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Approval Voting with Endogenous Candidates*

Abstract

We present a formal model of political competition under approval voting which allows for endogenous candidate entry. Our analysis yields a number of novel insights. First, we develop a notion of sincere voting behavior under approval voting, called *relative sincerity*. We then show that the relatively sincere voting behavior is consistent with the strategic calculus of voting. Second, we show that in a one-dimensional model with distance preferences, equilibria in relatively sincere strategies and without spoiler candidates always generate outcomes close to the median voter. Moreover, approval voting satisfies Duverger's Law in the sense that there are at most two winning positions! Finally, we extend our analysis to arbitrary policy spaces. In the general setting, approval voting is shown to be susceptible to the same kinds of problems as the plurality rule, such as the possibility of non-majoritarian outcomes, failure to elect the Condorcet winner and existence of spoiler candidates.

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1 Introduction

The objective of the paper is to study a model of endogenous candidacy under approval voting (hereafter AV) and to compare its outcomes to those under the plurality rule.[1]

AV is a non-rank, scoring method of voting. Under this method a citizen can vote for as many alternatives as he wishes with the restriction that each alternative can receive at most one vote. All votes count equally and the alternative receiving the highest number of votes is chosen to be the winner. AV is currently used by several academic and professional bodies to elect their office-bearers.[2] Many scholars of electoral systems have recommended that AV be used for political elections as well and regard it to be *the* electoral reform of the century (see Brams (1980); 105). There is a large body of literature that studies the electoral outcomes under AV and compares them vis-a-vis the plurality rule. However, the problem of endogenous candidacy has so far been ignored in the literature.[3] As we argue in the paper, this has some important implications for the relative performance of the two voting systems.

The present work seeks to fill this gap by incorporating the politicians' entry decisions into the electoral game. For this purpose, we use the 'citizen-candidate' approach to electoral competition.[4] Under this approach, the political process is modeled as a three-stage game. At the first stage, policy-motivated candidates decide whether or not to run for office at a cost. At the second stage, the citizens vote over the set of candidates. At the third stage, the winner implements his

[1] The plurality rule is an electoral rule where a citizen has one vote that he can cast for one (and only one) candidate, and where the candidate with the most votes wins the election. This electoral rule is used for example in US Congressional elections.

[2] For example, the Fellows of the Econometric Society and the President of the Social Choice and Welfare Society are elected under AV; the American Statistical Association or the Mathematical Association of America are some of the professional organizations who have adopted this method (see Fishburn and Brams (1988) and Brams and Nagel (1991) for a discussion of some of those elections); AV is also used in electing the Secretary General of the United Nations, in elections in some Eastern European countries and in voting over multiple-related initiatives in some US States. See Brams and Fishburn (2003) for a recent survey.

[3] Many scholars have recognized the importance of endogenizing the set of candidates, as Fishburn and Brams (1981b; 426) note: "Several people have also expressed concern about how Approval voting would affect who enters an election and how it would influence candidates' strategies. Although we do not address this concern, it surely deserves examination."

[4] The citizen-candidate approach was pioneered by Osborne and Slivinski (1996) and Besley and Coate (1997). Our model is closer in spirit to the latter since we model both candidate entry and voting as strategic decisions.

preferred policy. Both candidate entry and voting are modeled as strategic decisions made by rational agents.

There are three advantages to using the citizen-candidate approach for the comparative analysis of electoral systems. First, the citizen-candidate approach allows us to understand the effect of a voting rule on the incentives it creates for candidates to enter the political race. This is important given that every non-dictatorial voting procedure that satisfies unanimity is open to strategic entry or exit by candidates (see Dutta, Jackson and Le Breton (2001)). Second, it allows us to account for candidates' policy motivation. This is important given that there is empirical evidence that decision makers' policy preferences play an important role in their policy decisions (e.g., see Levitt (1996)). Third, while the Downsian analysis of AV for the most part has been limited to the one-dimensional model with three parties, the citizen-candidate approach enables us to go beyond this restrictive setting and makes it possible for us to handle the multi-dimensional policy space with an arbitrary number of potential candidates.

Our analysis yields a number of interesting results. The first set of results concerns the notion that AV encourages sincere voting. Under AV, the concept of sincerity is taken to mean that a citizen who votes for a candidate k must also vote for any other candidate j whom he prefers to k. We propose a further refinement of the voting behavior, viz. *relative sincerity.* An agent is said to vote relatively sincerely if, given others' voting strategies, he votes (does not vote, resp.) for all those candidates who give him a strictly higher (lower, resp.) payoff than the expected payoff from the outcome of the election.[5] We show that a voter's best-response set always contains a *relatively sincere voting strategy.* We also prove, in the one-dimensional model with single-peaked preferences, the existence of a voting equilibrium in relatively sincere strategies.

The second set of results concerns the electoral process under AV when the policy space is one-dimensional and the voters have distance preferences. Our analysis shows that, while there

[5] This refinement is intuitively plausible and conforms with observations made by some scholars regarding how people vote under AV (e.g., see Brams and Nagel (1991)).

may be a multiplicity of candidates under AV, in any equilibrium, there are at most two winning positions. We also show that the outcomes of equilibria in which there are no spoiler candidates (i.e. candidates who run in the race not to win but because their presence in the race leads to a different electoral outcome), and which satisfy relative sincerity, are arbitrarily close to the median voter's ideal position as the entry costs become small. This contrasts with the plurality rule where there always exist equilibria with extreme policy outcomes. However, this result, suggesting policy moderation under AV, does not extend to the equilibria involving spoiler candidates. Hence, our analysis suggests that the role of AV in generating policy moderation depends upon the plausibility of equilibria with spoiler candidates.

Our third set of results characterizes the political equilibria under AV in a general setting. We find that the outcomes under AV and those under the plurality rule are generally distinct, *even* in elections with only two candidates! In particular, neither is a subset of the other. This may come as a surprise, especially since AV and the plurality rule are equivalent methods when there are two *exogenously given* alternatives. When one considers the fact that candidate entry is endogenous, the equivalence no longer holds. This highlights a methodological problem in comparing electoral systems by studying their performance over an exogenous set of alternatives. We also find that AV is vulnerable to the same kinds of problems as the plurality rule such as non-majoritarian policy outcomes, failure to elect the Condorcet winner and presence of spoiler candidates.

The organization of the remainder of the paper is as follows. In section 2 we review the current literature. In section 3 we present the model and develop various refinements of voting behavior. In section 4 we study our model in the context of a one-dimensional policy space with distance preferences. Section 5 extends our analysis to arbitrary policy spaces. Section 6 concludes. All proofs are in the appendix.

2 Related Literature

The present work contributes to a large literature on the comparative properties of alternative electoral systems. There are two strands of this literature that compare AV vis-a-vis the plurality rule: (*i*) the literature on the properties of voting behavior over a fixed set of alternatives, and (*ii*) Downsian models of political competition.

The seminal paper under the first strand of the literature is Brams and Fishburn (1978). This paper compares different single-ballot voting rules under dichotomous and trichotomous preferences.[6] The authors show that when the preferences are dichotomous, each voter has a unique undominated strategy under AV and it always elects the Condorcet winner. This is not true of the other voting rules. Moreover, when the preferences are trichotomous, AV is the only system under which the set of sincere voting strategies is equivalent to the set of undominated strategies. Fishburn and Brams (1981a) show that AV dominates the plurality rule in the sense that if the latter system elects the Condorcet winner, then the former must elect it as well, but the converse is not true. They also show that whenever there is a Condorcet winner, there exists a sincere strategy profile under AV which elects it. Fishburn and Brams (1981b) and (1981c) show that in any optimal voting strategy, an expected utility maximizer should vote for all the serious contenders that give him a utility higher than the expected utility. Hence, voting must be sincere on the set of serious contenders.

Scholars have criticized the above approach on several grounds. Some argue that the preference structure that admits only dichotomous and trichotomous preferences is restrictive. Others have pointed out that even if the citizens voted sincerely, they still need to make strategic calculations in deciding how many candidates to vote for (e.g., see Tullock (1979) or Saari (2001)). Cox (1984), which looks at three-candidate elections in two-member districts in England between 1832 and

[6] A voter is said to have dichotomous preferences if the set of alternatives can be partitioned into two subsets, say M and L, such that all elements in M are strictly preferred to those in L and the voter is indifferent between all alternatives within the sets M and L. Under trichotomous (multichotomous, resp.) preferences, there are three (more than three, resp.) such partitions.

1867, finds evidence supporting strategic behavior by voters.[7] The importance of strategic voting is further made clear in De Sinopoli (1999) where the author shows that even though there exists under AV a sincere strategy that implements the Condorcet winner, such a strategy need not be consistent with sophisticated voting in the sense that it fails to survive iterated elimination of weakly dominated strategies. We draw on this critique and focus on equilibrium voting strategies.

The second strand of the literature uses the Downsian framework to study various electoral systems. Cox (1985) considers a one-dimensional, Downsian model with three political parties. He shows that under AV there exists a unique Nash equilibrium in which all the parties adopt the median voter's ideal position. However, this result does not generalize to more than three parties. Cox (1987) compares the outcomes under the plurality rule to those obtained under AV. He shows that in two-candidate elections, they all adopt the median voter's ideal policy under both systems, while in elections with at least three candidates, such a convergent equilibrium still exists under AV, but not under the plurality rule. Our analysis shows that even in two-candidate equilibria, the set of equilibrium outcomes differs under the two systems. Myerson and Weber (1993) also compare different electoral systems with Downsian parties. They look at equilibria under which voters hold rational expectations about their vote being pivotal over every possible pair of candidates. They find that, under AV, all the serious contenders are located at the position of the median voter. In contrast, the plurality rule imposes little restriction on the position of the winning candidates, making it possible for an extremist to be elected. Our analysis highlights that these results need not be true when the set of contenders is endogenous.[8]

3 The Model

Consider a polity $\mathcal{N}$ consisting of a finite number of citizens, indexed by $\ell = 1, ..., |\mathcal{N}|$. This polity must elect a public official who will be in charge of implementing a policy. Let $\mathcal{A}$ denote the finite

[7] Each voter could cast up to two votes without accumulating them and therefore the electoral rule was equivalent to AV over three candidates.

[8] Scholars have also studied the effect of different electoral systems on corruption (see Myerson (1993a) and Myerson (2002)) and the incentives to favor minority interests (see Myerson (1993b)).

and non-empty set of feasible policies. Citizens are policy-motivated in that their utility depends on the policy which is implemented. A citizen ℓ's preference ordering over the set of alternatives $\mathcal{A}$ is represented by a utility function $u^{\ell} : \mathcal{A} \rightarrow \Re$. Each citizen ℓ is assumed to have a unique ideal policy w_{ℓ}, where $w_{\ell} \equiv \underset{w \in \mathcal{A}}{\arg\max} \; u^{\ell}(w)$, although more than one citizen could have the same ideal policy.

There are three stages to the political process. At the first stage, each citizen chooses whether or not to enter the electoral race. In order to stand as a candidate, a citizen must incur a utility cost $\delta > 0$. At the second stage, each citizen decides which of the self-declared candidates to vote for. The election is held under AV. The candidate receiving the most votes is elected. In the event of a tie between several candidates, each of them is elected with an equal probability. At the third stage, the elected candidate chooses the policy to be implemented. If no candidate entered the race, a default outcome $x_0 \in \mathcal{A}$ is implemented. We now analyze each of these stages in reverse order.

