An axiomatic approach to the value in games with coalition structure.*

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March 31, 2008

Abstract
We study a value for transferable utility games with coalition structures. We provide an axiomatic characterization using the properties of efficiency, linearity, independence of null coalitions, balanced contributions inside a coalition, coordination, and equal sharing in unanimity games. By replacing the latter by a weighted version, we characterize the Owen value (Owen, 1977).

Keywords: coalition structure, coalitional value.

1 Introduction
Coalition structure is important in many real-world contexts, such as the formation of cartels or bidding rings, alliances or trading blocs among nation

*Financial support from the Spanish Ministerio de Ciencia y Tecnología and FEDER through grant SEJ2005-07637-C02-01/ECON and the Xunta de Galicia through grants PGIDIT06PXIC300184PN and PGIDIT06PXIB362390PR is gratefully acknowledged.
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states, research joint ventures, and political parties.

These situations can be modelled through transferable utility (TU, for short) games, in which the players partition themselves into coalitions for the purpose of bargaining. Given a coalition structure, bargaining occurs between coalitions and between players in the same coalition.

In this kind of games, it is of interest to study how to share the benefit between the coalitions and between the members inside each coalition. This problem has been treated in the framework of the theory of games from two different points of view. One of them is axiomatic and the other is non-cooperative.

The non-cooperative point of view focuses on studying allocation rules (or values) which arise in a given non-cooperative environment. The aim of the axiomatic approach is to identify each rule with a set of "desirable" properties. This helps to understand the nature of the different allocation rules and their applicability. Moreover it is useful to choose the more appropriate allocation rule in each particular framework.

Under both approaches the main idea is that the coalitions play among themselves as individual agents in a game among coalitions, and then, the profit obtained by each coalition is distributed among its members.

In this paper, we follow the axiomatic approach.

Aumann and Drèze (1974) proposed the first value for TU games with coalition structure.

Owen (1977) defined a new value for this kind of games and developed an axiomatic approach to his solution. Hart and Kurz (1983) provided an alternative axiomatic characterization of the Owen value. Other characterizations of the Owen value have been provided by Winter (1992), Calvo et al. (1996), Hamiache (1999, 2001), Peleg and Sudhölter (2003) and Albizuri and Zarzuelo (2004).

The Owen value has been non-cooperatively supported by Vidal-Puga and Bergantiños (2003) and Vidal-Puga (2005).

The Owen value is symmetric at the intercoalitional and intracoalitional levels of bargaining. Two coalitions that affect the game between coalitions in a symmetric way will receive the same aggregate payoff and any two players within a coalition that have the same effect on the game between the members inside each coalition will receive the same individual payoff. Nevertheless, this symmetry among coalitions may not be always a reasonable requirement for a value.

Levy and McLean (1989) provided several axiomatizations of values for
games with coalition structure in which intercoalitional and/or intracoalitional symmetry are dropped off.

Vidal-Puga (2006) defined a new value for this kind of games, $\zeta$ that takes into account the intercoalitional asymmetry. Moreover, he studied this value from a non-cooperative point of view.

In this paper we provide a characterization of $\zeta$. Some of the axioms used in the characterization are standard in the literature (such as efficiency and linearity), others (like independence of null coalitions or balanced contributions) are used in many different frameworks. Moreover, we introduce new properties in this kind of problems: coordination (which asserts that internal changes in a coalition which do no affect the game between coalitions, do not influence the final payment of the rest of the players) and equal sharing in unanimity games (which ensures that under the grand coalition unanimity game, each agent should receive the same payment). The latter is the only property among the presented that is not satisfied by the Owen value. Moreover, we provide a new characterization of the Owen value replacing this property by a new one. We also provide a characterization of an alternative value first proposed by Levy and McLean (1989).

The paper is organized as follows. In Section 2 we introduce the model. In Section 3 we present the properties used in the characterization and we prove that $\zeta$ satisfies all the properties. In Section 4 we present the characterization result. In Section 5 we prove that the properties are independent. In Section 6 we present two new characterizations of the Owen value and the value proposed by Levy and McLean (1989), respectively.

2 Notation

Let $U = \{1, 2, \ldots\}$ be a (may be infinite) set of potential players.

Given a finite subset $N \subset U$, let $\Pi(N)$ denote the set of all orders in $N$. Given $\pi \in \Pi(N)$, let $\text{Pre}(i, \pi)$ denote the set of the elements in $N$ which come before $i$ in the order given by $\pi$, i.e.

$$\text{Pre}(i, \pi) = \{ j \in N : \pi(j) < \pi(i) \}.$$

For any $S \subset N$, $\pi_S$ denotes the order induced in $S$ by $\pi$ (for all $i, j \in S$, $\pi_S(i) < \pi_S(j)$ if and only if $\pi(i) < \pi(j)$).

A game with transferable utility, or TU game, is a pair $(N, v)$ where $N \subset U$ is finite and $v : 2^N \to \mathbb{R}$ satisfies $v(\emptyset) = 0$. When $N$ is clear, we can
also denote \((N, v)\) as \(v\). Given a TU game \((N, v)\) and \(S \subseteq N\), \(v(S)\) is called the \textit{worth} of \(S\). Given \(S \subseteq N\), we denote the restriction of \((N, v)\) to \(S\) as \((S, v)\).

Given \(N \subseteq U\) finite, we call coalition structure over \(N\) a partition of the player set \(N\), i.e. \(\mathcal{C} = \{C_1, C_2, \ldots, C_m\} \subseteq 2^N\) is a coalition structure if it satisfies \(\bigcup_{C_q \in \mathcal{C}} C_q = N\) and \(C_q \cap C_r = \emptyset\) when \(q \neq r\). We also assume \(C_q \neq \emptyset\) for all \(q\).

A coalition structure \(\mathcal{C}\) over \(N\) is improper if either \(\mathcal{C} = \{\{i\}\}_{i \in N}\) or \(\mathcal{C} = \{N\}\).

For any \(S \subseteq N\), we denote the restriction of \(\mathcal{C}\) to the players in \(S\) as \(\mathcal{C}_S\), i.e. \(\mathcal{C}_S = \{C_q \cap S : C_q \in \mathcal{C} \text{ and } C_q \cap S \neq \emptyset\}\).

For simplicity, we write \(S \cup i\) instead of \(S \cup \{i\}\) and \(N \setminus i\) instead of \(N \setminus \{i\}\).

Given a TU game \((N, v)\) and a coalition structure \(\mathcal{C} = \{C_1, C_2, \ldots, C_m\}\) over \(N\), the game between coalitions is the TU game \((M, v/\mathcal{C})\) where \(M = \{1, 2, \ldots, m\}\) and \((v/\mathcal{C})(Q) = v\left(\bigcup_{q \in Q} C_q\right)\) for all \(Q \subseteq M\).

We denote the TU game \((N, v)\) with coalition structure \(\mathcal{C} = \{C_1, C_2, \ldots, C_m\}\) over \(N\) as \((N, v, \mathcal{C})\) or \((v, \mathcal{C})\). When \(\mathcal{C}\) is clear, we also write \(v\) instead of \((N, v, \mathcal{C})\).

We denote the set of all triples \((N, v, \mathcal{C})\) as \(CTU\).

Given \(G\) is a subset of \(CTU\), a \textit{coalitional value} (or simply a \textit{value}) in \(G\) is a correspondence \(\psi\) which assigns to each \((N, v, \mathcal{C}) \in G\) a vector \(\psi(v) \in \mathbb{R}^N\). With some abuse of notation, we say that \(\psi(v)\) is the value of \((N, v, \mathcal{C})\).

Given \(S \subseteq N\), \(S \neq \emptyset\), the \textit{unanimity game} with carrier \(S\) \((N, u_S^N)\) is defined as

\[
u_S^N(T) := \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{otherwise} \end{cases} \text{ for all } T \subseteq N.
\]

One of the most important values in TU games is the \textit{Shapley value} (Shapley (1953b)). A nonsymmetric generalization of the Shapley value is the \textit{weighted Shapley value} (Shapley (1953a), Kalai and Samet (1987, 1988)). We denote the Shapley value of the TU game \((N, v)\) as \(Sh(v) \in \mathbb{R}^N\); and given a vector of weights \(\omega \in \mathbb{R}^N_{++}\), we denote the weighted Shapley value as \(Sh^\omega(v) \in \mathbb{R}^N\).

Kalai and Samet (1987, Theorem 1) proved that the weighted Shapley value can be expressed as

\[
Sh^\omega(v) = \sum_{\pi \in \Pi(N)} p_\omega(\pi) [v(Pre(i, \pi) \cup i) - v(Pre(i, \pi))] \text{ for all } i \in N
\]

where
We now focus on TU games with coalition structure. Fix $\mathcal{C} = \{C_1, ..., C_m\}$ and let $M = \{1, ..., m\}$. Owen (1977) proposed a value based on Shapley’s which takes into account the coalition structure. We call this value the Owen value. It is defined as follows: Given $C_q \in \mathcal{C}$, the reduced TU game $(C_q, v_{C_q})$ is defined as

$$v_{C_q}(T) := Sh_q(v_{(N \setminus C_q) \cup T})$$

for all $T \subset C_q$.

