A Principal's Optimal Extorsion and Endogenous Empty Core Game: A General Result for TU Two-Person Games

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Abstract

We add a Principal (abstract enforcer) to any 2 person TU game in strategic form because of transaction costs or extorsion. The game yields in reasonable cases an *empty core* when there are no externalities in coalition formation and when the principal can choose any allocation on the frontier of the individual, rational and feasible set. The principal does so by inducing "The Simultaneous and Double Extorsion Game". Only one of the identical players and the principal collude, iff the identical agents combined "might" is more than a threshold of the principal's but less than 100%, i.e. *"divide and rule".* The threshold depends on the limit to the degree of double extorsion. Strikingly, he would prefer to induce an empty core game with an extreme level of extortion instead of a nonempty core one. The result, ie. that only an identical agent and the principal collude, is robust to allowing for variable principal "might" or harrasing ability that might induce a cooperative game with externalities in coalition formation. As solution concept, we use an extension of Myerson's (1978) Shapley value generalization to partition function games and the Aumann-Myerson link (1988) formation game.

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1 Introduction(coming soon, very soon)

We add a Principal (abstract enforcer) to any 2 person TU game in strategic form because of transaction costs or extorsion. The game yields in reasonable cases an *empty core* when there are no externalities in coalition formation and when the principal can extort by choosing any allocation on the frontier of the individual, rational and feasible set. The principal does so by inducing a "Simultaneous and Double Extorsion Game". Only one of the identical players and the principal collude, iff the identical agents combined "might" is more than a threshold of the principal's but less than 100%, i.e. *"divide and rule"*. The threshold depends on the limit to the degree of double extorsion. Strikingly, he would prefer to induce an empty core game with an extreme level of extortion instead of a nonempty core one. The result, ie. that only an identical agent and the principal collude, is robust to allowing for variable principal "might" or harrasing ability that might induce a cooperative game with externalities in coalition formation. Actually, the principal would prefer a game with positive externalities in coalition formation. As solution concept, we use an extension of Myerson's (1978) Shapley value generalization to partition function games and the Aumann-Myerson link (1988) formation game.

2 On Abstract Enforcer Games in Two-agent TU games without ECF

2.1 Definitions

A two-person strategic form game can be denoted as $\Gamma = (\{1,2\}, C_1, C_2, u_1, u_2)$

We denote its feasible set F as:

$F = \{y \in \mathbb{R}^2 | y_1 + y_2 \leq P\}$,

where P denotes the maximum transferable wealth that the players can jointly achieve. The feasible set F for any two-agent game with transferable utility will vary accordingly depending on binding contracts, moral hazard or adverse selection. Without loss of generality, let us say that binding agreeements are possible. Thus,

$F = \{(u_1(\mu), u_2(\mu)) \, | \mu \in \Delta(C)\}$,

where μ denotes a correlated strategy and $\Delta(C)$ represents the set of probability distributions over the cartesian product $C_1 \times C_2$. We can let,

$P = \max_{\mu \in \Delta(C)}(u_1(\mu) + u_2(\mu))$

In standard barganing theory, once the outside options v_1, v_2 are determined, the maximization of the Nash product denoted by $(y_1 - v_1)(y_2 - v_2)$ subject to $y_1 - v_1 \geq 0$ and $y_2 - v_2 \geq 0$ (IR) yields a unique bargaining solution that will be $\left(\frac{P}{2}, \frac{P}{2}\right)$ a point on the feasible Frontier.

Now imagine because of transaction costs , the two agents can not reach the frontier of the feasible set F (alternatively, even if transaction costs are zero, we could think that there is extorsion by a "principal", i.e., the two identical agents are intimidated by the latter when say bargaining over the frontier of F, thus, getting lower payoffs[1]). To solve this transaction cost problem, they call a third party, player 3 (or an abstract enforcer[2]), so that he can implement the Nash bargaining allocation in the frontier of F. Given that all players are selfish, we indentify the third party instead with a "principal" with all the bargaining ability. We allow this principal to harrase the identical agents by the use of differents kinds of mechanisms. More precisely, imagine that the principal would "harrase" them with an implied lower utility in terms of a lower payoff or wealth for any of the two identical agents. Actually, let us say that the harrasing will depend on the two identical agents being colluded or not when defending themselves against the principal. Also, the value of the coalition of the principal and any individual agent is endogenized. It is in this sense, that we say we add a "principal" with all the bargaining and destructive or harrasing ability. The outside options of the identical agents acting individually or colluding against the principal and even colluding with the principal are induced by the latter.The upper bound for the wealth obtained by any coalition is bounded by the total wealth P. Finally, we focus on an extreme case in which the two identical agents are defenseless in the sense that they can not influence the level of harrasing.

Formally, the situation above can be set up as an induced cooperative game. We will be interested in the core of this induced game and the coalitions that will form. To give necessary conditions for our results, i.e. that the introduction of a principal will change the completely the nature of the game, we will first set up and analize a given induced cooperative game with characteristic function v and then by backward induction we will solve the principal's problem: the selection of the optimal induced TU cooperative game (the optimal extorsion game) that will yield him the higher payoff. We begin with the case of no externalities in coalition formation.

2.2 Empty Cores in Enforcer Games without ECF

In this first subsection, we will assume for simplicity that the principal can choose only games without externalities in coalition formation (ECF) and derive sufficient conditions under which the induced games in question have empty cores.

Let us assume that because of some technological reason , the enforcer can choose mechanisms restricted additionally on $0 \leq y_i \leq a$, $i = 1, 2$,where$\frac{P}{2} < a \leq P$. We will come back to the case $0 \leq a \leq \frac{P}{2}$.

[1] As it will become clear later on, the two situations, transaction costs or extortion would lead to a game where the principal gets a transfer payment.

[2] The situation, where we have real enforcers is treated more in detail in Nieva (June 2003). In the latter paper, when transaction costs or extorsion devices are related to comunication noise, hidden actions, or adverse selection, we give conditions under which third party, enforcers, mediators, monitors and auditors, added to the corresponding two person game set up, are equivalent to abstract enforcer games as defined in the present paper; or games with a principal with all the barganing ability.

An induced extorsion game consistent with the assumption of no ECF with extorsion value a will have a characteristic function v as follows

$v1 = v2 = P - a$

$v12 = m$

$v13 = v23 = a$

In this paper, we assume that m is given technologically and it is such that we have only a superadditive cooperative game (and not strictly superadive[3]).

Loosely speaking, the principal may induce the following double and simultaneous extorsion game: If player 1 colludes with him, he will design a mechanism that would implement the allocation $(a, P - a) = (y_1, y_2)$ in figure 1. If instead the principal colludes with the other identical agent 2, he will implement the symmetric allocation on the frontier F $(P - a, a) = (y_1, y_2)$. In our cooperative game, these simultaneous "offers" would imply $v13 = v23 = a$ and $v1 = v2 = P - a$.

[3]Note that if assumption in proposition 1 ($m > 2(P-a)$) and harrasing costs of production would be positive then we would have a strictly superaditive game. Intuitively, in the grand coalition the principal doesn't harrase; thus, harrasing costs incurred would be zero and more output would be produced.