3.1. Policy Selection Stage. Because it is the last stage of the game, any candidate j who wins the election implements his preferred policy w_j.

Let v_j^{ℓ} be citizen ℓ's payoff when citizen j is in office, with $v_j^{\ell} \equiv u^{\ell}(w_j)$. Also, denote by v_0^{ℓ} the payoff received by ℓ when no citizen runs for office and the default outcome is implemented.

3.2. Voting Stage. Let $\mathcal{C} \subseteq \mathcal{N}$ denote a non-empty set of candidates who are running for office. Under AV, each citizen can vote for as many candidates as he wants, with the restriction that at most one indivisible vote can be cast per candidate. We describe citizen ℓ's voting decision by $\alpha^{\ell}(\mathcal{C}) \in \{0,1\}^{|\mathcal{C}|}$, whereby $\alpha_j^{\ell} = 1$ (0, resp.) means that citizen ℓ votes (does not vote, resp.) for candidate j. Let $\alpha(\mathcal{C}) = \left\{\alpha^1(\mathcal{C}), ..., \alpha^{|\mathcal{N}|}(\mathcal{C})\right\}$ denote the profile of voting decisions.[9]

We shall call the set of candidates who receive the most votes as the *Winning Set* and denote

[9] Whenever it is possible to do so without causing a confusion, we shall omit $\mathcal{C}$ and denote the voting decisions and voting profile as α^{ℓ} and α, respectively.

it by $W(\mathcal{C},\alpha)$. Formally,

$$W(\mathcal{C},\alpha) \equiv \left\{ j \in \mathcal{C} : \sum_{\ell \in \mathcal{N}} \alpha_j^\ell \geq \sum_{\ell \in \mathcal{N}} \alpha_k^\ell, \text{ for all } k \in \mathcal{C} \right\}.$$

Let $p_j(\mathcal{C},\alpha)$ denote the probability that a candidate $j \in \mathcal{C}$ becomes the policy-maker when the voting profile is α. Given our tie-breaking assumption, $p_j(\mathcal{C},\alpha) = \frac{1}{|W(\mathcal{C},\alpha)|}$ if $j \in W(\mathcal{C},\alpha)$ and 0 otherwise. We can now define a (pure strategy) voting equilibrium.

Definition 1 ***(Voting Equilibrium)*** *Given a non-empty set of candidates $\mathcal{C}$, a strategy profile $\alpha^*(\mathcal{C})$ is an equilibrium of the voting stage if for any citizen $\ell \in \mathcal{N}$,*
(i) $\sum_{j \in \mathcal{C}} p_j\left(\mathcal{C}; \alpha^{\ell *}, \alpha^{-\ell *}\right) v_j^\ell \geq \sum_{j \in \mathcal{C}} p_j\left(\mathcal{C}; \alpha^{\ell}, \alpha^{-\ell *}\right) v_j^\ell$, for all $\alpha^\ell \in \{0,1\}^{|\mathcal{C}|}$, and
(ii) $\alpha^{\ell *}$ is weakly undominated.

The first condition means that each citizen chooses his voting strategy in order to maximize his expected utility, taking into account others' voting strategies and anticipating the policy implemented by the winner. The second condition is a standard refinement used in the voting literature. Under the plurality rule, this condition requires that a citizen does not vote for any of his least-preferred alternatives. Under AV, it implies that a citizen does not vote for any of his least-preferred alternatives and that he votes for all his most-preferred alternatives. We state this formally in lemma 1 (due to Brams and Fishburn (1978)). But before, let us introduce the following notation. Consider a non-empty set of candidates $\mathcal{C}$. For any citizen $\ell \in \mathcal{N}$, let $G^\ell(\mathcal{C}) \equiv \left\{ j \in \mathcal{C} : v_j^\ell \geq v_k^\ell \text{ for all } k \in \mathcal{C} \right\}$ denote the set of citizen ℓ's most-preferred candidates, and $L^\ell(\mathcal{C}) \equiv \left\{ j \in \mathcal{C} : v_k^\ell \geq v_j^\ell \text{ for all } k \in \mathcal{C} \right\}$ the set of citizen ℓ's least-preferred candidates. The following lemma characterizes the set of weakly undominated voting strategies.

Lemma 1 *A voting strategy is weakly undominated for citizen ℓ if and only if $\alpha_j^\ell = 1$ for all $j \in G^\ell(\mathcal{C})$, and $\alpha_k^\ell = 0$ for all $k \in L^\ell(\mathcal{C})$.*

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3.3. Entry Stage. Each citizen must decide whether or not to stand for election at a utility cost $\delta > 0$. Let $s^\ell \in \{0,1\}$ be citizen ℓ's entry decision, where $s^\ell = 1$ (0, resp.) means that citizen

ℓ chooses to stand (not to stand, resp.) for election. The pure-strategy profile is denoted by $s = \left\{s^1, ..., s^{|\mathcal{N}|}\right\}$ and the set of candidates associated with it by $\mathcal{C}(s) \equiv \left\{\ell \in \mathcal{N} : s^\ell = 1\right\}$. Citizens anticipate others' entry strategies, the voting profile associated with each set of candidates, as well as the policy each candidate will implement if elected, and then strategically decide whether to enter in the race. Citizen ℓ's expected payoff for a given strategy profile s is thus

$$U^\ell\left(\mathcal{C}(s), \alpha\right) = \sum_{j \in \mathcal{N} \cup \{0\}} p_j\left(\mathcal{C}(s), \alpha\left(\mathcal{C}(s)\right)\right)\ v_j^\ell - s^\ell\ \delta$$

, where $p_0(.)$ denotes the probability that the set of candidates is empty.

Let $\sigma^\ell \in [0,1]$ denote the mixed strategy of agent ℓ, where σ^ℓ should be interpreted as the probability that ℓ enters the political race. A mixed strategy equilibrium of the entry stage - σ^* - is a profile of strategies such that for each citizen ℓ, $\sigma^{\ell *}$ is a best response to $\sigma^{-\ell *}$.

3.4. Political Equilibrium. We can now define a political equilibrium as a pair $(\sigma^*, \alpha^*(.))$ consisting of an entry decision profile σ^* and a voting strategy profile $\alpha^*(.)$ such that: (i) $\alpha^*(.)$ is a voting equilibrium for any non-empty set of self-declared candidates $\mathcal{C}$; and (ii) σ^* is an equilibrium of the entry game given the voting function $\alpha^*(.)$.

It is easy to show the existence of voting equilibria in pure strategies. Moreover, given the finiteness of the action space (i.e. {enter, not enter}), we know that there exists an equilibrium of the entry stage game. Hence, we get the following result.

Proposition 1 *A political equilibrium exists.*

Following Besley and Coate (1997), we are going to focus our attention on pure strategy equilibria, i.e. political equilibria $\{\sigma^*, \alpha^*(.)\}$ where citizens employ pure strategies at the entry stage.

3.5. Refining the Voting Behavior. As we saw in lemma 1, the standard voting refinement of weak undominance puts very little restriction on the permissible voting behavior. Not only does this mean that the notion of Nash equilibrium gives us little predictive power, it also fails to capture the intuitive (and empirically observed) way in which people vote under AV. Hence, we consider two plausible notions of sincere voting behavior: *sincere voting strategies* and *relatively sincere voting strategies*.[10] We then proceed to examine the sense in which these notions are consistent with the strategic calculus of voting. But first we need to introduce some notation.

Let $BR^{\ell}(\mathcal{C}, \alpha) \equiv \arg\max_{\alpha^{\ell} \in \{0,1\}^{|\mathcal{C}|}} U^{\ell}\left(\mathcal{C}; \alpha^{\ell}, \alpha^{-\ell}\right)$ be citizen ℓ's set of best responses to the strategy profile $\alpha^{-\ell}$. $BR^{\ell}(\mathcal{C}, \alpha)$ consists of the (pure) voting strategies that maximize citizen ℓ's expected utility given others' voting strategies and a set of candidates $\mathcal{C}$. We are now in a position to formally define our various notions of sincerity.

Definition 2 ***(Sincerity)*** *Let $|\mathcal{C}| \geq 2$. A citizen ℓ's voting strategy is said to be sincere if for all candidates j and k, with $j \neq k$ and $v_k^{\ell} > v_j^{\ell}$, the following two conditions are satisfied: (i) $\alpha_k^{\ell} = 1$ whenever $\alpha_j^{\ell} = 1$; and (ii) $\alpha_j^{\ell} = 0$ whenever $\alpha_k^{\ell} = 0$.*

This concept requires that if a citizen votes for a candidate j, then he must also cast a vote for every candidate k whom he strictly prefers over j. Similarly, if he does not vote for a candidate k, then he must not cast a vote for any candidate j who gives him a strictly lower utility compared to k. In any election with no more than three candidates, every weakly undominated voting strategy satisfies this definition.

However, for a large number of candidates, the concept of sincerity does not put any restriction on how many candidates a citizen should vote for. He could vote for only the members of $G^{\ell}(\mathcal{C})$, or he could vote for everyone outside of $L^{\ell}(\mathcal{C})$, or he could draw a cutoff at any intermediate candidate. The concept of relative sincerity provides a condition on where such a cutoff should be drawn. It requires a citizen to vote (not to vote, resp.) for all the candidates who give him a strictly higher (lower, resp.) utility than the expected utility from the outcome of the election ($U^{\ell}(\mathcal{C}; \alpha)$). More formally,

Definition 3 ***(Relative Sincerity)*** *Let $|\mathcal{C}| \geq 2$. A citizen ℓ's voting strategy α^{ℓ} is said to be sincere relative to others' voting strategies $\alpha^{-\ell}$ if the following two conditions hold whenever citizen ℓ is not indifferent between all candidates: (i) $\alpha_i^{\ell} = 1$ whenever $v_i^{\ell} > U^{\ell}\left(\mathcal{C}; \alpha^{\ell}, \alpha^{-\ell}\right)$; and (ii) $\alpha_i^{\ell} = 0$ whenever $v_i^{\ell} < U^{\ell}\left(\mathcal{C}; \alpha^{\ell}, \alpha^{-\ell}\right)$.*

[10] The first notion is due to Brams and Fishburn (1978).

The following lemma (due to Fishburn and Brams (1981c)) shows that the equilibrium voting strategies must be relatively sincere on the set of serious contenders.

Lemma 2 *Let $|W(\mathcal{C}, \alpha(.))| \geq 2$. In any equilibrium where a citizen ℓ is not indifferent between all the serious contenders, his voting strategy α^{ℓ} must be such that, for all $i \in W(\mathcal{C}, \alpha(.))$,*

(i) if $\alpha_i^{\ell} = 1$, then $v_i^{\ell} \geq \frac{1}{|W|} \sum\limits_{j \in W} v_j^{\ell}$; and

(ii) if $v_i^{\ell} > \frac{1}{|W|} \sum\limits_{j \in W} v_j^{\ell}$, then $\alpha_i^{\ell} = 1$.

