Pi_B^2(p_1, p_2)$ via a strictly monotonic transformation of firm 2's action, this implies that $G(p_1, q_2)$ is strictly quasi-concave in q_2. Therefore, $r_2(p_1)$ is a continuous function.

With the change of decision variable $z_2 = H^2(p_1, q_2)$ for firm 2, a similar two-step procedure as for firm 1 above show that $r_2(p_1)$ is increasing if (17) holds.

The existence of PSNE follows from our basic existence result (Theorem 15). ■

For a good understanding of the result, we now interpret the specific role played by each of the assumptions in familiar contexts, and then relate this Proposition to known results on the existence of PSNEs in the two standard oligopolies. Condition (16) is known to make the payoff of firm 1 in the standard Cournot game submodular, and thus firm 1's reaction curve decreasing. Condition (17) is known to make the payoff of firm 1 in the standard Bertrand game supermodular, and thus firm 1's reaction curve increasing. It follows that the three assumptions are quite natural since Cournot and Bertrand duopolies are typically games of strategic substitutes and complements respectively. Finally, (18) guarantees the quasi-concavity of firm 2's payoff in own price in the Bertrand game.

Significantly, it turns out that these familiar structural conditions on the two standard duopolies also constitute minimal sufficient conditions to make the price-quantity duopoly enjoy strategic quasi-complementarities and thus possess PSNEs according to the results of the present paper. It follows that the basic structure imposed by the approach to PSNE existence in the present paper is precisely as natural for the hybrid duopoly at hand as the familiar strategic complementarity and substitutability properties are to the standard (differentiated-product) Bertrand and Cournot duopolies respectively. This duality between pseudo-complementarities in hybrid duopoly on the one hand and strategic complementarity in Bertrand duopoly and substitutability in Cournot duopoly on the other is a novel insight of independent interest for oligopoly theory.

4.3 Bertrand Competition with increasing returns for one firm

Consider a Bertrand duopoly with differentiated substitute products wherein firms 1 and 2 choose prices x and y (in a price set $[0, \overline{p}]$) and face a demand system (D^1, D^2) for their products, such that $D^i_i < 0$ (Law of Demand) and $D^i_j > 0, i, j = 1, 2,\ i \neq j$ (goods are substitutes). With $C_1(\cdot)$ and $C_2(y) = c_2 y$ denoting the firms' cost functions, the profit function of firms 1 and 2 are

$$F(x, y) = xD^1(x, y) - C_1[D^1(x, y)]$$

and

$$G(x, y) = (y - c_2)D^2(x, y).$$

Using insights from supermodular games, Vives (1990) and Milgrom and Shannon (1994) derive sufficient conditions for the existence of PSNE via the strategic complementarity of the game.[18] Alternatively, for the classical approach using Brouwer's fixed point theorem, sufficient conditions for continuous reaction curves are well-known (e.g., Caplin and Nalebuff, 1991 or Vives, 1999). Both approaches require a convex cost function (or decreasing returns to scale) for both firms. In fact, with (at least one) general concave cost function, the above approaches do not extend, and thus, not surprisingly, no existence result is known so far. We provide the first such result, which imposes no restriction on one firm's cost function, so it may well be concave (locally or globally).

Assumptions

(A1) (i) $D^2(x,y)$ is log-submodular (i.e., $D^2 D^2_{12} - D^2_1 D^2_2 \leq 0$)

(ii) $\frac{1}{D^2(x,y)}$ is strictly convex in x for each y, or $D^2(x,y)D^2_{11}(x,y) - 2[D^2_1(x,y)]^2 < 0, \forall(x,y)$.

(A2) $D^1_2(D^1_1)^2 - D^1\left[D^1_1 D^1_{12} - D^1_2 D^1_{11}\right] > 0$ for all (x,y).

