Endogenous Cooperative Networks: Social Capital and the Small-World Property

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Abstract

This paper shows that the relationship between social networks and cooperative trading is a two-way street. The possibility of cooperative trading enforced by third-party (community) sanctions provides players with incentives to form costly links, and investing in costly links increases the power of third-party enforcement. In other words, the fact that trading is embedded in the social structure provides players with incentives to form costly links to internalize network externalities that arise from the fact that the structure of the network determines the interaction pattern and the information flows between players.

The paper formalizes in game theoretic model, the insight of Coleman’s concerning the importance of closure for the emergence of cooperative behavior and explains how and why clousere emerge. Based on this a formal definition of social capital is advanced and shows that pair-wise equilibrium networks have the small-world property.
1 Introduction

It is widely known that many beneficial economic transactions are enforced by informal means. Social networks and reputation are frequently used as informal enforcement mechanisms to substitute for formal means like courts and explicit contracts. For instance, Greif (1997) have documented that social norms were used to enhance exchange in Europe as early as the medieval period, and that informal enforcement mechanisms based on social networks are currently utilized in economies with relatively well developed legal systems such as the USA.

The literature studying informal enforcement mechanism in social networks is vast. This literature, more or less, concludes that cooperative trading can be enforced by third-party sanctions when either information concerning the history of play of different players flow to the rest of the society or when players adopt a contagious strategy (players react to a deviation by punishing subsequent partners regardless of their history of play) or both. This result has been derived on the assumption that social networks are already in place. That is, for a given pattern of play (usually the complete networks) that determines the information flows. People, however, are known to spend resources (e.g., time and effort) to create networks. So, we need to study pathways by which networks get formed and the reasons why they get formed. This paper does exactly so. In particular, it argues that the relationship between social networks and cooperative trading is a two-way street. The possibility of cooperative trading enforced by third-party (community) sanctions provides players with incentives to form costly links, and investing in costly links increases the power of third-party enforcement. In other words, the fact that trading is embedded in the social structure provides players with incentives to form costly links to internalize network externalities that arise from the fact that the structure of the network determines the interaction pattern and the information flows between players.

The framework studied is one where individuals in the early stage of their lives invest in costly social links that have a long-term nature. Once links are formed, indi-

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1 This literature is discussed in detail below.

2 For instance, this is the case when a supplier-customer relationship is formed, a joint-venture
individuals engage in economic transactions captured by a repeated Prisoner’s Dilemma game with changing partners a-la-Kandori (1992). A crucial aspect of the model concerns the relationship between information transmission and social structure. In particular, it is assumed that when a player, say $i$, cheats another player, say $k$, then player $j$ gets informed about player $i$’s defection right after the deviation occurs when $j$ and $k$ are linked. When they are not linked, but there is an $i$-excluding path of length $l$ connecting them, player $j$ gets informed about $i$’s defection $l/s$ periods after that took place, while when there is no such path, $j$ never becomes aware of $i$’s defection. Thus, in our model $1/s$ measures the speed at which the information travels through links$^3$.

In this setting, a strategy profile specifies the links that individuals have to form during the network formation phase and the actions they have to take during the interaction phase (hereinafter referred to as the continuation game). An equilibrium of the whole game is a sequential equilibrium strategy profile such that the proposed network is pair-wise stable—that is no pair of unlinked individuals has an incentive to form a link. In what follows, a pair-wise stable network that is sustained by a sequential equilibrium strategy profile is called a pair-wise equilibrium network.

The particular strategy profile studied here prescribes the following: (i) linked players cooperate in each encounter as long as they have no information that either of them has deviated in the past, while they defect in each encounter with those for whom they have information of a defection in the past; (ii) unlinked players defect in each encounter regardless of the history of play of their partners; and (iii) only deviations occurring during the network formation phase that give rise to networks in which cooperation among linked players is no longer a sequential equilibrium in the continuation game are punished with mutual defection forever thereafter. This strategy profile gives rise to pair-wise equilibrium networks that exhibit a high de-

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$^3$This formulation encompasses the two most emblematic repeated game models. When $s \geq 1$ and everyone is linked to everyone else, the model corresponds to the one in which everyone observes everyone else play in each period, while when $s = 0$, the model corresponds to the one in which no one else observes what happens between a given pair of individuals.
gree of clustering or network closure—dense connections between network members. In particular, in any pair-wise equilibrium network, each player either belongs to a complete component—that is, a group of mutually linked players—or has no links. The reason is that closure creates local common knowledge and increases the quality of third-party enforcement (ensures that information travels from cheated players to friends of defecting players). Thus, the paper not only formalizes, in game theoretic framework, Coleman’s insight concerning the importance of closure for the emergence of cooperative behavior, but also explains how closure emerges.

Pair-wise equilibrium networks, however, are not necessarily efficient. Mainly, an inefficiency arises since there are pair of players, say \(i\) and \(j\), that could increase their joint payoff but \(i\)'s best response to cooperation is cooperation while \(j\)'s best response to it is defection. This occurs, for example, when bilateral sanctions are enough for cooperation to be a best response for player \(i\)'s, but they are not for that to be player \(j\)'s best response or when player \(i\) and \(j\) belong to different components and thus player \(j\)'s links cannot punish \(i\) for deviating against \(j\).

**Social Capital.** The concept of social capital, broadly understood as referring to social norms, trust and social relations that facilitate the achievement of certain goals, has spread across variety of disciplines, as well as political and media contexts. In fact, this has been used to explain a wide range of sociological, political and economic phenomena, ranging from economic growth to quality of democracy and civic participation. Hundreds of paper have been published showing that some measure of social capital has a positive effect on a given outcome and thus public policy design should take it into account\(^4\). However, as argued by Durlauf and Fafchamps (2004), most empirical studies that attempt to measure the impact of social capital are methodologically flawed and make conclusions that are beyond of what the statistical exercises carried out suggest. One of the reasons for this to happen is that the concept of social capital has not been adequately defined and as a consequence of this, it is hard to identify empirical measures of it. In fact, the view of social capital expressed above has been criticized\(^5\): (i) for focusing on what social capital

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\(^4\)For an outstanding review of the social capital literature see Durlauf and Fafchamps (2004).

does rather than on what it is.; (ii) for being considered as a by-product of a largely unintentional process resulting from the existence social relations; and (iii) for being defined at the community rather than at the individual level.

This paper contributes to this debate by proposing an approach to the concept of social capital that acknowledges above criticisms. In particular, the results of the paper suggest that the amount of social capital that agent \( i \) draws from being part of a social network is equal to the number of \( i \)'s links that are linked to each other. In other words, it is not only who you know, but also who they know. This approach to social capital has the advantage that it is defined at the individual level, distinguishes the returns and costs of social capital from the capital itself, and highlights the importance of certain networks properties. In particular, that network closure is a necessary condition for social capital, and this together with density result in high levels of social capital.

Using this definition of social capital it is shown that within a society, an increase in social capital, understood as the sum of individuals’ social capital, leads to an increase in total well-being, while in a cross-section of societies, those with higher levels of social capital not necessarily have a higher well-being than those with lower levels of social capital.

**The small-world property.** Since Watts and Strogatz (1998) were able to generate highly clustered graphs (networks) with a small average path length, the empirical properties of large social, natural and economic networks have been extensively investigated by physicists\(^6\). They have been concerned with the statistical properties of large networks, and have shown that most networks are not well-characterized by random graphs. In fact, they have reached the conclusion that the topology and evolution of networks are governed by robust organizing principles, of which the small-world and the scale-free property are the most important\(^7\). The former, which is of

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\(^6\) For excellent surveys concerning the statistical properties of networks see Albert and Barabasi (2002) and Dorogostev and Mendes (2002).

\(^7\) The small-world property has sociological roots in experiments performed by the social psychologist Stanley Milgram (1967), who found that the median completed chain length between any two individuals in the US was six.
interest here, is the principle that networks are highly clustered (many of my friends are also friends with one another) relative to random graphs and have short average path length (most people are separated by few links).

Table No1
Research Collaboration Networks

<table>
<thead>
<tr>
<th>Network</th>
<th>Size</th>
<th>$y(g)$</th>
<th>$L(g)$</th>
<th>$L_{\text{rand}}(g)$</th>
<th>$C(g)$</th>
<th>$C_{\text{rand}}(g)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computer Science</td>
<td>13169</td>
<td>3.59</td>
<td>9.7</td>
<td>7.34</td>
<td>0.496</td>
<td>$3 \times 10^{-4}$</td>
</tr>
<tr>
<td>Mathematics</td>
<td>70975</td>
<td>3.9</td>
<td>9.5</td>
<td>8.2</td>
<td>0.59</td>
<td>$5.4 \times 10^{-5}$</td>
</tr>
<tr>
<td>Physics</td>
<td>56627</td>
<td>173</td>
<td>4.0</td>
<td>2.12</td>
<td>0.726</td>
<td>0.003</td>
</tr>
<tr>
<td>Biomedical</td>
<td>1520251</td>
<td>18.1</td>
<td>4.6</td>
<td>4.91</td>
<td>0.066</td>
<td>$1.1 \times 10^{-5}$</td>
</tr>
<tr>
<td>Economics</td>
<td>81217</td>
<td>1.67</td>
<td>9.47</td>
<td>22.04</td>
<td>0.157</td>
<td>$2 \times 10^{-5}$</td>
</tr>
<tr>
<td>Neuroscience</td>
<td>209293</td>
<td>11.5</td>
<td>6</td>
<td>5.01</td>
<td>0.76</td>
<td>$5.5 \times 10^{-5}$</td>
</tr>
<tr>
<td>Los Alamos</td>
<td>52209</td>
<td>9.7</td>
<td>5.9</td>
<td>4.79</td>
<td>0.43</td>
<td>$1.8 \times 10^{-4}$</td>
</tr>
</tbody>
</table>


Table 1 shows the clustering coefficient and characteristic path length for seven research collaborative networks for a similar period. It is easy to see that all of them exhibit the small-world property. For instance, for economics, the characteristic path length is 1.2 times that for a random graph while the clustering coefficient is 2389 times that for the equivalent random graph.