This lemma provides a simple characterization of agents' voting strategies over the winning set. It states that in the event of a tie, a citizen who is not indifferent between all serious contenders should vote (not vote, resp.) for all those who give him a utility strictly higher (lower, resp.) than the expected utility he derives from the electoral outcome. Note that this lemma implies that if *all* candidates are serious contenders, then the equilibrium voting behavior necessarily satisfies relative sincerity.

Our next result shows that the notion of relative sincerity is consistent with the strategic calculus of voting in the sense that each voter's best-response set contains such a strategy.

Proposition 2 *Let $|\mathcal{C}| \geq 2$. For any citizen ℓ and strategy profile $\alpha^{-\ell}$, there exists a strategy $\alpha^{\ell} \in BR^{\ell}(\mathcal{C}, \alpha)$ such that α^{ℓ} is sincere relative to $\alpha^{-\ell}$.*

Since the concept of relative sincerity is stronger than the concept of sincerity, proposition 2 implies that sincere strategies are contained in the agents' best-response sets as well.

In the next section, we prove the existence of equilibria in relatively sincere strategies in the context of a one-dimensional model with single-peaked preferences. Then we explore how this refinement helps us to characterize the plausible electoral equilibria under AV.

4 One-dimensional Policy Space

We now analyze a special case of the model developed in section 3 - one with a one-dimensional policy space and distance preferences. There are two motivations for doing so. First, since the previous literature on AV has focused, almost exclusively, on the one-dimensional model, the following analysis highlights the important differences between our approach vis-a-vis the previous ones. In particular, we show that there are important implications of endogenizing the set of candidates. Second, we are able to compare the equilibria under AV with those under the plurality rule and examine the effect of AV on policy moderation.

Consider the set of alternatives to be $[0,1]$. Citizens' preferences are represented by a utility function $u^{\ell}(w)$ which satisfies the following properties: (i) (Single-peakedness) given any two policies x and y such that $y \geq x \geq w_{\ell}$ or $w_{\ell} \geq x \geq y$, we must have $u^{\ell}(x) \geq u^{\ell}(y)$; (ii) (Concavity) for each ℓ, $u^{\ell}(a_D) \geq \frac{1}{|D|}\sum_{d \in D} u^{\ell}(d)$, where D is a set of policies and $a_D \equiv \frac{1}{|D|}\sum_{d \in D} d$, the average policy; and (iii) (Distance Condition) consider any citizens ℓ and k and any policies x and y, such that $k \geq \ell$ and $y \geq x$, then $u^{k}(y) \geq u^{k}(x)$ whenever $u^{\ell}(y) \geq u^{\ell}(x)$.[11]

We normalize $u^{\ell}(w_{\ell})$ to 0. Also, let m denote the median voter's ideal policy, and let $\mathcal{M} \equiv \{\ell \in \mathcal{N} : w_{\ell} = m\}$ be the set of citizens at that position. We assume the number of citizens on either side of m, $\frac{|\mathcal{N} \backslash \mathcal{M}|}{2}$, to be an integer. Also, we assume that when indifferent between voting and not voting over a subset of serious contenders, a citizen votes for all of them.[12] Now we proceed to characterize the set of political equilibria under AV.

4.1. Characterization of Political Equilibria. Our first result concerns the number of winning positions. We prove that in any voting equilibrium, irrespective of the number of candidates standing for election, there can be at most two winning positions. Furthermore, when there are two distinct winning positions, they must be such that the median voter m is indifferent between the two.

Proposition 3 *Suppose $|W(\mathcal{C}, \alpha(.))| \geq 2$, with $w_j \neq w_k$ for some j and k in the winning set $W(\mathcal{C}, \alpha(.))$. In any voting equilibrium, $w_i \in \{w_L, w_R\}$ for any candidate i in the winning set, with w_L and w_R such that $v_L^m = v_R^m$ and $0 \leq w_L < m < w_R \leq 1$.*

To understand the intuition behind proposition 3 consider the case of a tie between a left

[11] Our distance condition eliminates the implausible scenarios such as the one where a leftist voter prefers a rightist candidate, say R, over a leftist candidate, say L, and *at the same time* a rightist voter prefers L over R.

[12] Note that this assumption is not restrictive if preferences are strictly concave, or if there is no candidate whose ideal policy coincides with the mean platform of the serious contenders.

moderate, a right moderate and a left extremist. Also, suppose that the average policy lies between the two moderate positions. We know from lemma 2 that the left moderate must receive the votes of all the citizens to the left of him. Also, if a citizen to the right of the left moderate candidate is voting for the left extremist, he must be voting for the left moderate too. Hence if the two left candidates were to tie, the same citizens must be voting for them! By the same principle, each citizen must be voting either for all the candidates on the left of the average or all the candidates on the right of the average (but not both). By the distance property, if the median voter votes for the candidates on the left of the average policy, then everyone on the left of the median must be doing the same, and therefore there could not have been a tie between the policies on the left of the average and those on the right, a contradiction.

The above result is interesting because it answers peoples' concern that AV may obliterate the two-party system and lead to a proliferation of platforms at which winning candidates can be found. Our result shows that this is not the case. In fact, if any political equilibrium, the winning candidates will be located at one of at most two positions. Hence, AV satisfies a variant of Duverger's law. Duverger (1954) observed that the plurality rule favors a two-party system since the citizens, fearing that they might waste their vote, will choose not to cast their ballot for a potential entrant. Our analysis shows that a similar result emerges under AV as well.[13]

There are, however, two important differences. The first one concerns the number of candidates. While under the plurality rule, there will be only two serious contenders, AV does not prevent the entry of more than one candidate at each winning position. Indeed, *even when* citizens are exclusively policy-motivated, several may decide to run on the same platform in order to increase the probability that their ideal policy is implemented. The intuition behind this result is that a second candidate running on the leftist winning platform improves the prospect that a leftist candidate wins the election since the leftist voters will vote for both candidates.[14] In con-

[13] Casual empiricism confirms this result. In all the AV elections that the authors are aware of, there have been at most two frontrunners.

[14] For instance, Brams and Nagel (1991) show that citizens tend to vote for all the candidates who are similar

trast, under the plurality rule, the second candidate runs the risk of splitting the leftist votes, and thereby increases the chances of the rightist candidate's victory.[15] Hence, there can be under AV more than two serious contenders, but only two policy outcomes. The second difference concerns the logic driving the result. Under the plurality rule, two winning positions arise from the fear of 'wasting-the-vote'. Indeed, voters' tendency to ignore a third party is based on a self-fulfilling prophecy that a vote for that party would be wasted. AV has the potential to solve this problem since the citizens can vote for multiple candidates. Hence, the source of the two-position result under AV is a different one: the relatively sincere voting behavior over the set of serious contenders (lemma 2).

We are now in a position to characterize the set of 'serious political equilibria' (i.e. those involving no spoiler candidates). We know from the previous result that there are two classes of such equilibria - one-position and two-position equilibria. The following proposition characterizes the set of one-position equilibria. Note that in any one-position equilibrium, there is only one candidate. Indeed, a second candidate running on the same platform would be better off exiting the race since the policy outcome would be unchanged, while he would save on the entry cost δ.

Proposition 4 *There exists a political equilibrium where citizen i runs unopposed if and only if*
(i) citizen i prefers to run against the default outcome, i.e. $-v_0^i \geq \delta$; and
(ii) there does not exist another citizen h such that: (a) $v_h^m > v_i^m$ and $-v_i^h \geq \delta$; or (b) $v_h^m = v_i^m$ and $-v_i^h \geq 2\ \delta$.

The first condition requires that for candidate i, the utility gain from having his most-preferred policy instead of the default outcome exceeds the cost of running. The second condition implies that no other candidate could at least tie with candidate i (which would happen if he is (weakly) preferred to i by the median voter), and would want to enter (which would occur if the utility gain from having his most-preferred policy instead of i's exceeds the entry cost).

We now characterize a two-position, serious political equilibrium.

in their policy positions.

[15] A similar explanation has been forwarded for Al Gore's defeat in the 2000 US Presidential election, where Nader's candidacy split the leftist votes.

Proposition 5 *Consider a two-position, serious political equilibrium with* $|\mathcal{C}| = c \geq 2$*. Then the following conditions must be satisfied:*

(i) *there exist citizens whose ideal policies are* w_L *and* w_R*, with* $0 \leq w_L < m < w_R \leq 1$ *and* $v_L^m = v_R^m$*;*

(ii) *let* c_R *denote the number of candidates running at position* w_R*, then for every candidate* i *running at* w_L*,* $-\frac{c_R}{c\ (c-1)} v_R^i \geq \delta$*, and a similar condition holds for candidates at* w_R*;*

(iii) *for any* $i \notin \mathcal{C}$ *with ideal policy* w_L*,* $-\frac{c_R}{c\ (c+1)} v_R^i < \delta$*, and a similar condition holds for any* $i \notin \mathcal{C}$ *with ideal policy* w_R*; and*

(iv) *there does not exist a citizen* k *with* $w_k \in (w_L, w_R)$ *such that* $-\frac{1}{c} \sum\limits_{i \in \{L,R\}} c_i\ v_i^k \geq \delta$ *and* $\left|\left\{\ell \in \mathcal{N} : k \in G^\ell\left(\mathcal{C} \cup \{k\}\right)\right\}\right| \geq \frac{|\mathcal{N} \backslash \mathcal{M}|}{2}$*.*

Moreover, the above conditions are sufficient if for any citizen k *with* $w_k \in (w_L, w_R)$*,*

$$\left|\left\{\ell \in \mathcal{N} : k \in G^\ell\left(\mathcal{C} \cup \{k\}\right)\right\}\right| < \frac{|\mathcal{N} \backslash \mathcal{M}|}{2} - 1.$$

We know from proposition 3 that the two positions must be on either side of the median. Hence condition (i). It must also be true that the candidates who are running are worse off quitting (condition (ii)), and those who are not running will either not get sufficient votes or the increment in their payoff does not justify the cost of running (conditions (iii) and (iv)).

4.2. A Comparison between AV and the Plurality Rule. We now investigate the difference between the sets of serious political equilibria under AV and the plurality rule. Since our main focus is on which policy is implemented, we will consider a political equilibrium under the plurality rule equivalent to one under AV if they both generate the same set of policy outcomes.

Before proceeding, we need to characterize equilibria under the plurality rule. By extending Besley and Coate (1997) to allow for more than one citizen sharing an ideal position, it is easy to verify that political equilibria under the plurality rule are such that: (a) the necessary and sufficient conditions for the existence of one-candidate political equilibria under the plurality rule are identical to those in proposition 4, and thus that the set of one-position equilibria under AV is equivalent to the set of one-candidate equilibria under the plurality rule; (b) the first two conditions of proposition 5 are analogous to the necessary and sufficient conditions for a two-candidate political equilibrium under the plurality rule. Also the other conditions of proposition 5 are not necessary under the plurality rule. Indeed, there are two important ways in which the two-position equilibria under AV differ from those under the plurality rule. First, as noted above, AV reduces the barriers to entry by new candidates. While under AV several candidates running at the same position *help* each other by increasing the probability that one of them is appointed as the policy-maker, under the plurality rule they *hurt* one another by splitting the votes. Hence, condition (iii) of proposition 5 is necessary under AV, but not under the plurality rule. Second, under the plurality rule, candidates face no risk of entry from other citizens, even if these potential candidates are more centrist. Citizens may not want to vote for the centrist candidate because that will reduce the vote share of their preferred candidate, leading to a possible victory of their least-preferred candidate. Such a problem does not arise under AV and hence, a centrist candidate may find it worthwhile to enter (and win) if condition (iv) in the above proposition is violated; (c) there does not exist a three-candidate political equilibrium under the plurality rule; and (d) in political equilibria with four or more candidates, only one or two are winning.