Thus, each $v_{C_q}(T)$ is the Shapley value of the coalition $T$ in the game between coalitions assuming that the members of $C_q \setminus T$ are out.

**Definition 1** (Owen, 1977) Given a TU game with coalition structure $(N, v, \mathcal{C})$, the Owen value is defined as

$$Ow_i(v) := Sh_i(v_{C_q})$$

for all $i \in C_q \in \mathcal{C}$.

The interpretation of this definition is as follows: players in $C_q$ should divide $Sh_q(v/C)$, which is their value in the game between coalitions. In order to compute the contribution of each player a new game is defined. The worth of a subcoalition $T \subset C_q$ is the value that $T$ should get if the other players in $C_q$ are not present and $T$ plays the role of $q \in M$ in the game between coalitions.

When the coalition structure is improper, the Owen value coincides with the Shapley value.

Levy and McLean (1989) defined a family of weighted coalitional values with intracoalitional symmetry. These values are defined as follows: Given a vector of weights $\omega \in \mathbb{R}_+^M$ for the coalitions and $C_q \in \mathcal{C}$, the weighted reduced TU game $(C_q, v_{C_q}^{\omega N})$ is

$$v_{C_q}^{\omega N}(T) := Sh_q^{\omega}(v_{(N \setminus C_q) \cup T})$$

for all $T \subset C_q$. 

$$p_{\omega}(\pi) = \prod_{j=1}^{[N]} \frac{\omega_{\pi(j)}}{\sum_{k=1}^{N} \omega_{\pi(k)}}.$$
Definition 2 (Levy and McLean, 1989) Given a TU game with coalition structure \((N, v, \mathcal{C})\), the weighted coalitional value with intracoalitional symmetry, with weights given by \(\omega \in \mathbb{R}_+^M\), is defined as

\[
\varphi_i^\omega(v) := Sh_i(v^\omega_{C_q})
\]

for all \(i \in C_q \subseteq \mathcal{C}\).

The interpretation of this definition is as before. However, in this case, the coalitions are not treated symmetrically in the game between coalitions.

When \(\mathcal{C} = \{\{i\}\}_{i \in N}\), this value coincides with the weighted Shapley value. When \(\mathcal{C} = \{N\}\) it coincides with the Shapley value. When \(\omega_q = \omega_r\) for all \(q, r\), it coincides with the Owen value.

A natural value for each \(\omega_q\) is the size of coalition \(C_q\) (Kalai and Samet, 1987). However, we should be cautious with the definition of the weighted reduced TU game \((C_q, v^\omega_{C_q})\). Remark that \(v^\omega_{C_q}(T) = Sh_q^\omega(v/\mathcal{C}(N \setminus C_q) \cup T)\) is interpreted as the value that \(T\) would get should \(C_q \setminus T\) be not present and \(T\) play the role of \(q \in M\). However, coalition \(T\) has size \(|T| \leq |C_q|\) in \(\mathcal{C}(N \setminus C_q) \cup T\). Under this interpretation, if the players in \(C_q \setminus T\) are not present, there is no reason to assume that the weight of \(q \in M\) remains unchanged.

In order to take into account the real size of the subcoalitions, Vidal-Puga (2006) defined the following value for TU games with coalition structure.

Fix \(C_q \subseteq \mathcal{C}\). Let \(T \subseteq C_q\). Consider a weight system \(\lambda(T) \in \mathbb{R}_+^M\) given by \(\lambda_q(T) = |T|\) and \(\lambda_r(T) = |C_r|\) otherwise. The reduced TU game \((C_q, v^s_{C_q})\) is defined as:

\[
v^s_{C_q}(T) := Sh_q^{\lambda(T)}(v/\mathcal{C}(N \setminus C_q) \cup T)
\]

for all \(T \subseteq C_q\).

Each \(v^s_{C_q}(T)\) is the weighted Shapley value (with weights given by the size of each coalition) of coalition \(T\) in the game between coalitions assuming that the members of \(C_q \setminus T\) are out.

Definition 3 (Vidal-Puga, 2006) Given a TU game with coalition structure \((N, v, \mathcal{C})\), the value \(\zeta\) is defined as

\[
\zeta_i(v) := Sh_i(v^s_{C_q})
\]

for all \(i \in C_q \subseteq \mathcal{C}\).
For simplicity, we also write $\zeta_i^N$ instead $\zeta_i(v)$.

This value can be inductively computed as follows (Vidal-Puga (2006, Proposition 3.1.)):

$$\zeta_{i}^{\{i\}} = v(\{i\}) \text{ for all } i \in N. \text{ Assume that we know } \zeta^S \in \mathbb{R}^S \text{ for all } S \subseteq N. \text{ Then, } \zeta_{i}(v) = \zeta_{i}^{N}$$

$$= \frac{1}{|N|} \left[ v(N) + \sum_{j \in C_q \setminus i} \frac{|N|}{|C_q|} \left( \zeta^N_{i} - \zeta^N_{j} \right) + \sum_{C_r \in C \setminus C_q} \left( \frac{|C_r|}{|C_q|} \sum_{j \in C_q \setminus C_r} \zeta^N_{j \setminus C_r} - \sum_{j \in C_q \setminus C_r} \zeta^N_{j \setminus C_q} \right) \right]$$

for all $i \in C_q \in C$.

## 3 Properties

In this section we present some properties of the values. Moreover we provide several results.

**Efficiency (EFF)** For any game $(N, v, C)$, $\sum_{i \in N} f_{i}(N, v, C) = v(N)$.

That is, a value is efficient if it distributes the worth of the grand coalition.

**Definition 4** Given two games $(N, v, C)$, $(N, w, C)$, and real numbers $\alpha$ and $\beta$, the game $(\alpha v + \beta w)$ is defined as:

$$(\alpha v + \beta w)(S) = \alpha v(S) + \beta w(S)$$

for all $S \subset N$.

**Linearity (LIN)** Given $(N, v, C)$, $(N, w, C)$ and real numbers $\alpha$ and $\beta$, $f_{i}(N, \alpha v + \beta w, C) = \alpha f_{i}(N, v, C) + \beta f_{i}(N, w, C)$

for all $i \in N$ and all $C$.

Before introducing the next property we provide two definitions:

**Definition 5** We say that $i \in N$ is a null player if

$$v(T \cup i) = v(T)$$

for all $T \subset N \setminus i$. 

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Definition 6 We say that $C_q \in \mathcal{C}$ is a null coalition if all its members are null players.

Independence of Null Coalitions (INC) Given a game $(N, v, \mathcal{C})$ and a null coalition $C_q \subseteq \mathcal{C}$,

$$f_i (N, v, \mathcal{C}) = f_i (N \setminus C_q, v, \mathcal{C}_N \setminus C_q)$$

for all $i \in N \setminus C_q$.

This property and EFF imply that the aggregated payment of the agents in $C_q$ is zero. INC asserts that if a coalition in $\mathcal{C}$ is null, it does not influence the allocation within the rest of the players.

Balanced Contributions Among Players in the same coalition (BCAP) Given a game $(N, v, \mathcal{C})$. For all $i, j \in C_q \subseteq \mathcal{C}$,

$$f_i (N, v, \mathcal{C}) - f_i (N \setminus j, v, \mathcal{C}_N \setminus j) = f_j (N, v, \mathcal{C}) - f_j (N \setminus i, v, \mathcal{C}_N \setminus i).$$

This property states that for any two agents that belong to the same coalition in $\mathcal{C}$, the amount that each agent would gain or lose by the other’s withdrawal from the game should be equal.

The principle of Balanced Contributions is used in different contexts. Myerson (1977) used a similar property for games with graphs. He called it Fairness. Later, Myerson (1980) used a similar axiom to characterize the Shapley value. Calvo et al. (1996) defined this property for games with level structure (a level structure describes a hierarchy of cooperation between the players). They characterized the Owen value using this property. Recently, Bergantiños and Vidal-Puga (2005) characterized an extension of the Owen value with this principle and Alonso-Meijide et al. (2007) characterized a parametric family of coalitional values with a similar property.

Coordination (CO) For all $v, v'$ and $C_q \in \mathcal{C}$, if

$$v \left( T \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) = v' \left( T \cup \bigcup_{C_r \in \mathcal{R}} C_r \right)$$

for all $T \subseteq C_q$ and all $\mathcal{R} \subseteq \mathcal{C} \setminus C_q$, then,

$$f_i (N, v, \mathcal{C}) = f_i (N, v', \mathcal{C}) \text{ for all } i \in C_q.$$
This property says that, given a coalition \( C_q \), if there are changes inside other coalitions, but these changes do not affect to the worth of any subset of \( C_q \) with the rest of coalitions, then these internal changes do not affect the final payment of each agent in \( C_q \).

