Efficient and fair assignment mechanism is strongly group manipulable

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Abstract

This paper studies the allocation of indivisible objects to agents without using monetary transfers. Fairness requires social planners to use random assignments. However, I show that if a mechanism satisfies a minimum efficiency requirement and some mild fairness requirements, it must be manipulable by a group of agents in a strong sense: by misreporting preferences each agent of the group can obtain a lottery that strictly first-order stochastically dominates the lottery he would obtain in the truth-telling case. My result holds as long as there are at least three agents and at least three objects, no matter outside option exists or not. Non-manipulability results exist when there are only two objects and outside option does not exist.

Keywords: Random assignment, Efficiency, Fairness, Strong group manipulation

JEL Classification: C78; D71; D78

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1 Introduction

Optimal allocation of resources to agents is one of the core issues in economics. In many markets resources are indivisible and monetary transfers are not allowed. Examples include assigning medical residents to hospitals (Roth, 1984), assigning children to public schools (Abdulkadiroglu and Sönmez, 2003), assigning on-campus houses to students (Hylland and Zeckhauser, 1979), and so on. These markets have been studied a lot by the matching theory literature. This paper studies the simplest form of the problem called object allocation problem: agents have strict preferences on objects and each demands only one object, while objects do not have priority rankings of agents. In reality, social planners often first require agents to report ordinal preferences, then run a centralized procedure (a mechanism) to find an assignment.

Efficiency is a primary goal in economics. In solving the object allocation problem efficiency requires social planners to find assignments that are as good as possible for agents. Fairness is a normative goal that is often imposed by institutional or ethical constraints. Since objects are indivisible, fairness often requires social planners to use random assignments. That is, social planners have to use a mechanism that involves randomization in its procedure such that from an ex-ante view, each agent receives a lottery, which specifies his chance of obtaining each object, although ex-post each object is obtained by at most one agent.

My objective in this paper is to show that if a mechanism satisfies a minimum efficiency criterion and some mild fairness criteria, it must be manipulable by a group of agents in a strong sense: by misreporting preferences, each agent of a manipulating group can obtain a lottery that strictly first-order stochastically dominates the lottery he would obtain in the truth-telling case. My result holds no matter whether outside option exists or not.\footnote{When outside option exists, agents are allowed to report a preference ordering that lists the outside option above an object.} When outside option exists, the manipulating group can be as small as containing two agents.

To illustrate my result, let us consider Random Serial Dictatorship (RSD) and Probabilistic Serial (PS), two mechanisms that have been studied a lot in the literature.

- RSD: Randomly draw an ordering of agents from the uniform distribution. Then
let agents sequentially choose their most preferred objects from the remaining ones according to the drawn ordering.

• PS: Imagine all objects as “divisible” cakes. Let agents “eat” their most preferred objects with equal speeds. If an object is exhausted, let relevant agents eat their next most preferred objects among the remaining ones until each agent’s total consumption adds up to one. The fraction of an object that an agent eats is the probability that he obtains the object.

The fairness in RSD is achieved by drawing the ordering of agents uniformly at random, while the fairness in PS is achieved by giving all agents equal eating speeds. Now I show that both mechanisms are strongly group manipulable. Consider a problem consisting of three agents $i_1, i_2, j$ and three objects $o_1, o_2, o_3$. Consider the following two preference profiles:

\[
\begin{array}{ccc}
\succ^*_i & \succ^{*}_i & \succ^{*}_j \\
o_1 & o_1 & o_2 \\
o_2 & o_2 & o_3 \\
o_3 & o_3 & o_1 \\
\vdots & \vdots & \vdots \\
(a) \succ^*_f & & \\
\end{array}
\]

\[
\begin{array}{ccc}
\succ^0_i & \succ^{0}_i & \succ^{0}_j \\
o_1 & o_2 & o_2 \\
o_2 & o_1 & o_3 \\
o_3 & o_3 & o_1 \\
\vdots & \vdots & \vdots \\
(b) \succ^0_f & & \\
\end{array}
\]

For the two preference profiles, RSD and PS find same assignments shown below. At $\succ^*_f$, $i_1, i_2$ obtain the lottery $(1/2o_1, 1/3o_2, 1/6o_3)$, which strictly first-order stochastically dominates the lottery $(1/2o_1, 1/6o_2, 1/3o_3)$ they obtain at $\succ^*_f$. Therefore, $i_1, i_2$ can strongly group manipulate the two mechanisms at $\succ^*_f$ by reporting $(\succ^0_i, \succ^0_{i_2})$.

\[
\begin{array}{ccc}
& i_1 & i_2 & j \\
o_1 & 1/2 & 1/2 & \\
o_2 & 1/6 & 1/6 & 2/3 \\
o_3 & 1/3 & 1/3 & 1/3 \\
\vdots & \vdots & \vdots & \vdots \\
(a) \text{RSD}(\succ^*_f)=\text{PS}(\succ^*_f) \\
\end{array}
\]

\[
\begin{array}{ccc}
& i_1 & i_2 & j \\
o_1 & 1/2 & 1/2 & \\
o_2 & 1/3 & 1/3 & 1/3 \\
o_3 & 1/5 & 1/5 & 2/3 \\
\vdots & \vdots & \vdots & \vdots \\
(b) \text{RSD}(\succ^0_f)=\text{PS}(\succ^0_f) \\
\end{array}
\]

It is known that RSD is \textit{ex-post efficient}, while PS is \textit{ordinally efficient}. Ex-post efficiency requires a random assignment to be a probability distribution on Pareto efficient
deterministic assignments. It is weaker than ordinal efficiency and ex-ante efficiency. Since in practical applications Pareto efficiency is a minimum efficiency requirement for deterministic assignments, ex-post efficiency is a reasonable minimum efficiency requirement for random assignments. Therefore, I use ex-post efficiency as the efficiency criterion in my theorems.

RSD satisfies two fairness criteria known as \textit{equal treatment of equals} and \textit{weak envy-freeness}, while PS satisfies a stronger one known as \textit{envy-freeness}.\footnote{Equal treatment of equals requires that if any two agents report same preferences, they obtain equal probability of each object. Weak envy-freeness requires that any agent do not think any other agent’s lottery strictly first-order stochastically dominates his, while envy-freeness requires that each agent think his lottery weakly first-order stochastically dominates any other’s.} The fairness criteria in my theorems are a combination of these existing criteria and three new ones proposed by this paper. RSD and PS satisfy the three new criteria. Therefore, my theorems provide fundamental explanations for the strong group manipulability of RSD and PS, and uncover the tension between fairness and group incentive compatibility in the presence of minimum efficiency criterion. Below I simply introduce the three new fairness criteria.

\textit{Equal top-assignment of equal tops} requires that if any two agents report the same object as most preferred, then they obtain equal probability of the object. It is similar to equal treatment of equals, but its “equal” condition is only applied to agents’ top choice. \textit{Top advantage} requires that if an agent obtains positive probability of an object he does not most prefer, then any other agent who most prefers the object should obtain a higher probability of the object than the former agent. Therefore, top advantage requires that agents have advantage in the assignments of their top choice, and can be seen as a complement of equal top-assignment of equal tops. \textit{Uniform tail-assignment of uniform tails} requires that if all agents prefer a subset of objects to the remaining objects (i.e., agents have equal upper contour set of some object), and have equal preferences on the remaining objects (i.e., agents have equal preference tails), then they receive equal probability of each remaining object. In other words, the differences in agents’ lotteries should only exist in the upper contour set on which they have different preferences. Top advantage is independent of envy-freeness and other criteria, while the other two are weaker than envy-freeness and independent of equal treatment of equals or weak envy-freeness.