From those features, we can conclude that the sets of outcomes generated by the one-candidate equilibria are identical under the two systems. Moreover, the set of outcomes from the two-position, serious equilibria under AV is a subset of those obtained under the plurality rule. The following proposition goes further in characterizing the conditions under which the equilibrium set of policy outcomes under AV can be more moderate than the one arising under the plurality rule.

Proposition 6 *Consider the class of serious political equilibria. The set of outcomes from the political equilibria under AV is a subset of the set of outcomes under the plurality rule. In addition, when preferences are symmetric, the former set is more moderate than the latter in the sense that if the lottery that implements w_L and w_R is an equilibrium outcome under AV, then for every equilibrium under the plurality rule that implements $\widetilde{w}_L$ and $\widetilde{w}_R$, with $[\widetilde{w}_L, \widetilde{w}_R] \subset [w_L, w_R]$, there exists a political equilibrium under AV with the same outcomes.*

To appreciate the above proposition, consider the following example.

Example 1. Suppose there are 36 voters whose ideal points are distributed on the set $\{0, 1, ..., 9, 10\}$. Let there be 6 voters with ideal point 5 and $6 - k$ voters with ideal point $5 \pm k$. Voters have Euclidean preferences, denoted by $v_j^\ell = -\|w_\ell - w_j\|$.[16] Suppose that the cost of running for office is $\delta = 0.8$. We denote by $\{w_j, w_k\}$ an equilibrium under which candidates with ideal points w_j and w_k run against each other.

It is easy to see that $\{0, 10\}$, $\{1, 9\}$, $\{2, 8\}$, $\{3, 7\}$, and $\{4, 6\}$ are all two-candidate equilibria under the plurality rule. These equilibria are sustained by the anticipation on the part of the potential moderate candidates that they will not receive any votes. These expectations are rational since voters do not want to waste their vote on the moderate candidate only to lead the opposite extremist to win. For instance, consider the scenario in which 0 and 10 are running against each other. Suppose that a moderate candidate, say at 5, is considering whether to enter the race. A voter on the left of 5 will not want to vote for the candidate at 5 because that will lead to the candidate at 10 winning outright!

Under AV, potential entrants do not face this problem. Suppose that 0 and 10 are running against each other and 5 entered. For everybody with ideal points at 3, 4, 5, 6 and 7, voting for 5 is always a part of *any* weakly dominant strategy. This means that 5 will receive at least 24 votes while 0 and 10 will receive at most 15 votes. It follows that the equilibrium $\{0, 10\}$ is destroyed by the credible threat of entry by a candidate at 5! By a similar logic, one can show that only $\{4, 6\}$ survives as a two-position equilibrium under AV. $\square$

Proposition 6 and the above example provide underpinnings for the argument made by the advocates of AV that giving citizens the choice of casting more than one vote improves the electoral prospects of the centrist candidates. However, there are two qualifications to be kept in mind regarding proposition 6. First, the extent of moderation under AV need not be very substantial, and second, if preferences are asymmetric, then AV may not produce moderation. Our next two examples shed light on these qualifications.

Example 2. A community has to select a representative who, once elected, will have to choose

[16] We denote Euclidean preferences by $\|.\|$ in order to avoid any confusion with the cardinality sign $|.|$.

the share of a fixed budget allocated to a public good. Let the set of policy alternatives be $\left\{0, \frac{1}{100}, \frac{2}{100}, ..., 1\right\}$, and let citizens' ideal policies be uniformly distributed over this set, with five citizens of each type.[17] The median voter's ideal policy is $\frac{50}{100}$. Citizens have Euclidean preferences over the public good, $u^{\ell}(g) = -\|w_{\ell} - g\|$, and bear a cost $\delta = \frac{1}{20}$ if they decide to run for election.

First note that any pair of citizens, with $w_i + w_j = 1$ and $\|w_i - w_j\| \geq \frac{1}{10}$, running against each other is a political equilibrium under the plurality rule.

Now consider the citizens with ideal points at $\frac{3}{100}$ and $\frac{97}{100}$. It is easy to see that the first three necessary conditions of proposition 5 are satisfied when five citizens of each type are running for election. Also, for any citizen whose ideal point lies in-between, the set of citizens who would prefer her to the other candidates is at most $\frac{5\ (97-3)}{2} = 235$. In the same time, $\frac{|\mathcal{N} \backslash \mathcal{M}|}{2} - 1 = 249$. Hence, both the fourth and the sufficiency conditions are satisfied as well. We can then conclude that the lottery where $\frac{3}{100}$ and $\frac{97}{100}$ are each adopted with equal probabilities is an equilibrium policy outcome under AV. Furthermore, $\left\{\frac{4}{100}, \frac{96}{100}\right\}$, $\left\{\frac{5}{100}, \frac{95}{100}\right\}$, ..., and $\left\{\frac{45}{100}, \frac{55}{100}\right\}$ are equilibria as well. □

Example 3. Suppose preferences are now represented by the following utility function

$$\begin{aligned} u^{\ell}(w_i) &= -\frac{1}{2}\ \|w_{\ell} - w_i\| \quad \text{if } w_{\ell} \geq w_i, \\ u^{\ell}(w_i) &= -2\ \|w_{\ell} - w_i\| \quad \text{if } w_i \geq w_{\ell}. \end{aligned}$$

Then, under the plurality rule, the equilibrium lotteries over the policies are $\left\{\frac{2}{100}, \frac{62}{100}\right\}$, $\left\{\frac{6}{100}, \frac{61}{100}\right\}$, ..., and $\left\{\frac{34}{100}, \frac{54}{100}\right\}$, where in all of them, each policy is adopted with an equal probability.

Consider the equilibrium where, under the plurality rule, $\frac{34}{100}$ and $\frac{54}{100}$ are adopted with equal probabilities. It turns out that this lottery cannot be supported as a political equilibrium outcome under AV. The intuition is as follows. Suppose there is one citizen running at $\frac{34}{100}$ and another one at $\frac{54}{100}$.[18] Then, a second candidate will want to stand on the leftist platform (since his expected

[17] Note that the argument does not depend on the restriction that there are five citizens of each type. The rationale for this assumption will become clear in the next example.

[18] Two or more candidates at each position would violate condition (ii) of proposition 5.

benefit of running, net of the entry cost, is equal to $\frac{1}{60}$). In the same time, no other citizen at the rightist position will want to enter. This asymmetry comes from the difference in preferences. As a result, $\frac{34}{100}$ will be implemented with probability two thirds and $\frac{54}{100}$ with probability one third. This means that the benefit of running for the rightist candidate will now be $\frac{1}{30}$, less than the entry cost. Anticipating this, the rightist candidate will then choose not to enter in the first place. Thus, $\left\{\frac{34}{100}, \frac{54}{100}\right\}$ cannot be supported by a political equilibrium under AV. The same is true for $\left\{\frac{30}{100}, \frac{55}{100}\right\}$ and $\left\{\frac{22}{100}, \frac{57}{100}\right\}$.

Moreover, in the political equilibria where the other pairs are implemented, the leftist policy (and thus the most extreme one) is adopted with a higher probability than the rightist one. For example, $\left\{\frac{2}{100}, \frac{62}{100}\right\}$ is supported by a political equilibrium under AV as well, but with the set of candidates and the winning set both consisting of four candidates with $\frac{2}{100}$ as an ideal policy and one candidate with $\frac{62}{100}$. $\square$

The above example shows that when citizens differ in the intensity of their preferences over the policy outcomes, AV may cause a larger entry at some position. This, in turn, may reduce the incentive for candidates in other positions to enter the race. The net effect could be extremism rather than moderation.

To sum up, AV lowers the entry costs for new candidates. However, the location at which the new candidates may enter the race depends upon the distribution and intensity of citizens' preferences. In some cases, there is an incentive for new candidate entry at centrist positions. In other cases, the entry may be concentrated in extremist positions.

4.3. Refining the Voting Behavior. In section 3, we developed an intuitively plausible notion of voting behavior, viz. relative sincerity. Our analysis showed that relatively sincere strategies are compatible with rational behavior in the sense that voters' best-response sets always contain such strategies (see proposition 2). Our next result goes further to prove the existence of equilibria in relatively sincere strategies in the case of the one-dimensional model.

Lemma 3 *If the policy space is one-dimensional and the preferences are single-peaked, then for any non-empty set of candidates, there exists a voting equilibrium in relatively sincere strategies.*

This sub-section explores the implications of relatively sincere voting behavior on the degree of moderation attained under AV. Our next proposition shows that if voters play relatively sincere voting strategies, then the equilibrium outcome of any serious political equilibrium must be 'close' to the median voter's ideal policy.

Proposition 7 *Suppose that the voting behavior satisfies relative sincerity and $-v_0^m \geq \delta$. In any serious political equilibrium, the median voter does not want to run against any candidate i, i.e. $-v_i^m < \delta$.*

To understand the logic behind the above result, note that there are two types of serious equilibria - those with one winning position and those with two winning positions. It follows immediately from proposition 4 that the outcome under the first case cannot be 'too far' from the median. The second case is that of two winning positions. We know from proposition 3, that the two winning positions must be on either side of the median voter's ideal position. However, by concavity of preferences, *at least* a majority of voters derive more utility from the median position than the expected utility they obtain from the outcome of the election. This means that if citizens vote relatively sincerely, a candidate entering at the median position will win outright. Hence, the only two-position equilibria that survive are those which generate outcomes sufficiently close to the median voter's ideal policy such that the median voter does not find it worthwhile to enter.

To summarize, our analysis of the one-dimensional model shows that AV can lead to moderate outcomes if we look at equilibria without spoiler candidates. Moreover, the extent of moderation is substantial if we consider equilibria in relatively sincere strategies.

4.4. The Case of Spoiler Candidates. We have primarily focused our attention on the case of serious candidates. There is another class of equilibria - those involving spoiler candidates. Under such equilibria, there are candidates running in the race even though they do not stand a chance of winning. The presence of spoiler candidates is not without a consequence. Indeed, while proposition 7 puts a restriction on how far from the median the candidate positions can be, in presence of spoiler candidates such a restriction does not apply, as our next example illustrates.