Assumption (A1) is well known to yield a reaction curve that is downward-sloping (part i) and continuous (part ii), as seen in the proof. (A1) (i) is quite restrictive, as it requires D^2_{12} to be strongly negative.[19] However, given the paucity of existence results under increasing returns in oligopoly in general, one would not expect a high level of generality.

The new assumption is (A2). It clearly imposes a very mild restriction on the demand function. Indeed, the third term is always negative, and the first term has a high tendency for a negative sign. In addition, it is sufficient (but not necessary) for (A2) to hold to have $D^1_2 D^1_{11} - D^1_1 D^1_{12} < 0$. The latter is equivalent to D^1_1/D^1_2 being decreasing in x, which is a very general condition.

Importantly, the next existence result imposes no assumptions at all on firm 1's cost function.

Proposition 23 *Under Assumptions (A1)-(A3), the Bertrand game has a PSNE.*

Proof. For firm 2, by Assumption (A1)(ii), $G(x,y)$ is strictly quasi-concave in y for each x, and thus the reaction correspondence $f(y)$ of firm 2 is a continuous function (Caplin and Nalebuff, 1991). In addition, $\log G(x,y) = \log(y - c_2) + \log D^2(x,y)$ so (A1(i)) implies that $\log G(x,y)$ is supermodular, so the reaction curve $r_2(x)$ is decreasing (Milgrom and Shannon, 1994). For firm 1, the fact that (A2) implies that the reaction curve $r_1(y)$ is quasi-decreasing was already proved in Example 8. Existence of PSNE then follows from Theorem 16. ■

[18] For the case of price competition with homogeneous goods, see Prokopovych and Yannelis (2017).

[19] One example of a demand function satisfying this assumptions is (the verification details are left out)

$$D^2(x,y) = \frac{1}{(1+y)^2} + (1+x)e^{-y}.$$

A noteworthy point is that this existence result imposes no restrictions at all on the cost function of firm 1 (other than continuity). Therefore, a key implication of this result is that a quasi-decreasing reaction curve is a natural property for a Bertrand firm when its cost function has increasing returns to scale (even strong ones), either in a local or a global sense. In contrast, in such cases, in general, the other two common properties, upward-monotonicity and continuity, may easily fail to hold.

5 Appendix: missing proofs

This appendix provides all the proofs that were not given in the main text, in the chronological order in which the associated results appear in the text.

Proof of Proposition 2

The proofs of (i)-(ii) are straightforward. To prove (iii), as f is quasi-increasing, $\forall x \in X$,

$$\limsup_{y\uparrow x} f(y) \leqslant f(x) \leqslant \liminf_{y\downarrow x} f(y).$$

Since g is continuous and increasing,

$$\limsup_{y\uparrow x} g\left(f(y)\right) = g\left(\limsup_{y\uparrow x} f(y)\right) \leqslant g\left(f(x)\right) \leqslant g\left(\liminf_{y\downarrow x} f(y)\right) = \liminf_{y\downarrow x} g\left(f(y)\right).$$

Therefore, $g \circ f$ is quasi-increasing. Similarly,

$$\limsup_{y\uparrow x} f\left(g(y)\right) \leqslant f\left(\limsup_{y\uparrow x} g(y)\right) = f\left(g(x)\right) = f\left(\liminf_{y\downarrow x} g(y)\right) \leqslant \liminf_{y\downarrow x} f\left(g(y)\right).$$

Thus, $f \circ g$ is quasi-increasing.