My first result (Section 3) proves that in the simple environment of three agents and
at least three objects, no matter outside option exists or not, any ex-post efficient mechanism that satisfies equal top-assignment of equal tops, top advantage, and equal treatment of equals must be strongly group manipulable. Here equal treatment of equals can be replaced by weak envy-freeness. In this simple environment, it has been proved that RSD is the only ex-post efficient and strategy-proof mechanism that satisfies equal treatment of equals (Bogomolnaia and Moulin, 2001) or weak envy-freeness (Nesterov, 2017). Therefore, in this environment when ex-post efficiency and equal treatment of equals are present, individual incentive compatibility must result in strong group manipulability.

My second result (Section 4) studies the environment of at least four agents and at least three objects. If outside option exists, the first result still holds. It is because when some three agents prefer some three objects that the other agents regard as worse than outside option, then the three agents essentially form an isolated problem and the previous proof applies. However, if outside option does not exist, I prove that a mechanism is still strongly group manipulable if it further satisfies uniform tail-assignment of uniform tails. Since envy-freeness implies all the fairness criteria used in my results except for top advantage, there is a corollary that when there are at least three agents and at least three objects, no matter outside option exists or not, any ex-post efficient mechanism that satisfies envy-freeness and top advantage is strongly group manipulable.

My last result (Section 5) studies the environment of only two objects. This completes my analysis of all possible situations. Interestingly, I prove that when there are at least three agents and outside option does not exist, any ex-post efficient mechanism that satisfies equal treatment of equals and top advantage must be not strongly group manipulable. If the mechanism further satisfies uniform tail-assignment of uniform tails, I prove that it must be group strategy-proof. This is an interesting complement of Bade’s (2016a) result that when there are at least three agents and at least three objects, no matter outside option exists or not, any ex-post efficient mechanism that satisfies equal treatment of equals is not group strategy-proof. Nevertheless, if outside option exists, the previous manipulability results still hold in the two-object environment.

In Section 6 I discuss independence of axioms in my theorems, other restricted preference domains, and some other issues.

Related Literature
Group strategy-proofness has been studied a lot for deterministic mechanisms (Svensson,
1999; Pápai, 2000; Pycia and Ünver, 2017). However, it is rarely discussed for random mechanisms. To my best knowledge, Bade (2016a) provides the only closely related study. Bade proves that when there are at least three agents and at least three objects, no matter outside option exists or not, any ex-post efficient mechanism that satisfies equal treatment of equals is not group strategy-proof: by misreporting preferences, some members of a manipulating group obtain lotteries that are not strictly first-order stochastically dominated by the lotteries they would obtain in the truth-telling case, while the remaining members of the group obtain lotteries same as the truth-telling case. Group strategy-proofness is a strong requirement: no matter what cardinal utilities agents have behind their preferences, by misreporting preferences no member of any group can obtain higher expected utilities without hurting others in the group. Therefore, the statement that a mechanism is not group strategy-proof is weak. For example, PS is known to be ex-post efficient and satisfy equal treatment of equals, but not to be strategy-proof. Thus, Bade’s result does not provide new knowledge on the group incentive property of PS.

By contrast, by using more fairness criteria than Bade I prove stronger manipulability theorems. They imply that no matter what cardinal utilities agents have behind their preferences, all members of a manipulating group must obtain higher expected utilities than truth telling by misreporting preferences. In particular, although PS is not strongly manipulable by individuals (because PS is weakly strategy-proof), my theorems imply that PS is strongly manipulable by groups. Moreover, in the two-object environment I prove an interesting complement of Bade’s result by using new fairness criteria.

Multiple papers have identified conditions under which strategy-proofness is equivalent to weak group strategy-proofness for deterministic mechanisms. Barberà et al. (2016) study general private good economies that subsume matching, division, object allocation, and auction as special cases. The random assignment problem studied here can be accommodated by their model if we let lotteries be the outcomes assigned to agents. However, then the mechanisms they study would require agents to report complete preferences on lotteries. In other words, their mechanisms require more information than ordinal preferences, which is beyond the focus of this paper. So my results do not conflict with theirs. For earlier studies in this strand, the reader can refer to Barberà and Jackson (1995); Barberà et al. (2010). Alva (2017) identify conditions under which group manipulation is equivalent to individual and pair manipulation.
Zhou (1990); Bogomolnaia and Moulin (2001); Martini (2016); Nesterov (2017) have proved multiple impossibility theorems regarding the incompatibility of efficiency, fairness, and strategy-proofness. Zhou proves that when there are at least three agents, any ex-ante efficient and symmetric (agents of identical cardinal utilities obtain equal expected utility) mechanism is not strategy-proof. BM prove that when there are at least four agents, any ordinally efficient mechanism that satisfies equal treatment of equals is not strategy-proof. Martin strengthens BM's result by weakening ordinal efficiency to non-wastefulness. Nesterov proves that when there are at least three agents, any ex-post efficient and upper-envy-free mechanism is not upper-shuffle-proof. Upper-envy-freeness are stronger than equal-top assignment of equal tops, equal treatment of equals, and uniform tail-assignment of uniform tails, while upper-shuffle-proofness is weaker than strategy-proofness. Therefore, he has the corollary that any ex-post efficient mechanism that satisfies envy-freeness is not strategy-proof. It is interesting to compare this corollary to mine which obtains strong group manipulability by adding the top advantage requirement. Zhou, BM and Nesterov all assume that objects are as many as agents. This assumption allows them to easily construct an isolated small-size problem in a large-size environment. Martin does not make this assumption, but he assumes that outside option exists, which is indispensable for his result.

2 Definition

2.1 Object allocation problem

A finite set of heterogeneous indivisible objects $O$ is assigned to a finite set of agents $I$. I assume that there are at least two objects and at least three agents. That is, $|O| \geq 2$ and $|I| \geq 3$. Each $i \in I$ demands only one object and each $o \in O$ has only one copy. When an agent does not obtain an object, I say he obtains $\emptyset$. So $\emptyset$ plays the role of outside option. Let $\hat{O} \equiv O \cup \{\emptyset\}$. Each $i \in I$ has a strict preference ordering $\succ_i$ of $\hat{O}$, with the associated weak ordering denoted by $\succsim_i$. Let $SU(\succ_i, o) \equiv \{o' \in \hat{O} : o' \succ_i o\}$ be the strict upper contour set of $o$ at $\succ_i$. Let $\top(\succ_i)$ be the most preferred object of $\succ_i$. Let $\succ_i |_{O'}$ be the restriction of any $\succ_i$ to any $O' \subseteq \hat{O}$. That is, for any $o, o' \in O'$, $o \succ_i |_{O'} o'$ if and only if $o \succ_i o'$. Any $o$ is acceptable to $i$ if $o \succsim_i \emptyset$. For any $J \subseteq I$, let $\succ_J \equiv \{\succ_i\}_{i \in J}$ be the preference profile of $J$. 