This paper contributes to this literature by adding the incentives dimension to the study of network properties. In fact, it is shown that most pair-wise equilibrium networks exhibit the small-world property. This means that social networks that have this property are in part the outcome of rational agents purposefully investing in link formation to get the benefits from belonging to a highly clustered network. In addition, it is shown that networks that have the small-world property tend to be those with high-levels of social capital and vice-versa.
Related literature. This paper relates to two strands of the literature. First, our paper is related to the repeated game literature in which identical individuals, who may have different histories of play, belonging to a community are randomly matched to play a Prisoner’s dilemma game (Ellison 1994; Kandori, 1992; Okuno-Fujiwara and Postlewaite, 1995). The main result of this literature is that cooperation can be made self-sustainable by mean of strategies that either punish deviators or reward conformers or use a mix of both. This result hinges on either the existence of information flows from each pair-wise match concerning past actions to the rest of the society or a contagious strategy; that is, players react to a deviation by punishing subsequent partners regardless of the history of play. The main difference stands for the fact that in our approach information flows are endogenously determined by the links formed while in theirs they are exogenously given by assuming a fixed and global pattern of play. This allows us to investigate network effects on the possibility of self-enforcing cooperation through social norms (third-party sanctions) and vice-versa. Still in the area of repeated games, an additional related paper is Haag and Lagunoff (2000). They consider a complete network where ex-ante heterogeneous players with respect to their discount factor play a prisoners’ dilemma game with the whole population. They force players to play the same action—defect or cooperate—against each of their neighbors. Hence, the strategic implications of the exogenously given social network are the result of imposing players to play the same action rather than being the result of information transmission among an endogenously determined set of connected players.

Second, the paper borrows from the strategic network formation literature where individuals choose whether or not to form costly links (Jackson and Wolinsky 1996; Bala and Goyal, 2000; Jackson and Watts, 1999; and, Goyal and Joshi, 2003). None of these papers, however, deal with the issue of how social norms that induce self-enforcing cooperation affect the network formation process, and how then that affects individuals’ incentives to trade in cooperative terms. In fact, our paper is one of the few that examines the interplay between network formation and cooperation. As far as we know Vega-Redondo (2002) is the only other paper that combines the network formation issue with the issue of self-enforcing cooperation. It considers a situation
in which agents play idiosyncratic repeated prisoner’s dilemma games with each of their neighbors. As in this paper, networks have a bearing on cooperation since players can communicate, gradually via their links, behavioral information about their links. The modeling assumptions are quite different however. In his model, players are exogenously limited to have a maximum number of links, can delete and form links at no cost as time goes on, only stable links are maintained; that is, links in which cooperation is self-sustainable, and unstable links are exogenously deleted. He shows that under payoff volatility, the network converges in the long-run to a stable and dense network (that is, an ergodic stochastic process). Yet, the equilibrium network architecture cannot be analytically characterized. In fact, he can analytically characterize networks only under the assumption that there is no payoff volatility. In which case, he show that the network displays, almost surely, the maximum connectivity exogenously allowed. Not a highly plausible real outcome. Thus, Vega-Redondo’s paper can be seen as complementary to this since he studies a dynamic model in which the network changes over time exogenously while in this paper considers a network formation process in which individuals purposely invest in costly links. In addition, his treatment of social capital suffers from the same criticisms made to the usual treatment of social capital. That is, he focuses on a concept of social capital defined in terms of what it does and not what it is.

The remainder of the paper proceeds as follows. In the next section, Section 2, the model is presented. Next, in Section 3, the analysis is undertaken. This is carry-out in three stages. First concerns the repeated game equilibrium given a social network. Second, the equilibrium of the whole game is studies. That is, pair-wise equilibrium networks are characterized in terms of a set of architectures, and the main characteristics of the equilibrium architectures are discussed. And, in the third, the main assumptions are discussed in great detail. The last section, Section 4, discusses the concept of social capital and the small-world property. The last section concludes.


2 The Matching Game

2.1 The Static Game

The society consists of \( N + 1 \geq 3 \), with \( N \) odd, infinitely lived players who may interact through a collection of infinitely repeated games. At \( t = 0 \), before repeated interactions start, the \( N + 1 \) individuals form an undirected graph (network formation stage). A graph is composed of a set of nodes and a set of links; each node represents a player while each link indicates a bilateral relationship between two players. This process will be explained in more detail below. At each period \( t \geq 1 \), each player interacts with at most one player and thus two players can go several periods without meeting each other. The probability that any two individuals \( i \) and \( j \) meet in any given period when they belong to social network \( g \) is time independent and equal to \( p_{ij} (g) : G \rightarrow [0, 1] \), with \( p_{ij} (g) = p_{ji} (g) \) for all \( ij \in g \) and all \( g \in G \), where \( G \) is the set of all possible networks.

For each pair of players who actually interact, \( i, j \in g \), the stage game they play is a prisoner’s dilemma with a payoff matrix given by:

<table>
<thead>
<tr>
<th></th>
<th>D</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>( d_i, d_j )</td>
<td>( b_i, 0 )</td>
</tr>
<tr>
<td>C</td>
<td>( 0, b_j )</td>
<td>( c_i, c_j )</td>
</tr>
</tbody>
</table>

where, as is customary, \( D \) stands for defection and \( C \) for cooperation\(^8\).

The payoffs satisfy the following restrictions: \( b_i > c_i > d_i > 0 \) and \( c_i + c_j > b_i \) for all \( i, j \in N + 1 \)^9.

Each of the prisoner’s dilemma games is choice independent, in the sense that players’ decisions in the past do not restrict the feasible choices in the future. They need not be, however, strategically independent since the behavior in the future may be made contingent on information gained in the past. In fact, as explained shortly,

\(^8\)The result are robust for instance to payoffs that are \( i \) and \( j \) dependant. The gain in intuition from doing so is small relative the loss in simplicity.

\(^9\)This last assumption ensures that cooperation is the efficient outcome in the one shot game.
under the approach adopted here, information on how players have behaved in the past diffuses through the social network gradually.

Finally, each individual discounts the future with a discount factor equal to $\delta \in (0, 1)$, the payoff from being unmatched in any given period is normalized to 0, and everything about the game is common knowledge.

### 2.2 Social Networks

The social network or network of connections among individuals is described by a graph $g \in G \equiv \{g \mid g \subseteq g^{N+1}\}$ which is an $N+1 \times N+1$ matrix, where $G$ is the set of all undirected graphs of $N+1$. Each element of $g$ is denoted by $g_{ij}$ and $g$ is a symmetric matrix, i.e., $g_{ij} = g_{ji}$. The element $g_{ij} = 1$ means that $i$ and $j$ are linked to each other while $g_{ij} = 0$ means that they are not linked to each other. The set of $i$’s direct contacts is $N_i(g) \equiv \{j \neq i : g_{ij} = 1\}$, which is of size $n_i(g)$. Thus, the size of network $g$ is $n(g) = \sum_{i\in N} n_i(g) / 2$ and if $n_i(g) = v$ for all $i \in N+1$, then $g$ is a symmetric network of degree $v$, denoted by $g^v$. In contrast, when $n_i(g) \neq n_j(g)$ for some $i, j \in N+1$, then $g$ is an asymmetric network. Among the possible symmetric networks, two will be of crucial importance. These are the complete network $g^N$, i.e., $n_i(g) = N$ for all $i \in N+1$, and the empty network $g^0$, i.e., $n_i(g) = 0$ for all $i \in N+1$. In addition, $g + ij$ (resp. $g - ij$) denotes the network obtained by adding (resp. subtracting) the link formed by player $i$ and $j$ to (resp. from) $g$ and the set of all links is defined as $N(g) = \cup_i N_i(g)$.

It is said that there is a path between player $i$ and $k$ if there is a distinct set of players $\{j_1, j_2, \ldots, j_{i-1}, j_{i+1}, \ldots, j_n\}$ such that $g_{j_1i} = g_{j_2j_1} = g_{j_3j_2} = \ldots = g_{jk} = 1$. Then, a network is connected if there exists a path between any pair $ij \in N+1$. A network $g' \subset g$ is a component of $g$ if for all $i, j \in g'$, $i \neq j$, there exists a path in $g'$ connecting $i$ and $j$, and for all $i \in g'$ and $k \in g$, $g_{ik} = 1$ implies that $k \in g'$. A component is complete if $g_{ij} = 1$ for all $i, j \in g'$. In what follows the $m$-complete component belonging to network $g$ is denoted by $M_m(g)$ and its degree by $v_m$, for $m \in \{1, 2, 3, \ldots\}$.

The exclusive group architecture will be of special importance in this paper. This
is characterized by a group of isolated players, denoted by $M_0(g)$, and $m$ distinct groups of completely connected players (complete components), $M_1(g), ..., M_m(g)$, each having degree $v_m$. Thus, $n_i(g) = 0$, for $i \in M_0(g)$, while $n_j(g) = v_m$, for all $j \in M_m(g)$ for $m \in \{1, 2, 3, \ldots \}$. A special case of this architecture is the dominant group network in which there is only one complete component.

### 2.3 Link Formation

Before the matching game starts each individual announces the links he wants to form. For all $i, j \in N + 1$, $s_{ij} = 1$ if $i$ wants to form a link with $j$, and $s_{ij} = 0$ otherwise. By convention $s_{ii} = 0$. A link is created when $s_{ij} * s_{ji} = 1$. Thus, $g_{ij} = 1$ if and only if $s_{ij} * s_{ji} = 1$. Links are thus created by mutual consent.