To prove (iv), let f and g be quasi-increasing functions. Using the fact that (a) $\liminf_{y\uparrow x} \lambda f(x) = \lambda \liminf_{y\uparrow x} f(x)$ and that a similar property holds for $\limsup$, and (b) the subadditivity of the $\limsup$ operation and the superadditivity of the $\liminf$ operation, we have:

$$\limsup_{y\uparrow x} \left(\lambda f(y) + g(y)\right) \leqslant \lambda \limsup_{y\uparrow x} f(y) + \limsup_{y\uparrow x} g(y) \leqslant \lambda f(x) + g(x),$$

and

$$\liminf_{y\downarrow x} \left(\lambda f(y) + g(y)\right) \geqslant \lambda \liminf_{y\downarrow x} f(y) + \liminf_{y\downarrow x} g(y) \geqslant \lambda f(x) + g(x).$$

This establishes (vi). ∎

Proof of Lemma 5

(a) Let $\{y_n\}_{n\in\mathbb{N}}$ be such that $y_n \uparrow x$ and $f(y_n) \to \limsup_{y\uparrow x} f(y)$. Since $y_n \uparrow x$ and r is increasing, $r(y_1) \leqslant r(y_n) \leqslant r(x)$. Since r is increasing and $\{y_n\}_{n\in\mathbb{N}}$ is an increasing sequence, $\{r(y_n)\}_{n\in\mathbb{N}}$ converges, so that $r(y_n) \to \bar{r} \leqslant r(x)$. Since β is continuous, $f(y_n) = \beta(r(y_n), y_n) \to \beta(\bar{r}, x)$ and since β is increasing in its first coordinate, $\beta(\bar{r}, x) \leqslant \beta(r(x), x) = f(x)$, that is, $\limsup_{y\uparrow x} f(y) \leqslant f(x)$.

Taking $\{y_n\}_{n\in\mathbb{N}}$ to be such that $y_n \downarrow x$ and $f(y_n) \to \liminf_{y\downarrow x} f(y)$ and repeating the same arguments, we conclude that $\liminf_{y\downarrow x} f(y) \geqslant f(x)$. Thus, f is quasi-increasing.

(b) For any $y' > y$, since β is increasing in its first argument and r is increasing, we have

$$\frac{f(y') - f(y)}{y' - y} = \frac{\beta(r(y'), y') - \beta(r(y), y)}{y' - y} \geqslant \frac{\beta(r(y), y') - \beta(r(y), y)}{y' - y} \geqslant K.$$

Hence f is K-lower-Liptschitz.

(c) Since β is continuously differentiable in s for each x, and Y and X are compact, $\partial\beta/\partial y$ is uniformly bounded. Hence β is lower-Liptschitz in y for each x. The result follows from part (b). ∎

The following result will be useful in the proof of Proposition 13.

Lemma 24 *Assume that $f : [a, b] \to [a, b]$ satisfies (13) and $\bar{x} \in [a, b]$ is a fixed point of f. Then, if $y \in (\bar{x}, b]$ is sufficiently close to $\bar{x}$, $f(y) < y$. Similarly, if $y \in [a, \bar{x})$ is sufficiently close to $\bar{x}$, $f(y) > y$.*

Proof. Let $\bar{x}$ be a fixed point of f. Because of (13), there is a neighborhood U of $\bar{x}$ such that $y \in U \cap ([a, b] \setminus \{\bar{x}\}) \Longrightarrow \frac{f(y)-\bar{x}}{y-\bar{x}} < 1$. Then if $y \in U \cap (\bar{x}, b]$, $\frac{f(y)-\bar{x}}{y-\bar{x}} < 1$ implies $f(y) < y$. Similarly, if $y \in U \cap [a, \bar{x})$, $\frac{f(y)-\bar{x}}{y-\bar{x}} < 1$ implies $f(y) > y$. ∎

Proof of Proposition 13

(a) Suppose that $\bar{x}, \bar{y} \in [a, b]$ are both fixed points of f, with (say) $\bar{x} < \bar{y}$. Using Lemma 24 twice, we can pick $y_1, y_2 \in (\bar{x}, \bar{y})$ such that $y_1 < y_2$ and $f(y_1) < y_1$ and $f(y_2) > y_2$.