Example 4. Consider the preferences and population described in example 2. $\mathcal{C} = \left\{0, \frac{30}{100}, \frac{70}{100}, 1\right\}$ and $W(\mathcal{C}, \alpha(.)) = \left\{\frac{30}{100}, \frac{70}{100}\right\}$ can be sustained as a four-candidate political equilibrium in the above example. For instance, 0 believes that his exit will cause everyone to the left of $\frac{70}{100}$ to vote for $\frac{70}{100}$. This will lead to $\frac{70}{100}$ winning outright. Similarly, 1 believes that his exit will lead the candidate at $\frac{30}{100}$ to win outright. Moreover, every potential entrant believes that his entry would trigger a voting behavior that would lead to either $\frac{30}{100}$ or $\frac{70}{100}$ getting elected outright. This could deter the potential entrants from running. Thus the above behavior constitutes an equilibrium. □

These equilibria, though possible in theory, rely on self-fulfilling prophecies, which raises the question of their plausibility. Empirically, we find that in the elections held under AV, there have been instances of candidates running even when they did not stand a chance to win. But whether their presence favored the prospects of non-moderate candidates is an open question. It would be an interesting project to design and conduct voting experiments which could throw light on this question.

5 Some Results in the General Policy Space

In this section, we extend our analysis to the case of general preferences and policy spaces. One advantage of our approach vis-a-vis the Downsian one is our ability to obtain equilibrium predictions in such a general setting. We saw in the earlier section that the set of possible candidates running under AV can be a subset of those running under the plurality rule. As our next example shows there are outcomes which can arise as political equilibria under AV but cannot arise as outcomes under the plurality rule.

Example 5. A community must elect a local official who is in charge of undertaking a public investment. There are three possible public investment projects that we characterize by their scale (although they may differ along other dimensions as well) - small, medium and large. If nobody runs for office, no investment is undertaken and citizens get payoff 0. The community is divided into three types of citizens whose (reduced-form) preferences over the scale of the project are presented below.

Investment Project

	small	medium	large
Group 1	12	7	0
Group 2	7	12	0
Group 3	7	7	12

Let n_i denote the number of type-i citizens and assume $n_1 = n_2 = \frac{n_3}{2}$. Also, let the cost of running for office be $\delta = 2$.

Note that a type-1 citizen running against a type-2 cannot be an equilibrium configuration under the plurality rule since a type-3 citizen will find it beneficial to stand as a candidate. Indeed, if one type-3 citizen also runs, he will get the votes of his fellows (since they will not abstain given that $v_3^3 > v_1^3 = v_2^3$, when in the same time weak undominance will require $\alpha_1^3 = \alpha_2^3 = 0$), while the best that type-1 and type-2 citizens can do is all to vote for one of the other two candidates. So the type-3 candidate will either win outright or tie with one of the other candidates. He will then get an expected utility of at least $\frac{1}{2}(7+12) - 2 = \frac{15}{2}$ (which occurs in the event of a tie) compared to the 7 he would get by not entering. Hence he is indeed better off standing for election.

However, there exists a voting strategy profile under AV that sustains a type-1 running against a type-2 as an equilibrium. If the set of candidates consists of citizens of types 1 and 2, the only weakly undominated (and equilibrium) voting strategy is the type-1 and type-2 citizens voting for their candidate and the type-3 citizens abstaining.

Suppose now that a citizen of type 3 enters the race and let $\alpha^h(\{1,2,3\}) = \{1,2\}$ for $h \in \{1,2\}$ and $\alpha^3(\{1,2,3\}) = \{3\}$. This candidate then gets the votes of all his fellows while the other citizens cast a vote for each of the other two. All three contenders thus tie. The type-3 candidate would then get an expected utility of $\frac{20}{3}$, less than the 7 he gets by not running. This means that he has no incentive to stand for election. Now we need to check that this voting function is indeed an equilibrium one, i.e. no citizen would gain by changing her vote. For a type-3 citizen, this is the unique admissible voting strategy. A type-1 citizen would get a lower expected utility if she decides not to vote for the type-2 candidate ($\frac{12+0}{2} = 6$ *vs* $\frac{12+7+0}{3} = \frac{19}{3}$) and *a fortiori* for her fellow candidate. Hence, she has no incentive to deviate. The same is true for the type-2 citizens.

It remains to show that neither of the two candidates wants to exit and that no other citizen wants to enter the electoral race. If the type-1 candidate decides to step out, he will get a utility equal to 7 (since the type-2 candidate is then going to win outright), lower than the $\frac{15}{2}$ he was getting. Hence, he has no incentive to exit the race. The same is true for the type-2 candidate.

If a citizen of type $h \in \{1, 2\}$ decides to enter, he will tie with the other two candidates. His expected utility will then be equal to $\frac{25}{3}$, less than the $\frac{19}{2}$ he was getting. So he will not enter. And we have shown above that a type-3 citizen will not want to stand for election.

Hence, there exists under AV, but not under the plurality rule, a political equilibrium where one citizen of type 1 and one of type 2 are running against each other. Moreover, this equilibrium satisfies relative sincerity. □

From proposition 6 and example 5 we can conclude that

Proposition 8 *In general, there is no equivalence between the sets of $c\,(\geq 2)$-candidate political equilibria under AV and the plurality rule.*

This proposition highlights the fact that different electoral rules create different incentives for candidates to enter the political race. There have been several attempts to compare electoral rules by 'reconstructing' the outcomes of an election under various voting rules.[19] The above proposition warns us that such attempts may be misleading since electoral rules make a difference at the candidate entry stage as well as at the voting stage.

[19] See for example Brams and Merrill (1994).

Endogenizing candidate entry also calls for a re-examination of the desirability properties of voting systems which are based on an analysis over an exogenous set of alternatives. There are two Condorcet criteria which have received attention in the literature. A voting system is said to satisfy the Condorcet Winner Criterion if it has a voting equilibrium that elects a Condorcet winner (if one exists), i.e. an alternative that beats any other alternative in a pairwise majority vote. Similarly, a voting system is said to satisfy the Condorcet Loser Criterion if it does not have a voting equilibrium that elects a Condorcet loser (if one exists), i.e. the alternative that is defeated by every other alternative in a pairwise majority vote.

One of the criticisms of the plurality rule is that it fails to satisfy the Condorcet criteria. Our next example shows that AV is susceptible to the very same problem, and may even elect the Condorcet loser while a Condorcet winner exists.

Example 6. Consider again an economy which has to decide on a public investment. There are three projects - small, medium and large. There are four preference profiles in the community. Utility levels are

	Project size		
	small	medium	large
Type 1	10	6	2
Type 2	2	10	2
Type 3	2	6	10
Type 4	10	0	0

Suppose that the distribution of citizens between these four groups is such that $n_1 + n_4 > n_2 + n_3 > n_4 + 1$ and $n_3 > n_1 + n_2$, with at least one citizen of each type. Also let the entry cost $\delta \in (1, 4)$.

It is easy to see that the small project is the Condorcet winner while the medium one is the Condorcet loser.

Now consider the situation where one citizen of each of the first three types are running against each other, i.e. $\mathcal{C} = \{1, 2, 3\}$. If elected the type-1 candidate will implement the small project, the type-2 candidate the medium one and the type-3 the large one. Hence, the Condorcet winner is the type-1 candidate and the Condorcet loser the one of type 2. Now, there exists an equilibrium under AV where the type-2 candidate is the only serious contender, the other two being spoiler candidates. This equilibrium is supported by the following voting behavior: (i) type-1 citizens vote for both the type-1 and type-2 candidates; (ii) type-2 citizens vote for their fellow candidate; (iii) type-3 citizens vote for both the type-2 and type-3 candidates; and (iv) type-4 citizens vote for the type-1 candidate. □

There are several things to note about this example. First, it is also an equilibrium under the plurality rule, where the type-2 candidate receives the votes of all citizens except the ones of type-4 citizens who cast their ballot for the type-1 candidate. Second, this equilibrium also satisfies relative sincerity. Hence, even imposing the sincerity refinement on the voting behavior does not guarantee that AV satisfies the Condorcet criteria. Finally, note that from a Rawlsian point of view, the medium size project is the least-desirable outcome while the small project is the most-desirable one. The same is true from a utilitarian point of view if $2\ n_1 + 5\ n_4 > 4\ n_2 + 2\ n_3$ and $n_3 > n_1 + 2\ n_2$.[20]

6 Conclusion

In this paper we developed a model of political competition that enabled us to study the electoral process under AV and to compare it with the plurality rule. While the existing studies of alternative systems have focused on a fixed set of alternatives, we adopted the citizen-candidate framework to endogenize candidate entry.

We first examined the notion that AV encourages sincere voting behavior. To this end, we developed a refinement of voting behavior - relative sincerity - which is consistent with the intuitive

[20] For instance, $n_1 = n_4 = 6$, $n_2 = 1$ and $n_3 = 9$ is one such distribution.

notion of sincere voting behavior under AV. We showed that relatively sincere voting behavior is consistent with the rational calculus of voting.

We then developed a one-dimensional model of political competition with distance preferences. This set up enabled us to examine the claim that AV leads to more centrist policies as compared to the plurality rule. We found that the outcomes under AV are more moderate than those that may arise under the plurality rule if we focus our attention on 'serious' equilibria in relatively sincere strategies. Hence our analysis found precise conditions, viz. no spoiler candidates and relatively sincere voting behavior, under which AV leads to more centrist outcomes as compared to the plurality rule. However, we also find that there need be little policy moderation if these two conditions did not hold. Hence, we are cautious in our support of AV over the plurality rule until we have empirical evidence on how prevalent these two conditions are. We also showed that when the policy space is one-dimensional, there may be a variety of candidates running for office, but all the serious contenders are clustered at no more than two positions! This result, although contrary to the popular intuition that AV may lead to a large number and variety of candidates, seems to be confirmed by casual empiricism.

We also were able to highlight the methodological contribution of the citizen-candidate approach to comparing electoral rules. We showed that in general there is no equivalence between the set of candidates running for election under AV and the plurality rule. Moreover, various properties of voting rules which are based on an analysis over a fixed set of alternatives may not hold when we allow for endogenous entry.

There are two possible extensions of this work. On the empirical side, we would like to examine whether AV satisfies the conditions that we showed are sufficient for policy moderation. On the theoretical side, we would like to expand the citizen-candidate framework to studying other voting rules such as the Borda Rule, negative voting and so on.

References

Besley, T. and S. Coate, 1997. An economic model of representative democracy. *Quarterly Journal of Economics* 112, 85-114.

Brams, S., 1980. Approval voting in multicandidate elections. *Policy Studies Journal* 9, 102-108.

Brams, S. and P. Fishburn, 1978. Approval voting. *American Political Science Review* 72, 831-847.

Brams, S. and P. Fishburn, 2003. Going from Theory to Practice: The Mixed Success of Approval Voting. *Social Choice and Welfare*, forthcoming.

Brams, S. and S. Merrill, 1994. Would Ross Perot have won the 1992 presidential election under Approval voting?. *PS: Political Science and Politics* 27, 39-44.

Brams, S. and J. Nagel, 1991. Approval voting in practice. *Public Choice* 71, 1-17.

Cox, G., 1984. Strategic electoral choice in multi-member districts: Approval voting in practice?. *American Journal of Political Science* 28, 722-738.

Cox, G., 1985. Electoral equilibrium under Approval voting. *American Journal of Political Science* 29, 112-118.

Cox, G., 1987. Electoral equilibrium under alternative voting institutions. *American Journal of Political Science* 31, 82-108.

De Sinopoli, F., 1999. Two examples of strategic equilibria in Approval voting games. *CORE Discussion Paper* DP9931.