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A deterministic assignment is a function $\pi : I \rightarrow \hat{O}$ such that $\pi(i) = \pi(j)$ for any distinct $i, j \in I$ if and only if $\pi(i) = \pi(j) = \emptyset$. Let $\Pi$ be the set of all deterministic assignments. A random assignment is a probability distribution $p$ on $\Pi$. For any $p, p_i \in \Delta\hat{O}$ is the lottery assigned to $i$, and $p_{i,o}$ is the probability that $i$ obtains $o$.

Hence, $\Delta\Pi$ is the set of all random assignments. For any two lotteries $p_i, q_i \in \Delta\hat{O}$, $p_i$ strictly first-order stochastically dominates $q_i$ for any $i$, denoted by $p_i >_i q_i$, if $\sum_{o \succ_i o'} p_i(o') \geq \sum_{o' \succ_i o} q_i(o')$ for all $o \in \hat{O}$ and $\sum_{o' \succ_i o} p_i(o') > \sum_{o' \succ_i o} q_i(o')$ for some $o \in \hat{O}$. I use $p_i \succeq_i q_i$ to denote that either $p_i >_i q_i$ or $p_i = q_i$, and use $p_i \not>_i q_i$ and $p_i \not\succeq_i q_i$ to denote that $p_i >_i q_i$ and $p_i \succeq_i q_i$ do not hold respectively.

Let $\mathcal{P}$ be any domain of agents’ preferences. In this paper I focus on two preference domains: the unrestricted domain $\mathcal{R}$, which contains all strict preference orderings of $\hat{O}$, and the no outside-option domain $\mathcal{Q}$, which contains all strict preference orderings of $\hat{O}$ that rank $\emptyset$ as worst. Given any $\mathcal{P}$, a random mechanism is a function $\rho : \mathcal{P}^{\mid I\mid} \rightarrow \Delta\Pi$ such that $\rho(\succ_I)$ is the random assignment found by $\rho$ for any $\succ_I$.

## 2.2 Group manipulation

Given a mechanism, some agents may want to manipulate the mechanism by reporting non-truthful preferences. The literature has discussed a lot about individual manipulation. In this paper I study group manipulation and define two related concepts. In the first one, a group of agents is said to weakly manipulate a mechanism if by misreporting preferences some members of the group obtain new lotteries that are not strictly first-order stochastically dominated by the lotteries they obtain in the truth-telling case, while the other members obtain lotteries same with the truth-telling case. In the second one, a group is said to manipulate a mechanism if by misreporting preferences every member of the group obtains a new lottery that strictly first-order stochastically dominates the lottery he obtains in the truth-telling case.

Formally, a group $J \subseteq I$ weakly group manipulate a mechanism $\rho$ at $\succ_I$ if by reporting some $\succ'_J$, the set $\{i \in J : \rho_i(\succ_I) \neq \rho_i(\succ_{I \setminus J}, \succ'_J)\}$ is nonempty, and for every $i$ in the set, $\rho_i(\succ_I) \not>_i \rho_i(\succ_{I \setminus J}, \succ'_J)$. $\rho$ is group strategy-proof if it is not weakly group manipulable. When $J$ is restricted to be singleton, group strategy-proofness reduces to the usual notion of strategy-proofness. A group $J \subseteq I$ strongly group manipulate a mechanism $\rho$ at $\succ_I$ if by reporting some $\succ'_J$, $\rho_i(\succ_{I \setminus J}, \succ'_J) >_i \rho_i(\succ_I)$ for all $i \in J$. $\rho$
is **minimally group strategy-proof** if it is not strongly group manipulable. When $J$ is restricted to be singleton, minimal group strategy-proofness reduces to the notion of **weak strategy-proofness** defined by Bogomolnaia and Moulin (2001).

Strong group manipulation is a much stronger concept than weak group manipulation. If agents have von Neumann–Morgenstern utilities behind their preferences, then strong group manipulation requires that for all vNM utilities consistent with agents’ preferences, by misreporting preferences each member of the manipulating group can have a higher expected utility than truth telling, while weak group manipulation only requires that for some consistent vNM utilities, by misreporting preferences some members of the manipulating group can have higher expected utilities than truth telling, while the other members receive same lotteries as truth telling.

In fact, strong group manipulation is the strongest manipulation concept I can figure out in the ordinal environment. To illustrate it, I define two intermediate concepts. At any $\succ_I$, a group $J$ is said to $\alpha$-**group manipulate** a mechanism $\rho$ at $\succ_I$ if by reporting some $\succ'_J$, $\rho_i(\succ_I) \not\geq_i \rho_i(\succ_{I \setminus J}, \succ'_J)$ for all $i \in J$. This means that for some consistent vNM utilities, every member of $J$ can have a higher expected utility than truth telling by misreporting preferences. Therefore, this concept is stronger than weak group manipulation. But it is weaker than strong group manipulation. At any $\succ_I$, a group $J$ is said to $\beta$-**group manipulate** a mechanism $\rho$ at $\succ_I$ if by reporting some $\succ'_J$, $\rho_i(\succ_{I \setminus J}, \succ'_J) \geq_i \rho_i(\succ_I)$ for all $i \in J$, and $\rho_j(\succ_{I \setminus J}, \succ'_J) >_j \rho_j(\succ_I)$ for at least one $j \in J$. This means that for all consistent vNM utilities, all members of $J$ have weakly higher expected utilities than truth telling, and at least one member has a strict higher expected utility than truth telling. Therefore, this concept is weaker than strong group manipulation. But it is stronger than weak group manipulation. So their relations can be summarized as follows.

\[
\text{strong group manipulation} \quad \overset{\alpha}{\leftarrow} \quad \text{\alpha-group manipulation} \quad \overset{\beta}{\rightarrow} \quad \text{\beta-group manipulation} \quad \overset{\rho}{\rightarrow} \quad \text{weak group manipulation}
\]

In the following I will define ex-post efficiency and several fairness criteria. A mechanism $\rho$ is said to satisfy an efficiency or fairness criterion if $\rho(\succ_I)$ satisfies the criterion for all $\succ_I$ in the given preference domain.
2.3 Ex-post efficiency

For any $\succ_i$, a deterministic assignment $\pi$ is **Pareto efficient (PE)** if there does not exist another $\pi'$ such that $\pi'(i) \succeq_i \pi(i)$ for all $i$ and $\pi'(j) \succ_j \pi(j)$ for some $j$. A random assignment $p$ is **ex-post efficient (ExPE)** if for any $\pi$ such that $p(\pi) > 0$, $\pi$ is Pareto efficient. In an ExPE mechanism, every agent must only obtain positive probabilities of acceptable objects. In the literature there are stronger efficiency criteria such as ordinal efficiency and ex-ante efficiency. In this paper I choose ExPE for two reasons. First, because PE is often a basic requirement for deterministic assignments, ExPE is often a basic requirement for random assignments and pursued by social planners. Second, since ExPE is the minimum efficiency requirement, sticking to it can highlight the tension between fairness and group incentive compatibility.

2.4 Fairness

For any random assignment $p$ and any $\succ_i$, the following are three fairness criteria that have been focused in the literature.

1. $p$ is **envy-free (EF)** if $p_i \geq_i p_j$ for any distinct $i, j$. That is, every $i$ thinks that his lottery is weakly better than any other $j$’s.

2. $p$ is **weakly envy-free (wEF)** if $p_j \not\succ_i p_i$ for any distinct $i, j$. That is, every $i$ does not think that any other $j$’s lottery is strictly better than his.