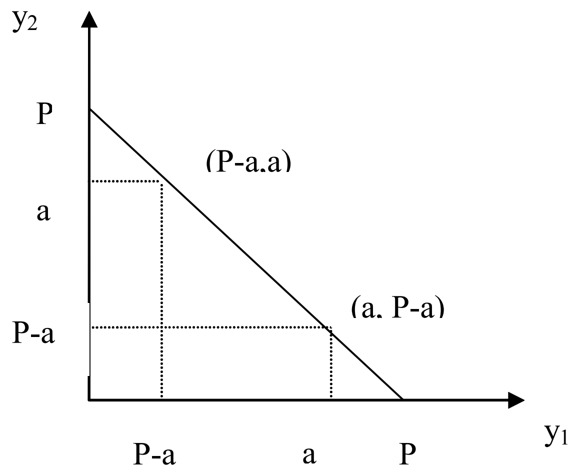

Figure 1:

Proposition 1: If $m > 2(P - a)$ and $\frac{P}{2} < a \leq P$, the associated cooperative game has an empty core:

Proof: If an allocation y_1 y_2 y_3 in a three person game is in the core, then $v(123) \geq \frac{v(12)+v(13)+v(23)}{2}$

In our case $P \geq \frac{m+a+a}{2}$. After rearranging we get $2(P - a) \geq m$. Thus, the claim follows.■

Loosely speaking, if there are increasing returns to the fighting abilities of the identical agents and no ECF, the core of the cooperative game in question is empty provided that the coalition of the principal and any agent yields strictly more than half of the total wealth in the economy, i.e. $\frac{P}{2} < a \leq P$.

2.3 Coalition Formation

The enforcer cares for his payoff in a given induced cooperative game. For predicting this payoff that will depend on which coalitions form, we use the Shapley-Aumann-Myerson (1988) solution. We now give give necessary and sufficient conditions for the formation of coalition structures different than the grand coalition and the implied payoffs.

For this purposes let denote the graph structures as follows:The first type of graph for the case for three players is the empty graph without links. The complete graph

g^N with a maximum of three links in our case is represented as:

```
    3
   / \
  1——2
graph N
```

The one-link graphs representation is:

```
   3             3          3
  /               \
 1    2        1    2     1——2
graph 1        graph 2    graph 3
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The two-links graphs:

```
   3              3         3
  / \              \       /
 1    2        1—— 2      1——2
graph 4        graph 5    graph 6
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Let's say for now that a is fixed and the characteristic function is the same as above. It will be useful to state this lemma proved in the appendix:

Lemma 1:

Assume we have 3 person normalized superadditive games that are non-trivial (i.e. value of 2 person coalitions are strictly positive) or where $v(i) + v(j) < v(ij)$ for $i \neq j$ and $i, j \in (1, 2, 3)$.

Let a 1 link graph structure (graphs 1, 2 and 3 above) be a blocking graph if the Myerson values for the respective two players linked is strictly greater than their Shapley values in the complete graph

Lemma 1a:

Without loss of generality (wlg), if the graph implied by 13 is a blocking graph, then it is the unique subgame perfect equilibrium outcome.

If there are two blocking graphs, say associated (w.l.g.) with links 13 and 23, we have the following 2 lemmas:

Lemma 1b:

(W.l,g), if $\phi_{i=3}^{g=1} > \phi_{i=3}^{g=2}$, then the one-link graph implied by 13, graph 1, is the unique subgame perfect equilibrium outcome.

Lemma 1c:

Wlg, if $\phi_{i=3}^{g=1} = \phi_{i=3}^{g=2}$, then it is a subgame perfect equilibrium outcome for both 13 or 23 to be the only graphs to form.

Lemma 1d:

In contrast, the complete graph and a one-link graph may form, iff there is no one-link blocking graph

Proofs:

Omitted. See appendix■

Proposition 2: Let a satisfy $P \geq a > \frac{P}{2}$. Only a rancher and a enforcer collude in any subgame perfect equilibrium, iff agents' combined "might" m is such that $\vartheta(a) = \frac{3P-2a}{2} < m < P$

Proof: We need to check that for m on this range the assumptions of lemma 1 hold for such a coalition structure to form. Additionally, we need to rule out that the 2 identical agents do not collude in any equilibrium.

First, note that any m that we check will be consistent with the assumption of superadditivity, because if $\frac{P}{2} < a$, then $\frac{3P-2a}{2} > 2(P-a)$.

For the specialized enforcer, we will pick the lower bound (when $m = \frac{3P-2a}{2}$) and show that her Shapley value is equal to its Myerson value in graph 1 or 2[4] $\phi_{i=3}^{g=1,2}(v(m)) = a - \frac{P}{2}$ (where $\phi_i^g(v(m))$ is the Myerson value, given m, corresponding to agent i if the graph that formed is $g \in \{1,2,3,4,5,6,N\}$). Note, that the latter Myerson value is independent of m. Finally, we need to show that the enforcer's Shapley value is a decreasing function of m on $\left]\frac{3P-2a}{2}, P\right[$.

By definition the Shapley value for the enforcer is[5]:
$\frac{[v(123)-v(12)]+[v(13)-v(1)]+[v(213)-v(21)]+[v(23)-v(2)]+2v(3)}{6}$

Plugging in known values we get
$\frac{[P-m]+[a-(P-a)]+[P-m]+[a-(P-a)]+0+0}{6} = \frac{2a-m}{3}$,

and $m = \frac{3P-2a}{2}$ solves $\frac{2a-m}{3} = a - \frac{P}{2}$. Additionally, the enforcer's Shapley value is clearly a decreasing function of m. Note that this argument holds for $P \geq a > \frac{P}{2}$

As the assumptions of lemma 1 require, we now show that the agents' Shapley values are an increasing function of m on the interval $(3,4)$, but lower than $\phi_{i=1,2}^{g=i}(v(m)) = \frac{P}{2}$ when $m = P$. The Shapley value of the enforcer is $\frac{2a-m}{3}$, it follows that the agents' Shapley value is an increasing function. If $m = P$ given that the enforcer's Shapley value is $\frac{2a-m}{3} = \frac{2a-P}{3}$, any rancher's value is $\frac{P-\frac{2a-P}{3}}{2} = \frac{3P-2a+P}{6} = \frac{2P-a}{3} < \frac{P}{2} = \phi_{i=1,2}^{g=i}(v(m))$.

Finally, we need to rule out that the two identical agents collude. Suppose they do collude. Given that it is a subgame perfect equilibrium for any agent and the enforcer to accept a bilateral link with any agent and reject any link after that (with payoffs $\phi_i^{g=i}(v(m)) = \frac{P}{2}$, for $i = 1,2$ and $\phi_{i=3}^{g=1,2}(v(m)) = a - \frac{P}{2}$ for the agent and the enforcer respectively), there is a deviation by the agent who gets to propose next (twice consecutively) together with the enforcer and prefers to accept a link with the latter and not with the other identical rancher to begin with. This deviation exists for all m on the interval $\left]\frac{3P-2a}{2}, P\right[$, because $\frac{m}{2} = \phi_{i=1,2}^{g=3}(v(m)) < \frac{P}{2} = \phi_i^{g=i}(v(m))$, for $i = 1,2$. In words, the Myerson value of an identical agent is higher if colluded with the enforcer than when linked with the other rancher only; thus, one rancher

[4] Recall the equal gains principle by the two associated agents of the Myerson value when a bilateral link is added (Myerson 1977). Let $c = \phi_i^g(v(m))$ be any agent i's (where $g = i = 1,2$) Myerson value of the graph where only one of the agents is linked with the enforcer, and $d = \phi_{i=3}^{g=1,2}(v(m))$ the Myerson value value for the enforcer. Thus,.$c - (P-a) = d - 0$ where $d + c = a$. After simplifying, we get $c = \frac{P}{2}$ and $d = a - \frac{P}{2}$

[5] To give intuition for the Shapley value, imagine players enter with equal probability in a room in the six possible following orders and their weighted marginal contribution is computed accordingly: 123, 132, 213, 231, 312, 321

will deviate and reject the latter link. It follows that the two identical agents will never collude.