Define the function $g(y) = f(y) - y$ on $[y_1, y_2]$. Since f is quasi-decreasing and $y \mapsto (-y)$ is continuous, hence quasi-decreasing, g is quasi-decreasing by Proposition 2(iv) adapted to quasi-decreasing functions. Moreover, $g(y_1) < 0 < g(y_2)$. We can apply Tarski's intersection point theorem (Theorem 3) to the function g thus defined and the constant function $c(y) = 0, \forall y$, which is continuous and thus quasi-increasing. By that Theorem, the supremum of $\{y \in [y_1, y_2] : 0 \geqslant g(y)\}$ is the largest element of $\{y \in [y_1, y_2] : 0 = g(y)\}$. Denote by $\mathring{y} \in [y_1, y_2]$ this largest element of $\{y \in [y_1, y_2] : 0 = g(y)\}$, that is, $\mathring{y}$ is the largest fixed point of f. It is clear that $\mathring{y} < y_2$, since $g(y_2) > 0$ and $g(\mathring{y}) = 0$.

Using again Lemma 24, we can find $\hat{y} \in (\mathring{y}, y_2)$ such that $f(\hat{y}) < \hat{y}$. Defining as before $\hat{g} : [\hat{y}, y_2] \to \mathbb{R}$ by $\hat{g}(y) = f(y) - y$, we see that it is a quasi-decreasing function satisfying the assumptions of Theorem 3. Therefore, there exists $\tilde{y} \in (\hat{y}, y_2)$, such that $0 = \hat{g}(\tilde{y}) = f(\tilde{y}) - \tilde{y}$, that is, $\tilde{y} \in [y_1, y_2]$ is a fixed point of f and $\tilde{y} > \hat{y} > \mathring{y}$, which contradicts the fact that $\mathring{y}$ is the highest fixed point of f in $[y_1, y_2]$. This contradiction establishes the claim of part (a).

(b) Towards eventual contradiction, assume instead that the correspondence f has (at least) two distinct fixed points; call them $x_1 \in f(x_1)$ and $x_2 \in f(x_2)$. Define a (single-valued) selection $\widehat{f}$ of f such that $\widehat{f}(x_1) = x_1$, $\widehat{f}(x_2) = x_2$ and $\widehat{f}(x)$ is arbitrary for all the other x's. Then, by assumption, $\widehat{f}$ satisfies the assumptions in (a). Therefore, $\widehat{f}$ has at most one fixed point by part (a), a contradiction. This proves part (b). ∎

Proof of uniqueness of PSNE in Example 21

Consider an arbitrary selection of f, denoted $\widehat{f}$, and any candidate fixed point $y_0 = \widehat{f}(y_0)$. The proof is divided into two separate cases. Assume first that $\widehat{f}$ is neither continuous from the left nor from the right at y_0. Since $\widehat{f}$ is quasi-decreasing, it must have a downward jump from the left and from the right at y_0. Then the fact that (13) holds at $x = y_0$ follows directly.

Next, consider the second case, where $\widehat{f}$ is one-sided continuous (say from the right) at y_0. Since $D(\widehat{f}(y), y)$ is increasing in y, it is differentiable a.e. in y (with respect to Lebesgue measure). For any sequence $y_n \searrow y_0$ such that $D(\widehat{f}(y), y)$ is differentiable at y_n for each n, one has

$$D_1(\widehat{f}(y_n), y_n)\widehat{f}'(y_n) + D_2(\widehat{f}(y_n), y_n) \geqslant 0 \text{ or } f'(y_n) \leqslant -\frac{D_2(\widehat{f}(y_n), y_n)}{D_1(\widehat{f}(y_n), y_n)}.$$

Taking limsup on both sides yields (in view of the fact that $f(y_n) \longrightarrow f(y_0)$ and (15))

$$\limsup_{n\to\infty} \widehat{f}'(y_n) \leqslant -\frac{D_2(y_0, y_0)}{D_1(y_0, y_0)} < 1. \tag{20}$$

For any sequence $y_n \searrow y_0$, due to the downward jump at y_0, (13) clearly holds.