**Equal Sharing in Unanimity Games (ESUG)** For any \( C \),

\[
f_i \left( N, u^N_N, C \right) = f_j \left( N, u^N_N, C \right)
\]

for all \( i, j \in N \).

This property asserts that under the unanimity game with carrier \( N \), each agent should receive the same payment.

Frequently, it is interpreted that players form coalition structures in order to improve their bargaining strength (Hart and Kurz, (1983)). However, as Harsanyi (1977) points out, the bargaining strength does not improve in general. An individual can be worse off bargaining as a member of a coalition than bargaining alone. This is what is known as the "Harsanyi paradox".

ESUG avoids the "Harsanyi paradox" in the case where all the agents are symmetric. In the unanimity game with carrier \( N \) all the agents are necessary to obtain a positive payment. Hence it seems reasonable that their assignment should be independent of the coalitional structure.

**Null Player (NP)** Given a null player \( i \in N \) in a game \( (N, v, C) \),

\[
f_i \left( N, v, C \right) = 0.
\]

**Independence of Null Players (INP)** Given a null player \( i \in N \) in a game \( (N, v, C) \),

\[
f_j \left( N, v, C \right) = f_j \left( N \setminus i, v, C_{N \setminus i} \right)
\]

for all \( j \in N \setminus i \).

This property asserts that if agent \( i \) is a null player, no agent \( j \) will get a different payment if \( i \) is removed from the game.

**Proposition 7** \( Sh^\omega \) satisfies INP for any \( \omega \).
Proof. Let $i \in N$ and let $j \in N \setminus i$ be a null player in $(N, v, C)$.

It is clear that

$$Sh^\omega_i (N, v) = \sum_{\pi \in \Pi(N)} p_\omega(\pi)[v(Pre(i, \pi) \cup i) - v(Pre(i, \pi))] =$$

$$\sum_{\pi^{-j} \in \Pi(N \setminus j)} \left[ \sum_{\pi \in \Pi(N): \pi_{N \setminus j} = \pi^{-j}} p_\omega(\pi)[v(Pre(i, \pi) \cup i) - v(Pre(i, \pi))] \right]$$

where $\pi_{N \setminus j}$ denotes the order $\pi \in \Pi(N)$ after removing agent $j$.

Moreover,

$$Sh^\omega_i (N \setminus j, v) = \sum_{\pi^{-j} \in \Pi(N \setminus j)} p_\omega(\pi^{-j})[v(Pre(i, \pi^{-j}) \cup i) - v(Pre(i, \pi^{-j}))].$$

We prove two claims:

Claim 8 For each $\pi^{-j} \in \Pi(N \setminus j)$,

$$\sum_{\pi \in \Pi(N): \pi_{N \setminus j} = \pi^{-j}} p_\omega(\pi) = p_\omega(\pi^{-j}).$$

Proof. Let $\pi^{-j} \in \Pi(N \setminus j)$. We will prove this claim applying an induction argument over the cardinality of $N$.

If $|N| = 2$, the result is trivial.

Consider that the result is true for $|N| \leq k$. We prove it for $|N| = k + 1$. Let $N = \{j_1, \ldots, j_k, j\}$ where $\pi^{-j}(j_1) < \ldots < \pi^{-j}(j_k)$. Hence,

$$\sum_{\pi \in \Pi(N): \pi_{N \setminus j} = \pi^{-j}} p_\omega(\pi) = \sum_{\pi \in \Pi(N): \pi_{N \setminus j} = \pi^{-j}, \pi(j) < \pi(j_k)} p_\omega(\pi) + \sum_{\pi \in \Pi(N): \pi_{N \setminus j} = \pi^{-j}, \pi(j) > \pi(j_k)} p_\omega(\pi)$$

$$= \sum_{\pi \in \Pi(N \setminus j_k): \pi_{N \setminus j_k} = \pi^{-j}} p_\omega(\pi_{N \setminus j_k}) \frac{\omega_{j_k}}{\omega_j + \sum_{r=1}^k \omega_{j_r}}$$

$$+ p_\omega(\pi^{-j}) \frac{\omega_j}{\omega_j + \sum_{r=1}^k \omega_{j_r}}.$$

By induction hypothesis,

$$\sum_{\pi \in \Pi(N \setminus j_k): \pi_{N \setminus j_k} = \pi^{-j}} p_\omega(\pi_{N \setminus j_k}) = p_\omega(\pi_{N \setminus j_k}^{-j}).$$
Therefore, replacing this expression in equation above, we have that

$$
\sum_{\pi \in \Pi(N) : \pi_N \setminus j = \pi^{-j}} p_{\omega}(\pi) = p_{\omega}(\pi_{N \setminus j}^{-j}) \frac{\omega_{jk}}{\omega_{j} + \sum_{r=1}^{k} \omega_{jr}} + p_{\omega}(\pi^{-j}) \frac{\omega_{j}}{\omega_{j} + \sum_{r=1}^{k} \omega_{jr}}
$$

(1)

Moreover,

$$
p_{\omega}(\pi^{-j}) = p_{\omega}(\pi_{N \setminus j}^{-j}) \frac{\omega_{jk}}{\omega_{j} \sum_{r=1}^{k} \omega_{jr}}, \text{ that is,}
$$

$$
p_{\omega}(\pi_{N \setminus j}^{-j}) = p_{\omega}(\pi^{-j}) \frac{\omega_{jk}}{\omega_{j} \omega_{jr} \sum_{r=1}^{k} \omega_{jr}}.
$$

Replacing this expression in (1),

$$
\sum_{\pi \in \Pi(N) : \pi_N \setminus j = \pi^{-j}} p_{\omega}(\pi) = p_{\omega}(\pi^{-j}) \frac{\sum_{r=1}^{k} \omega_{jr}}{\omega_{j} \omega_{jr} \sum_{r=1}^{k} \omega_{jr}} \frac{\omega_{jk}}{\omega_{j} + \sum_{r=1}^{k} \omega_{jr}} + p_{\omega}(\pi^{-j}) \frac{\omega_{j}}{\omega_{j} + \sum_{r=1}^{k} \omega_{jr}}
$$

$$
= p_{\omega}(\pi^{-j}).
$$

Claim 9 \(v(Pre(i, \pi) \cup i) - v(Pre(i, \pi)) = v(Pre(i, \pi^{-j}) \cup i) - v(Pre(i, \pi^{-j}))\) for each \(\pi^{-j} \in \Pi(N \setminus j)\) and \(\pi \in \Pi(N)\) with \(\pi_{N \setminus j} = \pi^{-j}\).

Proof. It follows from \(j\) being a null player. ■

So, \(Sh^\omega_i(N, v) = Sh^\omega_i(N \setminus j, v)\) is a consequence of both claims. ■

Proposition 10 \(\zeta\) satisfies EFF, LIN, INC, BCAP, CO and ESUG.

Proof. \(\zeta\) satisfies EFF: Both \(sh\) and \(Sh^\lambda(T)\) satisfy EFF (Shapley (1953) and Kalai and Samet (1987), respectively). Thus, from the definition of \(\zeta\), we conclude that \(\zeta\) satisfies EFF.

\(\zeta\) satisfies LIN: From the inductive formula of \(\zeta\) and by induction on the number of agents, is straightforward to check that \(\zeta\) satisfies LIN.

\(\zeta\) satisfies INC: Let \(C = \{C_1, ..., C_m\}\) and let \(C_q \in C\) be a null coalition. Denote \(M = \{1, 2, ..., m\}\).

To prove that \(\zeta_i(N, v, C) = \zeta_i(N \setminus C_q, v, C_N \setminus C_q)\) for all \(i \in N \setminus C_q\) is enough to prove that \(v^\pi_{C_q}(T) = v^\pi_{C_q}(T)\) for all \(T \subset C_r\) and all \(C_r \in C \setminus C_q\).
By definition
\[ v_{C_q}^N(T) = Sh_r^\lambda(T)(v/\mathcal{C}(N\setminus C_r)\cup T) \]
and \( q \in M \) is a null player in the game between coalitions. Since \( Sh^\lambda(T) \) satisfies INP (Proposition 7), we have
\[ Sh_r^\lambda(T)(v/\mathcal{C}(N\setminus C_r)\cup T) = Sh_r^\lambda(T)(v/\mathcal{C}(N\setminus (C_q\cup C_r))\cup T) \]
and by definition,
\[ Sh_r^\lambda(T)(v/\mathcal{C}(N\setminus (C_q\cup C_r))\cup T) = v_{C_q}^N(T). \]

Combining the three last expressions we obtain the result.