Individual $i$’s cost of forming a link with individual $j$ is $r_i$. Thus, the cost of each link is assumed to be independent of the network structure and the partner, but varies across individuals. In addition, it is assumed that before the matching phase starts, everyone observes the number of links that everyone else has formed and with whom they have been formed\(^{10}\).

### 2.4 Information Flows

The information flows are as follows. Player $j$ becomes informed about player $i$’s defection against $k$ in any given period when either $j$ is linked to $k$ or when there is an $i$-excluding path connecting player $k$ (the cheated player) and player $j$. When $j$ and $k$ are linked, then player $j$ gets informed about player $i$’s defection right upon the deviation, while if player $j$ is not linked to $k$, but there is a path connecting them, then how fast player $j$ gets informed depends on how many links away from $k$ he is, and the speed at which the information travels through links, $s$. Formally, let $l_i(j, k)$ be the $i$-excluding distance between player $k$ and $j$ defined as the shortest path joining $k$ and $j$ that does not involve player $i$. As is standard, it is assumed that $l_i(j, j) = 0$ for any $j \in N_i(g)$ while if no $i$-excluding path exists between $k$

\(^{10}\)This assumption is more stringent than needed. It will be enough that players know only about the links held by their direct social contacts.
and $j$, it will be convenient to assume that $l_i(j, k) = \infty$. Then, let us define the function $v_i(j, k) = 1$ if $g_{kj} = 1$ and $v_i(j, k) = \max \left\{ 1, \left\lfloor \frac{l_i(j, k)}{s} \right\rfloor \right\}$ otherwise, where $[x]$ is the largest integer smaller or equal to $x$. The interpretation of this distance is straightforward: it is the number of steps (and therefore periods in the repeated game) which are required for an information held by $k$ ($j$) to reach $j$ ($k$) without the participation of player $i$. When $v_i(j, k) = 1$, any information learned by $j$ last period is learned by $k$ in the next period, while if $v_i(j, k) = l_i(j, k)$, then information learned by $j$ last period is learned by $k$ in $l_i(j, k)$ periods ahead. For instance, when $s = 0$, there is no information transmission between the cheated player and the rest of the network (private monitoring) while when $s = N + 1$, then any player connected to $k$ either directly or through an $i$-excluding path gets immediately informed about $i$’s defection against $k$ (public monitoring)\textsuperscript{11}.

2.5 The Equilibrium Concept

The equilibrium concept used is pairwise equilibrium network in pure strategies. That is, the strategy profile $(s, a) = ((s_1, a_1), (s_2, a_2), ..., (s_{N+1}, a_{N+1}))$, where $(s_i, a_i) \equiv (s_{i1}, s_{i2}, ..., s_{iN}, a_{ij}^0, a_{ij}^1, ..., a_{ij}^t)$ and $a_{ij}^t$ is player $i$’s action when playing the stage game with player $j$ in period $t$, is a pairwise equilibrium network if $(s, a)$ is a sequential equilibrium in the whole game and $s$ is pairwise stable—that is no pair of players gain by altering the current configuration of links by adding a link. Thus, $g$ is said to be a pairwise equilibrium network if and only if there is a sequential equilibrium strategy profile $s$ which supports $g$ in a pair-wise stable fashion, and $a$ is a sequential equilibrium in the continuation game\textsuperscript{12}.

\textsuperscript{11}Notice that the longest possible path in any network $g$ with $N + 1$ players has length $N$ and thus if $s > N$, then all players linked to $k$ through a path gets informed one period after player $i$ deviates against $k$.

\textsuperscript{12}The concept of Nash equilibrium by itself is too weak a concept. In fact the empty network is always a Nash equilibrium. More generally, for any pair $i$ and $j$, it is always mutual best response for players to offer to form no links.
Let \( g \in G \). Then, individual \( i \)'s expected payoff is given by:

\[
U_i(g) \equiv (1 - \delta) \sum_{t=0}^{\infty} \delta^t \left\{ \sum_{j \in N + 1 / \{i\}} p_{ij}(g) u_i(\alpha_{ij}, \alpha_{ji}) \right\} - n_i(g) r_i,
\]

where the strategy profile \((s_i, a_i)\) has been omitted to save on notation.

Formally, pairwise stability coupled with sequential rationality implies that for all \( g_{ij} = 0 \), if \( U_i(g + ij) > U_i(g) \) then \( U_j(g + ij) < U_j(g) \) and for any set of existing links for player \( i \), \( X_i(g) \subseteq N_i(g) \), \( U_i(g) \geq U_i(g - X_i(g)) \). That is, no player gains by unilaterally deleting any subset of existing links \( X_i(g) \) and no pair of players gain by altering the current configuration of links by forming a link.

Finally, a network \( g \) is efficient when \( \sum_i U_i(g) \geq \sum_i U_i(g') \) for all \( g' \in G \).

3 The Analysis

3.1 The Repeated Game Phase

This sub-section focuses on the continuation game starting after the network formation phase has ended.

Notice that the continuation game is stationary in nature and therefore the focus is on stationary equilibria in which players rely on a “grim” trigger strategy that supports cooperation between linked players whenever it is self-sustainable. That is, I abstract from the more complex issue of how individuals choose and arrive at the cooperative equilibrium instead of playing any other possible sequential equilibria. Given that an individual may not be a bystander-observer of other people’s play, the grim-trigger strategy considered in the continuation game, denoted by \( \sigma \), is as follows: player \( i \) when playing \( j \), plays \( D \) when \( i \) and \( j \) are not linked, and plays \( C \) if: (i) \( i \) and \( j \) are linked, and (ii) \( i \) either has no information that \( j \) played \( D \) before, be it against himself or against some third party \( h \) or \( j \) has never defected before. If player \( i \) gets information that \( j \) has defected (against \( i \) or \( h \)), then \( i \) chooses defection against \( j \).

\[ \text{Sometimes the first condition is referred as link addition proof and the second as strong link deletion proof.} \]
in all interactions with $j$ after receiving the information on $j$’s defection. If $j$ cheats $i$, but $k$ does not observe this, then it is a non-event as far as $k$ is concerned, and it does not affect her conduct towards $j$.

It is worthwhile to remark that in the trigger strategy studied here, it is only cheating that triggers a punishment. In other words, mutual defections do not evoke sanctions when they are part of the prescribed pattern of behavior. That is, if $k$ defects against $j$ in order to punish the latter for cheating $i$, then $k$ is not cheating but rather carrying out a prescribed punishment, so others becoming aware of the defection would not punish $k$ in turn. It is also assumed that if $k$ learns player $j$’s history, then this is common knowledge between $k$ and $j$.

Let $g$ be the network formed in period 0. Then player $i$’s expected payoff in the continuation game when cooperating in each encounter with linked players and defecting against unlinked players when everyone else is following $\sigma$ is

\[ V_{ik}(\sigma \mid g) \equiv (1 - \delta) c_i + \delta \sum_{j \in N+1/\{i\}} p_{ij}(g) \left[ g_{ij} c_i + (1 - g_{ij}) d_i \right]. \]

Player $i$’s expected payoff from deviating during the simultaneous move game against the current partner, say $k$, when he has never defected before and then conforming to the grim-trigger strategy $\sigma$ forever thereafter when everyone else is following $\sigma$ is

\[ V_{ik}(D, \sigma_{-i} \mid g) \equiv (1 - \delta) b_i + \delta p_{ik}(g) d_i \]
\[ + \delta \sum_{j \in N+1/\{i,k\}} p_{ij}(g) \left\{ g_{ij} \left[ \sum_{t=0}^{\nu_{ij}(j,k)-1} (1 - \delta)^t c_i + \delta^\nu_{ij}(j,k) d_i \right] + (1 - g_{ij}) d_i \right\}. \] (1)

The first term is self-explanatory. The second one is the long-run payoff from being matched with player $k$ with probability $p_{ik}(g)$ since they play mutual defection forever thereafter against each other. The third term is the long-run payoff from being matched with a player other than $k$, which is composed of two distinct terms. The first is the expected payoff when the next period partner, say $j$, does not learn that $i$ defected against $k$ in the last period. This is equal to the payoff from mutual cooperation. The second is the expected payoff when $j$ learns that $i$’s defected against $k$ in the last period. This corresponds to the payoff from mutual defection forever
thereafter since that is common knowledge between $i$ and $j^{14}$. The last term is the payoff from being matched with an unlinked player, which is the payoff from mutual defection forever thereafter.

After a few steps of simple algebra, equation (1) becomes,

$$V_{ik}(D, \sigma_{-i} | g) \equiv (1 - \delta) b_i + \delta p_{ik}(g) d_i$$

$$\sum_{j \in N_i(g) \setminus \{i, k\}} p_{ij}(g) \left\{ g_{ij} \left[ \delta \left( 1 - \delta^\nu_{i}(j, k) \right) c_i + \delta^\nu_{i}(j, k) + 1 d_i \right] + (1 - g_{ij}) d_i \right\}.$$

Because player $i$ cooperates in each encounter when everyone else is following $\sigma$ if and only if $V_{ik}(\sigma | g) \geq V_{ik}(D, \sigma_{-i} | g)$, cooperation is player $i$’s best response to $\sigma_{-i}$ if and only if

$$\delta \left[ \frac{p_{ik}(g) (c_i - d_i)}{\text{gain from avoiding bilateral sanction}} + \sum_{j \in N_i(g) \setminus \{k\}} p_{ij}(g) \delta^\nu_{i}(j, k) (c_i - d_i) \right] \geq \frac{(1 - \delta) (b_i - c_i)}{\text{short-run gain from deviation}},$$

(2)

After rearranging terms in equation (2), the following result is obtained.