A similar argument with $y_n \nearrow y_0$ works if $\widehat{f}$ is left-continuous at y_0 (and is thus left out). ∎

6 Conclusion

By building on an intersection point theorem due to Tarski (1955), the main result of this paper demonstrates that a pure-strategy Nash equilibrium exists in two-player games when one reaction curve is continuous and increasing and the other has no downward jumps (though it may well have upward jumps). We elaborate in some detail on functions with the latter property, called quasi-increasing in Tarski (1955), by deriving a number of results on natural operations involving such functions. In particular, these results include sufficient conditions on an objective function for quasi-increasing functions to arise as argmax's of parametric optimization problems. While this is obtained via from the standard case via a simple change of variable, it allows for non-monotone comparative statics. Some novel equilibrium uniqueness results are proved, which rely on a local (instead of the common global) contraction property, and require quasi-increasingness instead of continuity of best responses. The special case of symmetric n-player games is also covered, thus unifying some existing results dealing mostly with Cournot oligopoly.

In an important part of the paper, we argue that our results have a promising scope of application for a wide variety of economic models, including a growth model with increasing returns, a hybrid duopoly model (of price and quantity competition), and a Bertrand model with increasing returns for one firm. We illustrate in elementary ways all the various steps needed to actually apply our results for each of these models, tacitly establishing that strategic quasi-complementarity (or a quasi-increasing reaction curve) forms a convenient relaxation of strategic complementarity, and arises naturally in well-known economic models.

As can be illustrated with other examples, the results of the present paper also apply in various models for which they require alternative sufficient conditions on the primitives that are not be nested with known conditions from supermodular games. For instance, one can easily apply our results to the standard Cournot and Bertrand models with differentiated products and obtain sufficient conditions on demand and costs that are not nested with existing ones (this is available from the authors upon request). Finally, one can also apply our results to some of the models proposed by Monaco and Sabarwal (2016) as games with strategic complements for one player and strategic substitutes for the other player. One such example is the inspector-inspectee class of games studied by Leshem and Tabbach (2012), where the structural asymmetry betwen the properties of the two reaction curves needed for our result holds in a natural way.

References

[1] Acemoglu, D. and M.K. Jensen (2009). Aggregate comparative statics, *Games and Economic Behavior*, 81, 27-49.

[2] Amir, R. (1996a), Cournot oligopoly and the theory of supermodular games, *Games and Economic Behavior*, 15, 132-148.

[3] Amir, R. (1996b), Sensitivity analysis in multisector optimal economic dynamics, *Journal of Mathematical Economics*, 25, 123-141.

[4] Amir, R. (1996c), Continuous stochastic games of capital accumulation with convex transitions, *Games and Economic Behavior,* 15, 111-131.

[5] Amir R. and V. Lambson (2000), On the effects of entry in Cournot markets, *Review of Economic Studies*, 67, 235-254.

[6] Caplin, A. and B. Nalebuff (1991). Aggregation and imperfect competition: On the existence of equilibrium, *Econometrica*, 59, 25-59.

[7] Curtat, L. (1996). Markov equilibria in stochastic games with complementarities. *Games and Economic Behavior*, 17, 177–199.

[8] Dechert, WD and K Nishimura (1983), A complete characterization of optimal growth paths in an aggregated model with a non-concave production function, *Journal of Economic Theory*, 31, 332-354.

[9] Dierker, E. (1972). Two Remarks on the number of equilibria of an economy, *Econometrica*, 40, 951-953.

[10] Echenique, F. (2002). Comparative statics by adaptive dynamics and the correspondence principle, *Econometrica* 70, 833-844.

[11] Echenique, F. (2004), A characterization of strategic complementarities, *Games and Economic Behavior*, 46, 325-347.

[12] Echenique, F. and A. Edlin (2004), Mixed equilibria in games of strategic complements are unstable, *Journal of Economic Theory,* 118, 61-79.