3. $p$ satisfies **equal treatment of equals (ETE)** if $\succ_i = \succ_j$ for any distinct $i, j$ implies $p_i = p_j$. That is, any two agents of equal preferences obtain equal lotteries.

EF is a strong requirement since it totally eliminates (either strong or weak) envies, while wEF only eliminates strong envies. By contrast, ETE is derived from a different idea that in a fair assignment agents should be treated symmetrically. A basic symmetry requirement is that if agents are “equal” (identified by equal preferences), then they should receive equal lotteries.

Below I introduce three new fairness criteria. As proved in my theorems, they are crucial for efficient and fair mechanisms to be strongly group manipulable.

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3I say $i$ strongly envies $j$ in a random assignment $p$ if $p_j >_i p_i$, while $i$ weakly envies $j$ if $p_i \not>_i p_j$
4. \( p \) satisfies **equal top-assignment of equal tops (ETAET)** if \( \text{top}(\succ_i) = \text{top}(\succ_j) \) for any distinct \( i, j \) implies that \( p_{i,\text{top}(\succ_i)} = p_{j,\text{top}(\succ_j)} \). That is, if any two agents mostly prefer the same object, then they obtain equal probabilities of the object.

5. \( p \) satisfies **top advantage (TA)** if \( \text{top}(\succ_i) \neq \text{top}(\succ_j) \) and \( p_{j,\text{top}(\succ_i)} > 0 \) for any distinct \( i, j \) imply that \( p_{i,\text{top}(\succ_i)} > p_{j,\text{top}(\succ_i)} \).

That is, if any \( j \) obtains positive probability of some object that he does not most prefer, then any \( i \) who most prefers the object must obtain a higher probability of the object than \( j \).

6. \( p \) satisfies **uniform tail-assignment of uniform tails (UTAUT)** if \( SU(\succ_i, o) = SU(\succ_j, o) \) and \( \succ_i |_{O \setminus SU(\succ_i, o)} = \succ_j |_{O \setminus SU(\succ_j, o)} \) for some \( o \in \hat{O} \) and every distinct \( i, j \) imply that \( p_{i,o'} = p_{j,o'} \) for all \( o' \in \hat{O} \setminus SU(\succ_i, o) \).

That is, if all agents prefer the objects in \( SU(\succ_i, o) \) to the objects in \( \hat{O} \setminus SU(\succ_i, o) \), and their preferences on \( \hat{O} \setminus SU(\succ_i, o) \) are equal, then they obtain equal probabilities of each object in \( \hat{O} \setminus SU(\succ_i, o) \).

ETAET is also derived from the symmetry idea aforementioned, but it only requires symmetry in the assignment of agents’ top choices. It is independent of ETE and wEF, and is implied by EF.

TA can be seen as a complement of ETAET. It requires that agents have advantage at the assignments of their top choices than other agents who have different top choices. To see that it is also a fairness notion, note that by being asked to submit preferences, agents often believe that social planners attempt to satisfy their preferences. Agents have equal chances to report preferences: each agent can list an equal number of objects in his preference ordering, and if he reports an object as top choice, he has to give up reporting other objects as top choice. Therefore, if \( i \) reports an object \( o \) as top choice while \( j \) reports a different top choice, then it is arguably fair for \( i \) to claim higher probability of \( o \) than \( j \) since \( j \) spends his top choice on another object.

In my theorems I often impose ExPE, ETAET and TA simultaneously. ExPE guarantees that any object that most preferred by some agent must be assigned to agents. ETAET and TA together guarantee that every agent must obtain positive probability of his most preferred object, which is greater than the probability obtained by any other agent who most prefers a different object. This fact plays an important role in my proofs.
UTAUT requires that if all agents’ preferences only differ in ordering of the strict upper contour set of some object, then their assignments also only differ in the assignment of the strict upper contour set. It is different from the above criteria in that it imposes a condition on the whole preference profile, instead of any two agents’ preferences.

To better explain the new fairness criteria, below I relate them to other notions in the literature. Figure 1 summarizes the relations.

Figure 1: Relations between multiple notions (those in blue are used in my theorems)

**Relations to other criteria**

Nesterov (2017) introduce two related fairness criteria. Simply speaking, a random assignment $p$ satisfies strong equal treatment of equals (SETE) if any two agents with equal preferences from their most preferred objects down to some particular object obtain equal probabilities of the objects from their most preferred to the particular one; $p$ is upper envy-free (UEF) if any two agents with preferences of equal strict upper contour sets of some object $o$ obtain equal probability of $o$.

It is easy to see that ETE and ETAET are two extreme special cases of SETE by applying the “equal” condition to any two agents’ top choices or to any two agents’ whole preferences. UTAUT is a special case of UEF by applying the “equal strict upper contour set” condition to all agents’ preferences. Nesterov shows that EF implies UEF, which further implies SETE.

Hashimoto et al. (2014) introduce the notion of ordinal fairness (OF). It requires that if an agent obtains positive probability of any $o$, then his surplus at $o$ (the total

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4Formally, $p$ is upper envy-free if $SU(\succ_i, o) = SU(\succ_j, o)$ implies that $p_{i,o} = p_{j,o}$, while $p$ satisfies strong equal treatment of equals if $SU(\succ_i, o) = SU(\succ_j, o)$ and $\succ_i | SU(\succ_i, o) \Rightarrow \succ_j | SU(\succ_j, o)$ imply that $p_{i,o'} = p_{j,o'}$ for all $o' \in SU(\succ_i, o) \cup \{o\}$. 

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probability of objects weakly better than \( o \) he obtains) should be no greater than any other agent’s surplus at \( o \).\(^5\) It is easy to see that OF implies SETE, so OF also implies ETAET and ETE. When each agent obtains positive probability of his most preferred object, OF also implies TA.

Kojima and Ünver (2014) introduce the notion of favoring higher ranks (FHR) for deterministic assignments. It requires that any object preferred by any agent to his assignment should be assigned to another agent who ranks the object at least as high as the former agent.\(^6\) If defining a notion similar to FHR for random assignments, then it would require that any object most preferred by some agent must be assigned only to agents who most prefer it. Then it implies TA.

3 Manipulability theorems for \(|I| = 3\) and \(|O| \geq 3\)

I first consider the simple environment of three agents and at least three objects. My first theorem proves that no matter outside option exists or not, an ex-post efficient mechanism that satisfies equal treatment of equals (or weak envy-freeness), equal top-assignment of equal tops, and top advantage, must be strongly group manipulable.

**Theorem 1.** When \(|I| = 3\) and \(|O| \geq 3\), in \( Q \) (therefore also in \( R \)), an ex-post efficient mechanism that satisfies equal treatment of equals (or weak envy-freeness), equal top-assignment of equal tops, and top advantage is strongly group manipulable.

The proof of the theorem is as follows. Let \( I = \{i_1, i_2, j\} \) and \( O = \{o_1, o_2, o_3, \ldots\} \). Consider the two preference profiles \( \succ^*_I \) and \( \succ^*_J \) in Figure 2. All preferences in them are contained in \( R \) and \( Q \). In both preference profiles all agents prefer \( o_1, o_2, o_3 \) to other objects, and \( i_1, i_2 \) always have equal preferences.