**Proposition 1** The grim-trigger strategy $\sigma$ is a sequential equilibrium in the continuation game if and only if $p_{ik}(g) \geq \max \{ \bar{p}_{ik}(g), \bar{p}_{ki}(g) \}$, where

$$\bar{p}_{ik}(g) \equiv p_i(\delta) - \sum_{j \in N_i(g) \setminus \{k\}} p_{ij}(g) \delta^\nu_{i}(j, k)$$

(3)

and $p_i(\delta) \equiv \frac{(1 - \delta)(b_i - c_i)}{\delta(c_i - d_i)}$.

This result says that cooperation between $i$ and $k$ is self-sustainable either when they are matched sufficiently often so that they suffer each other’s punishment frequently (sufficiency of bilateral sanctions) or when the information transmission from the cheated player to the links of the cheating player is such that a deviating player is punished by several of his own links. For instance, when the information transmission technology is such that $s = 0$, cooperation between $i$ and $k$ is self-sustainable only

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14 It is worthwhile to remark that player $h$ can never observe mutual defections between any two players, say $i$ and $j$, unless he is linked to both of them.
when bilateral sanctions are powerful enough, while when the technology is such that \( s \geq 1 \), cooperation between linked players might be self-sustainable despite the fact that bilateral sanctions are not powerful enough.

### 3.2 The Network Formation Phase

In this sub-section, the network formation phase is studied. In order to do so, the following simplifying assumptions are made:

- (A1): \( p_{ij}(g) = \frac{1}{N} \) for all \( i, j \in N + 1 \) and all \( g \in G \).
- (A2): \( s = N + 1 \).

Assumption (A1) implies uniform random matching\(^{15}\). That is, a linked player is equally likely to be matched with a player with whom he has a link as well as with a player with whom he has no link.

Assumption (A2) says that information takes one period to travel to nodes that are one link away as well as to those that are more than one link away. This implies that \( v_i(j, k) = 1 \) when either \( g_{jk} = 1 \) or there is an \( i \)-excluding path connecting \( j \) and \( k \). That is, information flows immediately from player \( k \) to any player who is connected to \( k \) through an \( i \)-excluding path irrespective of the length of the path. This can be interpreted as an information transmission technology in which there is no perceptible delay and thus information reaches any node immediately regardless of the number of links through which the information has to travel. The consequences of these two assumptions are discussed in great detail in the next section.

Furthermore, in order to make the problem interesting, the following two assumptions are made.

- (A3): \( \frac{1}{N} (c_i - d_i) \geq r_i \) for all \( i \in N + 1 \).
- (A4): The discount factor \( \delta \) is sufficiently large so that \( \delta \geq \frac{1}{(N-1)} \max \{ Np_i(\delta) - 1, 0 \} \) for all \( i, k \in N + 1 \).

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\(^{15}\)This is standard in the repeated game literature that take the network as given.
The former says that for each player \( i \), the expected payoff from cooperating relative to that from defecting exceeds the cost of forming a link, and the latter ensures that in a complete network, cooperation between any pair of linked players is self-enforcing.

The strategy profile studied from here onwards prescribes the following: (i) to form a set of links that results in a network of social contacts \( \tilde{g} \in G(\sigma) \), where

\[
G(\sigma) \equiv \{ g \in G \mid \sigma \text{ is a sequential equilibrium in the continuation game} \};
\]

(ii) to defect forever thereafter in each encounter when the network formed \( g \notin G(\sigma) \) is observed; and (iii) to follow \( \sigma \) from period 1 onwards whenever the network formed is such that \( \sigma \) is a sequential equilibrium in the continuation game –that is, \( g \in G(\sigma) \).

It is worthwhile to note that this strategy profile prescribes a punishment only when, upon a deviation in the network formation phase, the grim trigger strategy \( \sigma \) is no longer a sequential equilibrium in the continuation game, and that the punishment prescribed by it, is a sequential equilibrium in the continuation game (out-of-the-equilibrium path) starting in period 1. This has the advantage that only individuals’ incentive to cooperate on-the-equilibrium path must be considered.

It readily follows from equation (2), assumptions A1 and A2 that cooperation between \( i \) and \( k \) is self-enforcing when everyone else is following \( \sigma \) when

\[
\sum_{j \in N_\alpha(g)/(\beta)} \delta^{\alpha}(j,\beta) \geq \max \{ Np_\alpha(\delta) - 1, 0 \} \text{ for } \alpha, \beta = i, k \text{ and } \alpha \neq \beta. \tag{4}
\]

The first thing worthy of note is that player \( i \) by forming a cooperative link (a link where cooperation is self-enforcing) with player \( k \), can neither destroy cooperation between \( h \) and any other player nor it can destroy cooperation between \( i \) and any of his links (including \( k \))\(^{16}\). Consider figure 1, where panel (a) represents the original situation. Suppose that in this case cooperation between \( i \) and \( j \) is not self-enforcing

\[^{16}\text{Formally, this readily follows from noticing that}
\sum_{j \in N_\alpha(g)/(\ell)} \delta^{\alpha}(j,\ell) - \sum_{j \in N_\alpha(g)/(k)} \delta^{\alpha}(j,k) = \delta^{\alpha}(l,k) \geq 0 \text{ for all } h \in N + 1.\]
while cooperation between $i$ and $k$ is so by bilateral sanctions. Then, if $i$ creates a new a link with $k$, player $k$ can punish $i$ when he deviates against $j$ since there is an $i$-excluding path between $j$ and $k$. Thus, cooperation between $i$ and $j$ may become self-enforcing.

The marginal gross return to individual $i$ from adding a link with individual $h$ given a network $g \in G(\sigma)$ is

$$\triangle V_i(g + ih, g) \equiv \begin{cases} -\frac{n_i(g)}{N} (c_i - d_i) & \text{if } I_{ih}(g + ih) = 0, \\ \frac{1}{N} (c_i - d_i) & \text{if } I_{ih}(g + ih) = 1, \end{cases}$$

where $I_{ih}(g + ih) = 1$ when cooperation between $i$ and $h$ is self-enforcing under network $g + ih$ and $I_{ih}(g + ih) = 0$ otherwise.

Because linked as well as unlinked players can interact with one another, when a cooperative link is added, the opportunity cost of it is $\frac{1}{N} d_i$ while the benefit of it is $\frac{1}{N} c_i$. While when a non-cooperative link is added, the strategy profile prescribes defection forever thereafter, and thus the opportunity cost of it is $\frac{n_i(g)}{N} c_i + \frac{1}{N} d_i$ while the benefit of it is $\frac{n_i(g)+1}{N} d_i$. This explains the payoffs in equation 5. It readily follows from this and assumption A3 that all cooperative links must be mutually linked and that no pair of players has an incentive to add a non-cooperative link to a network $g \in G(\sigma)$. 

Figure 1:
The marginal gross return to individual $i$ from unilaterally deleting a subset of existing links $X_i(g)$ given a network $g \in G(\sigma)$ is

$$\Delta V_i(g - X_i(v), g) \equiv \begin{cases} -\frac{1}{N}n_i(g)(c_i - d_i) & \text{if } I_{ij}(g - X_i(g)) < I_{ij}(g) \text{ for some } ij, \\ -\frac{1}{N}x_i(g)(c_i - d_i) & \text{if } I_{ij}(g - X_i(\tilde{g})) = I_{ij}(g) \text{ for all } ij, \end{cases}$$

(6)

where $x_i(g)$ is the cardinality of the set $X_i(g)$.

Because $g \in G(\sigma)$, it contains only cooperative links, then the opportunity costs and benefits to deleting a subset of existing links depends on whether cooperation between the remaining links is self-sustainable or not. If this is not the case, then the strategy profile prescribes defection forever thereafter, and thus the opportunity cost is $\frac{n_i(g)}{N}c_i$ while the benefit is $\frac{n_i(g)}{N}d_i$. While when the opposite occurs, the opportunity cost is $\frac{x_i(g)}{N}c_i$ while the benefit is $\frac{x_i(g)}{N}d_i$. This explains the payoffs in 5. It then follows from this and assumption A3 that no player has an incentive to delete any subset of existing links from any network $g \in G(\sigma)$.

Let us define the set of all players for whom cooperation is a best response regardless of the network architecture by $\Upsilon(N) = \{i \in N + 1 : Np_i(\delta) - 1 \leq 0\}$. Notice that cooperation between any two players belonging to this set is self-enforcing by bilateral sanctions alone.

The next proposition characterizes pair-wise equilibrium networks in this case.

**Proposition 2** Suppose assumptions A1, A2, A3, and A4 hold. Then, (1) a non-empty pair-wise equilibrium network $g$ always exists; (2) Suppose that $g$ is a pair-wise equilibrium network. Then $g$ has the exclusive groups architecture in which there are $m = 1, 2, 3, \ldots$ complete components, each having a degree of at least $v_m \geq \max_{i \in M_m(g)} \left\{ 1 + \frac{1}{2} \left[ Np_i(\delta) - 1 \right] \right\}$, and no two players, say $i$ and $j$, belonging to $\Upsilon(N)$ can be in different components, and the rest are isolated players. If $v_m = v$ for all $m$, then $g$ is symmetric otherwise is asymmetric; (3) The complete network $g^N$ is the unique efficient (pair-wise equilibrium) network.