Because the enforcer is indifferent between accepting a link with one rancher today or another one with a different rancher tomorrow there is always subgame perfect equilibria with two types of graphs where the enforcer colludes with any rancher, i.e, only coalitions with all players either in the set $\{1,3\}$ or $\{23\}$ form. Alternatively, see lemma 1c■.

Corollary 1: If ranchers are as strong or stronger than the enforcer, i.e. $m = P$, then any 2 agent coalition can form including that of the two identical ranchers.

Proof:

The argument in the proof of claim 2 doesn't work any more. The proof is straightforward using backward induction as in Aumann and Myerson (1988). Or see Nieva (October 2002)■.

Corollary 2:

Only the complete graph and the implied grand coalition forms with payoffs $\left(\frac{3P-2a+m}{6}, \frac{3P-2a+m}{6}, \frac{2a-m}{3}\right)$, iff we have instead m such that[6], $2(P-a) \leq m < \vartheta(a) = \frac{3P-2a}{2}$

Proof:

From lemma 1, we need to check that at least one agent's Myerson value in any coalition of 2, cannot beat their corresponding Shapley value in the complete graph for the range in question.

Recall from proposition 2 that for the case of the enforcer, when $m = \frac{3P-2a}{2}$, the two values in question are equal, i.e., $\phi_{i=3}^{g=1,2}(v(m)) = \phi_{i=3}^{g=N}(v(m)) = a - \frac{P}{2}$. Given that $\phi_{i=3}^{g=1,2}(v(m))$ is a constant function and $\phi_{i=3}^{g=N}(v(m))$ is decreasing in m,the claim any 2 person coalition that contains the enforcer will not form.

For any identical agent, we have instead that $\phi_{i=1,2}^{g=N}(v(m))$ is increasing in m and equal to $\frac{3P-2a+m}{6}$. If $m = \frac{3P-2a}{2}$, $\frac{m}{2} = \phi_i^{g=3}(v(m)) = \phi_{i=1,2}^{g=N}(v(m)) = \frac{3P-2a+m}{6}$. Given that $\phi_i^{g=3}(v(m))$ decreases faster, we have that the claim holds for any identical agent when colluded with the other identical one■.

2.4 The Optimal Induced Double and Simultaneous Extorsion Game without ECF

Let m and a be given such that $a > \frac{P}{2}$. Formally, as we explained before, imagine that the enforcer can induce the following "Double and Simultaneous Extorsion Game": After selecting a value for θ such that $a > \theta > \frac{P}{2}$, he induces the following cooperative game: He extorts both identical agents with payoffs to the coalition (enforcer-identical agent) and the leftout rancher of $(i,j) = (\theta, P-\theta) = (1-\ell)(a, P-a) + \ell(P-a, a)$,

[6] The case of $m = \vartheta(a)$ can be characterized by backward induction.

where $\ell \in \left[0, \frac{1}{2}\right[$ and i and j are the payments for the coalition (enforcer and agent) and left out identical respectively. Actually, we are ruling out by assumption (a technological assumption) that for example the enforcer cannot choose $(i, j) = (P, 0)$.

Let m is such that $\vartheta(a) = \frac{3P-2a}{2} < m < P$

Theorem 1: Assume the enforcer induces a game without ECF as described above. Then,

a. the enforcer would pick $\ell = 0$ or equivalently $\theta = a$.

b. The latter would happen also if he can pick any *feasible* a such that $a < \frac{P}{2}$.

Proof:

a. Recall the Myerson value for the principal when colluded with one of the identical agents, $\phi_{i=3}^{g=1,2}(v(m)) = a - \frac{P}{2}$, is increasing in a.

b. All the Myerson values in any graph (recall, we don't need to look at 2 link graphs) are increasing in a for the enforcer■

In other words, the principal would prefer to induce an empty core game insted of a nonempty core one.

2.5 Results when there are ECF

Theorem 2: Assume the enforcer induces a game with ECF. He would prefer a game with positive externalities in coalition formation. Actually, the optimal value of $\varepsilon = m - P$, where ε is externality factor as $v1 = v2 = P - a + \varepsilon$.

Proof:

See example and lemmas 4 and 5 in the appendix.

Theorem 3: Let a be given. If m is variable and with lower bound $\vartheta(a)$, the principal would prefer a game with positive ECF than one without ECF. Actually, the optimal $m \longrightarrow \vartheta(a)$ and the optimal value of $\varepsilon \longrightarrow \vartheta(a) - P = \frac{P-2a}{2}$.

Proof:

See appendix■

3 Appendix

3.1 Proof of Lemma 1

Lemma 1(Proofs need some fine tuning still!!):

Assume we have 3 person normalized superadditive games that are non-trivial (i.e. value of 2 person coalitions are strictly positive) or where $v(i) + v(j) < v(ij)$ for $i \neq j$ and $i, j \in (1, 2, 3)$.

Let a 1 link graph structure (graphs 1, 2 and 3 above) be a blocking graph if the Myerson values for the respective two players linked is strictly greater than their Shapley values in the complete graph

Lemma 1a:

Without loss of generality (wlg), if the graph implied by 13 is a blocking graph, then it is the unique subgame perfect equilibrium outcome.

Proof:

First, note that if there is only one link with the above characteristics, then by backward induction, it is a subgame perfect equilibrium for that two linked players not to accept a second link. By assumption $v(i)+v(j) < v(ij)$ for $i \neq j$ and $i, j \in (1, 2, 3)$. So, by definition of the Myerson value, the players not linked form the third link as their payoffs increase; thus, there is a deviation and the complete graph would follow. Additionallly, rejecting the link will yield them their Shapley values. Thus accepting their link and not accepting any link afterwards is a subgame perfect equilibrium strategy outcome. Finally, any other graph structure has a deviation and would lead to the complete graph. For the latter case in turn, we would have a deviation of one player in its first link graph to form if 13 is still out there to propose. From above it follows that if 13 is before in the rule of order, 13 will be the last to form in anticipation of the formation of the complete graph■.

If there are two blocking graphs, say associated (w.l.g.) with links 13 and 23, we have the following cases:

1st case:

Lemma 1b:

(W.l,g), if $\phi_{i=3}^{g=1} > \phi_{i=3}^{g=2}$, then the one-link graph implied by 13, graph 1, is the unique subgame perfect equilibrium outcome.

Proof:

Suppose 13 is formed. If anyone accepts a second link the third forms and the complete graph forms, thus this would not happen. Now, we need to show that both 1 and 3 find it optimal to accept their link and not reject it. If they reject, we know that the only possible graphs to form are the complete graph or the 23 graph. The best it can happen for 3 is that link 13 is called upon to propose again afer every subsequent pair of players to propose rejected. The same occurs to 1 (note that 1's Myerson value in graph 2 is $v1$ strictly lower than $\phi_{i=1}^{g=2}$). Thus, it is optimal for both to accept.

Claim 2 follows because of three reasons

1. link 23 will not form because (according to a given rule of order) either player 3 in link 23 rejects if 13 is still out there to propose, or, if 13 has not proposed yet, players in 13 would not accept any more links in anticipation of 23 accepting.