[13] Edlin, A. and C. Shannon (1998), Strict monotonicity in comparative statics, *Journal of Economic Theory*, 81, 201-219.

[14] Granot, F. and A. Veinott (1985). Substitutes, complements, and ripples in network flows, *Mathematics of Operations Research*, 10, 471-497.

[15] Hoernig, S. (2003), Existence of equilibrium and comparative statics in differentiated goods Cournot oligopolies, *International journal of industrial organization* 21, 989-1019.

[16] Leshem, S. and A. Tabbach (2012). Commitment versus Flexibility in Enforcement Games, *B.E. Journal of Theoretical Economics (Contributions)*, 12, 1935-1704.

[17] McManus, M. (1962), Number and size in Cournot equilibrium, *Yorkshire Bulletin of Economic and Social Research*, 14, 14-22.

[18] McManus, M. (1964), Equilibrium, number and size in Cournot equilibrium, *Yorkshire Bulletin of Economic and Social Research,* 16, 68–75.

[19] Milgrom, P. and J. Roberts (1990), Rationalizability, learning, and equilibrium in games with strategic complementarities, *Econometrica*, 58, 1255-1278.

[20] Milgrom, P. and J. Roberts (1994), Comparing equilibria, *American Economic Review*, 84, 441-459.

[21] Milgrom, P. and C. Shannon (1994), Monotone comparative statics, *Econometrica*, 62, 157-80.

[22] Monaco, A. and T. Sabarwal (2016), Games with strategic complements and substitutes, *Economic Theory*, 62, 65-91.

[23] Nash, J. (1951), Non-cooperative games, *Annals of Mathematics,* 54, 286-295.

[24] Novshek, W. (1985), On the existence of Cournot equilibrium, *Review of Economic Studies*, L II, 85-98.

[25] Quah, J.K.-H. and B. Strulovici (2009). Comparative statics, informativeness, and the interval dominance order. *Econometrica* 77, 1949–1992.

[26] Quah, J.K.-H. and B. Strulovici (2012). Aggregating the single crossing property, *Econometrica*, 80, 2333-2348.

[27] Prokopovych, P. and N. Yannelis (2017), On strategic complementarities in discontinuous games with totally ordered strategies, *Journal of Mathematical Economics*, 70, 147-153.

[28] Reinganum, J. F. (1982). Strategic Search Theory, *International Economic Review*, 23, 1-17.

[29] Roberts, J. and H. Sonnenschein (1976), On the existence of Cournot equilibrium without concave profit functions, *Journal of Economic Theory*, 13, 112-117.

[30] Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave N-person games, *Econometrica*, 3, 520-534.

[31] Roy, S. and T. Sabarwal (2012). Characterizing stability properties in games with strategic substitutes, *Games and Economic Behavior*, 75, 337-353.

[32] Shannon, C. (1995). Weak and strong monotone comparative statics, *Economic Theory*, 5, 209-227.

[33] Singh, N. and X. Vives (1984), Price and quantity competition in a differentiated duopoly, *The RAND Journal of Economics*, 15, 546-554.

[34] Tarski, A. (1955), A lattice-theoretic fixed point theorem and its applications, *Pacific Journal of Mathematics,* 5, 285-309.

[35] Topkis, D. (1978), Minimizing a submodular function on a lattice, *Operations Research*, 26, 305-321.

[36] Topkis, D. (1979), Equilibrium points in non-zero sum n-person submodular games, *SIAM J. of Control and Optimization*, 17, 773-787.

[37] Topkis, D. (1998), *Supermodularity and Complementarity*, Princeton University Press.

[38] Vives, X. (1990), Nash equilibrium with strategic complementarities, *Journal of Mathematical Economics*, 19, 305-321.

[39] Vives, X. (1999). *Oligopoly pricing: Old ideas and new tools*, The MIT Press.