The driving force behind this result is that in any network players belonging to the same component have incentives to deviate and to form links. The intuition is as
follows. Suppose that $g$ is a pair-wise equilibrium network in which there is a path between player $l$ and $m$, denoted by $\{m_1, m_2, ..., m_n\}$, but $l$ and $m$ are not linked. Because $g$ is a pair-wise equilibrium network, cooperation between $l$ ($m$) and each of his links must be self-sustainable. This, however, implies that cooperation between $l$ and $m$ is also self-sustainable. To see this notice the following: (i) cooperation between player $l$ and $m_1$ must be self-enforcing since $g$ is a pair-wise equilibrium network; (ii) there is an $l$-excluding path between $m$ and any of the players that enforce cooperation between $m_1$ and $l$; and (iii) the relevant information flows from $m_1$ to his links at the same speed as from $m$ to the links of $m_1$. These three things together imply that the same players that punish $l$ for defecting against $m_1$ will punish $l$ for defecting against $m$, and thus $g$ cannot be a pair-wise equilibrium network unless $l$ and $m$ are mutually linked. This can be seen more clearly in figure 2.

Cooperation between $m_1$ and $l$ is enforced by themselves in conjunction with players $a$ and $b$. Now suppose that $m$ and $l$ form a link—the dotted line in the figure—and that $l$ deviates against $m$. Because there is an $i$-excluding path between $m$ and $m_1$, there is also one between $m$ and $a$ and $b$, which implies that before next period starts players $a$, $b$ and $m_1$ become aware of such deviation and thus they punish player $l$. Because $l$'s best response to cooperation by player $m_1$ is cooperation, then $l$'s best response to cooperation by player $m$ must also be cooperation since if he deviates, he will be punished not only by $a$, $b$, and $m_1$, but also by $m$.

It is interesting to comment on the fact that closure is the main characteristic of pair-wise equilibrium networks. As in Chwe (2001), closure aids at locally creating
common knowledge of information—that is knowledge of other people’s knowledge—that is essential for cooperation. In other words, closure permits that the relevant information travels from cheated players to the friends of defecting players and thus it increases the efficiency of third-party enforcement. Thus, the model not only formalizes Coleman’s insight in a game theoretic framework, but also shows that closure is a natural given on pair-wise stable networks when the strategy profile prescribes that players cooperate with linked players only.

This result also shows that weak links—understood as path between two players—are not pair-wise stable in networks in which the speed at which the information travels through links is high, while strong links—understood as direct connection between two individuals—are so.

Finally, it is worthwhile to comment on the fact that the efficient network is complete. This implies that there is a pair-wise inefficiency that arises when a pair of players, say $i$ and $j$, could increase their joint payoff but $i$’s best response to cooperation is cooperation, while $j$’s best response to it is defection. This occurs, for example, when bilateral sanctions are enough for player $i$, but they are not for player $j$ or when player $i$ and $j$ belong to different components.

### 3.3 Robustness

In this section the model robustness is discussed. First, it is assumed that only linked players can be matched with one another and, second, that the speed at which the information travels through links depends on the distance between links.

- (A5): $p_{ij}(g) = \frac{n_{ij}}{N}$ for all $i, j \in N + 1$; and (ii) $c_i + r_i \geq \frac{N+1}{N}d_i$ for all $i \in N + 1$.

This assumption means that matching occurs only among linked players and thus when the network is incomplete the probability that a player stays unmatched in any given period is positive. The probability that player $i$ finds a match in any given period is: $\sum_{j \in N_i(g)} \frac{n_{ij}}{N} = \frac{n_i(g)}{N} < 1$ for all $g \neq g^N$.

The main difference with the case analyzed before is that adding a link has an extra marginal benefit which is to rise the probability of being matched in any given period.
As result of this, the marginal return from adding a link is higher than before, while the marginal return from deleting a subset of existing link is even lower than before. In fact, individual $i$'s marginal gross return from adding a link with individual $h$ given a network $g \in G(\sigma)$ is $\Delta V_i (g + ih, g) + \frac{1}{N} d_i$, while his marginal gross return from unilaterally deleting any subset of existing links $X_i (g)$ is $\Delta V_i (g - X_i (g), g) - \frac{2}{N} d_i$. Because $\Delta V_i (g - X_i (g), g) < 0$, no player has an incentive to unilaterally delete any subset of existing links. In contrast to the case in which matching occurs between linked as well unlinked players, they might have an incentive to add links even when that destroys cooperation. This occurs when between pair of players that have too few cooperative links. The reason is that their cost of destroying cooperation is small relative to the increase in the probability of finding a partner and trading with him in non-cooperative terms. An immediate consequence of this is that in any pair-wise equilibrium network $g$, players cannot have too few links. We then have the following result.

**Proposition 3** Suppose assumptions A2, A3, A4 and A5 hold. Then, (1) a non-empty pair-wise equilibrium network always exists.

(2) Suppose that $g$ is a pair-wise equilibrium network. Then $g$ has the exclusive groups architecture in which there are $m = 1, 2, 3...$ complete components, each having a degree of at least $v_m \geq \left\{ 1 + \frac{1}{\delta} \left[ N p_i (\delta) - 1 \right], \frac{d_i - r_i}{\delta (c_i - d_i)} \right\}$, and no two players, say $i$ and $j$, belonging to $\Upsilon (N)$ can be in different components, and the rest are isolated players. If $v_m = v$ for all $m$, then $g$ is symmetric otherwise is asymmetric;

(3) The complete network $g^N$ is the unique efficient (pair-wise equilibrium) network.

The driving force behind this result is the same as the one in proposition 2 (in any pair-wise equilibrium network a component must be complete). Furthermore, the same divergency between a pair-wise equilibrium and an efficient network is present here. The main difference is that in any pair-wise equilibrium network no pair of players can have too few links. Thus, the results in proposition 2 are fairly robust to the assumption that linked as well as unlinked players can be matched with one another.
Next the case in which the information travels one link at the time, but linked as well unlinked players can be matched with one another is analyzed.

• (A6): $s = 1$.

This assumption implies that when $i$ deviates against $k$, before the next period begins only $k$’s links become aware of such deviation, then two periods ahead of the deviation, the links of player $k$’s links become informed of $i$’s deviation, and so on and so forth. Thus, when a player is $l$ links away from player $k$, it takes him $l$ periods to become informed of player $i$’s deviation against $k$.

It readily follows from equation (2), assumptions A2 and A6 that cooperation between $i$ and $k$ is self-enforcing when everyone else is following $\sigma$ if and only if

\[
\sum_{j \in N_{\alpha}(g)/\{\beta\}} \delta^{\alpha,(j,\beta)} \geq \max \{Np_{\alpha}(\delta) - 1, 0\} \text{ for } \alpha, \beta = i, k \text{ and } \alpha \neq \beta. \tag{7}
\]

The marginal gross payoff from either adding a link or deleting a subset of existing links is the same as when information travel at a speed $s = N + 1$. Thus, in any pair-wise equilibrium network, a pair of players, say $i$ and $j$, form a link whenever cooperation is self-enforcing. This means that any two players for whom bilateral punishments are enough to enforce cooperation must be mutually linked. In contrast to the case in which $s = N + 1$, it does not mean that any two players, say $i$ and $j$, that are further away from each other but connected through a path, have an incentive to form a link. To see why this might be the case let us go back to figure 2. In that figure, when information travels to nodes that are more than link away as fast as to those that are only one link away, $l$ and $m$ have an incentive to form a link since the same players ($a$, $b$, and $m_1$) that punish $l$ from the next period onwards when he deviates against $m_1$, also punish $l$ from the next period onwards when he deviates against $m$. In contrast, when in each period information travels only to nodes that are one link away, these players no longer punish $l$ from the next period onwards when he deviates against $m$. In fact, they begin punishing $l$ only 4 periods after $l$ deviates, and thus third-party enforcement is not as efficient as when $s = N + 1$. 22
Before characterizing pair-wise equilibrium networks, let define an $m$-circle network or component, where $m$ is the number of nodes in the circle, as one in which for each player $i$ belonging to the $m$-circle network there is a path from $i$ to $i$ and each player $i$ has only two links: player $i+1$ and player $i-1$ (see, figure 3). Then, the following result is derived.

**Proposition 4** Suppose assumptions A1, A2, A3 and A6 hold. Then, (1) a network $g$ in which $n_i(g) = 1$ for at least some $i \in \mathbb{N} + 1$ cannot be a pair-wise equilibrium network for all $|\Upsilon(N)| > 2$; (2) a network $g$ in which there is an $m$-circle component cannot be a pair-wise equilibrium network; and (3) the exclusive groups architecture in which there are $m$ complete components, each having a degree of at least $\nu_m \geq \max_{i \in M_m(g)} \left\{ 1 + \frac{1}{8} [N p_i(8) - 1] \right\}$, and no two players, say $i$ and $j$, belonging to $\Upsilon(N)$ can be in different components, and the rest are isolated players is a pair-wise equilibrium network.

Part (1) rules out any architecture in which a player has one link only. In particular, no network that have components with the following architectures: the line, the tree, the star and the inter-linked star, can be pair-wise stable. Part (2) rules networks or components that have the $m$-circle architecture. If a network with a $m$-circle component were to be a pair-wise equilibrium network, then it would mean that cooperation between $i$ and $i+1$ and $i$ and $i-1$ is self-enforcing. In this is then cooperation between $i$ and any other player in between $i+1$ and $i-1$ must also be self-enforcing. The reason is that players $i+1$ and $i-1$ would take less time to become
informed of \( i \)' deviation against the new link that it takes them to become informed of \( i \)'s deviation against one of them. Thus, \( i \) and the in between link would have an incentive to deviate and form a new link. While part (3) shows that the dominant group and exclusive groups architectures are pair-wise equilibrium networks under the same conditions as when \( s = N + 1 \). They are so because their structure result in \( l_i(j, k) = v_i(j, k) = 1 \) for each pair of linked players. In other words, each player belonging to a component is one link away from any other player belonging to the same component and thus it becomes immediately informed of any deviation that occurs within the component regardless of the speed at which the information travels through links.