Dutta, B., M. Jackson and M. Le Breton, 2001. Strategic candidacy and voting procedures. *Econometrica* 69(4), 1-22.

Duverger, M., 1954. *Political Parties*. Wiley, New York.

Fishburn, P. and S. Brams, 1981a. Approval voting, Condorcet's principle, and runoff elections. *Public Choice* 36, 89-114.

Fishburn, P. and S. Brams, 1981b. Efficacy, power and equity under Approval voting. *Public Choice* 37, 425-434.

Fishburn, P. and S. Brams, 1981c. Expected utility and Approval voting. *Behavioral Science* 26, 136-142.

Fishburn, P. and S. Brams, 1988. Does Approval voting elect the lowest common denominator?. *Political Science and Politics* 21, 277-284.

Levitt, S., 1996. How do senators vote? Disentangling the role of voter preferences, party affiliation, and senator ideology. *American Economic Review* 86, 425-441.

Myerson, R., 1993a. Effectiveness of electoral systems for reducing government corruption: a game-theoretic analysis. *Games and Economic Behavior* 5, 118-132.

Myerson, R., 1993b. Incentives to cultivate favored minorities under alternative electoral systems. *American Political Science Review* 87, 856-869.

Myerson, R., 2002. *Bipolar Multicandidate Elections With Corruption.* mimeo.

Myerson, R. and R. Weber, 1993. A theory of voting equilibria. *American Political Science Review* 87, 102-114.

Osborne, M. and A. Slivinski, 1996. A model of political competition with citizen-candidates. *Quarterly Journal of Economics* 111, 65-96.

Saari, D., 2001. Analyzing a nail-biting election. *Social Choice and Welfare* 18, 415-430.

Tullock, G., 1979. Comment on Brams and Fishburn and Balinski and Young. *American Political Science Review* 73, 551-552.

7 Appendix

Proof of Lemma 1. (Sufficiency) Let α^ℓ be a weakly dominated strategy. Then we have to show that there exists another strategy $\widehat{\alpha}^\ell$ such that for all $\alpha^{-\ell}$, $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) \geq U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with a strict inequality for some $\alpha^{-\ell}$. But before proceeding, let $R\left(\mathcal{C};\alpha\right) \equiv \{i \in \mathcal{C} : \left(\sum_{\ell\in\mathcal{N}} \alpha_i^\ell\right) + 1 = \sum_{\ell\in\mathcal{N}} \alpha_k^\ell$ for all $k \in W\left(\mathcal{C};\alpha\right)\}$ be the set of candidates who are one vote short to tie for election. There are two cases to consider:

Case 1: $\alpha_i^\ell = 0$ for some $i \in G^\ell\left(\mathcal{C}\right)$. Pick $\widehat{\alpha}^\ell$ such that $\widehat{\alpha}_i^\ell = 1$ and $\widehat{\alpha}_k^\ell = \alpha_k^\ell$ for all $k \neq i$. First, if $W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$, then $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$. Second, if $i \in W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ and $\left|W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)\right| \geq 2$, then $W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = \{i\}$ and $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) \geq U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with a strict inequality when $\alpha^{-\ell}$ is such that there exists $k \in W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with $k \notin G^\ell\left(\mathcal{C}\right)$. Finally, if $i \in R\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$, then $W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right) \cup \{i\}$ and $p_k\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) < p_k\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ for all $k \in W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$. Then, $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) \geq U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with a strict inequality when $\alpha^{-\ell}$ is such that there exists $k \in W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with $k \notin G^\ell\left(\mathcal{C}\right)$.

Case 2: $\alpha_i^\ell = 1$ for some $i \in L^\ell\left(\mathcal{C}\right)$. Pick $\widehat{\alpha}^\ell$ such that $\widehat{\alpha}_i^\ell = 0$ and $\widehat{\alpha}_k^\ell = \alpha_k^\ell$ for all $k \neq i$. First, if $W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$, then $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$. Second, if $i \in W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ and $\left|W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)\right| \geq 2$, then $i \notin W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right)$ and $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) \geq U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with a strict inequality when $\alpha^{-\ell}$ is such that there exists $k \in W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right)$ with $k \notin L^\ell\left(\mathcal{C}\right)$. Finally, if $W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right) = \{i\}$ and $R\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right) \neq \emptyset$, then $W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) = W\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right) \cup R\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$, and $p_k\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) > p_k\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ for all $k \in R\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ and $p_i\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) < p_i\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$. Then, $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) \geq U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$ with a strict inequality when $\alpha^{-\ell}$ is such that there exists $k \in W\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right)$ with $k \notin L^\ell\left(\mathcal{C}\right)$.

(Necessity) Let $\widehat{\alpha}^\ell$ be a weakly undominated strategy. For any arbitrary strategy $\alpha^\ell \left(\neq \widehat{\alpha}^\ell\right)$, it suffices that there exists such a strategy profile $\alpha^{-\ell}$ for $-\ell$ such that $U^\ell\left(\mathcal{C};\widehat{\alpha}^\ell,\alpha^{-\ell}\right) > U^\ell\left(\mathcal{C};\alpha^\ell,\alpha^{-\ell}\right)$. There are two cases to consider:

Case 1: $\widehat{\alpha}_i^\ell = 1$ and $\alpha_i^\ell = 0$ for some $i \in \mathcal{C}$. Since $\widehat{\alpha}^\ell$ is weakly undominated, it must be that

$i \notin L^{\ell}(\mathcal{C})$. Pick any $k \in L^{\ell}(\mathcal{C})$. We know that $\widehat{\alpha}_k^{\ell} = 0$. Choose $\alpha^{-\ell}$ such that $\sum_{-\ell} \alpha_i^{-\ell} = \sum_{-\ell} \alpha_k^{-\ell}$ and $\sum_{-\ell} \alpha_j^{-\ell} < \left(\sum_{-\ell} \alpha_i^{-\ell} - 1\right)$ for all $j \neq i, k$. Under $\left(\alpha^{\ell}, \alpha^{-\ell}\right)$, there is either a tie between i and k or k wins outright. On the other hand, under $\left(\widehat{\alpha}^{\ell}, \alpha^{-\ell}\right)$, candidate i is the outright winner.

Case 2: $\widehat{\alpha}_i^{\ell} = 0$ and $\alpha_i^{\ell} = 1$ for some $i \in \mathcal{C}$. Since $\widehat{\alpha}^{\ell}$ is weakly undominated, it must be that $i \notin G^{\ell}(\mathcal{C})$. Pick any $k \in G^{\ell}(\mathcal{C})$. We know that $\widehat{\alpha}_k^{\ell} = 1$. Choose $\alpha^{-\ell}$ such that $\sum_{-\ell} \alpha_i^{-\ell} = \sum_{-\ell} \alpha_k^{-\ell}$ and $\sum_{-\ell} \alpha_j^{-\ell} < \left(\sum_{-\ell} \alpha_i^{-\ell} - 1\right)$ for all $j \neq i, k$. Under $\left(\alpha^{\ell}, \alpha^{-\ell}\right)$, there is either a tie between i and k or i wins outright. On the other hand, under $\left(\widehat{\alpha}^{\ell}, \alpha^{-\ell}\right)$, candidate k is the outright winner. **Q.E.D.**

Proof of Lemma 2. In order to simplify notation, denote the winning set $W(\mathcal{C}, \alpha(.))$ by W. Also, let

$$H^{\ell}(W) \equiv \left\{ i \in W : v_i^{\ell} \geq \frac{1}{|W| - 1} \sum_{j \in W \setminus \{i\}} v_j^{\ell} \right\}$$

and

$$E^{\ell}(W) \equiv \left\{ i \in W : v_i^{\ell} > \frac{1}{|W| - 1} \sum_{j \in W \setminus \{i\}} v_j^{\ell} \right\}$$

be the sets of candidates citizen ℓ (strictly) prefers to the lottery over all other winning candidates.

First, let $\alpha_i^{\ell} = 1$. We have to show that $i \in H^{\ell}(W)$. Suppose not. Then, $v_i^{\ell} < \frac{1}{|W|-1} \sum_{j \in W \setminus \{i\}} v_j^{\ell}$. Now, construct $\widehat{\alpha}^{\ell}$ such that $\widehat{\alpha}_i^{\ell} = 0$ and $\widehat{\alpha}_j^{\ell} = \alpha_j^{\ell}$ for all $j \neq i$. Since $i \in W$ and $|W| \geq 2$, $\widehat{W} = W \setminus \{i\}$, where $\widehat{W} \equiv W\left(\mathcal{C}; \widehat{\alpha}^{\ell}, \alpha^{-\ell}\right)$. As a result, $U^{\ell}\left(\mathcal{C}; \widehat{\alpha}^{\ell}, \alpha^{-\ell}\right) > U^{\ell}\left(\mathcal{C}; \alpha^{\ell}, \alpha^{-\ell}\right)$. Hence, $\alpha^{\ell} \notin BR^{\ell}(\mathcal{C}, \alpha)$, a contradiction.

Second, suppose that $i \in E^{\ell}(W)$ and $\alpha_i^{\ell} = 0$. Construct $\widehat{\alpha}^{\ell}$ such that $\widehat{\alpha}_i^{\ell} = 1$ and $\widehat{\alpha}_j^{\ell} = \alpha_j^{\ell}$ for all $j \neq i$. Since $i \in W$, we have that $\widehat{W} = \{i\}$. Moreover, since $i \in E^{\ell}(W)$, $U^{\ell}\left(\mathcal{C}; \widehat{\alpha}^{\ell}, \alpha^{-\ell}\right) = v_i^{\ell} > U^{\ell}\left(\mathcal{C}; \alpha^{\ell}, \alpha^{-\ell}\right)$. Hence, $\alpha^{\ell} \notin BR^{\ell}(\mathcal{C}, \alpha)$, a contradiction. **Q.E.D.**

Proof of Proposition 2. Given a voting profile α, let $R(\mathcal{C}, \alpha)$ denote the set of candidates in $\mathcal{C}$ who receive exactly one vote less than those in the set $W(\mathcal{C}, \alpha)$. For $\ell \in \mathcal{N}$, suppose that $\alpha^{\ell} \in BR^{\ell}(\mathcal{C}, \alpha)$ and there exists a candidate j such that $v_j^{\ell} > U^{\ell}(\mathcal{C}, \alpha)$ and $\alpha_j^{\ell} = 0$.

Claim 1 $j \notin W(\mathcal{C},\alpha) \cup R(\mathcal{C},\alpha)$.

Proof of Claim 1. Define $\widetilde{\alpha}^{\ell}$ such that $\widetilde{\alpha}^{\ell}_j = 1$ and $\widetilde{\alpha}^{\ell}_k = \alpha^{\ell}_k$ for all $k \neq j$. To verify this claim consider the two possibilities:

(i) Suppose $j \in W(\mathcal{C},\alpha)$. Then $W\left(\mathcal{C},\widetilde{\alpha}^{\ell},\alpha^{-\ell}\right) = \{j\}$ and hence, $v^{\ell}_j \equiv U^{\ell}\left(\mathcal{C},\widetilde{\alpha}^{\ell},\alpha^{-\ell}\right) > U^{\ell}(\mathcal{C},\alpha)$. But then $\alpha^{\ell} \notin BR^{\ell}(\mathcal{C},\alpha)$.