**Lemma 1.** When \(|I| = 3\) and \(|O| \geq 3\), in \( Q \) (therefore also in \( R \)), an ex-post efficient mechanism \( \rho \) that satisfies equal treatment of equals is not strongly group manipulable only if \( \rho(\succ^*_I) = \rho(\succ^*_J) \).

\(^5\)Hashimoto et al. (2014) prove that PS is the only mechanism that satisfies ordinal fairness and non-wastefulness.

\(^6\)Kojima and Ünver (2014) characterize the Boston mechanism using favoring higher ranks, consistency, resource monotonicity, rank-respecting invariance.
If I denote \( \rho \) as not strongly group manipulable only if each preference profile. That is, if \( \rho \) is ExPE, it must be that \( \rho_{j, o_1}(\succ_i) = \rho_{j, o_1}(\succ_j^o) = 0 \).

If \( \rho \) further satisfies ETE, then each of \( i_1, i_2 \) must obtain \( o_1 \) with probability 1/2 in each preference profile. That is, \( \rho_{i_1, o_1}(\succ_i^o) = \rho_{i_2, o_1}(\succ_i^o) = \rho_{i_1, o_1}(\succ_j^o) = \rho_{i_2, o_1}(\succ_j^o) = 1/2 \).

If I denote \( \rho_{i_1, o_2}(\succ_i^o) = \rho_{i_2, o_2}(\succ_i^o) = x \) and \( \rho_{i_1, o_2}(\succ_j^o) = \rho_{i_2, o_2}(\succ_j^o) = y \), then \( \rho(\succ_i^o) \) and \( \rho(\succ_j^o) \) can be represented as follows.

<table>
<thead>
<tr>
<th></th>
<th>( i_1 )</th>
<th>( i_2 )</th>
<th>( j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( o_1 )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( 1 - 2x )</td>
</tr>
<tr>
<td>( o_2 )</td>
<td>( x )</td>
<td>( x )</td>
<td>( 1 - 2x )</td>
</tr>
<tr>
<td>( o_3 )</td>
<td>( \frac{1}{2} - x )</td>
<td>( \frac{1}{2} - x )</td>
<td>( 2x )</td>
</tr>
</tbody>
</table>

(a) \( \rho(\succ_i^o) \)

<table>
<thead>
<tr>
<th></th>
<th>( i_1 )</th>
<th>( i_2 )</th>
<th>( j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( o_1 )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{2} )</td>
<td>( 1 - 2y )</td>
</tr>
<tr>
<td>( o_2 )</td>
<td>( y )</td>
<td>( y )</td>
<td>( 1 - 2y )</td>
</tr>
<tr>
<td>( o_3 )</td>
<td>( \frac{1}{2} - y )</td>
<td>( \frac{1}{2} - y )</td>
<td>( 2y )</td>
</tr>
</tbody>
</table>

(b) \( \rho(\succ_j^o) \)

Since \( i_1, i_2 \) always obtain 1/2 of \( o_1 \) in both preference profiles, whenever \( \rho(\succ_i^o)(\pi_1) \neq \rho(\succ_j^o)(\pi_1) \), the lotteries of \( i_1, i_2 \) in one preference profile must strictly first-order stochastically dominate their lotteries in another preference profile for both \( (\succ_i^o, \succ_j^o) \) and \( (\succ_i^o, \succ_j^o) \). Therefore, if \( x > y \), \( i_1, i_2 \) can strongly group manipulate \( \rho \) at \( \succ_j^o \) by reporting \( (\succ_i^o, \succ_j^o) \). If \( x < y \), \( i_1, i_2 \) can strongly group manipulate \( \rho \) at \( \succ_i^o \) by reporting \( (\succ_i^o, \succ_j^o) \). So \( \rho \) is not strongly group manipulable only if \( x = y \), which is equivalent to \( \rho(\succ_i^o) = \rho(\succ_j^o) \). \( \square \)

If \( \rho \) further satisfies equal top-assignment of equal tops and top advantage, it must be that \( x < 1/3 \) and \( y = 1/3 \). Therefore, \( \rho \) must be strongly group manipulable.

The following lemma proves that in this simple environment, an ex-post efficient
mechanism that satisfies equal top-assignment of equal tops and weak envy-freeness must also satisfy equal treatment of equals. So in the theorem equal treatment of equals can be replaced by weak envy-freeness.

**Lemma 2.** When $|I| = 3$ and $|O| \geq 3$, in $Q$ (therefore also in $R$), an ex-post efficient assignment that satisfies equal top-assignment of equal tops and weak envy-freeness must satisfy equal treatment of equals.

**Proof.** Suppose any two agents $i, j$ report equal preferences and most prefer object $o_1$. Suppose $p$ is the assignment found by the mechanism. Equal top-assignment of equal tops requires that $p_{i,o_1} = p_{j,o_1}$. If only $o_1$ is acceptable to $i, j$, then $i, j$ obviously obtain equal lotteries. If their preference ordering is $o_1 \succ o_2 \succ \emptyset \succ \cdots$, then weak envy-freeness requires that $p_{i,o_2} \geq p_{j,o_2}$ and $p_{i,o_2} \leq p_{j,o_2}$. So $i, j$ must still obtain equal lotteries. If their preference ordering is $o_1 \succ o_2 \succ o_3 \succ \cdots \succ \emptyset$, since there are only three agents, ex-post efficiency requires that $\sum_{k=1}^{3} p_{i,o_k} = \sum_{k=1}^{3} p_{j,o_k} = 1$. If $p_{i,o_2} > p_{j,o_2}$, then it must be that $p_i >_j p_j$, which violates weak envy-freeness. Therefore, it must be that $p_{i,o_2} = p_{j,o_2}$. So $i, j$ obtain equal lotteries and $p$ satisfies equal treatment of equals.

In this simple environment Bogomolnaia and Moulin (2001) and Nesterov (2017) respectively prove that RSD is the only ex-post efficient and strategy-proof mechanism that satisfies equal treatment of equals or weak envy-freeness. Therefore, I have the following corollary.

**Corollary 1.** When $|I| = 3$ and $|O| \geq 3$, in $Q$ (therefore also in $R$), an ex-post efficient mechanism that satisfies equal treatment of equals (or weak envy-freeness) cannot be strategy-proof and minimally group strategy-proof simultaneously.

Thus, in the presence of minimum efficiency and weak fairness requirements, a mechanism with good individual incentive property must have a poor group incentive property.

### 4 Manipulability theorems for $|I| \geq 4$ and $|O| \geq 3$

Now I consider the general environment of at least four agents and at least three objects. When the preference domain is $R$, I can construct preference profiles that subsume those in the proof of Theorem 1 by requiring that some three agents prefer some three objects
to the others, and the other agents do not accept the three objects. Then the previous
theorem still holds in this section. Similar proof methods are also used by other papers
that assume outside option exists or objects are as many as agents (Erdil, 2014; Martini,

**Theorem 2.** When $|I| \geq 4$ and $|O| \geq 3$, in $\mathcal{R}$, an ex-post efficient mechanism that
satisfies equal treatment of equals (or weak envy-freeness), equal top-assignment of equal
tops, and top advantage is strongly group manipulable.