This suggests that the results in proposition 2 may not be entirely robust to changes in the speed at which the information travels the network \( s \), yet it is interesting to highlight that architectures in which closure is an important characteristic are still pair-wise equilibrium networks.

The assumption that information transmission is not strategic has not been discussed yet. Then, it is worthwhile to do so here before ending this section. First, it is clearly obvious that a cheating player, say \( i \), has no incentive to reveal to other players that he cheated a partner. Thus, assuming that information travels the network only when there is an \( i \)-excluding path is unimportant. Second, it was assumed that a cheated player always communicates to his links that he was cheated by player \( i \). Then, the relevant question is if he has incentives to do so. Given that transmitting information is costless, doing so is at least a weakly dominant strategy for each player.

4 Social Capital and the Small-World Property

In this section in light of the results obtained so far the concepts of social capital and Small-World property are discussed.
4.1 Social Capital

The speed at which the “social capital” concept has spread across such a variety of disciplines points to its tremendous appeal. Despite of this the definition of social capital has remained elusive. It is undoubtedly the case that the vagueness of the concept of social capital may have promoted its popularity, it is also the case that this has contributed to a failure to identify the true role of social capital at both, the theoretical and empirical level. Among the most popular definitions of social capital is the one proposed by Coleman (1990), who defines it as “the shared knowledge, understanding, norms, rules and expectations about patterns of interaction that groups of individuals bring to a recurrent activity.” Other well known definition are the ones by Lin (2001), who defines social capital as “resources embedded in social networks and accessed and used by actors for actions”, and Putman (1993), who, by analogy with notions of physical capital and human capital, defines social capital “as features of social organization, such as networks, norms, and trust, that facilitate coordination and cooperation for mutual benefit”\textsuperscript{17}.

The underlying ideas behind these definitions are: (i) the community as whole stands to gain from social capital; (ii) the benefit comes from the existence of trust, norms of behavior, and adequate expectations; and (iii) social capital is to what it does and not what it is.

As pointed out by several authors such as Portes and Landlot (1988), Woolcock (1988), and Durlauf and Fafchamps (2004), this view of social capital is troublesome since it leads to a kind of circular reasoning or tautology that makes hard to evaluate the importance of social capital. For instance, saying that developed nations became so because they posses social capital while undeveloped ones remain such because they do not have it, clearly leads to a concept with no distinct meaning. This problem, together with the fact these authors see social capital as something people are borned with or can acquire at no cost, seems inconsistent with any definition of capital being that social, human or physical. In addition, the fact that social capital is routed on

\textsuperscript{17}There are several more definitions of social capital, but most of them are combination of the ideas underlying the definitions given here.
concepts like trust, social norms and expectations is problematic from an empirical perspective since this cannot be measured or instrumented accurately.

From an economist point of view, social capital should, at least, be defined at the individual level and obtained through personal costly investments oriented to the institutionalization of group relations that can be used to obtain other personal, as oppose to group, benefits. In fact, sociologist Pierre Bourdieu, who presented the first contemporary analysis of social capital, writes “the network of relationships is the product of investment strategies, individual or collective, consciously or unconsciously aimed at establishing or reproducing social relationships that are directly usable in the short or long term.”

Bourdieu’s formulation considers social capital as a personal attribute that cannot be evaluated without knowing the social network in which the individual is involved. The extent to which an individual can gain by being member of a certain social network depends on their connections (who are his friend, but also who are the friends of his friends), the strength of this connections, and the gains from interacting with those connections.

In light of this discussion and the model’s results an definition of social capital is proposed. The amount of social capital that agent \( i \) draws from being part of social network \( g \), denoted by \( SC_i (g) \), is the number of his own links that are linked to each other. Formally, social capital is

\[
SC_i (g) = \frac{1}{2} \sum_{j \in N_i(g)} \sum_{k \in N_i(g)} g_{jk}.
\]

Figure 4—panel (a) shows a network with 4 agents in which each agent has two links, but no social capital, while panel (b) shows a different architecture with the same 4 agents, but now social capital is equal to 1.

This definition of social capital acknowledges the criticisms above. In fact, it is defined at the individual level, it is not associated to the outcome of it but to its source, and has clearly defined benefits and costs. Mainly, player \( i \)’s social capital \( SC_i (g) \) requires a personal investment equal to \( \sum_{j \in N_i(g)} r_i \) and yields a gross expected return equal to \( R_i (g) = \frac{1}{N} \sum_{j \in N_i(g)} (c_i - d_i) \).
This definition of social capital highlights the importance of network density and network closure. Without closure there is no social capital and without density but closure the level of social capital that a player $i$ can draw from being part of a network is bound to be low. Thus, closure and density are both needed for having a network with high levels of social capital. This suggests that two particular conditions under which social structures create beneficial outcomes, and thus they can orient public policy concerning social networks.

One can also define the aggregated amount of social capital for any given network $g$, denoted by $SC(g)$, as follows:

$$SC(g) = \sum_{i \in N+1} SC_i(g).$$

Because social capital, as defined here, is not equated with the outcome of it, societies with more social capital do not always have a larger total welfare than their counterparts with less of it. To see this, let us consider two societies $A$ and $B$, and denote a pair-wise equilibrium network in society $A$ by $g^A$ and that in society $B$ by $g^B$. Furthermore, suppose that $g^A$ and $g^B$ have the dominant group architecture with degree $v^A < v^B$ and that the benefit from cooperation is higher among members of society $A$, i.e., $c^A > c^B$.

It is easy see that social capital in society $s = A, B$ is $SC(g^S) = v^S \left( (v^S)^2 - 1 \right)$ and total welfare is $W(g^S) = \sum_{i \in N+1} \left( v^S (c^S - d) + d - v^S r \right)$. It readily follows from this and the fact that $v^A < v^B$ and $c^A > c^B$, that $SC(g^A) < SC(g^B)$ and
This discussion leads to the following proposition.

**Proposition 5** (i) Within a society an increase in aggregated social capital implies an increase in total welfare; and (ii) societies with more social capital do not always have a larger total welfare.

Before ending this section, it is worthwhile to comment on the fact that the discussion has been cast in terms of cooperative trading as something that benefits the society as a whole. Since the prisoner’s dilemma problem may be capturing situations in which the cooperative outcome is beneficial for the partners, but detrimental for the society, more social capital may mean less total welfare. For instance, illegal trading in drugs, children, and arms, crime, collusive agreements to restrain output are all examples of transactions that can be sustained by informal enforcement mechanism based on social networks that are detrimental for the society as whole.

### 4.2 The Small-World Property

The small-world phenomenon is the principle that individuals in a social network are all linked by short chains of acquaintances, or six degrees of separation. This phenomenon has its roots in experiments performed by the social psychologist Stanley Milgram. In his now-classic 1967 paper, he described an experiment he performed involving letters that were passed from acquaintance to acquaintance to trace out short paths through the social network of the United States. He asked participants to forward a letter to a target person living near Boston, with the restriction that each participant could advance the letter only by forwarding it to a single acquaintance. Milgram found that the median completed chain length was six.

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\[ W(g^A) \leq W(g^B) \]

The condition for \( W(g^A) > W(g^B) \) is the following:

\[
 c^A \geq N v^B \left( \frac{c^B - d}{N} - r \right) + rN + d. 
\]
Physicists Watts and Strogatz (1998) were the first to succeed in generating graphs with the so called small-world property. Roughly, these are graphs with a high clustering coefficient $C(g)$ and a small characteristic path length $L(g)$. The characteristic path length represents the average shortest distance from any node in the graph to any other node in it. It measures on average how many individuals a player has to go through to reach another player in the community. The clustering coefficient measures how related are a set of nodes. For a given node, it is the ratio of the actual number of links among the neighbors of that node to the maximum number of possible links between these neighboring nodes. The clustering coefficient for a graph is defined as the average of the clustering coefficient of all its nodes.

Formally, the individual clustering coefficient in network $g$ is then formally defined as:

$$C_i(g) = \frac{1}{n_i(g) (n_i(g) - 1)} \sum_{j \in N_i(g)} \sum_{k \in N_i(g)} g_{jk}.$$  

To understand this, suppose that player $i$ has $n_i(g)$ links, then the maximum amount of links that can exist among player $i$’s links is $\frac{n_i(g) (n_i(g) - 1)}{2}$. This occurs when every link of $i$ is connected to every other link of $i$. Then $C_i(g)$ is the fraction of these links that actually exits and that are shared by player $i$. Thus, $C_i(g)$ reflects the extent to which the friends of $i$ are also friends to each other and thus the cliquishness of player $i$’s neighborhood. In fact, when $C_i(g) = 0$, none of the $i$’s friends are friends with each other and thus there are no information flows among player $i$’s friends, while when $C_i(g) = 1$, all of the $i$’s friends are linked with one another and thus information flows well within player $i$’s friends.

The network clustering coefficient is defined as follows:

$$C(g) = \frac{\sum_{i \in N+1} \sum_{j \in N_i(g)} \sum_{k \in N_i(g)} g_{jk}}{\sum_{i \in N+1} n_i(g) (n_i(g) - 1)} \in [0, 1],$$

which is the probability that two acquaintances of a randomly chosen person are themselves acquainted in the social network $g$. Thus, $C(g)$ is a measure of the cliquishness (closure) of the social network $g$. 