2. Following the same reasoning as in (1), link 12 will not form because in the worst of cases if 23 proposes before or after 12 gets to propose, at the end 13 will finally form before or after 23 get to propose (according to a given rule of order).

3. In an analogous way, the complete graph will not form because the first link to form will always be prevented as 13 follows or precedes proposers 23 or 12.

To clarify, we will proof (1). If link 23 is the last to be proposed it will be accepted and no more links will be formed. If 12 is behind 13 in the rule of order, the link 13 will form (thus, 12 will be rejected and 13 will be accepted) in anticipation of 23's action. If 13 is behind 12, 13 will be accepted because otherwise 12 will be rejected in anticipation of the formation of 23.

If after 23 we have 13 to be the last one to propose according to the rule of order 3 will reject waiting for 13 to be accepted. If 12 is next instead, then player 2 would not accept last link 12. Of course 13 in anticipation would not accept any more links and thus, 23 would not form to begin with in the latter case. It is obvious that if 12 and 13 are still out there to propose player 3 in link 23 would reject to begin with.

Lemma 1c:

Wlg, if $\phi_{i=3}^{g=1} = \phi_{i=3}^{g=2}$, then it is a subgame perfect equilibrium outcome for both 13 or 23 to be the only graphs to form.

Proof:

Suppose 13 is formed. If anyone accepts a second link the third forms and the complete graph forms, thus this would not happen. Now, we need to show that both 1 and 3 find it optimal to accept their link and not reject it. If they reject, we know that the only possible graphs to form are the complete graph or the 23 graph. The best it can happen for 3 is that link 13 or 23 are called upon to propose again and accepted after every subsequent pair of players to propose rejected. The same occurs to 1 (note that 1's Myerson value in graph 2 is $v1$ strictly lower than $\phi_{i=1}^{g=2}$). Thus, it is optimal for both to accept. A similar reasoning applies if we 23 forms.

Two show that either link will form, it is clear that link 12 cannot be the unique graph to form because they would accept links so that the complete graph would form. This would be prevented or 1 or 2 would reject link If 13 or 23 propose before or after.

Claim 3 follows because 3 is indifferent between 13 or 23. Intuitively, let us say that even when indifferent in terms of payoffs 3 prefers on an apriori basis 13. Then following claim 2 only link 13 would be the unique subgame perfect equilibrium outcome. But the same argument follows if she prefers instead 23. Thus, there exist to subgame perfect equilibrium outcomes: Either link 13 or 23 form$\blacksquare$.

Lemma 1d:

In contrast, the complete graph and a one-link graph may form, iff there is no one-link blocking graph

Proof:

As in 1a because $v(i)+v(j)<v(ij)$ for $i\neq j$ a two link graph never forms.■.

3.2 Examples

For illustration, let $a=3$ and $P=4$. We compute the Myerson values associated with each graph for different initial conditions (different values for m) for the value of the coalition of the 2 ranchers $v(\{1,2\})$ and find that only in example 4, when $m=3\frac{3}{9}$ (consistent with proposition 2 as $P=4>m=3\frac{3}{9}>\vartheta(a)=\frac{3P-2a}{2}=\vartheta(3)=\frac{3*4-2*3}{2}=3$), that the Myerson value of the stable graph is a "reasonable" prediction of the equilibrium payoffs for the game. In the latter case the two agents united are very strong. Note that in all the four examples but example 1 these games have an empty core. In Nieva (Novermber 2002), we conjecture that an extension of the Aumann-Myerson solution should never yield the grand coaliton whenever the core is emtpy.

3.2.1 No ECF

Example 1 with a non-empty core.

We assume that the enforcer by hassling the two agents can make them get each in average a payoff of 1 even if they collude. Hence we get:

$v(1)=1=P-a$

$v(2)=1$

$v(\{3\})=0$

$v(12)=2(P-a)$

$v(13)=3=a$

$v(23)=3$

$v(123)=4=P$

If an allocation x_1 x_2 x_3 in a three person game is in the core then $v(123)\geq\frac{v(12)+v(13)+v(23)}{2}$. Example 1 satisfies the later weak inequality with equality. The allocation (1,1,2) is in the core. Recall the core is the set of allocations x that satisfy the following conditions:

$x_1\geqq v(1)\qquad x_2\geqq v(2)\qquad x_3\geqq v(3)$

$x_1+x_2\geqq v(12)\qquad x_1+x_3\geqq v(13)\qquad x_2+x_3\geqq v(23)$

$x_1+x_2+x_3\geq v(123)$

The Myerson value for graph 1 is $(2,1,1)$

The Myerson value for graph 2 is $(1,2,1)$

The Myerson value for graph 3 is $(1,1,0)$

The Myerson value for graph 4 is $(1\frac{1}{3},1\frac{1}{3},1\frac{1}{3})$

The Myerson value for graph 5 is $(1,2,1)$

The Myerson value for graph 6 is $(2,1,1)$

The Myerson value for the complete graph is $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$, which is also the Shapley value of this game.

Claim 1a: The endogenous game of link formation with the Myerson value as fixed valuation has subgame perfect equilibria where the complete graph with payoffs $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$ forms. If there are equilibria with incomplete graphs then the payoffs are also $(1\frac{1}{3}, 1\frac{1}{3}, 1\frac{1}{3})$

Proof:

By backward induction. See Nieva (October 2002)

Example 2 (empty core)

Let us now consider instead a modified characteristic function in which by colluding, the two agents can get more than 2. Hence $v(123) \geq \frac{v(12)+v(13)+v(23)}{2}$. For this class of games, we always get an empty core as long as $v(12) > 2$.

$v(1) = 1$

$v(2) = 1$

$v(3) = 0$

$v(12) = 2\frac{1}{6}$

$v(13) = 3$

$v(23) = 3$

$v(123) = 4$

The Myerson value for graph 1 is $(2, 1, 1)$

The Myerson value for graph 2 is$(1, 2, 1)$

The Myerson value for graph 3 is $(1\frac{1}{12}, 1\frac{1}{12}, 0)$

The Myerson value for graph 4 is $(1\frac{12}{36}, 1\frac{12}{36}, 1\frac{12}{36})$

The Myerson value for graph 5 is$(1\frac{1}{36}, 2\frac{4}{36}, \frac{34}{36})$

The Myerson value for graph 6 is $(2\frac{4}{36}, 1\frac{1}{36}, \frac{34}{36})$

The Myerson value for the complete graph or Shapley value is $(1\frac{13}{36}, 1\frac{13}{36}, 1\frac{10}{36})$

As in case 1 the complete graph will form.

Example 3 (empty core)

Here, we decrease the bargaining power of the principal by increasing the value of the two-identical-agent coalition and get the threshold above which the grand coalition never forms. We have the threshold $\vartheta(a = 3) = m = 3$.

$v(\{p1\}) = 1$

$v(\{p2\}) = 1$

$v(\{p3\}) = 0$

$v(\{p1, p2\}) = 3 = \vartheta$

$v(\{p1, p3\}) = 3$

$v(\{p2, p3\}) = 3$

$v(\{p1, p2, p3\}) = 4$

The Myerson value for graph 1 is $(2, 1, 1)$

The Myerson value for graph 2 is $(1, 2, 1)$

The Myerson value for graph 3 is $(1\frac{3}{6}, 1\frac{3}{6}, 0)$

The Myerson value for graph 4 is $(1\frac{2}{6}, 1\frac{2}{6}, 1\frac{2}{6})$
The Myerson value for graph 5 is $(1\frac{1}{6}, 2\frac{1}{6}, \frac{4}{6})$
The Myerson value for graph 6 is $(2\frac{1}{6}, 1\frac{1}{6}, \frac{4}{6})$
The Myerson value for the complete graph is $(1\frac{1}{2}, 1\frac{1}{2}, 1)$

Example 4 (empty core)

We claim that by further increasing $v(12)$, we get to cases where using the Myerson value is an intuitively adequate approach for solving these games. Thus by using this solution concept we predict only coalitions of a rancher and the enforcer.