\( \zeta \) satisfies BCAP: Fix \( C_q \in \mathcal{C} \) and \( i, j \in C_q \).

By definition of \( \zeta_i \),
\[ \zeta_i(N, v, \mathcal{C}) - \zeta_i(N\setminus j, v, \mathcal{C}_N\setminus j) = Sh_i(C_q, v_{C_q}^N) - Sh_i(C_q\setminus j, v_{C_q\setminus j}^N) \]
By definition of the reduced game,
\[ v_{C_q}^N(T) = v_{C_q\setminus j}^N(T) \] for all \( T \subseteq C_q \setminus j \). Thus, \( (C_q\setminus j, v_{C_q\setminus j}^N) \) coincides with \( (C_q\setminus j, v_{C_q\setminus j}^N) \) and so, expression above can be restated as
\[ \zeta_i(N, v, \mathcal{C}) - \zeta_i(N\setminus j, v, \mathcal{C}_N\setminus j) = Sh_i(C_q, v_{C_q}^N) - Sh_i(C_q\setminus j, v_{C_q\setminus j}^N). \]

Since \( Sh \) satisfies Balanced Contributions (Myerson, 1980. Theorem 1), we have that
\[ \zeta_i(N, v, \mathcal{C}) - \zeta_i(N\setminus j, v, \mathcal{C}_N\setminus j) = Sh_i(C_q, v_{C_q}^N) - Sh_i(C_q\setminus j, v_{C_q\setminus j}^N). \]
Reasoning as before, we have that \( (C_q\setminus i, v_{C_q}^N) \) coincides with \( (C_q\setminus i, v_{C_q\setminus i}^N) \), so by the expression above, and from the definition of \( \zeta_i \),
\[ \zeta_i(N, v, \mathcal{C}) - \zeta_i(N\setminus j, v, \mathcal{C}_N\setminus j) = Sh_j(C_q, v_{C_q}^N) - Sh_j(C_q\setminus i, v_{C_q\setminus i}^N) = \zeta_j(N, v, \mathcal{C}) - \zeta_j(N\setminus i, v, \mathcal{C}_N\setminus i) \]
and hence the result.

\( \zeta \) satisfies CO: Let \( \mathcal{C}, v \) and \( v' \) such that
\[ v\left( T \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) = v'\left( T \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) \] for all \( T \subseteq C_q \) and all \( \mathcal{R} \subseteq \mathcal{C}\setminus\{C_q\} \).

It is enough to prove that \( v_{C_q}^N(T) = v_{C_q}^N(T) \) for all \( T \subseteq C_q \).
By the condition satisfied by \( v \) and \( v' \) we have that,
\[ (v/\mathcal{C}(N\setminus C_q)\cup T) = (v'/\mathcal{C}(N\setminus C_q)\cup T) \] for all \( T \subseteq C_q \).
Moreover, the weight system $\lambda(T)$ depends only on the number of agents in each problem. Since the number of agents in $(v/\mathcal{C}(N\backslash C_q)\cup T)$ coincides with the number of agents in $(v'/\mathcal{C}(N\backslash C_q)\cup T)$, by definition of $\text{Sh}^{\lambda(T)}$, it is straightforward to check that:

$$\text{Sh}^{\lambda(T)}(v/\mathcal{C}(N\backslash C_q)\cup T) = \text{Sh}^{\lambda(T)}(v'/\mathcal{C}(N\backslash C_q)\cup T) \text{ for all } T \subset C_q.$$ (2)

By definition of $v^*_C$ and $v^*_{C_q}$,

$$v^*_C(T) = \text{Sh}^{\lambda(T)}(v/\mathcal{C}(N\backslash C_q)\cup T) \text{ and } v^*_{C_q}(T) = \text{Sh}^{\lambda(T)}(v'/\mathcal{C}(N\backslash C_q)\cup T) \text{ for all } T \subset C_q.$$

Hence, by expression (2),

$$v^*_C(T) = v^*_{C_q}(T).$$

**$\zeta$ satisfies ESUG:** Let $\mathcal{C}$ and $u^N(T)$. Fix $i \in C_q \in \mathcal{C}$. By definition of $\zeta$,

$$\zeta_i(u^N) = \frac{1}{|N|} \left[ u^N(N) + \sum_{j \in C_q \backslash i} \frac{|N|}{|N_q|} \left( \zeta^N_{N\backslash j}(u^N_{N\backslash j}) - \zeta^N_{N\backslash i}(u^N_{N\backslash i}) \right) + \sum_{C_r \subset \mathcal{C} \subset C_q} \left( \frac{|C_r|}{|C_q|} \sum_{j \in C_q} \zeta^N_{N\backslash C_r}(u^N_{N\backslash C_r}) - \sum_{j \in C_r} \zeta^N_{N\backslash C_q}(u^N_{N\backslash C_q}) \right) \right].$$

By definition of $u^N$, we have that

$$\zeta_i(u^N_{N\backslash j}) = \zeta_j(u^N_{N\backslash i}) = \zeta_j(u^N_{N\backslash C_q}) = 0 \text{ for all } j \in C_r \subset \mathcal{C} \backslash C_q.$$

Thus, from expression before:

$$\zeta_i(u^N_{N\backslash i}) = \frac{1}{|N|} u^N(N) = \frac{1}{n}.$$ 

Thus, $\zeta_i(u^N) = \zeta_j(u^N) = \frac{1}{n}$ for all $i, j \in N$. □

**4 Characterization**

**Theorem 11** A value $\phi$ over the set of TU games with coalition structure satisfies EFF, LIN, INC, BCAP, CO and ESUG, if and only if $\phi = \zeta$.
Proof. Let \( C = \{C_1, \ldots, C_m\} \) be a coalition structure. Let \( M = \{1, \ldots, m\} \).

Let \( \phi^1 \) and \( \phi^2 \) be two values satisfying EFF, LIN, INC, BCAP, CO and ESUG. We will prove that \( \phi^1_i(N, v, C) = \phi^2_i(N, v, C) \) for all \( i \in N \).

We prove the result by induction over the number of players, \(|N|\).

If \(|N| = 1\), since \( \phi^1 \) and \( \phi^2 \) satisfy EFF, \( \phi^1(N, v, C) = \phi^2(N, v, C) \).

Assume the result holds for \(|N| < k\). Now we prove that the result holds for \(|N| = k\).

Shapley (1953b) proved that every TU game can be expressed as a linear combination of unanimity games. Since \( \phi^1 \) and \( \phi^2 \) satisfy LIN we can restrict our proof to unanimity games.

Let \( S \subset N, S \neq \emptyset \). Let consider the game \( u^S_N \).

First, we will prove that it is enough to restrict the proof to the case where all the coalitions intersect the carrier, \( S \). To prove that, suppose that there exists some coalition, say \( C_m \in C \) that does not intersect the carrier, that is, \( S \cap C_m = \emptyset \).

Clearly, \( C_m \) is a null coalition. Thus, by INC,
\[
\phi^1_i(N, u^S_N, C) = \phi^1_i(N \setminus C_m, u^S_{N \setminus C_m}, C_{N \setminus C_m}) \text{ and }
\phi^2_i(N, u^S_N, C) = \phi^2_i(N \setminus C_m, u^S_{N \setminus C_m}, C_{N \setminus C_m}) \text{ for all } i \in N \setminus C_m.
\]

And by induction hypothesis,
\[
\phi^1_i(N \setminus C_m, u^S_{N \setminus C_m}, C_{N \setminus C_m}) = \phi^2_i(N \setminus C_m, u^S_{N \setminus C_m}, C_{N \setminus C_m}) \text{ for all } i \in N \setminus C_m.
\]

Moreover, as an implication of INC and EFF,
\[
\sum_{i \in C_m} \phi^1_i(N, u^S_N, C) = 0 \text{ and }
\sum_{i \in C_m} \phi^2_i(N, u^S_N, C) = 0.
\]

Now we prove that every agent in \( C_m \) receives zero under both values.

If \(|C_m| = 1\), from expression above, it is clear that \( \phi^1_i(N, u^S_N, C) = \phi^2_i(N, u^S_N, C) = 0 \), \( i \in C_m \).

If \(|C_m| = l, l > 1\), from BCAP,
\[
\phi^1_i(N, u^S_N, C) - \phi^2_i(N, u^S_{N \setminus j}, C_{N \setminus j}) = \phi^2_i(N, u^S_N, C) - \phi^2_i(N \setminus i, u^S_{N \setminus i}, C_{N \setminus i}),
\]
for all \( i, j \in C_m, x = 1, 2 \).

By induction hypothesis, \( \phi^1_i(N \setminus j, u^S_{N \setminus j}, C_{N \setminus j}) = \phi^2_i(N \setminus j, u^S_{N \setminus j}, C_{N \setminus j}) \), for all \( i, j \in C_m, x = 1, 2 \).