**Proof.** Let $I \equiv \{i_1, i_2, j, k_1, \ldots, k_n\}$ and $O \equiv \{o_1, o_2, \ldots, o_m\}$ with $n \geq 1$ and $m \geq 3$.
Consider the two preference profiles in Figure 3 that generalize those in the proof of
Theorem 1. In both preference profiles $o_1, o_2, o_3$ are not acceptable to $k_1, \ldots, k_n$, but

\[
\begin{array}{cccccccc}
\succ^{*}_{i_1} & \succ^{*}_{i_2} & \succ^{*}_{j} & \succ^{*}_{k_1} & \cdots & \succ^{*}_{k_n} \\
o_1 & o_1 & o_2 & o_4 & \cdots & o_4 \\
o_2 & o_2 & o_3 & \vdots & \cdots & \vdots \\
o_3 & o_3 & o_1 & o_m & \cdots & o_m \\
\vdots & \vdots & \vdots & \emptyset & \cdots & \emptyset \\
\end{array}
\quad
\begin{array}{cccccccc}
\succ^{0}_{i_1} & \succ^{0}_{i_2} & \succ^{0}_{j} & \succ^{0}_{k_1} & \cdots & \succ^{0}_{k_n} \\
o_2 & o_2 & o_2 & o_4 & \cdots & o_4 \\
o_1 & o_1 & o_3 & \vdots & \cdots & \vdots \\
o_3 & o_3 & o_1 & o_m & \cdots & o_m \\
\vdots & \vdots & \vdots & \emptyset & \cdots & \emptyset \\
\end{array}
\]

(a) $\succ^{*}_{I}$ \quad (b) $\succ^{0}_{I}$

Figure 3: Two preference profiles

are preferred by $i_1, i_2, i_3$ to other objects. So $\{i_1, i_2, i_3, o_1, o_2, o_3\}$ essentially constitute an
isolated sub-problem. Then I can use previous arguments to prove the theorem.

If the preference domain is $\mathcal{Q}$, agents must report all objects as acceptable. When
there are many objects, each agent’s preference ordering will be long. This causes two
differences. First, the proof for Theorem 2 does not hold now since the preference profiles
in the proof are not allowed in $\mathcal{Q}$. Second, when agents’ preference orderings are long, the
fairness criteria in Theorem 2 do not have much restrictions on the assignments. In the
following I show that if a mechanism further satisfies uniform tail-assignment of uniform
tails, I can generalize the previous proof.

**Theorem 3.** When $|I| \geq 4$ and $|O| \geq 3$, in $\mathcal{Q}$, an ex-post efficient mechanism that
satisfies equal treatment of equals (or weak envy-freeness), equal top-assignment of equal
tops, top advantage, and uniform tail-assignment of uniform tails is strongly group ma-
nipulable.
Proof. Let $I = \{i_1, i_2, \ldots, i_n\}$ and $O = \{o_1, o_2, \ldots, o_m\}$ with $n \geq 3$ and $m \geq 3$. Consider the two preferences profiles in Figure 4.

\[
\begin{array}{cccc}
\succ^{*}_{i_1} & \succ^{*}_{i_2} & \cdots & \succ^{*}_{i_n} & \succ^{*}_{j} \\
o_1 & o_1 & \cdots & o_1 & o_2 \\
o_2 & o_2 & \cdots & o_2 & o_3 \\
o_3 & o_3 & \cdots & o_3 & o_1 \\
o_4 & o_4 & \cdots & o_4 & o_4 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
o_m & o_m & \cdots & o_m & o_m \\
\emptyset & \emptyset & \cdots & \emptyset & \emptyset \\
\end{array}
\]

(a) $\succ_{j}^{*}$

\[
\begin{array}{cccc}
\succ^{o}_{i_1} & \succ^{o}_{i_2} & \cdots & \succ^{o}_{i_n} & \succ^{o}_{j} \\
o_2 & o_2 & \cdots & o_2 & o_2 \\
o_1 & o_1 & \cdots & o_1 & o_3 \\
o_3 & o_3 & \cdots & o_3 & o_1 \\
o_4 & o_4 & \cdots & o_4 & o_4 \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
o_m & o_m & \cdots & o_m & o_m \\
\emptyset & \emptyset & \cdots & \emptyset & \emptyset \\
\end{array}
\]

(b) $\succ_{j}^{o}$

Figure 4: Two preference profiles

Since all agents prefer all objects to $\emptyset$, UTAUT requires that $\rho_{i,\emptyset}(\succ_{j}^{*}) = \rho_{j,\emptyset}(\succ_{j}^{*})$, and $\rho_{i,\emptyset}(\succ_{j}^{o}) = \rho_{j,\emptyset}(\succ_{j}^{o})$ for all distinct $i, j$. Therefore, $\sum_{k=1}^{m} \rho_{i,o_k}(\succ_{j}^{*}) = \sum_{k=1}^{m} \rho_{j,o_k}(\succ_{j}^{*})$ and $\sum_{k=1}^{m} \rho_{i,o_k}(\succ_{j}^{o}) = \sum_{k=1}^{m} \rho_{j,o_k}(\succ_{j}^{o})$ for all distinct $i, j$. ExPE requires that if $m \leq n + 1$, then $o_1, \ldots, o_m$ must be exhausted; if $m > n + 1$, then $o_1, \ldots, o_{n+1}$ must be exhausted. Therefore, $\sum_{k=1}^{m} \rho_{o_k}(\succ_{j}^{*}) = \sum_{k=1}^{m} \rho_{o_k}(\succ_{j}^{o}) = \min\{\frac{m}{n+1}, 1\}$ for all $i \in I$. Since all agents prefer $o_1, o_2, o_3$ to $o_4, \ldots, o_m$ and have equal preferences on $o_4, \ldots, o_m$, UTAUT requires that $\rho_{o_k}(\succ_{j}^{*}) = \rho_{o_k}(\succ_{j}^{o}) = \frac{1}{n+1}$ for all $i \in I$ and all $k = 4, \ldots, \min\{m, n+1\}$.

However, $i_1, \ldots, i_n$ always prefer $o_1$ to $o_3$, while $j$ always prefers $o_3$ to $o_1$. So in both preference profiles ExPE requires that $o_1$ must be assigned only to $i_1, \ldots, i_n$. ETE, ETAET, and TA together require that for all $i \in J \equiv \{i_1, i_2, \ldots, i_n\}$, $\rho_{i,o_1}(\succ_{j}^{*}) = \frac{1}{n+1}$, $\rho_{i,o_2}(\succ_{j}^{*}) < \frac{1}{n+1}$, $\rho_{i,o_1}(\succ_{j}^{o}) = \frac{1}{n}$, and $\rho_{i,o_2}(\succ_{j}^{o}) = \frac{1}{n+1}$. So if letting $\rho_{i,o_2}(\succ_{j}^{*}) = z$, then $\rho(\succ_{j}^{*})$ can be represented by Table 4. Note that $z < \frac{1}{n+1}$. Similarly, $\rho(\succ_{j}^{o})$ can be represented by Table 5. It is easy to see that the lottery obtained by every $i \in J$ in $\rho(\succ_{j}^{*})$ strictly first-order stochastically dominates the lottery obtained by $i$ in $\rho(\succ_{j}^{o})$. So $J$ can strongly group manipulate $\rho$ at $\succ_{j}^{*}$ by reporting $\succ_{j}^{o}$. As before, I can prove that ETE can be replaced by wEF.

□

Since envy-freeness implies all fairness criteria in Theorem 1-3 except for top advan-
Corollary 2. When $|I| \geq 3$ and $|O| \geq 3$, in $Q$ (therefore also in $R$), an ex-post efficient mechanism that satisfies envy-freeness and top advantage is strongly group manipulable.