$v(\{1\}) = 1$
$v(2) = 1$
$v(3) = 0$
$v(12) = 3\frac{3}{9}$
$v(13) = 3$
$v(23) = 3$
$v(123) = 4$
The Myerson value for graph 1 is $(2, 1, 1)$
The Myerson value for graph 2 is $(1, 2, 1)$
The Myerson value for graph 3 is $(1\frac{6}{9}, 1\frac{6}{9}, 0)$
The Myerson value for graph 4 is $(1\frac{3}{9}, 1\frac{3}{9}, 1\frac{3}{9})$
The Myerson value for graph 5 is $(1\frac{2}{9}, 2\frac{2}{9}, \frac{5}{9})$
The Myerson value for graph 6 is $(2\frac{2}{9}, 1\frac{2}{9}, \frac{5}{9})$
The Myerson value for the complete graph is $(1\frac{5}{9}, 1\frac{5}{9}, \frac{8}{9})$

Claim 1b: The endogenous game of link formation with the Myerson value as fixed valuation has 2 natural subgame perfect equilibria where *either* a coalition with all players in the set $\{p1, p3\}$ or $\{p2, p3\}$ form.

Proof:

By backward induction or using lemma 1 above■

3.2.2 Robustness when Externalities in Coalition Formation:

Let us assume that what coalitions can achieve depends on sets of links (or graphs as in Aumann and Myerson (1988) or Jackson and Wolinsky (1996)) among players and on the coalition structure following the partition function approach in which spillovers in coalition formation are possible (see Bloch (1996) Ray and Vohra (1996) and Myerson(1978)). Loosely speaking, the extended Myerson value that allows externalities in coalition formation is a extended Shapley value or weighted average of contributions of players to coalitions taking into account also the corresponding contributions in different coalition structures. We assume that utility is transferable.

Before we describe the relationship between coalitions worth and links and coalitions structures, let N be a finite set of players. Given N, let CL be the set of all coalitions(nonempty subsets) of N,

$CL = \{S | S \subseteq N, S \neq \varnothing\}$

Let PT the set of partitions of N, so
$\{S^1, ..., S^l\} \in PT$ iff:
$U_{i=1}^{l} S^i = N$, $\forall j \; S^j \neq \varnothing, \forall k S^j \cap S^k = \varnothing$ if $k \neq j$.

Let ECL be the set of embedded coalitions, that is the set of coalitions with specifications as to how the other player are aligned. Formally:

$ECL = \{(S, Q) | S \in Q \in PT\}$

For any finite set L, let $\mathbb{R}^L$ denote the set of real vectors indexed on the members of L.

For our case wth three players, we will begin with the complete graph g^N or, equivalently, with the original game where everyone is linked:

3
/ \
1——2
graph N

For this graph we will have a game in partition function form that would correspond to a vector in $w^{g=N} \in \mathbb{R}^{ECL}$**.** For any such $w^N \in \mathbb{R}^{ECL}$ and any embedded coalition $(S, Q) \in ECL$, $w^N_{S,Q}$, the (S, Q) component of w^N is interpreted as the wealth, measured in units of transferable utility, which the coalition S would have to divide among its members if all the players were aligned into the coalitions of partition Q. In general, we would have $w^g \in \mathbb{R}^{ECL}$, where $g \in \{1, 2, 3, 4, 5, 6, N\}$). Note that ECL is independent of the graph structure.

Negative Externalities in coalition formation We use a straightforward extension of the Shapley value in partition function form as in Myerson (1978) allowing for cooperation structures. We assume for the numerical example that the value of the two identical agent coalition has a lower bound of $3\frac{3}{9} = \frac{30}{9} = m$, that is given by technology. Thus, we want to find out if the enforcer can do better by hasszling the singletons, where the minimun level of hazzle yields $\frac{15}{9}$ for the identical agents fighting separately, half of what they can achieve united.

Let us have for the partition that consists of $\{\{1\}, \{2\}, \{3\}\}$ the singleton agents' value to be 1.3. Note that for values for the singleton agents such that $1 < v(1) = v(2) < \frac{15}{9}$, we get negative externalities in coalition formation. Let (both for our numerical example and in general)

$w^N_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1.3 = P - a + \varepsilon$
$w^N_{\{2\},\{\{1\},\{2\},\{3\}\}} = 1.3 = P - a + \varepsilon$
$w^N_{\{3\},\{\{1\},\{2\},\{3\}\}} = 0$

Where ε is such that $w^N_{\{1\},\{\{1\},\{2\},\{3\}\}} + w^N_{\{2\},\{\{1\},\{2\},\{3\}\}} \leq m = 3\frac{3}{9}$. In the particular case analized, we have $\varepsilon = 0.3$

If the two ranchers collude:

$w^N_{\{1,2\},\{\{1,2\},\{3\}\}} = 3\frac{3}{9} = m$
$w^N_{\{3\},\{\{1,2\},\{3\}\}} = 0$

If rancher i colludes with the enforcer we get:
$w^N_{\{i,3\},\{\{i,3\},\{j\}\}} = 3 = a$
$w^N_{\{j\},\{\{i,3\},\{j\}\}} = 1 = P - a$, for $i = 1, 2$ $j \in \{1, 2\}$ $i \neq j$
Thus, we have externalities in coalition formation as $P - a = 1 = w^N_{\{j\},\{\{i,3\},\{j\}\}} \neq w^N_{\{j\},\{\{i\},\{j\},\{3\}\}} = P - a + \varepsilon$, i.e. the value of player i acting alone is dependent on the coalition structure.
The last partition, the grand coalition has one element, itself that is worth 4.
$w^N_{\{1,2,3\},\{\{1,2,3\}\}} = 4 = P$
Let $\Phi_1(w^N)$ be the extended Shapley value for games with externalities in coalition formation for player 1 for the complete graph N. Following Myerson (1978) we have:
$\Phi_1(w^N) = \frac{1}{3}w^N_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^N_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^N_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^N_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^N_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{2}{3}w^N_{\{1\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^N_{\{2,3\},\{\{2,3\},\{1\}\}}$
$+\frac{1}{6}w^N_{\{2\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^N_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^N_{\{1\},\{\{1\},\{2\},\{3\}\}}$
Plugging in values we get in general:
$\Phi_1(w^N) = \frac{1}{3}P + \frac{1}{6}m - \frac{1}{3}0 + \frac{1}{6}a - \frac{1}{3}(P-a) + \frac{2}{3}(P-a) - \frac{1}{3}a + \frac{1}{6}(P-a+\varepsilon) + \frac{1}{6}0 - \frac{1}{3}(P-a+\varepsilon)$
$\Phi_1(w^N) = \frac{3P+m-2a-\varepsilon}{6}$ (a decreasing function of ε)
In particular for the parameter values above:
$\Phi_1(w^N) = \frac{1}{3}4 + \frac{1}{6}3\frac{3}{9} - \frac{1}{3}0 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{2}{3}1 - \frac{1}{3}3 + \frac{1}{6}1.3 + \frac{1}{6}0 - \frac{1}{3}1.3 = 1.5056$
After Checking for consistency of the particular and the general case, we have:
$\frac{12+3\frac{3}{9}-6-.3}{6} = 1.505\,6$
Similarly for $\Phi_2(w^N)$, we get:
$\Phi_2(w^N) = \frac{1}{3}w^N_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^N_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^N_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^N_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^N_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^N_{\{2\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^N_{\{1,3\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^N_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^N_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^N_{\{2\},\{\{1\},\{2\},\{3\}\}}$
$\Phi_2(w^N) = \frac{1}{3}4 + \frac{1}{6}3\frac{3}{9} - \frac{1}{3}0 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{2}{3}1 - \frac{1}{3}3 + \frac{1}{6}1.3 + \frac{1}{6}0 - \frac{1}{3}1.3 = 1.505\,6$
Finally for $\Phi_3(w^N)$ we have:
$\Phi_3(w^N) = \frac{1}{3}w^N_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^N_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^N_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^N_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^N_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^N_{\{3\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^N_{\{1,2\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^N_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^N_{\{2\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^N_{\{3\},\{\{1\},\{2\},\{3\}\}}$
Thus:

$\Phi_3(w^N) = \frac{1}{3}4 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{2}{3}0 - \frac{1}{3}3\frac{3}{9} + \frac{1}{6}1.3 + \frac{1}{6}1.3 - \frac{1}{3}0 = 0.988\,89$

After summing all individual extended Shapley values, we get: $2 * 1.505\,6 + 0.988\,89 = 4.0001$

Summarizing:

In particular:

Φ^N=$\Phi(w^N) = (1.505\,6, 1.505\,6, 0.988\,89)$

From above, it is straightforward to show lemma 2:

Lemma 2:

$\Phi_3(w^N)$ is an increasing function of ε, because $\Phi_1(w^N) = \frac{3P+m-2a-\varepsilon}{6}$, is a decreasing function of ε.

We want to calculate the corresponding Shapley values for different graph structures. First, for one link graphs we have:

3	3	3
/	\	
1 2	1 2	1——2
graph 1	graph 2	graph 3

For graph 1 we will have a game in partition function form that would correspond to a vector in $w^{g=1} \in \mathbb{R}^{ECL}$.

We will have in our example for the partition that consists of $\{\{1\},\{2\},\{3\}\}$:

$w^1_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1.3 = (P - a + \varepsilon)$

$w^1_{\{2\},\{\{1\},\{2\},\{3\}\}} = 1.3 = (P - a + \varepsilon)$

$w^1_{\{3\},\{\{1\},\{2\},\{3\}\}} = 0$

If the two ranchers collude:

$w^1_{\{1,2\},\{\{1,2\},\{3\}\}} = 2.6 = 2(P - a + \varepsilon)$

$w^1_{\{3\},\{\{1,2\},\{3\}\}} = 0$

If rancher 1 colludes with the enforcer we get:

$w^1_{\{1,3\},\{\{1,3\},\{2\}\}} = 3 = a$

$w^1_{\{2\},\{\{1,3\},\{2\}\}} = 1 = P - a$

Thus, we have externalities in coalition formation as $P - a = 1 = w^1_{\{2\},\{\{1,3\},\{2\}\}} \neq w^1_{\{2\},\{\{1\},\{2\},\{3\}\}} = 1.3 = (P - a + \varepsilon)$, i.e. the value of player i acting alone is dependent on the coalition structure.

If rancher 2 colludes with the enforcer we get:

$w^1_{\{2,3\},\{\{2,3\},\{1\}\}} = 1.3 = (P - a + \varepsilon)$

$w^1_{\{1\},\{\{2,3\},\{1\}\}} = 1.3 = (P - a + \varepsilon)$

Thus, we have dont have externalities in coalition formation as $(P - a + \varepsilon) = 1.3 = w^1_{\{1\},\{\{2,3\},\{1\}\}} = w^1_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1.3 = (P - a + \varepsilon)$, i.e. the value of player i acting alone is dependent on the coalition structure.

The last partition, the grand coalition has one element, itself that is worth 4.

$w^1_{\{1,2,3\},\{\{1,2,3\}\}} = 4 = P$

Now we compute the extended Myerson values for graph 1:

$\Phi_1(w^1) = \frac{1}{3}w^1_{\{1,2,3\},\{\{1,2,3\}\}}$

$+\frac{1}{6}w^1_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^1_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^1_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^1_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{2}{3}w^1_{\{1\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^1_{\{2,3\},\{\{2,3\},\{1\}\}}$
$+\frac{1}{6}w^1_{\{2\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^1_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^1_{\{1\},\{\{1\},\{2\},\{3\}\}}$
$\Phi_1(w^1) = \frac{1}{3}P + \frac{2}{6}(P-a+\varepsilon) - \frac{1}{3}0 + \frac{1}{6}a - \frac{1}{3}(P-a) + \frac{2}{3}(P-a+\varepsilon) - \frac{1}{3}(P-a+\varepsilon)$
$+\frac{1}{6}(P-a+\varepsilon) + \frac{1}{6}0 - \frac{1}{3}(P-a+\varepsilon) = \frac{1}{3}P + \frac{1}{6}a - \frac{1}{3}(P-a) + \frac{3}{6}(P-a+\varepsilon)$
$\Phi_1(w^1) = \frac{P+\varepsilon}{2}$, thus $\Phi_1(w^1)$ is an increasing function of ε.
For our numerical example, we have:
$\Phi_1(w^1) = \frac{1}{3}4 + \frac{1}{6}2.6 - \frac{1}{3}0 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{2}{3}1.3 - \frac{1}{3}1.3 + \frac{1}{6}1.3 + \frac{1}{6}0 - \frac{1}{3}1.3 = 2.15$
Similarly for $\Phi_2(w^1)$, we get:
$\Phi_2(w^1) = \frac{1}{3}w^1_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^1_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^1_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^1_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^1_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^1_{\{2\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^1_{\{1,3\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^1_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^1_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^1_{\{2\},\{\{1\},\{2\},\{3\}\}}$
$\Phi_2(w^1) = \frac{1}{3}4 + \frac{1}{6}2.6 - \frac{1}{3}0 + \frac{1}{6}1.3 - \frac{1}{3}1.3 + \frac{2}{3}1 - \frac{1}{3}3 + \frac{1}{6}1.3 + \frac{1}{6}0 - \frac{1}{3}1.3 = 1.0$
Finally for $\Phi_3(w^1)$ we have:
$\Phi_3(w^1) = \frac{1}{3}w^1_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^1_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^1_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^1_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^1_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^1_{\{3\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^1_{\{1,2\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^1_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^1_{\{2\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^1_{\{3\},\{\{1\},\{2\},\{3\}\}}$
Thus, for the general case we get:
$\Phi_3(w^1) = \frac{1}{3}P + \frac{1}{6}a - \frac{1}{3}(P-a) + \frac{1}{6}(P-a+\varepsilon) - \frac{1}{3}(P-a+\varepsilon) + \frac{2}{3}0 - \frac{2}{3}(P-a+\varepsilon)$
$+\frac{1}{6}(P-a+\varepsilon) + \frac{1}{6}(P-a+\varepsilon) - \frac{1}{3}0 = \frac{2a-P-\varepsilon}{2}$,
In particular:
$\Phi_3(w^1) = \frac{1}{3}4 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{1}{6}1.3 - \frac{1}{3}1.3 + \frac{2}{3}0 - \frac{1}{3}2.6 + \frac{1}{6}1.3 + \frac{1}{6}1.3 - \frac{1}{3}0 = 0.85$