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If a network is connected—that is there are no isolated players, the characteristic path length of network \( g \) is formally defined as follows:

\[
L(g) = \frac{1}{N(N+1)} \sum_{i \in N+1} \sum_{j \in N+1} d(i, j) \geq 1,
\]

where \( l(i, j) \) is the geodesic distance between \( i \) and \( j \) or the shortest distance between them. If there is no path connecting \( i \) and \( j \), then \( l(i, j) = \infty \). This implies that if \( g \) is not connected, then the characteristic path length is formally speaking infinite, i.e., \( L(g) = \infty \). In order to avoid this problem, it is assumed that \( L(g) \) is the average among all distances \( l(i, j) \) for pairs \( i \) and \( j \), both belonging to giant component (see, for instance, Goyal et al. 2005, Newman, 2001 for the same assumption)\(^\text{19}\).

In general, there is no precise formal definition of what a small-world exactly means. Following Watts (1999), in this paper a network will have the small-world property when the following conditions are met:

1. The network is numerically large as compared to the average number of links \( v(g) \), i.e., \( N + 1 \gg v(g) \).
2. The network is decentralized in that there is no dominant central vertex to which most other vertices are directly connected. This implies a stronger condition than sparseness: not only must the average degree \( v(g) \) be much less than \( N+1 \), but the maximal degree \( v_{\text{max}} \) over all vertices must also be much less than \( N+1 \).
3. The network is highly clustered, in that most friendship circles are strongly overlapping. That is, many friends of our friends are also friends with each other.
4. The characteristic path-length between the nodes of the giant component is small. In particular, \( L(g) \) for that component is of order of at most \( \ln N + 1 \).

All four criteria are necessary for the small-world property to be remarkable. If a network did not contain many people, then it would not be surprising if they were

\(^{19}\)Other papers assume that for any pair \( ij \) with \( g_{ij} = 0 \), \( d(i, j) \) is the largest possible distance \( (N + 1) \).
all closely linked (as in a small town). If most people knew most other people then, once again, it would not be surprising to find that two strangers had an acquaintance in common. If the network were highly centralized—for instance, it has the start architecture—then an obvious short path would exist through the center of the star between all pairs of vertices. Finally, if the network were not clustered—that is, if each individual chooses their friends independently of any of their friends’ choices—then it follows from random-graph theory that most people would be only a few degrees of separation apart even for very large networks20. The last two conditions in above definition are harder to be proven, but in the light of everyday experience they seem quite plausible. Some people are clearly more outgoing than others, but even the most gregarious individuals are constrained by time and energy to know only a tiny fraction of the entire population. While it might be difficult to determine in practice how many friends of an individual are also friends with each other, and even more difficult to measure this for a large network, common sense tells us that whatever this fraction is, it is much larger than that which we would expect for a randomly connected network. In fact, if the world were randomly connected, then any one’s acquaintances would be just as likely to come from a different country, occupation, and socioeconomic class as one’s own. Clearly this is not the case in real life.

In the next proposition conditions are found under which pair-wise equilibrium networks exhibit the small-world property.

**Proposition 6** Suppose that assumptions (A1), (A2), (A3) and (A4) hold. Then, any pair-wise equilibrium networks exhibit the Small-world phenomena when \( v_m (v_m + 1) \ll (N + 1)^2 \) holds for each component \( m \).

**Proof.** Notice first that the average degree of a network is given by: 
\[
y(g) = \frac{2z(g)}{N+1},
\]
where \( z(g) \) is the number of links in network \( g \). Suppose that a pair-wise equilibrium network has the dominant group architecture, where the complete component has degree \( v \). Then, 
\[
z(g) = \frac{v(v+1)}{2}
\]
and thus average number of links in this network is 
\[
\frac{v(v+1)}{(N+1)}.\]
The clustering coefficient for this network is \( C(g) = 1 \) since \( C_i(g) = 1 \) for

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20 In fact, a random graph is a close approximation to the smallest possible graph for any given \( N \) and \( v \) (where \( v_{\text{max}} \) and the variance in \( v \) is not too large).
all \( i \in M(g) \) and \( n_i(g) = 0 \) for all \( i \notin M(g) \), and the characteristic path length of the giant component, which in this case is the complete component, is \( L(g) = \frac{1}{N(N+1)} \sum_{i \in M(g)} \sum_{j \in M(g)} d(i, j) = 1 \). Thus, the clustering coefficient takes the highest possible value and the characteristic path length the smallest possible value. Thus if \( v(v+1) \ll (N+1)^2 \), properties 1, 2, 3 and 4 of the definition of small-world are satisfied.

Consider now the exclusive groups architecture in which there are \( m \) components each having degree \( v_m \). Then \( C(g) = 1 \) and \( L(g) = \sum_{m} \frac{1}{v_m(v_m+1)} \sum_{i \in N+1} \sum_{j \in N+1} d(i, j) = 1 \). Thus, if \( v_m(v_m+1) \ll (N+1)^2 \), properties 1, 2, 3 and 4 of the definition of small-world are satisfied.

This result shows that pair-wise equilibrium networks in which the largest component is not too large as compared to the size of the network has the small-world property. It is worthy of note that a crucial aspect of the small-world property is a high clustering coefficient. This suggests that there is a positive relationship between social capital and small-world networks. In fact, individual \( i \)'s social capital \( SC_i(g) \) is equal to \( n_i(g)(n_i(g) - 1)C_i(g) \). Since total welfare increases with the amount of social capital and this increases with the clustering coefficient, it is plausible to think that there is a positive relationship between small-world networks and welfare. In fact, the table in the introduction shows that cooperative research networks have the small-world property. This suggests the small-world property in social networks is the outcome of rational agents investing in costly links to maximize their payoffs.

### 5 Conclusions

This paper has contributed to understanding the emergence of self-enforcing cooperative behavior in different communities and how the possibility of self-sustainable cooperative behavior shapes the social structure of a community. In a sense, it is an explanation of how trust emerges in a society is provided (i.e. self-enforcing cooperative behavior) and how the society is shaped by that possibility. In particular, it was shown that individuals are willing to invest resources to form links in order to take...
advantage of third-party or community enforcement at least up to the point where cooperation can be made self-sustainable. This leads to communities or social networks that are neither too spare nor too dense. That is, individuals form, at least, the minimal number of links needed for cooperation to be sustained in a self-enforcing fashion and never form more links than those under which cooperation is self-sustainable. In short, social networks are formed so that linked as well as unlinked individuals trust each other.

Furthermore, in equilibrium not only do social networks emerge in which individuals take advantage of network externalities created by information transmission between linked players, but also the network structures that emerge are, under a trusting strategy at least, Pareto efficient. Thus, individuals are by themselves able to solve the difficult problem of devising an efficient social network structure or they internalize the strong externalities associated to link formation at a personal cost.

I will end by briefly discussing an extension which I believe is worth pursuing. This concerns the study of social norms based on ascriptive features. Given the prevalence throughout history of social norms in which individuals condition their behavior on group identity or personal traits like ethnicity and religion, the study of these types of social norms is of a great empirical importance. For instance, Greif (1993) presents an interesting example of social norms based on personal traits used by Maghribi traders in the Mediterranean during the 11th century. He documents that the lack of formal institutions able to enforce overseas trading, induced the Maghribi traders to adopt the following social norm: no Maghribi trader would trade with another Maghribi trader who had cheated a Maghribi trader before. The study of ascriptive social norms will help us to understand another feature of social networks, which is segregation by ascriptive features like race and religion even when those are payoff-irrelevant. In fact, work done by the author in Balmaceda (2005b) suggests that total welfare of adopting a within-group social norm is larger than that when a between-group social norm is adopted, but a within-group norm gives rise to a more segregated society than a between-group norm. Thus, the prevalence of within group social norms and segregation by ascriptive features may be rationalized in terms of efficiency by following the interplay between network formation and cooperation proposed in this
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Appendix

Proof of proposition 2.

Lemma 7 (i) In any pair-wise equilibrium network $g$, any two players, say $i$ and $k$, such that $\sum_{j \in N_{\alpha}(g+ik)/\{\beta\}} \delta^{\alpha}(j,\beta) \geq \max\{Np_{\alpha}(\delta) - 1, 0\}$ for $\alpha, \beta = i, k$ and $\alpha \neq \beta$, must be mutually linked; (ii) in any pair-wise equilibrium network $g$, it cannot be the case that $n_i(g) = n_k(g) = 1$ and $g_{ik} = 0$; and (iii) in any pair-wise equilibrium network $g$, any component must be complete.

Proof. Part (i). Suppose that $g$ is pair-wise equilibrium network and $g_{ih} = 0$. In addition, suppose that $\sum_{j \in N_{\alpha}(g+ih)/\{\beta\}} \delta^{\alpha}(j,\beta) \geq \max\{Np_{\alpha}(\delta) - 1, 0\}$ for $\alpha, \beta = i, k$ and $\alpha \neq \beta$. Then by forming a link, cooperation is never destroyed and for those which $\sum_{j \in N_{\alpha}(g+ih)/\{\beta\}} \delta^{\alpha}(j,\beta) \geq \max\{Np_{\alpha}(\delta) - 1, 0\}$ for $\alpha, \beta = i, k$ and $\alpha \neq \beta$ it becomes a best-response. Thus, player $i$’s marginal net return from the link with $k$ is at least $\frac{1}{N}(c_i - d_i)$ and player $k$’s net marginal return from the link with $i$ is at least $\frac{1}{N}(c_i - d_i)$. By assumption A3, $i$ and $h$ has an incentive to form a link and thus $g$ cannot be a pair-wise stable network.

Part (ii). Suppose that $g$ is pair-wise equilibrium network in which $n_i(g) = n_k(g) = 1$. Because $i$ and $k$ have one link only, then it must be true that $Np_{\alpha}(\delta) - 1 \leq 0$ for $\alpha = i, k$. Then, cooperation between $i$ and $k$ is self-enforcing and thus part (i) implies that they must be mutually linked.