(ii) Suppose $j \in R(\mathcal{C},\alpha)$. Then $W\left(\mathcal{C},\widetilde{\alpha}^{\ell},\alpha^{-\ell}\right) = W(\mathcal{C},\alpha) \cup \{j\}$ and hence, $U^{\ell}\left(\mathcal{C},\widetilde{\alpha}^{\ell},\alpha^{-\ell}\right) = \frac{|W|}{|W|+1}U^{\ell}(\mathcal{C},\alpha) + \frac{1}{|W|+1}v^{\ell}_j > U^{\ell}(\mathcal{C},\alpha)$. But then $\alpha^{\ell} \notin BR^{\ell}(\mathcal{C},\alpha)$.

Given that $j \notin W(\mathcal{C},\alpha) \cup R(\mathcal{C},\alpha)$, we have $W\left(\mathcal{C},\widetilde{\alpha}^{\ell},\alpha^{-\ell}\right) = W(\mathcal{C},\alpha)$ and hence, $\widetilde{\alpha}^{\ell} \in BR^{\ell}(\mathcal{C},\alpha)$. Hence, we can replace α^{ℓ} with $\widetilde{\alpha}^{\ell}$ such that $\widetilde{\alpha}^{\ell}_j = 1$ for all $j \in \mathcal{C}$ satisfying $v^{\ell}_j > U^{\ell}(\mathcal{C},\alpha)$ and $\widetilde{\alpha}^{\ell}_j = \alpha^{\ell}_j$ otherwise and have $\widetilde{\alpha}^{\ell} \in BR^{\ell}(\mathcal{C},\alpha)$. ■

Now suppose that $\alpha^{\ell} \in BR^{\ell}(\mathcal{C},\alpha)$ and there exists a candidate j such that $v^{\ell}_j < U^{\ell}(\mathcal{C},\alpha)$ and $\alpha^{\ell}_j = 1$.

Claim 2 $j \notin W(\mathcal{C},\alpha)$.

Proof of Claim 2. The proof is analogous to that of claim 1. Hence, we can replace $\widetilde{\alpha}^{\ell}_j = 0$ in place of $\alpha^{\ell}_j = 1$ for all $j \in \mathcal{C}$ such that $v^{\ell}_j < U^{\ell}(\mathcal{C},\alpha)$ and the new strategy $\widetilde{\alpha}^{\ell} \in BR^{\ell}(\mathcal{C},\alpha)$. ■

We have thus constructed a relatively sincere strategy which belongs to $BR^{\ell}(\mathcal{C},\alpha)$. **Q.E.D.**

Proof of Proposition 3. Consider a winning set W with at least 3 different candidates in it (i.e. h, j and k with $w_h \neq w_j \neq w_k$). Note that this means that for each voter i, $G^i(W) \neq L^i(W)$. Hence, he must vote for at least one candidate and *not* vote for at least one candidate. For any $t \in W$, let $F_t(D)$ denote the set of voters in the interval D who are voting for t. Let $A \equiv \frac{1}{|W|}\sum_{t \in W} t$ denote the average policy over the set W.

Claim 1 *For any elements y and z such that $A \geq z > y$, $F_y([0,1]) = F_z([0,1])$ (i.e. the same people must be voting for both y and z).*

Proof of Claim 1. Since $A \geq z$, by relative sincerity and concavity, $[0,z] \subset F_z([0,1])$ and therefore, $F_y([0,z]) \subseteq F_z([0,z])$. Also, if a citizen i with $w_i > z$ votes for y then he must vote for z as well. It follows that $F_y((z,1]) \subseteq F_z((z,1])$. Hence, $F_y([0,1]) \subseteq F_z([0,1])$. But since $|F_y([0,1])| = |F_z([0,1])|$ (since both y and z are in W), it must be that $F_y([0,1]) = F_z([0,1])$. ■

What the above claim states is that voters can be partitioned into two disjoint and exhaustive sets: those voting for *all* the candidates below A and those voting for *all* the candidates strictly above A.

Claim 2 *Let m be the median voter. Without loss of generality, suppose that m votes for $h > A$. Then, every citizen i* with $w_i > m$ *votes for h as well.*

Proof of Claim 2. Suppose that m votes for some $h > A$, while i does not. Then i must be voting for some k with $A \geq k$. But since $w_i > m$, we have $v_k^m > v_h^m$ (by property (ii) of the preferences). But since m is voting for h, he must vote for k as well (by sincerity). That means m votes for all the elements in W, a contradiction. ■

The above claim implies that if m votes for $h > A$, then all the candidates strictly above A get strictly more than half of the votes while those below A get strictly less than half. This means that they cannot be in W. And hence we have established that there cannot be more than two elements in W.

Hence, winners are at two points, say w_L and w_R. Without loss of generality, let $w_L < w_R$. Now, by lemma 2 and single-peaked, concave preferences, there exists $A \in (w_L, w_R)$ such that for all h and $j \in W$ with $w_h = w_L$ and $w_j = w_R$,

$$\begin{aligned}
\alpha_h^\ell &= 1 \text{ and } \alpha_j^\ell = 0 \text{ for all } \ell \text{ with } w_\ell < A, \\
\alpha_h^\ell &= 0 \text{ and } \alpha_j^\ell = 1 \text{ for all } \ell \text{ with } w_\ell > A, \text{ and} \\
\alpha_h^\ell &= \alpha_j^\ell \text{ for all } \ell \text{ with } w_\ell = A
\end{aligned}$$

, where $A = m$ and $v_h^m = v_j^m$. Indeed, if $m < A$ or $m > A$, then $|F_h([0,1])| \neq |F_j([0,1])|$, a contradiction. Also, if $v_h^m \neq v_j^m$, then $\alpha_h^m \neq \alpha_j^m$ by lemma 2 and we would get the same contradiction. **Q.E.D.**

Proof of Proposition 4. Trivial. **Q.E.D.**

Proof of Proposition 5. (Necessity) Condition (i) follows from proposition 3 and $W(\mathcal{C}, \alpha(.)) = \mathcal{C}$.

It has to be the case that no candidate i wants to exit the race. If he chooses to deviate, i.e. $\widehat{s}_i = 0$, then $\widehat{W}(\mathcal{C}\backslash\{i\}, \alpha(.)) = \mathcal{C}\backslash\{i\}$ and he will get an expected utility $\widehat{U}_i = \frac{c_{k(i)}}{c-1} v^i_{k(i)}$, where $k(i) = L$ if $w_i = w_R$ and R if $w_i = w_L$. Hence, candidate i does not want to deviate if

$$U_i \geq \widehat{U}_i \Leftrightarrow -v^i_{k(i)} \geq \frac{c(c-1)}{c_{k(i)}}\delta.$$

It must also be the case that no other citizen with $w_i \in \{w_L, w_R\}$ wants to enter the race. Indeed, suppose $\widehat{s}_i = 1$. Then $\widehat{W}(\mathcal{C} \cup \{i\}, \alpha(.)) = \mathcal{C} \cup \{i\}$ and citizen i gets an expected utility $\widehat{U}_i = \frac{c_{k(i)}}{c+1} v^i_{k(i)} - \delta$. Rather, if he does not enter, his expected utility is $U_i = \frac{c_{k(i)}}{c} v^i_{k(i)}$. Then, citizen i does not want to enter if

$$U_i \geq \widehat{U}_i \Leftrightarrow \frac{c(c+1)}{c_{k(i)}}\delta \geq -v^i_{k(i)}.$$

Finally, there cannot be a citizen with an ideal point different from w_L and w_R, who is guaranteed to win outright or tie and who wants to enter the race. Remember that weak undominance requires $\alpha^\ell_i = 1$ for all $\ell \in \mathcal{N}$ and $i \in \mathcal{C}$ such that $i \in G^\ell(\mathcal{C})$. Hence, min $|F_i([0,1])| = \left|\left\{\ell \in \mathcal{N} : i \in G^\ell(\mathcal{C})\right\}\right|$ for all $i \in \mathcal{C}$. Also from proposition 3, we have that $|F_i([0,1])| = \frac{|\mathcal{N}\backslash\mathcal{M}|}{2}$ for all $i \in \mathcal{C}$ with $w_i \in \{w_L, w_R\}$. First, note that $\left|\left\{\ell \in \mathcal{N} : i \in G^\ell(\mathcal{C})\right\}\right| < \frac{|\mathcal{N}\backslash\mathcal{M}|}{2}$ for all $i \in \mathcal{C}$ with $w_i < w_L$ since $\left\{\ell \in \mathcal{N} : i \in G^\ell(\mathcal{C})\right\} \subset \{\ell \in \mathcal{N} : w_L \geq w_\ell\}$ and $\{\ell \in \mathcal{N} : w_\ell = w_L\} \neq \emptyset$. In other words, any candidate with a position to the left of w_L cannot be the most-preferred candidate of citizens at and to the right of w_L. This implies that $\left|\left\{\ell \in \mathcal{N} : i \in G^\ell(\mathcal{C})\right\}\right| < \frac{|\mathcal{N}\backslash\mathcal{M}|}{2}$ for any such candidate. The same is true for all i such that $w_i > w_R$. However, it is not necessarily true for $i \in \mathcal{N}$ with $w_i \in (w_L, w_R)$. If such a citizen enters the race, then any candidate j at w_L or w_R will be the least-preferred one for the voters at the median. Hence $\alpha^m_j = 0$ and max $|F_j([0,1])| = \frac{|\mathcal{N}\backslash\mathcal{M}|}{2}$ for all j with $w_j \in \{w_L, w_R\}$. Now, if $\left|\left\{\ell \in \mathcal{N} : i \in G^\ell(\mathcal{C})\right\}\right| \geq \frac{|\mathcal{N}\backslash\mathcal{M}|}{2}$, then candidate i either wins outright, or at least tie with the other candidates. Thus it must be him who does not want to enter. This will necessarily be the case if

$$\frac{1}{c}\left(c_L v_L^i + c_R v_R^i\right) > v_i^i - \delta \Leftrightarrow -\frac{1}{c}\sum_{k\in\{L,R\}} c_k v_k^i < \delta.$$

(Sufficiency) It remains to show that these conditions are sufficient. Let $\alpha\left(\mathcal{C}\cup\{i\}\right)$ be a voting rule such that for all $\ell \in \mathcal{N}\backslash\mathcal{M}$,

$$\begin{aligned}
\alpha_j^\ell\left(\mathcal{C}\cup\{i\}\right) &= \alpha_j^\ell\left(\mathcal{C}\right) \text{ for all } j\in\mathcal{C}, \text{ and} \\
\alpha_i^\ell\left(\mathcal{C}\cup\{i\}\right) &= 1 \text{ if } i\in G^\ell\left(\mathcal{C}\cup\{i\}\right) \text{ and 0 otherwise.}
\end{aligned}$$

For all $\ell\in\mathcal{M}$, let $\alpha_j^\ell\left(\mathcal{C}\cup\{i\}\right)=1$ for all $j\in\mathcal{C}$ and $\alpha_i^\ell\left(\mathcal{C}\cup\{i\}\right)=0$ if $w_i\notin[w_L,w_R]$. Otherwise, let $\alpha_j^\ell\left(\mathcal{C}\cup\{i\}\right)=0$ for all $j\in\left(\mathcal{C}\cup\{i\}\right)$ with $w_j\in\{w_L,w_R\}$ and $\alpha_j^\ell\left(\mathcal{C}\cup\{i\}\right)=1$ when $w_j\in(w_L,w_R)$.