In a related result, Nesterov (2017) proves that when there are at least three agents and objects are as many as agents, any ex-post efficient mechanism that satisfies envy-freeness is not strategy-proof. Here by adding the top advantage requirement I obtain a strong impossibility theorem regarding group incentive compatibility. These two results together seemingly suggest that envy-freeness is a too strong fairness requirement to achieve good individual or group incentive property.

5 (Non-)manipulability theorems for $|I| \geq 3$ and $|O| = 2$

The manipulability results in previous sections are all proved in the environment of at least three objects. A complete analysis would require me also to consider the environment.

Table 4: $\rho(\succ_i^j)$

<table>
<thead>
<tr>
<th>$o_1$</th>
<th>$\frac{1}{n}$</th>
<th>$\frac{1}{n}$</th>
<th>$\cdots$</th>
<th>$\frac{1}{n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o_2$</td>
<td>$z$</td>
<td>$z$</td>
<td>$\cdots$</td>
<td>$z$</td>
</tr>
<tr>
<td>$o_3$</td>
<td>$\frac{2n-1}{n+1} - z$</td>
<td>$\frac{2n-1}{n+1} - z$</td>
<td>$\cdots$</td>
<td>$\frac{2n-1}{n+1} - z$</td>
</tr>
<tr>
<td>$o_4$</td>
<td>$\frac{1}{n+1}$</td>
<td>$\frac{1}{n+1}$</td>
<td>$\cdots$</td>
<td>$\frac{1}{n+1}$</td>
</tr>
<tr>
<td>$o_{\text{min}(m,n+1)}$</td>
<td>$\frac{1}{n+1}$</td>
<td>$\frac{1}{n+1}$</td>
<td>$\cdots$</td>
<td>$\frac{1}{n+1}$</td>
</tr>
</tbody>
</table>

Table 5: $\rho(\succ_i^0)$

<table>
<thead>
<tr>
<th>$o_1$</th>
<th>$\frac{1}{n}$</th>
<th>$\frac{1}{n}$</th>
<th>$\cdots$</th>
<th>$\frac{1}{n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o_2$</td>
<td>$\frac{1}{n+1}$</td>
<td>$\frac{1}{n+1}$</td>
<td>$\cdots$</td>
<td>$\frac{1}{n+1}$</td>
</tr>
<tr>
<td>$o_3$</td>
<td>$\frac{2n-1}{n+1} - \frac{1}{n+1}$</td>
<td>$\frac{2n-1}{n+1} - \frac{1}{n+1}$</td>
<td>$\cdots$</td>
<td>$\frac{2n-1}{n+1} - \frac{1}{n+1}$</td>
</tr>
<tr>
<td>$o_4$</td>
<td>$\frac{1}{n+1}$</td>
<td>$\frac{1}{n+1}$</td>
<td>$\cdots$</td>
<td>$\frac{1}{n+1}$</td>
</tr>
<tr>
<td>$o_{\text{min}(m,n+1)}$</td>
<td>$\frac{1}{n+1}$</td>
<td>$\frac{1}{n+1}$</td>
<td>$\cdots$</td>
<td>$\frac{1}{n+1}$</td>
</tr>
</tbody>
</table>
of only two objects. Interestingly, I show that when outside option does not exist, ex-post efficiency and some fairness criteria used in previous results can guarantee that a mechanism is (minimally) group strategy-proof. The reason is that, when outside option does not exist and there are only two objects, every agent has only two possible preference orderings. Then a proper combination of ex-post efficiency and fairness criteria can have strict restriction on the assignments.

**Theorem 4.** When \(|I| \geq 3\) and \(|O| = 2\), in \(Q\), any ex-post efficient mechanism that satisfies equal treatment of equals and top advantage must be minimally group strategy-proof.

All proofs in this section can be found in the appendix. As before, equal treatment of equals can be replaced by the combination of weak envy-freeness and equal top-assignment of equal tops.

In the following I show that if a mechanism further satisfies uniform tail-assignment of uniform tails, then it must be group strategy-proof. This is an interesting complement of Bade’s manipulability result, which is proved in the environment of at least three objects.

**Theorem 5.** When \(|I| \geq 3\) and \(|O| = 2\), in \(Q\), any ex-post efficient mechanism that satisfies equal treatment of equals, top advantage, and uniform tail-assignment of uniform tails must be group strategy-proof.

However, if outside option exists, the manipulability results in previous sections still hold. So outside option causes a significant difference in this section.

**Theorem 6.** When \(|I| \geq 3\) and \(|O| = 2\), in \(R\), an ex-post efficient mechanism that satisfies equal treatment of equals (or weak envy-freeness), equal top-assignment of equal tops, and top advantage is strongly group manipulable.

6 Discussion

6.1 RSD and PS

RSD and PS are two popular mechanisms to solve the object allocation problem.\(^7\) RSD is ex-post efficient and strategy-proof, and satisfies weak envy-freeness and equal treat-

\(^7\)Competitive Equilibrium from Equal Income (Hylland and Zeckhauser, 1979) is also a well-known mechanism, but it requires agents to report cardinal utilities and is known to be manipulable.
Bade (2016b) proves that any random mechanism defined by uniformly randomizing the roles of agents in a Pareto efficient, strategy-proof and non-bossy deterministic mechanism is equivalent to RSD. Pycia and Troyan (2016) prove that RSD is the only mechanism that is obviously strategy-proof, ex-post efficient and symmetric. These results make RSD stand out from random mechanisms.

PS satisfies a stronger efficiency criterion (ordinal efficiency) than RSD. Although it is not strategy-proof, it is proved to be weakly strategy-proof (Bogomolnaia and Moulin, 2001), and asymptotically equivalent to RSD (Che and Kojima, 2010). It also satisfies the strong requirement of envy-freeness. In the introduction I have shown that RSD and PS are strongly group manipulable, and I state that my theorems can provide the fundamental reason. In the following I prove that the two mechanisms satisfy the three new fairness criteria I introduce.

**Proposition 1.** RSD and PS satisfy equal top-assignment of equal tops, top advantage, and uniform tail-assignment of uniform tails.

### 6.2 Independence of axioms

A complete analysis requires me to show that the efficiency/fairness criteria in my theorems are necessary. For the two non-manipulability theorems it is easy to do so since I can easily propose a mechanism that does not satisfy a specific criterion and prove it is strongly group manipulable. For the manipulability theorems ex-post efficiency is obviously necessary: the mechanism that always assigns all agents the virtual object $\emptyset$ trivially satisfies all fairness criteria and is group strategy-proof. However, it is difficult to show that each fair criterion is necessary, since it is difficult to prove a mechanism is not strongly group manipulable. For deterministic mechanisms it is known that group strategy-proofness is equivalent to the combination of strategy-proofness and non-bossiness (Pápai, 2000). Non-bossiness is an easy-to-check condition. But for random mechanisms, there is no known similar equivalence between (minimal) group strategy-proofness and the combination of (weak) strategy-proofness and some easy-to-check condition. So to prove minimal group strategy-proofness, I need to verify all possible ma-

---

8 Simply speaking, a mechanism is symmetric if its underlying game is anonymous to the roles of agents. It implies equal treatment of equals.