Summarizing:

In particular,
$\Phi^1 = \Phi(w^1) = (2.15, 1, .85)$
In general, we can claim:

Lemma 3

$\Phi_3(w^1) = \frac{2a-P-\varepsilon}{2}$ is a decreasing function of ε and $\Phi_1(w^1)$ is an increasing function of ε

For graph 3, **w**e will have in our example for the partition that consists of $\{\{1\},\{2\},\{3\}\}$:

$w^3_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1.3$
$w^3_{\{2\},\{\{1\},\{2\},\{3\}\}} = 1.3$
$w^3_{\{3\},\{\{1\},\{2\},\{3\}\}} = 0$

If the two ranchers collude:
$w^3_{\{1,2\},\{\{1,2\},\{3\}\}} = 3\frac{3}{9}$
$w^3_{\{3\},\{\{1,2\},\{3\}\}} = 0$

If rancher 1 colludes with the enforcer we get:
$w^3_{\{1,3\},\{\{1,3\},\{2\}\}} = 1.3$
$w^3_{\{2\},\{\{1,3\},\{2\}\}} = 1.3$

Thus, we have dont have externalities in coalition formation as $1.3 = w^3_{\{2\},\{\{1,3\},\{2\}\}} = w^3_{\{2\},\{\{1\},\{2\},\{3\}\}} = 1.3$, i.e. the value of player 2 acting alone is independent on the coalition structure.

If rancher 2 colludes with the enforcer we get:
$w^3_{\{2,3\},\{\{2,3\},\{1\}\}} = 1.3$
$w^3_{\{1\},\{\{2,3\},\{1\}\}} = 1.3,$

Thus, we have dont have externalities in coalition formation as $1.3 = w^3_{\{1\},\{\{2,3\},\{1\}\}} = w^3_{\{1\},\{\{1\},\{2\},\{3\}\}} = 1.3$, i.e. the value of player i acting alone is independent on the coalition structure.

The last partition, the grand coalition has one element, itself that is worth 4.
$w^3_{\{1,2,3\},\{\{1,2,3\}\}} = 3\frac{3}{9}$

Now we compute the extended Myerson values for graph 3:
$\Phi_1(w^3) = \frac{1}{3}w^3_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^3_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^3_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^3_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^3_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{2}{3}w^3_{\{1\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^3_{\{2,3\},\{\{2,3\},\{1\}\}}$
$+\frac{1}{6}w^3_{\{2\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^3_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^3_{\{1\},\{\{1\},\{2\},\{3\}\}}$
$\Phi_1(w^3) = \frac{1}{3}3\frac{3}{9} + \frac{1}{6}3\frac{3}{9} - \frac{1}{3}0 + \frac{1}{6}1.3 - \frac{1}{3}1.3 + \frac{2}{3}1.3 - \frac{1}{3}1.3 + \frac{1}{6}1.3 + \frac{1}{6}0 - \frac{1}{3}1.3 = 1.6667$

Similarly for $\Phi_2(w^3)$, we get:
$\Phi_2(w^3) = \frac{1}{3}w^3_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^3_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^3_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^3_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^3_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^3_{\{2\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^3_{\{1,3\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^3_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^3_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^3_{\{2\},\{\{1\},\{2\},\{3\}\}}$
$\Phi_2(w^3) = \frac{1}{3}4 + \frac{1}{6}2.6 - \frac{1}{3}0 + \frac{1}{6}1.3 - \frac{1}{3}1.3 + \frac{2}{3}1 - \frac{1}{3}3 + \frac{1}{6}1.3 + \frac{1}{6}0 - \frac{1}{3}1.3 = 1.0$

Finally for $\Phi_3(w^3)$ we have:
$\Phi_3(w^3) = \frac{1}{3}w^3_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^3_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^3_{\{2\},\{\{1,3\},\{2\}\}}$

$+\frac{1}{6}w^3_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^3_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^3_{\{3\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^3_{\{1,2\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^3_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^3_{\{2\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^3_{\{3\},\{\{1\},\{2\},\{3\}\}}$
Thus:
$\Phi_3(w^3) = \frac{1}{3}3\frac{3}{9} + \frac{1}{6}1.3 - \frac{1}{3}1.3 + \frac{1}{6}1.3 - \frac{1}{3}1.3 + \frac{2}{3}0 - \frac{1}{3}3\frac{3}{9} + \frac{1}{6}1.3 + \frac{1}{6}1.3 - \frac{1}{3}0 = -1.8974 \times 10^{-19} + 1.6667 * 2 = 3.333$

Summarizing:

In particular:
$\Phi^3 = \Phi(w^3) = (1\frac{2}{3}, 1\frac{2}{3}, 0)$

In general we have that the Myerson value for the identical ranchers is constant and equal to $\frac{m}{2}$ as in the case without ECF. This follows as before from the fairness axiom of the Myerson value.

Thus,

Lemma 4

$\Phi_{i=1,2}(w^3) = \frac{m}{2}$

Lemma 5

Given a and m such that $\vartheta(a) = \frac{3P-2a}{2} < m$,the enforcer would prefer a game without ECF to one with negative ECF.

Proof: Following lemma 2, for ε big enough the principal, player 3 would prefer to accept another link (and thus the complete graph will form) because his shapley value would be higher than his Myerson value if only colluded with one of the identical agents. In other words, the assumptions of lemma 1 would not hold for links 13 or 23 any more. As $\frac{m}{2} = \Phi_{i=1,2}(w^3) > \Phi_{i=1,2}(w^N) = \frac{3P+m-2a-\varepsilon}{6}$ for all ε (given that $\Phi_1(w^N)$ is a decreasing function of ε) for ε big enough only the two identical agent coalition forms.

Given that in any case $\Phi_3(w^1(\varepsilon = 0))$is strictly bigger than $\Phi_3(w^N(\varepsilon) > \Phi_3(w^3(\varepsilon)) = 0$ for all relevan ε, it is clear that the enforcer would not induce a game where the two identical agents collude as for lemma 4 his payoff would be zero, i.e., he would not choose a high enough ε. Neither he would choose a lower ε,because if the grand coalition forms his Myerson value in $\Phi_3(w^1) = a - \frac{P}{2} > \Phi_3(w^N) = \frac{2a+\varepsilon-m}{3}$ for all ε with that characteristic, Thus, he would choose $\varepsilon = 0$■

Positive Externalities in Coalition Formation Now we want to analize values for the singletons such that they are strictly lower than 1, say .7 and .4. We will have for the partition that consists of $\{\{1\},\{2\},\{3\}\}$:
$w^N_{\{1\},\{\{1\},\{2\},\{3\}\}} = 0.7$
$w^N_{\{2\},\{\{1\},\{2\},\{3\}\}} = 0.7$
$w^N_{\{3\},\{\{1\},\{2\},\{3\}\}} = 0$
If the two ranchers collude:
$w^N_{\{1,2\},\{\{1,2\},\{3\}\}} = 3\frac{3}{9}$

$w^N_{\{3\},\{\{1,2\},\{3\}\}} = 0$

In rancher i colludes with the enforcer we get:

$w^N_{\{i,3\},\{\{i,3\},\{j\}\}} = 3$

$w^N_{\{j\},\{\{i,3\},\{j\}\}} = 1$, for $i = 1, 2$ $j \in \{1, 2\}$ $i \neq j$

Thus, we have externalities in coalition formation as $1 = w^N_{\{j\},\{\{i,3\},\{j\}\}} \neq w^N_{\{j\},\{\{i\},\{j\},\{3\}\}} = 0.7$, i.e. the value of player i acting alone is dependent on the coalition structure.