Hence, combining both expressions, we have that
\( \phi_i(N, u_N^S, C) = \phi_i^x(N, u_N^S, C) \), for all \( i, j \in C_m \), and \( x = 1, 2 \). Moreover, since \( \sum_{i \in C_m} \phi_i(N, u_N^S, C) = 0 \), we obtain that \( \phi_i(N, u_N^S, C) = 0 \) for all \( i \in C_m \) and \( x = 1, 2 \).

From now on, we assume that \( S \cap C_q \neq \emptyset \) for all \( C_q \in \mathcal{C} \).

Fix \( C_q \in \mathcal{C} \) and \( i \in C_q \). We should prove that \( \phi_i^1(N, u_N^S, C) = \phi_i^2(N, u_N^S, C) \).

Let \( T := (C_q \cap S) \cup (N \setminus C_q) \).

**Claim 12**

\[
\begin{align*}
 u_N^S \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) &= u_N^T \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right)
\end{align*}
\]

for all \( T' \subset C_q \) and all \( \mathcal{R} \subset \mathcal{C} \setminus C_q \).

**Proof.** Let us denote \( S^1 = C_q \cap S \). Fix \( T' \subset C_q \). We distinguish three cases:

**Case 1:** \( S^1 \subset T' \) and \( \mathcal{R} = \mathcal{C} \setminus C_q \). In this case,

\[
S \subset \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) \text{ and } T \subset \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right),
\]

Thus by definition of \( u_N^S \) and \( u_N^T \), we have that

\[
\begin{align*}
u_N^S \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) &= u_N^T \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right).
\end{align*}
\]

**Case 2:** \( S^1 \nsubseteq T' \). Hence, there exists some \( i \in S^1 \) such that \( i \notin T' \), and so,

\[
S \nsubseteq \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) \text{ and } T \nsubseteq \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right).
\]

Hence,

\[
\begin{align*}
u_N^S \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) &= u_N^T \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) = 0.
\end{align*}
\]

**Case 3:** \( \mathcal{R} \neq \mathcal{C} \setminus C_q \). Thus there exists some \( C_k \in \mathcal{C} \setminus C_q \) such that \( C_k \nsubseteq \mathcal{R} \). Since by hypothesis, \( C_r \cap S \neq \emptyset \) for all \( C_r \in \mathcal{C} \), we have that

\[
S \nsubseteq \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) \text{ and } T \nsubseteq \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right). \]

Hence,

\[
\begin{align*}
u_N^S \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) &= u_N^T \left( T' \cup \bigcup_{C_r \in \mathcal{R}} C_r \right) = 0. \quad \blacksquare
\end{align*}
\]

We are under the assumptions of \( CO \) (Claim 12), hence,

\[
\phi_i^1(N, u_N^S, C) = \phi_i^1(N, u_N^T, C) \text{ and } \phi_i^2(N, u_N^S, C) = \phi_i^2(N, u_N^T, C).
\]
Hence, it is enough to prove that \( \phi_i^1(N, u_N^T, C) = \phi_i^2(N, u_N^T, C) \).

By BCAP,
\[
\phi_i^1(N, u_N^T, C) - \phi_i^1(N \backslash j, u_{N \backslash j}^T, C_{N \backslash j}) = \phi_j^1(N, u_N^T, C) - \phi_j^1(N \backslash i, u_{N \backslash i}^T, C_{N \backslash i})
\]
for all \( j \in C_q \backslash i \).

Hence,
\[
\sum_{j \in C_q \backslash i} \left( \phi_i^1(N, u_N^T, C) - \phi_i^1(N \backslash j, u_{N \backslash j}^T, C_{N \backslash j}) \right) = \sum_{j \in C_q \backslash i} \left( \phi_j^1(N, u_N^T, C) - \phi_j^1(N \backslash i, u_{N \backslash i}^T, C_{N \backslash i}) \right).
\]

Rearranging terms:
\[
(|C_q| - 1) \phi_i^1(N, u_N^T, C) = \sum_{j \in C_q \backslash i} \left( \phi_j^1(N, u_N^T, C) - \phi_j^1(N \backslash i, u_{N \backslash i}^T, C_{N \backslash i}) \right) + \phi_i^1(N \backslash j, u_{N \backslash j}^T, C_{N \backslash j}).
\]

(3)

Consider now the unanimity game \( u_N^N \).

Claim 13 Given \( C_r \in C \backslash C_q \),
\[
u_N^T \left( T' \cup \bigcup_{C_t \in Q} C_t \right) = u_N^N \left( T' \cup \bigcup_{C_t \in Q} C_t \right)
\]
for all \( T' \subset C_r \) and all \( Q \subset C \backslash C_r \).

Proof. Analogous to those for Claim 12. \( \blacksquare \)

We are under the assumptions of CO (Claim 13), and so,
\[
\phi_j^1(N, u_N^N, C) = \phi_j^1(N, u_N^T, C) \text{ for all } j \in N \backslash C_q.
\]

(4)

On the other hand, ESUG and EFF imply that
\[
\phi_j^1(N, u_N^N, C) = \frac{1}{|N|} \text{ for all } j \in N.
\]

(5)

Hence,
\[
\sum_{j \in N \backslash C_q} \phi_j^1(N, u_N^T, C) = \sum_{j \in N \backslash C_q} \phi_j^1(N, u_N^N, C) = \frac{|N \backslash |C_q|}{|N|}.
\]
Moreover, by $E_F$,

$$
\sum_{j \in C_q \setminus i} \phi_1^j(N, u_N^T, C) = u_N^T(N) - \phi_1^1(N, u_N^T, C) - \frac{|N| \setminus |C_q|}{|N|}.
$$

(6)

Since $u_N^T(N) = 1$, we have that $u_N^T(N) - \frac{|N| \setminus |C_q|}{|N|} = |C_q|/|N|$. Replacing this expression in (6):

$$
\sum_{j \in C_q \setminus i} \phi_1^j(N, u_N^T, C) = \frac{|C_q|}{|N|} - \phi_1^1(N, u_N^T, C).
$$

And replacing this expression in (3):

$$
(\frac{|C_q|}{|N|} - 1) \phi_1^1(N, u_N^T, C) = \sum_{j \in C_q \setminus i} \phi_1^j(N \setminus i, u_N^T, C_{N \setminus i}) + \sum_{j \in C_q \setminus i} \phi_1^j(N \setminus j, u_N^T, C_{N \setminus j}).
$$

Rearranging terms:

$$
\frac{|C_q|}{|N|} - \sum_{j \in C_q \setminus i} \phi_1^j(N \setminus i, u_N^T, C_{N \setminus i}) - \sum_{j \in C_q \setminus i} \phi_1^j(N \setminus j, u_N^T, C_{N \setminus j}) = \phi_1^1(N, u_N^T, C).
$$

And so,

$$
\phi_1^1(N, u_N^T, C) = \frac{1}{|C_q|} \left[ \frac{|C_q|}{|N|} - \sum_{j \in C_q \setminus i} \phi_1^j(N \setminus i, u_N^T, C_{N \setminus i}) + \sum_{j \in C_q \setminus i} \phi_1^j(N \setminus j, u_N^T, C_{N \setminus j}) \right].
$$

Since $\phi_2^2$ satisfies the same properties as $\phi_1^1$, we can proceed in the same way as for $\phi_1^1$, and we obtain that:

$$
\phi_2^2(N, u_N^T, C) = \frac{1}{|C_q|} \left[ \frac{|C_q|}{|N|} - \sum_{j \in C_q \setminus i} \phi_2^j(N \setminus i, u_N^T, C_{N \setminus i}) + \sum_{j \in C_q \setminus i} \phi_2^j(N \setminus j, u_N^T, C_{N \setminus j}) \right].
$$

But by induction hypothesis:
\[ \phi_1^1(i, u_{N-i}, C_{N-i}) = \phi_1^2(i, u_{N-i}, C_{N-i}) \text{ and} \]
\[ \phi_1^1(j, u_{N-j}, C_{N-j}) = \phi_1^2(j, u_{N-j}, C_{N-j}) \text{ for all } j \neq i. \]

Hence we conclude that \( \phi_1^1(N, u_N, C) = \phi_1^2(N, u_N, C). \]

## 5 Independence of the axioms

In this section we show that the axioms used in Theorem 11 are independent.

The following value satisfies LIN, INC, BCAP, CO, ESUG and fails EFF. This is the value proposed by Aumman and Drèze (1974) and it is defined as follows.

Let \( (N, v, C) \),

\[ AD_i(v) = Sh_i(C_q, v|C_q) \]

for all \( i \in C_q \), where \( (v|C_q)(T) = v(T) \) for all \( T \subset C_q \).