Note that weak undominance is satisfied and $\alpha(.)$ is a voting equilibrium. In addition, citizen i does not want to enter. **Q.E.D.**

Proof of Proposition 6. Note that the first two necessary conditions in proposition 5 are necessary and sufficient for a two-candidate political equilibrium under the plurality rule. Hence, the set of policy outcomes from political equilibria under AV where $W\left(\mathcal{C},\alpha(.)\right)=\mathcal{C}$ is a subset of the set of policy outcomes from the two-candidate political equilibria under the plurality rule.

Consider a lottery which implements w_L and w_R and satisfies the conditions of proposition 5. Now suppose that $\{\widetilde{w}_L,\widetilde{w}_R\}$ is the policy outcome of a two-candidate political equilibrium under the plurality rule. By proposition 5(i) (which is the same under both AV and the plurality rule), we have $\widetilde{v}_L^m=\tilde{v}_R^m$. Hence, condition (i) of proposition 5 holds.

The sufficient condition in proposition 5 is also satisfied. To see this, take any $k\in\mathcal{N}$ with $w_k\in(\widetilde{w}_L,\widetilde{w}_R)$. We will show that $\left\{\ell\in\mathcal{N}:k\in G^\ell\left(\{\widetilde{w}_L,w_k,\widetilde{w}_R\}\right)\right\}\subseteq\left\{\ell\in\mathcal{N}:k\in G^\ell\left(\{w_L,w_k,w_R\}\right)\right\}$. First, consider $\ell\in\mathcal{N}$ such that $k\in G^\ell\left(\{\widetilde{w}_L,w_k,\widetilde{w}_R\}\right)$. This implies that $v_k^\ell\geq\max\left\{\widetilde{v}_L^\ell,\tilde{v}_R^\ell\right\}$. Since $[\widetilde{w}_L,\widetilde{w}_R]\subset[w_L,w_R]$ and preferences are single-peaked and concave, we have for those citizens that $\tilde{v}_i^\ell>v_i^\ell$ for all $i\in\{L,R\}$. Hence, $\max\left\{\widetilde{v}_L^\ell,\tilde{v}_R^\ell\right\}>\max\left\{v_L^\ell,v_R^\ell\right\}$. Then, $v_k^\ell>\max\left\{v_L^\ell,v_R^\ell\right\}$

and $k \in G^{\ell}(\{w_L, w_k, w_R\})$.

Second, consider $\ell \in \mathcal{N}$ such that $k \in G^{\ell}(\{w_L, w_k, w_R\})$. If $w_\ell \notin (\widetilde{w}_L, \widetilde{w}_R)$, then either $w_k > \tilde{w}_L \geq w_\ell$ or $w_\ell \geq \tilde{w}_R > w_k$. In the first case, $\tilde{v}_L^{\ell} > v_k^{\ell}$ while $\tilde{v}_R^{\ell} > v_k^{\ell}$ in the second case. As a result, $k \notin G^{\ell}(\{\widetilde{w}_L, w_k, \widetilde{w}_R\})$. Rather, if $w_\ell \in (\widetilde{w}_L, \widetilde{w}_R)$, then either $v_k^{\ell} \geq \tilde{v}_i^{\ell} > v_i^{\ell}$ for all $i \in \{L, R\}$, in which case $k \in G^{\ell}(\{\widetilde{w}_L, w_k, \widetilde{w}_R\})$ or $\tilde{v}_i^{\ell} > v_k^{\ell} \geq v_i^{\ell}$ for some $i \in \{L, R\}$ in which case $k \notin G^{\ell}(\{\widetilde{w}_L, w_k, \widetilde{w}_R\})$.

As a result, $\left\{\ell \in \mathcal{N} : k \in G^{\ell}(\{\widetilde{w}_L, w_k, \widetilde{w}_R\})\right\} \subseteq \left\{\ell \in \mathcal{N} : k \in G^{\ell}(\{w_L, w_k, w_R\})\right\}$ and $\mid \{\ell \in \mathcal{N} : k \in G^{\ell}(\{\widetilde{w}_L, w_k, \widetilde{w}_R\})\} \mid < \frac{|\mathcal{N} \setminus \mathcal{M}|}{2} - 1$.

The same is true for condition (iv) of proposition 5 since we have shown that $\{\ell \in \mathcal{N} : k \in G^{\ell}(\{\widetilde{w}_L, w_k, \widetilde{w}_R\})\} \subseteq \left\{\ell \in \mathcal{N} : k \in G^{\ell}(\{w_L, w_k, w_R\})\right\}$ and we know that $\tilde{v}_i^{k} > v_i^{k}$ for all $i \in \{L, R\}$. Hence, if condition (iv) is satisfied for $\{w_L, w_R\}$, it must also hold for $\{\widetilde{w}_L, \widetilde{w}_R\}$.

It remains to show that conditions (ii) and (iii) are satisfied. We will proceed starting from the two-candidate political equilibrium under the plurality rule, adding candidates at $\tilde{w}_L$ and/or $\tilde{w}_R$ until condition (iii) is satisfied, then showing that condition (ii) holds. Since $\{\widetilde{w}_L, \widetilde{w}_R\}$ is the policy outcome under the plurality rule, we know that $\widetilde{c}_L = \widetilde{c}_R = 1$ and $\tilde{v} \equiv \tilde{v}_R^{L} = \tilde{v}_L^{R} \geq 2\delta$ (the inequality by condition (ii) of proposition 5, which is the same under the plurality rule, and the equality by symmetry). Either $\nexists\, i \in \mathcal{N} \setminus \mathcal{C}$ with $w_i \in \{\widetilde{w}_L, \widetilde{w}_R\}$ such that -$\tilde{v} > 6\delta$. Then, condition (iii) of proposition 5 is satisfied, as well as condition (ii). Suppose rather that there exists $i \in \mathcal{N} \setminus \mathcal{C}$ with $w_i \in \{\widetilde{w}_L, \widetilde{w}_R\}$ such that -$\tilde{v} > 6\delta$. In that case, this citizen chooses to enter the race and $\widetilde{c}_i = 2$, while $\widetilde{c}_{k(i)} = 1$. Now, either $12\delta \geq -\tilde{v}$ or $\widetilde{c}_i = n_i$ (where $n_i \equiv |\{\ell \in \mathcal{N} : w_\ell = w_i\}|$, i.e. all the citizens with ideal point w_i are now running) in which case no other citizen with the same ideal point wants to enter the race. Condition (iii) now holds. Also, since we had $-\tilde{v} > 6\delta$ in the first place and that $\widetilde{c}_i = 2$ and $\widetilde{c}_{k(i)} = 1$, condition (ii) is satisfied. Thus there exists a three-candidate political equilibrium under AV with $\{\widetilde{w}_L, \widetilde{w}_R\}$ as policy outcome. If rather, $-\tilde{v} > 12\delta$ and $\tilde{n}_i < c_i$ for some $i \in \{L, R\}$, then we proceed in the same way, adding one more candidate. **Q.E.D.**

Proof of Lemma 3. If there are two or less than two positions where candidates are located, then every voting strategy is relatively sincere. Consider the non-trivial possibility of three or more positions at which candidates are situated. Consider an intermediate position t (i.e. there is at least one position to the right of t and one to the left of t). Consider the following voting strategy- for each citizen ℓ, $\alpha_i^\ell = 1$ if and only if $v_i^\ell \geq v_t^\ell$. Such a strategy is weakly undominated for each ℓ. It is easy to see that under this strategy profile the candidate(s) at position t get(s) at least two votes more than any other alternatives. Hence it is a voting equilibrium. The voting strategies are, by construction, relatively sincere. **Q.E.D.**

Proof of Proposition 7. Consider one-candidate political equilibria. Note first that there exists an equilibrium in which the median citizen runs unopposed. Indeed, by assumption, $-v_0^m \geq \delta$ which corresponds to condition (i) of proposition 4. Moreover, no other citizen h with $w_h = m$ wants to enter the race, while for all $h \in \mathcal{N}\backslash\mathcal{C}$ with $w_h \neq m$, we have $v_h^m < v_m^m = 0$ (by single-peakedness), which satisfy condition (ii) of the same proposition.

Second, note that there cannot exist a political equilibrium in which citizen i with $-v_i^m \geq \delta$ runs unopposed. Indeed, if a citizen at the median position enters, he will win outright and get a utility $\widehat{U}_m = v_m^m - \delta = -\delta$. If rather he does not enter, he gets a utility $U_m = v_i^m$. Then he will want to enter since $\widehat{U}_m \geq U_m$, a contradiction. Hence, in any one-candidate political equilibrium (and we have shown that there exists at least one), candidate i's position must be such that $-v_i^m < \delta$.

Now consider political equilibria with at least two candidates. By proposition 3, we know that candidates are located at exactly two positions, w_L and w_R, with $v_L^m = v_R^m$. Let without loss of generality $m \geq \overline{w}$, where $\overline{w} = \frac{1}{c}\left(c_L w_L + c_R w_R\right)$ (a similar argument would apply for $\overline{w} > m$). Now, for all $\ell \in \mathcal{N}$,

$$v_{\overline{w}}^\ell \geq \frac{1}{c}\left(c_L v_L^\ell + c_R v_R^\ell\right)$$

, by Jensen's inequality. Since $m \geq \overline{w}$, we have $v_m^\ell \geq v_{\overline{w}}^\ell$ for all $\ell \in \mathcal{N}$ with $w_\ell \geq m$. Hence,

$$v_m^\ell \geq \frac{1}{c}\left(c_L v_L^\ell + c_R v_R^\ell\right)$$

, for all such citizens. Then, if a citizen with the median ideal point decides to enter, he will get a vote total $|F_m([0,1])| \geq \frac{|\mathcal{N} \backslash \mathcal{M}|}{2} + |\mathcal{M}|$, while $|F_L([0,1])|$ and $|F_R([0,1])|$ will be at most $\frac{|\mathcal{N} \backslash \mathcal{M}|}{2}$ (since $\{L\} \in L^\ell(\{L,R,m\})$ for all $\ell \in \mathcal{N}$ with $w_\ell \geq m$ and $\{R\} \in L^\ell(\{L,R,m\})$ for all $\ell \in \mathcal{N}$ with $m \geq w_\ell$.) Hence, if a citizen at the median enters the race, he is going to win outright. Now, he will want to enter if

$$v_m^m - \delta \geq \frac{1}{c}\left(c_L v_L^m + c_R v_R^m\right) \Leftrightarrow -v_i^m \geq \delta$$

, where $i \in \{L,R\}$ (the second inequality comes from $v_m^m = 0$ and $v_L^m = v_R^m$).

As a result, for m not to enter the race it must be that $-v_i^m < \delta$ for all $i \in \mathcal{C}$. **Q.E.D.**