Part (iii). Clearly a component of two players is complete and so let us consider components that have at least 3 players. Let $M(g)$ be a non-singleton component with 3 or more players in the pair-wise equilibrium network $g$. Then $\sum_{j \in N_{\alpha}(g)/\{\beta\}} \delta^{\alpha}(j,\beta) \geq \max\{Np_{\alpha}(\delta) - 1, 0\}$ for $\alpha, \beta = i, k \in M(g)$ with $\alpha \neq \beta$ and $g_{ik} = 1$. Take any two players $l, m \in M(g)$ with $g_{lm} = 0$. If they form a link and cooperation is self-enforcing, the marginal net payoff is $\frac{1}{N}(c_{\alpha} - d_{\alpha}) - r_{\alpha}$ for $\alpha = l, m$. Because
the distinct set of players easy to check that cooperation between \( l \) and \( m \) is self-enforcing. That is, \( \sum_{j \in N_{l}(g+lm)/(m)} \delta^{v_{l}(j,m)} \geq \max \{ Np_{l}(\delta) - 1, 0 \} \) and \( \sum_{j \in N_{m}(g+lm)/(l)} \delta^{v_{m}(j,l)} \geq \max \{ Np_{m}(\delta) - 1, 0 \} \). By pairwise stability, \( \sum_{j \in N_{a}(g)/(k)} \delta^{v_{a}(j,k)} \geq \max \{ Np_{a}(\delta) - 1, 0 \} \) for all \( k \in M(g) \) with \( g_{\alpha k} = 1 \) and for \( \alpha = l, m \). Because \( l, m \in M(g) \), there is a path linking \( l \) and \( m \). Let the distinct set of players \( \{ m_{1}, m_{2}, ..., m_{n} \} \) such that \( g_{mi} = g_{m_{2}m_{1}} = g_{m_{3}m_{2}} = ... = g_{m_{n}m_{n-1}} = 1 \) be that path. Take the node \( m_{1} \). Since \( g_{m_{1}l} = 1 \), pairwise stability implies that \( \sum_{j \in N_{l}(g)/(m_{1})} \delta^{v_{l}(j,m_{1})} \geq \max \{ Np_{l}(\delta) - 1, 0 \} \) and \( \sum_{j \in N_{m_{1}}(g)/(l)} \delta^{v_{m_{1}}(j,l)} \geq \max \{ Np_{m_{1}}(\delta) - 1, 0 \} \). Because there is an \( l \)-excluding path between any \( j \in N_{l}(g) / \{ m_{1} \} \) such that \( v_{l}(j,m_{1}) = 1 \) and \( m_{1} \), \( v_{l}(j,m) = 1 \). This implies that, \( \sum_{j \in N_{l}(g+l)/(m)} \delta^{v_{l}(j,m)} \geq \sum_{j \in N_{l}(g)/(m_{1})} \delta^{v_{l}(j,m_{1})} \geq \max \{ Np_{l}(\delta) - 1, 0 \} \). Similarly for player \( m \) and thus cooperation between \( l \) and \( m \) is self-enforcing.

Part (1). Existence is guaranteed by (A3), (A4) and 7.

Part (2). Recall that each component must be complete. It is enough to show that there could be more than one complete component. Suppose that there are two components \( M_{1}(g) \) and \( M_{2}(g) \) and that there is a player \( i \in M_{1}(g) \) and \( j \in M_{2}(g) \) such that \( Np_{l}(\delta) \leq 1 \) for all \( \alpha = i, j \). Then by part (ii) in lemma 7 these two players must be mutually linked. This implies that if there are two components, all the players for whom cooperation is enforcing by bilateral punishments should belong to the same component.

Suppose next that there are two components \( M_{1}(g) \) and \( M_{2}(g) \) and that all \( i \in N + 1 \) such that \( Np_{i}(\delta) \leq 1 \) belongs to the first component. Then take a player \( i \in M_{1}(g) \) and \( k \in M_{2}(g) \). Because they belong to different components \( g_{ik} = 0 \). Suppose now that they deviate and form a link with each other. This does not trigger a punishment when \( \sum_{j \in N_{a}(g)/(\{ \beta \})} \delta^{v_{a}(j,\beta)} \geq \max \{ Np_{a}(\delta) - 1, 0 \} \) for \( \alpha, \beta = i, k \). It is easy to check that \( \sum_{j \in N_{l}(g)/(k)} \delta^{v_{l}(j,k)} = \sum_{j \in N_{k}(g)/(l)} \delta^{v_{k}(l,k)} = 0 \) since there is no \( i \)-excluding path between \( k \) and \( j \) for all \( j \in N_{l}(g) / \{ k \} \). Similarly for player \( k \). Thus, \( i \) and \( k \) has no incentive to add a link.

Part (4). Note that in any pair-wise equilibrium network \( g \), \( W(g) = \sum_{i \in N+1} \left( \frac{n_{i}(g)}{N} c_{i} + \frac{N-n_{i}(g)}{N} d_{i} - n_{i} \right) \). Because \( \frac{1}{N} (c_{i} - d_{i}) - r_{i} \geq 0 \) for all \( i \) and in any pair-wise equilibrium network \( g \), \( n_{i}(g) > 0 \) if and only if cooperation is self-sustainable between \( i \) and his partners,
$W(g)$ is maximized by having a complete network. Assumption (A4) guarantees that if the complete network is formed, cooperation is self-enforcing between any pair of players. 

Proof of proposition 3.

**Lemma 8** (i) In any pair-wise equilibrium network $g$, any two players, say $i$ and $k$, such that $\sum_{j \in N_{\alpha}(g+ik)/\{\beta\}} \delta^{\alpha(j, \beta)} \geq \max \{Np_\alpha(\delta) - 1, 0\}$ for $\alpha, \beta = i, k$ and $\alpha \neq \beta$, must be mutually linked; (ii) in any pair-wise equilibrium network $g$, $n_i(\tilde{g}) \geq \frac{1}{N} \left[ d_i - n_i(\tilde{g}) (c_i - d_i) \right]$ for all $i \in N + 1$; (iii) in any pair-wise equilibrium network $g$, it cannot be the case that $n_i(g) = n_k(g) = 1$ and $g_{ik} = 0$; and (iv) in any pair-wise equilibrium network $g$, any component must be complete.

**Proof.** of lemma 8.

Part (i) is exactly as part (i) in lemma 7. Part (ii) follows immediately from the fact that $\Delta V_i(\tilde{g} + ih, \tilde{g}) - r_i = \frac{1}{N} [d_i - n_i(\tilde{g}) (c_i - d_i)] - r_i$ if $I_{ih}(g + ih) = 0$. Thus, player $i$ and $h$ do not have an incentive to add a non-cooperative link as long as $n_\alpha(\tilde{g}) \geq \frac{1}{N(c_i - d_i)}$ for either $\alpha = i$ or $\alpha = h$ or both.

Part (iii). Suppose that $g$ is pair-wise equilibrium network in which $n_i(g) = n_k(g) = 1$ and $g_{ik} = 0$. Because $i$ and $k$ have one link only and $g$ is pair-wise equilibrium network, then it must be true that $1 - Np_\alpha(\delta) \geq 0$ for $\alpha = i, k$. Then, cooperation between $i$ and $k$ is self-enforcing by bilateral sanctions only and thus part (i) implies that they must be mutually linked. This implies that a component of two players cannot be pair-wise stable.

Part (iv) is identical to part (iii) in lemma 7.

Part (2) follows from symmetry and lemma 8.

Part (3) and (4) are identical to parts (3) and (4) in the proof of proposition 2. 

**Proof.** of proposition 4.

In any pair-wise equilibrium network $g$ only cooperative links are allowed. Suppose that in $g$ there is a player $i$ with $n_i(g) = 1$ and $g_{ik} = 1$. Then, $i$’s best response to $k$ is cooperation if and only if $Np_i(\delta) \leq 1$. Furthermore, the fact that player $i$ is linked to $k$ only implies that player $k$’s best response to cooperation $i$ is
cooperation if and only if $Np_k(\delta) \leq 1$. This follows from the fact that there is no path between $i$ and any player $j$ linked to $k$ that could punish $k$ for deviating against $i$. Thus, if $n_i(g) = 1$, then it must be true that $Np_\alpha(\delta) \leq 1$ and $\frac{1}{N} (c_\alpha - d_\alpha) \geq r_\alpha$ for $\alpha = i, k$. If $b(N) > 3$, then this players must be mutually linked while if $b(N) = 1$, cooperation between $i$ and $k$ is not self-enforcing. If $b(N) = 2$ and $i, k \in B(N)$, then the network in which player $i$ and $k$ are linked and the rest are isolated players is a pair-wise equilibrium network. Deleting links is never profitable and adding a link destroys cooperation since no cooperative link can be added.

Suppose that $g$ is pair-wise equilibrium network and that there is a cycle that has $m$ players. Because $g$ is a pair-wise equilibrium network, then it must be true that for any player $i$ belonging to the $m$-cycle, cooperation between $i$ and $i+1$ and $i$ and $i-1$ is self-enforcing. Thus, for any player $i$ belonging to the $m$-circle component, $\delta^{m-1} \geq \max\{Np_i(\delta) - 1, 0\}$. Take now, player $i+2$, then cooperation with this player is self-enforcing if and only if $\delta^{m-2} + \delta \geq \max\{Np_i(\delta) - 1, 0\}$ and $2\delta \geq \max\{Np_{i+2}(\delta) - 1, 0\}$. Because, $\min\{2\delta, \delta^{m-2} + \delta\} > \delta^{m-1}$, then cooperation between $i$ and $i+2$ is self-enforcing and thus they have an incentive to add a link. The same holds between $i$ and $i+3$ and so on and so forth.

This follows immediately from proposition 2 and the fact that in a complete component $l_i(j,k) = 1$ for all $i, j, k \in M(g)$. ■

References


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