**Proposition 14** \( AD \) satisfies LIN, INC, BCAP, CO and ESUG.

**Proof.** \( AD \) satisfies LIN: See Aumman and Drèze (1974, Theorem 3, page 220).

\( AD \) satisfies INC: It is clear following the definition of \( AD \).

\( AD \) satisfies BCAP: \( Sh \) satisfies Balanced Contribution, (Myerson, 1980, Theorem 1), hence it is clear that \( AD \) satisfies BCAP.

\( AD \) satisfies CO: It is straightforward following the definition of \( AD \).

\( AD \) satisfies ESUG: Let \( (N, u_N, C) \). If \( C = \{N\} \), \( AD_i(u_N^N) = \frac{1}{n} \) for all \( i \in N \). If \( C \neq \{N\} \), \( AD_i(u_N^C) = 0 \) for all \( i \in N \). So we have the result. ■

Moreover, it is clear that \( AD \) fails EFF.

Now we present a value that satisfies EFF, LIN, INC, BCAP, ESUG and fails CO. This is the Shapley value (Shapley, 1953b).

**Proposition 15** \( Sh \) satisfies EFF, LIN, INC, BCAP and ESUG.

**Proof.** It is well known that \( Sh \) satisfies EFF and LIN (Shapley, 1953b).

\( Sh \) satisfies INC: \( Sh \) satisfies INP, hence it is straightforward to check that \( Sh \) satisfies INC.

\( Sh \) satisfies BCAP: See Myerson (1980, Theorem 1).

\( Sh \) satisfies ESUG: Consider the game \( (N, u_N^N, C) \). By definition of \( u_N^N \), all the agents are symmetric under \( u_N^N \). Since \( Sh \) satisfies Symmetry (Shapley, 1953b),
Consider the following example:

**Example 16** Let \( N = \{1, 2, 3\} \) and \( \mathcal{C} = \{C_1, C_2\} \) where \( C_1 = \{1\} \) and \( C_2 = \{2, 3\} \). Let \( v(S) = 1 \), for all \( S \subseteq 2^N \setminus \{\emptyset, \{3\}, N\} \), \( v(\{3\}) = 0 \) and \( v(N) = 2 \).

In this case, \( Sh(v) = (\frac{5}{6}, \frac{5}{6}, \frac{5}{6}) \) and \( \zeta(v) = (1, \frac{3}{4}, \frac{1}{4}) \).

Since the \( Sh \) is different from \( \zeta \) it is clear that \( Sh \) fails CO.

Before presenting the next value we give a definition:

**Definition 17** Let \( \mathcal{C} = \{C_1, ..., C_m\} \) and let \( M = \{1, ..., m\} \). Given a weight system \( \omega \in \mathbb{R}_+^M \), we define the weighted egalitarian value with weights given by \( \omega \), \( E^\omega \) as:

\[
E^\omega_q(v/\mathcal{C}) = \begin{cases} 
\frac{\omega_q}{|M|} v(N) & \text{if } \omega_q \neq 0 \text{ for some } r \in M \\
\frac{1}{|M|} v(N) & \text{otherwise}
\end{cases}
\]

for all \( q \in M \).

Now we present a value that satisfies EFF, INC, BCAP, CO, ESUG and fails LIN.

This is also a value defined in two stages. In the game between coalitions a weighted egalitarian value is applied and in the intracoalitional game the benefic is assigned among the agents following the Shapley value. Formally,

Let \( \mathcal{C} = \{C_1, ..., C_m\} \) and let \( M = \{1, ..., m\} \). Fix \( C_q \in \mathcal{C} \).

Let \( T \subset C_q \). Consider a weight system \( \delta(T) \in \mathbb{R}_{++}^M \) given by:

\[
\delta_q(T) = \begin{cases} 
0 & \text{if } q \text{ is null in the game } (v/\mathcal{C}_{(N\setminus C_q)\cup T}) \\
|T| & \text{otherwise}
\end{cases}
\]

\[
\delta_r(T) = \begin{cases} 
0 & \text{if } r \text{ is null in the game } (v/\mathcal{C}_{(N\setminus C_q)\cup T}) \text{ if } r \neq q, \\
|C_r| & \text{otherwise}
\end{cases}
\]

A reduced TU game \((C_q, \hat{v}_q^N)\) is defined as

\[
\hat{v}_q^N(T) := E^\delta(T)(v/\mathcal{C}_{(N\setminus C_q)\cup T})
\]

for all \( T \subset C_q \).
Definition 18 Given a TU game with coalition structure \((N, v, \mathcal{C})\), the value \(\phi^1\) is defined as:
\[ \phi^1_i(v) := Sh_i(\hat{v}^N_{C_q}) \]
for all \(i \in C_q \in \mathcal{C} \).

Proposition 19 \(\phi^1\) satisfies EFF, INC, BCAP, CO and ESUG.

Proof. \(\phi^1\) satisfies EFF: both the weighted Egalitarian, \(E^g(T)\), and \(Sh\) satisfy EFF, hence, \(\phi^1\) satisfies EFF.

\(\phi^1\) satisfies INC: Let \(\mathcal{C} = \{C_1, \ldots, C_m\}\) and let \(C_q \in \mathcal{C}\) such that \(C_q\) is a null coalition. In this case, by definition of null coalition, it is satisfied that
\[ v \left( C_q \cup \bigcup_{C_i \in R} C_i \right) = v \left( \bigcup_{C_i \in R} C_i \right) \quad \text{for all} \quad R \subset C \setminus C_q, \quad \text{hence,} \quad \delta_q(T) = 0. \]

To prove that \(\phi^1\) satisfies INC it is enough to prove that the reduced games, \(\hat{v}^N_{C_r}(T)\) and \(\hat{v}^{N\setminus C_q}_{C_r}(T)\) coincide for all \(T \subset C_r\) and all \(C_r \subset C \setminus C_q\).

Fix \(C_r \in C \setminus C_q\). Let \(T \subset C_r\).

Case 1: There exists some \(l \in M\) such that \(\delta_l(T) \neq 0\).

Hence, by definition,
\[ \hat{v}^N_{C_r}(T) = \frac{\delta_q(T)}{\sum_{l \in M} \delta_l(T)} v((N \setminus C_r) \cup T). \]

Since \(\delta_q(T) = 0\) and \(C_q\) is a null coalition,
\[ \hat{v}^N_{C_r}(T) = \frac{\delta_q(T)}{\sum_{l \in M} \delta_l(T)} v((N \setminus C_r) \cup T) = \frac{\delta_q(T)}{\sum_{l \in M} \delta_l(T)} v((N \setminus (C_r \cup C_q)) \cup T) = E^g(T)(v/ \mathcal{C}((N \setminus (C_r \cup C_q)) \cup T)) = \hat{v}^{N\setminus C_q}_{C_r}(T). \]

Case 2: \(\delta_l(T) = 0\) for all \(l \in M\).

In this case, it is clear that \(\hat{v}^N_{C_r}(T) = 0 = \hat{v}^{N\setminus C_q}_{C_r}(T)\).

\(\phi^1\) satisfies BCAP: the proof is analogous to the proof for \(\zeta\) because by the definition of the reduced game \(\hat{v}^N_{C_q}\), it satisfies that \(\hat{v}^N_{C_q}(T) = \hat{v}^{N\setminus \cup_{j\neq q}}_{C_q}(T)\) for all \(T \subset C_q \setminus j\) and \((C_q \setminus i, \hat{v}^N_{C_q})\) coincides with \((C_q \setminus i, \hat{v}^{N\setminus j}_{C_q \setminus i})\). Moreover, \(Sh\) satisfies Balanced Contributions.

\(\phi^1\) satisfies CO: Let \(\mathcal{C}, v\) and \(v'\) such that
\[ v \left( T \cup \bigcup_{C_r \in R} C_r \right) = v' \left( T \cup \bigcup_{C_r \in R} C_r \right) \quad \text{for all} \quad T \subset C_q \quad \text{and all} \quad R \subset C \setminus C_q. \]
It is enough to prove that $\tilde{v}_N^N(T) = \tilde{v}_q^N(T)$ for all $T \subset C_q$.

It is clear that the weight systems for the game $v$ and for the game $v'$ coincide. Moreover, by the condition satisfied by $v$ and $v'$, $(v/C_{(N\setminus C_q) \cup T}) = (v'/C_{(N\setminus C_q) \cup T})$ for all $T \subset C_q$, thus, it is clear that $\tilde{v}_q^N(T) = \tilde{v}_q^N(T)$.