The last partition, the grand coalition has one element, itself that is worth 4.

$w^N_{\{1,2,3\},\{\{1,2,3\}\}} = 4$

Let $\Phi_1(w^N)$ be the extended Shapley value for games with externalities in coalition formation for player 1 for the complete graph N. Following Myerson (1978) we have:

$\Phi_1(w^N) = \frac{1}{3}w^N_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^N_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^N_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^N_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^N_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{2}{3}w^N_{\{1\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^N_{\{2,3\},\{\{2,3\},\{1\}\}}$
$+\frac{1}{6}w^N_{\{2\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^N_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^N_{\{1\},\{\{1\},\{2\},\{3\}\}}$

Plugging in values we get:

$\Phi_1(w^N) = \frac{1}{3}4 + \frac{1}{6}3\frac{3}{9} - \frac{1}{3}0 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{2}{3}1 - \frac{1}{3}3 + \frac{1}{6}0.7 + \frac{1}{6}0 - \frac{1}{3}0.7 = 1.6056$

For $v(1) = v(2) = 0.4$, we have

$\Phi_1(w^N) = \frac{1}{3}4 + \frac{1}{6}3\frac{3}{9} - \frac{1}{3}0 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{2}{3}1 - \frac{1}{3}3 + \frac{1}{6}0.4 + \frac{1}{6}0 - \frac{1}{3}0.4 = 1.6556$

Similarly for $\Phi_2(w^N)$, we get:

$\Phi_2(w^N) = \frac{1}{3}w^N_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^N_{\{1,2\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^N_{\{3\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^N_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^N_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^N_{\{2\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^N_{\{1,3\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^N_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^N_{\{3\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^N_{\{2\},\{\{1\},\{2\},\{3\}\}}$

$\Phi_2(w^N) = \frac{1}{3}4 + \frac{1}{6}3\frac{3}{9} - \frac{1}{3}0 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{2}{3}1 - \frac{1}{3}3 + \frac{1}{6}0.7 + \frac{1}{6}0 - \frac{1}{3}0.7 = 1.6056$

Finally for $\Phi_3(w^N)$ we have:

$\Phi_3(w^N) = \frac{1}{3}w^N_{\{1,2,3\},\{\{1,2,3\}\}}$
$+\frac{1}{6}w^N_{\{1,3\},\{\{1,3\},\{2\}\}} - \frac{1}{3}w^N_{\{2\},\{\{1,3\},\{2\}\}}$
$+\frac{1}{6}w^N_{\{2,3\},\{\{2,3\},\{1\}\}} - \frac{1}{3}w^N_{\{1\},\{\{2,3\},\{1\}\}}$
$+\frac{2}{3}w^N_{\{3\},\{\{1,2\},\{3\}\}} - \frac{1}{3}w^N_{\{1,2\},\{\{1,2\},\{3\}\}}$
$+\frac{1}{6}w^N_{\{1\},\{\{1\},\{2\},\{3\}\}} + \frac{1}{6}w^N_{\{2\},\{\{1\},\{2\},\{3\}\}}$
$-\frac{1}{3}w^N_{\{3\},\{\{1\},\{2\},\{3\}\}}$

Thus:

$\Phi_3(w^N) = \frac{1}{3}4 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{1}{6}3 - \frac{1}{3}1 + \frac{2}{3}0 - \frac{1}{3}3\frac{3}{9} + \frac{1}{6}0.7 + \frac{1}{6}0.7 - \frac{1}{3}0 = 0.78889$

Summarizing: $\Phi^N = \Phi(w^N) = (1.6056, 1.6056, 0.78889)$

For $v(1) = v(2) = 0.4$, we have

$\Phi^N = \Phi(w^N) = (1.655\,6, 1.655\,6, 0.6888)$

We want to calculate the corresponding Shapley values for different graph structures.

For graph 1 we will have a game in partition function form that would correspond to a vector in $w^{g=1} \in \mathbb{R}^{ECL}$.

Summarizing: $\Phi^1 = \Phi(w^1) = (1.85, 1, 1.15)$

For $v(1) = v(2) = 0.4$, we have $\Phi^1 = \Phi(w^1) = (1.7, 1, 1.233\,3)$

For graph 3 we will have:

Summarizing: $\Phi^3 = \Phi(w^3) = (1\frac{2}{3}, 1\frac{2}{3}, 0)$

For $v(1) = v(2) = 0.4$, we have also $\Phi^3 = \Phi(w^3) = (1\frac{2}{3}, 1\frac{2}{3}, 0)$

Thus, with variable harrasing ability, the principal would induce a game with positive externalities in coalition formation. It is easy to check that for $v(1) = v(2) = 0$, the shapley value for the principal in the grand coalition is the lowest. Thus, in any case the grand coalition would never form. As for our numerical results, the best the principal can do is to set aproximately $v(1) = v(2) = 0.4$ (or $\varepsilon = .6$; see lemma 6 for the optimal ε). Also, note that even when the individual Myerson Value for the identical agents in graph 3 are better than their corresponding values in the Shapley value of the grand coalition,one of the agents in link 12 will block and again only one of the identical agenst will end up colluded with the enforcer (as in the proof of proposition 2).

Lemma 6: The enforcer would prefer a game with positive ECF than one without ECF. Actually, the optimal value of $\varepsilon = m - P$.

Proof:

As for lemma 2 and 3 there there will be a unique ε that maximizes the payoffs for the enforcer if colluded with one of the indentical agents. The value of ε is such that $\frac{m}{2} = \Phi_{i=1,2}(w^3) = \Phi_1(w^1) = \frac{P+\varepsilon}{2}$. Thus $\varepsilon = m - P$■

As an striking illustration, the optimal payoffs where the enforcer maximizes his in our example yield the following triplet.$(2\frac{1}{3}1, 1\frac{1}{3})$(only link 13 forms). Recall $1\frac{1}{3}$ is the payoff when $m = 2$, the core was non empty and and the grand coalition formed!!!. In general we have theorem 3:

Theorem 3: Let a be given. If m is variable and with lower bound $\vartheta(a)$, the principal would prefer a game with positive ECF than one without ECF. Actually, the optimal $m \longrightarrow \vartheta(a)$ and the optimal value of $\varepsilon \longrightarrow \vartheta(a) - P = \frac{P-2a}{2}$, with payoffs $\Phi_3(w^1) = \frac{2a-m}{2} \longrightarrow \frac{6a-3P}{4}$.

Proof:

It follows from the fact that the Myerson value of the principal in his coalition with any identicagent is decreasing in m.As from lemma 3, $\Phi_3(w^1) = \frac{2a-m}{2}$ when ε is optimal given m.i.e., $\varepsilon = m - P$■

In our example $\Phi_3(w^1) \longrightarrow 2\frac{1}{2}$.

It would be interesting to see if this latter result (that the principal prefers games with positive externalities in coalition formation) holds with the extension of the

Aumann-Myerson solution in Nieva (December 2002). Finally, in thinking of picking from the individual rational and feasible set, it is clear that in the numerical example, the principal will choose the values 3 and 1 for her coalition with any identical agent and the value of the left out rancher respectively in the extortion game (again we are assuming a technological barrier as we did in the first appendix). The argument for the general case is analogous as when there are no ECF.

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