$\varphi^1$ satisfies ESUG: Let $u_N^N$. Fix $C_q \in \mathcal{C}$. By definition, under $u_N^N$, the weight system $\delta(T)$ coincides with the weight system $\lambda(T)$. Therefore, by definition of the weighted egalitarian value,

$$E_q^\lambda(u_N^N) = \frac{\delta(T)}{\sum_{r \in M} \delta_r(T)} u_N^N(N) = \frac{|C_q|}{\sum_{r \in M} |C_r|} u_N^N(N) = \frac{|C_q|}{|N|}.$$ 

Since $Sh$ satisfies Symmetry (Shapley, 1953b) and all the agents are symmetric under $u_N^N$,

$$\varphi^1_i(u_N^N) = E_q^{\lambda(T)}(u_N^N) = \frac{|C_q|}{|N|} \frac{1}{|C_q|} = \frac{1}{|N|} \text{ for all } i \in C_q.$$

The same reasoning is valid for any $j \in C_r \in \mathcal{C} \setminus C_q$.
Hence, $\varphi^1_i(u_N^N) = \varphi^1_j(u_N^N) = \frac{1}{|N|}$ for all $i, j \in N$.

In Example 16, $\varphi^1(v) = \left(\frac{8}{12}, \frac{11}{12}, \frac{5}{12}\right)$. Since $\zeta(v) = \left(1, \frac{3}{4}, \frac{1}{4}\right)$, it follows that $\varphi^1$ fails LIN.

Now we present a value that satisfies EFF, LIN, BCAP, CO, ESUG and fails INC.

This value is similar to the previous one, the only difference is the weight system for the coalitions. More specifically, let $C = \{C_1, ..., C_m\}$ and let $M = \{1, ..., m\}$. Fix $C_q \in \mathcal{C}$.

Let $T \subset C_q$. Consider the weight system $\lambda(T) \in \mathbb{R}^M_{++}$. A reduced TU game $(C_q, \tilde{v}_N^N_q)$ is defined as:

$$\tilde{v}_N^N_q(T) := E_q^\lambda(v/C_{(N\setminus C_q) \cup T})$$

for all $T \subset C_q$.

**Definition 20** Given a TU game with coalition structure $(N, v, \mathcal{C})$, the value $\varphi^2$ is defined as:

$$\varphi^2_i(v) := Sh_i(\tilde{v}_N^N_q)$$

for all $i \in C_q \in \mathcal{C}$.
Proposition 21 \( \varphi^2 \) satisfies EFF, LIN, BCAP, CO and ESUG.

Proof. \( \varphi^2 \) satisfies EFF: Both \( E^{\mathcal{L}(T)} \) and \( Sh \) satisfy EFF, hence, \( \varphi^2 \) satisfies EFF.

\( \varphi^2 \) satisfies LIN: Let \( (N,v,C) \), \( (N,w,C) \) and real numbers \( \alpha \) and \( \beta \). Fix \( C_q \in \mathcal{C} \).

From the definition of \( E^{\mathcal{L}(T)} \),
\[
(\alpha v + \beta w)^N_{C_q}(C_q) = E^{\mathcal{L}(T)}((\alpha v + \beta w)/\mathcal{C}) = \frac{\lambda_q(T)}{\sum_{r \in M} \lambda_r(T)} (\alpha v + \beta w)(N) = \\
\frac{\alpha \lambda_q(T)}{\sum_{r \in M} \lambda_r(T)} v(N) + \frac{\beta \lambda_q(T)}{\sum_{r \in M} \lambda_r(T)} w(N) = \\
\alpha E^N_{q}(v/\mathcal{C}) + \beta E^N_{q}(w/\mathcal{C}) = \\
\alpha \tilde{v}^N_{C_q}(C_q) + \beta \tilde{w}^N_{C_q}(C_q).
\]

Moreover, \( Sh \) also satisfies LIN. Hence, it is no difficult to see that \( \varphi^2 \) satisfies LIN.

\( \varphi^2 \) satisfies BCAP: the proof is analogous to the proof for \( \zeta \) because by the definition of the reduced game \( \tilde{v}^N_{C_q}(T) = \tilde{v}^N_{C_q \cup \{i\}}(T) \) for all \( T \subset C_q \cup \{i\} \) and \( \left( C_q \cup \{i\}, \tilde{v}^N_{C_q} \right) \) coincides with \( \left( C_q, \tilde{v}^N_{C_q \cup \{i\}} \right) \). Moreover, \( Sh \) satisfies BCAP.

\( \varphi^2 \) satisfies CO: the proof is analogous to the proof for \( \zeta \), because the weight system \( \mathcal{L}(T) \) depends only on the number of agents in \( (N \backslash C_q) \cup T, v, \mathcal{C}_{(N \backslash C_q) \cup T} \) and in \( (N \backslash C_q) \cup T, v', \mathcal{C}_{(N \backslash C_q) \cup T} \) and the number of agents coincides in both problems. Moreover, the rest of conditions also hold.

\( \varphi^2 \) satisfies ESUG: The proof is analogous to the proof for \( \varphi^1 \) because under \( u^N_N \) both values coincide.

In Example 16, we have that \( \varphi^2(v) = \left( \frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right) \). Since \( \zeta(v) = \left( 1, \frac{2}{3}, \frac{1}{3} \right) \), it follows that \( \varphi^2 \) fails INC.

Now we present a value that satisfies EFF, LIN, INC, CO, ESUG and fails BCAP. This is the weighted coalitional value with intracoalitional symmetry (Levy and McLean, 1989) with \( \omega_q = |C_q| \) for all \( C_q \in \mathcal{C} \).

Proposition 22 \( \varphi^\omega \) satisfies EFF, LIN, INC, CO and ESUG.

Proof. \( \varphi^\omega \) satisfies EFF: See Levy and McLean (1989, Lemma 3A).

\( \varphi^\omega \) satisfies LIN: See Levy and McLean (1989, Lemma 3A).

\( \varphi^\omega \) satisfies INC: the proof is analogous to the proof for \( \zeta \) because \( Sh \) satisfies INP (Proposition 7).
\(\varphi^\omega\) satisfies CO: the proof is similar to the proof for \(\zeta\), because the weight system \(\omega\) depends only on the number of agents in each coalition in \(((N\setminus C_q)\cup T, v, C_{(N\setminus C_q)\cup T})\) and in \(((N\setminus C_q)\cup T, v', C_{(N\setminus C_q)\cup T})\) and this number of agents coincides in both problems. Moreover, the rest of conditions also hold.

\(\varphi^\omega\) satisfies ESUG: Let \(u^N\). Hence (Levy and McLean, (1989, Lemma 2A)),

\[
\varphi^\omega_i(u^N) = \frac{1}{|N|} \text{ for all } i \in N.
\]

In Example 16, \(\varphi^\omega(v) = (1, \frac{2}{3}, \frac{1}{3})\). Since \(\zeta(v) = (1, \frac{3}{4}, \frac{1}{4})\) it follows that \(\varphi^\omega\) fails BCAP.

Now we present a value that satisfies EFF, LIN, INC, BCAP, CO and fails ESUG. This is the Owen value, Ow (Owen, 1977).

**Proposition 23** Ow satisfies EFF, LIN, INC, BCAP and CO.

**Proof.** It is well-known that Ow satisfies EFF and LIN.

Ow satisfies INC: The proof is similar to the proof that \(\zeta\) satisfies INC, because \(Sh\) satisfies INP.

Ow satisfies BCAP: See Calvo et al. (1996).

Ow satisfies CO: the proof is similar to the proof for \(\zeta\), because the games between coalitions \(v\) and \(v'\) coincide. And so, by definition of \(Sh\), the respective reduced games coincide too.

**Example 24** Let \(N = \{1, 2, 3\}\) and \(C = \{C_1, C_2\}\) where \(C_1 = \{1\}\) and \(C_2 = \{2, 3\}\). Let \(u^N\). In this case, \(Ow(u^N) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\) and \(\zeta(u^N) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\).

Hence, it is clear that \(Ow\) fails ESUG.

In the following table we summarize the results presented in this Section:

<table>
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<tr>
<th></th>
<th>EFF</th>
<th>LIN</th>
<th>INC</th>
<th>BCAP</th>
<th>CO</th>
<th>ESUG</th>
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<tr>
<td>(\varphi^1)</td>
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</tr>
<tr>
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<td>×</td>
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</tr>
<tr>
<td>(\varphi^\omega)</td>
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<tr>
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6 Others results

In this section we present two new characterizations: one of the Owen value (Owen, 1977) and the other of a value presented by Levy and McLean (1989).

As we present in section before, the Owen value satisfies all the properties used in Theorem 11 except ESUG. The new characterization is obtained replacing this property for the following one:

**Inversed Proportional Sharing in Unanimity Games (IPSUG)** For any game \((N, u^N, C)\), and any coalitions \(C_q, C_r \in C\),

\[
|C_q| f_i (N, u^N, C) = |C_r| f_j (N, u^N, C)
\]

for all \(i \in C_q\) and \(j \in C_r\).

This property asserts that under the unanimity game with carrier \(N\), each agent should receive a payment inversely proportional to the size of the coalition he belongs to.

**Proposition 25** \(Ow\) satisfies IPSUG.

**Proof.** Let \((N, u^N, C)\) and \(C_N = \{C_1, ..., C_m\}\). Let \(M = \{1, ..., m\}\). Consider \(C_q, C_r \in C\). Let \(i \in C_q\) and \(j \in C_r\). Hence (Owen, 1977):

\[
Ow_i (u^N) = \frac{1}{|C_q||M|} \quad \text{and} \quad Ow_j (u^N) = \frac{1}{|C_r||M|}.
\]

Thus, \(|C_q| Ow_i (u^N) = |C_r| Ow_j (u^N)\). ■

Now we present the new characterization:

**Theorem 26** A value \(\phi\) over the set of TU games with coalition structure satisfies EFF, LIN, INC, BCAP, CO and IPSUG if and only if \(\phi = Ow\).

**Proof.** IPSUG and EFF imply that, \(\phi_i (N, u^N, C) = \frac{1}{|C_q||C|}\) for all \(i \in C_q \in C\). The rest of the proof is analogous to those of Theorem 11 and we omit it. ■

Now we present a new characterization of the value presented by Levy and McLean (1989) with weights given by the size of the coalitions, replacing the property of BCAP by a new one and adding the property of Intracoalitional Symmetry. Before present the characterization we introduce both properties:

Given \((N, v, C)\) and \(i \in N\), we denote by \((N, v^{-i}, C)\) the game given by \(v^{-i} (S) = v (S \cap (N \setminus i))\) for all \(S \subseteq N\).
Balanced v-Contributions Among Players in the same Coalition (v-BCAP)

Given a game \((N, v, C)\). For all \(i, j \in C_q \subseteq C\),

\[
f_i(N, v, C) - f_i(N, v^{-j}, C) = f_j(N, v, C) - f_j(N, v^{-i}, C) .
\]

This property states that for any two players that belong to the same coalition in \(C\), the amount that each agent would gain or lose if the other is turned into a null player should be equal.

Note that in this new property the agents do not leave the game.

Before introducing the next property we provide a definition:

**Definition 27** We say that \(i, j \in N\) are symmetric players if

\[
v(S \cup i) = v(S \cup j)
\]

for all \(S \subset N \setminus \{i, j\}\).

**Intracoalitional Symmetry (IS)** Given a game \((N, v, C)\). For all \(i, j \in C_q \subseteq C\), such that \(i\) and \(j\) are symmetric players,

\[
f_i(N, v, C) = f_j(N, v, C).
\]

**Proposition 28** \(\varphi^\omega\) with \(\omega_q = |C_q|\) for all \(q \in M\) satisfies v-BCAP and IS.

**Proof.** Let \(C = \{C_1, ..., C_m\}\) and \(M = \{1, ..., m\}\).

\(\varphi^\omega\) satisfies IS: see Levy and McLean (1989, Lemma 3A).

\(\varphi^\omega\) satisfies v-BCAP: assume first we are working under the unanimity game with carrier \(S\). Given \(C_q \subseteq C\), let \(S_q = C_q \cap S\) and \(S^c_q = C_q \cap (N \setminus S)\).

If \(i, j \in S_q\) or \(i, j \in S^c_q\), since \(\varphi^\omega\) satisfies Intracoalitional Symmetry it follows that

\[
\varphi^\omega_i(u^S_N) - \varphi^\omega_j(u^S_N) = \varphi^\omega_i((u^S_N)^{-j}) - \varphi^\omega_j((u^S_N)^{-i}).
\]

If \(i \in S_q\) and \(j \in S^c_q\), by Levy and McLean (1989, Lemma 2A),

\[
\varphi^\omega_i(u^S_N) - \varphi^\omega_j(u^S_N) = \sum_{r \in I^S_q} \omega_r \frac{1}{|C_q \cap S|} - 0
= \varphi^\omega_i((u^S_N)^{-j}) - \varphi^\omega_j((u^S_N)^{-i}).
\]

(7)

where \(I^S_q = \{r \in M : C_r \cap S \neq \emptyset\}\).
In general, we have to prove:

\[ \varphi^\omega_i(v) - \varphi^\omega_j(v^{-j}) = \varphi^\omega_j(v) - \varphi^\omega_i(v^{-i}) \]

(8)

Given \( v = \sum_{S \subseteq N} a_S u^S_N \); it is straightforward to check that \( v^{-j} = \sum_{S \subseteq N} a_S (u^S_N)^{-j} \).

Under LIN, (8) is equivalent to

\[ \sum_{S \subseteq N} a_S \varphi^\omega_i(u^S_N) - \sum_{S \subseteq N} a_S \varphi^\omega_j((u^S_N)^{-j}) = \sum_{S \subseteq N} a_S \varphi^\omega_j(u^S_N) - \sum_{S \subseteq N} a_S \varphi^\omega_i((u^S_N)^{-i}) \]

equivalently,

\[ \sum_{S \subseteq N} a_S \varphi^\omega_i(u^S_N) - \sum_{S \subseteq N} a_S \varphi^\omega_j(u^S_N) = \sum_{S \subseteq N} a_S \varphi^\omega_j((u^S_N)^{-j}) - \sum_{S \subseteq N} a_S \varphi^\omega_i((u^S_N)^{-i}) \]

and the result follows from (7).

Now we present the new characterization.

**Theorem 29** A value \( \phi \) over the set of TU games with coalition structure satisfies EFF, LIN, INC, v-BCAP, CO, ESUG and IS if and only if \( \phi = \varphi^\omega \) with \( \omega_q = |C_q| \), for all \( q \in M \).

**Proof.** Let \( \phi^1 \) and \( \phi^2 \) be two values satisfying EFF, LIN, INC, v-BCAP, CO, ESUG and IS.

Following the same reasoning as in the proof of Theorem 11, it is enough to prove that \( \phi^1_i(N, u^T_N) = \phi^2_i(N, u^T_N) \) for all \( i \in C_q \), where \( T = S_q \cup (N \setminus C_q) \) for some \( S_q \subseteq C_q \), \( S_q \neq \emptyset \). As in the proof of Theorem 11, CO implies \( \phi^\omega_i(N, u^T_N) = \phi^\omega_i(N, u^T_N) \) for all \( i \in N \setminus C_q \) and \( x = 1, 2 \).

Under EFF and ESUG, we have \( \sum_{i \in C_q} \phi^1_i(N, u^T_N) = \sum_{i \in C_q} \phi^2_i(N, u^T_N) \).

Under IS, we have \( \phi^\omega_i(N, u^T_N) = \phi^\omega_j(N, u^T_N) \) for all \( i, j \in S_q \) (respectively, \( i, j \in C_q \setminus S_q \)) and \( x = 1, 2 \). Hence it is enough to prove \( \phi^\omega_i(N, u^T_N) = 0 \) for all \( i \in C_q \setminus S_q \), \( x = 1, 2 \). This is clear for \( S_q = C_q \). Let \( i \in S_q \) and \( j \in C_q \setminus S_q \). Player \( j \) is a null player in \( (N, u^T_N) \) and hence \( (N, u^T_N) = (N, (u^T_N)^{-j}) \).

Under v-BCAP,

\[ 0 = \phi^\omega_i(N, u^T_N) - \phi^\omega_j(N, (u^T_N)^{-j}) = \phi^\omega_j(N, u^T_N) - \phi^\omega_j(N, (u^T_N)^{-i}) \]

Obviously, \( (N, (u^T_N)^{-i}) \) is the null game \( (u^T_N)^{-i} \) \( S = 0 \) for all \( S \subseteq N \) and thus EFF and IS imply \( \phi^\omega_j(N, (u^T_N)^{-i}) = 0 \). Thus, \( \phi^\omega_j(N, u^T_N) = 0 \) for \( x = 1, 2 \).
Remark 30 We need to add IS in order to avoid an infinite family of non-symmetric values. Take for example the value $F$ given by $F(N; v, \mathcal{C}) = \varphi^\omega(N; v, \mathcal{C})$ if $\{1, 2\} \notin \mathcal{C}$ or $\{3\} \notin \mathcal{C}$, with $\omega_q = |C_q|$ for all $C_q \in \mathcal{C}$. When $\{1, 2\}, \{3\} \in \mathcal{C}$, take $F_i(N; v, \mathcal{C}) = \varphi^\omega_i(N; v, \mathcal{C})$ for all $i \in N \setminus \{1, 2\}$ and moreover

$$F_1(N, v, \mathcal{C}) = \varphi^\omega_1(N, v, \mathcal{C}) + v(\{3\})$$
$$F_2(N, v, \mathcal{C}) = \varphi^\omega_2(N, v, \mathcal{C}) - v(\{3\}).$$

This value satisfies EFF, LIN, INC, $v$-BCAP, CO and ESUG, but fails IS.

References