9 For example, RSD is strategy-proof and non-bossy, but it is not minimally group strategy-proof.
manipulations of all possible groups, which is a difficult task when there are at least three agents and at least three objects.

Below I show that each efficiency/fairness criterion in the two non-manipulability theorems is necessary.

(1) Ex-post efficiency. The mechanism that assigns all agents $\emptyset$ for any preference profile $\succ_I$ and assigns the outcome of RSD for other preference profiles obviously satisfies equal treatment of equals, top advantage and uniform tail-assignment of uniform tails. However, it is strongly group manipulable by $I$ at $\succ_I$ by reporting any other preference profile.

(2) Equal treatment of equals. Consider the mechanism $\rho$ that finds the outcome of RSD for all other preference profiles but finds the following assignments for the two particular preference profiles shown below. In the following tables $o^q$ means that the corresponding agent obtains probability $q$ of $o$.

\[
\begin{array}{ccc}
\succ^*_{i_1} & \succ^*_{i_2} & \succ^*_{j} \\
\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\
\frac{1}{3} & \frac{2}{3} & 0 \\
0 & 0 & 0 \\
\end{array} \quad \begin{array}{ccc}
\succ^0_{i_1} & \succ^0_{i_2} & \succ^0_{j} \\
\frac{2}{9} & \frac{5}{9} & \frac{2}{3} \\
\frac{1}{9} & \frac{1}{9} & \frac{0}{1} \\
\frac{0}{1} & \frac{0}{1} & \frac{0}{1} \\
\end{array}
\]

(a) $\rho(\succ_I^*)$  \hspace{2cm} (b) $\rho(\succ_I^0)$

It is obvious that $\rho$ satisfies top advantage and uniform tail-assignment of uniform tails. But $\rho$ does not satisfy equal treatment of equals. Then $i_1$ can strongly manipulate $\rho$ at $\succ_I^*$ by reporting $\succ_{i_1}^0$.

(3) Top advantage. Consider the mechanism $\rho$ that finds the outcome of RSD for all other preference profiles but finds the following assignments for two particular preference profiles shown below.

\[
\begin{array}{ccc}
\succ^*_{i_1} & \succ^*_{i_2} & \succ^*_{j} \\
\frac{2}{9} & \frac{2}{9} & \frac{1}{9} \\
\frac{4}{9} & \frac{4}{9} & \frac{5}{9} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{array} \quad \begin{array}{ccc}
\succ^0_{i_1} & \succ^0_{i_2} & \succ^0_{j} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{array}
\]

(a) $\rho(\succ_I^*)$  \hspace{2cm} (b) $\rho(\succ_I^0)$

It is obvious that $\rho$ satisfies equal treatment of equals and uniform tail-assignment of uniform tails. But $\rho$ does not satisfy top advantage. Then $i_1, i_2$ can strongly group manipulate $\rho$ at $\succ_I^*$ by reporting $(\succ_{i_1}^0, \succ_{i_2}^0)$.
(4) Uniform tail-assignment of uniform tails. Consider the mechanism $\rho$ that finds the outcome of RSD for all other preference profiles but finds the following assignments for two particular preference profiles shown below.

<table>
<thead>
<tr>
<th>$\succsim_{i_1}^*$</th>
<th>$\succsim_{i_2}^*$</th>
<th>$\succsim_{j}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o_1^{4/9}$</td>
<td>$o_1^{4/9}$</td>
<td>$o_2^{8/9}$</td>
</tr>
<tr>
<td>$o_2^{1/18}$</td>
<td>$o_2^{1/18}$</td>
<td>$o_1^{1/9}$</td>
</tr>
</tbody>
</table>

(a) $\rho(\succsim_I)$

<table>
<thead>
<tr>
<th>$\succsim_{i_1}^o$</th>
<th>$\succsim_{i_2}^o$</th>
<th>$\succsim_{j}^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o_1^{4/3}$</td>
<td>$o_1^{1/3}$</td>
<td>$o_2^{1/3}$</td>
</tr>
<tr>
<td>$o_2^{1/3}$</td>
<td>$o_1^{1/3}$</td>
<td>$o_1^{1/3}$</td>
</tr>
</tbody>
</table>

(b) $\rho(\succsim_I)$

It is obvious that $\rho$ satisfies equal treatment of equals and top advantage. But $\rho$ does not satisfy uniform tail-assignment of uniform tails. Then $i_1, i_2$ can weakly group manipulate $\rho$ at $\succsim_I$ by reporting $(\succsim_{i_1}^o, \succsim_{i_2}^o)$.

### 6.3 Restricted preference domain

In the paper I only consider the unrestricted preference domain and the no outside-option domain. In some applications agents’ preferences may belong to a further restricted domain. Then we may hope that the manipulability theorems do not hold in these domains. However, a careful examination of my proofs should convince the reader that only the following three preference orderings are essential for Theorem 1 and 3. The three orderings differ only in the rank of the top three objects. So as long as a restricted domain contains the three orderings, the manipulability theorems must hold. Such restricted domains could be the single-peaked preference domain, which often appears in spatial competition and voting theory, and the single-dipped preference domain.

$$
\begin{array}{ccc}
  o_1 & o_2 & o_2 \\
  o_2 & o_1 & o_3 \\
  o_3 & o_3 & o_1 \\
  \vdots & \vdots & \vdots \\
\end{array}
$$

### 6.4 Manipulating group size

Group manipulation implicitly assumes that the agents in the manipulating group can coordinate their strategies. However, coordination often becomes more difficult when the
group is larger. Therefore, it is worthwhile to know the size that a manipulating group has to be in this paper. From my proofs it is easy to see that the manipulating group can be as small as containing two agents when (1) $|I| = 3$ and $|O| \geq 3$, or (2) $|I| > 3$, $|O| \geq 3$ and the preference domain is $\mathcal{R}$. Coordination between two agents is often much easier than that among multiple agents. In some markets there exist couples (Roth, 1984). When the manipulating group is a couple, coordination happens naturally. In the other cases of my proofs, the manipulating group have to be large. But the agents in the manipulating group I construct have equal preferences and use equal strategies. This feature can facilitate their coordination.

6.5 Stronger efficiency criteria

This paper sticks to ex-post efficiency to highlight the tension between fairness and group incentive compatibility. It is interesting to examine whether the combination of a stronger efficiency and some weaker fairness criteria can still result in (non-)manipulability theorems. This is left for future research.

Appendix

Proof of Theorem 4

Let the two objects be $o_1, o_2$. I prove the theorem by contradiction. Suppose at some preference profile $\succ_I$, some group $J \subsetneq I$ strongly group manipulates the mechanism $\rho$ by reporting $\succ'_J$. Suppose there are $x > 0$ agents in $J$ and $y$ agents in $I \setminus J$. Among $J$ there are $x_1$ agents who prefer $o_1$ to $o_2$, while among $I \setminus J$ there are $y_1$ agents who prefer $o_1$ to $o_2$. Since there are only two objects, all agents in $J$ must switch their preference ranking of $o_1, o_2$ in their misreported preferences. So the true and misreported preference profiles can be denoted as follows:

Thus, at $\succ_I$ there are $x_1 + y_1$ agents who prefer $o_1$ to $o_2$, while at $(\succ'_J, \succ_{I \setminus J})$ there are $(x - x_1) + y_1$ agents who prefer $o_1$ to $o_2$. Since $\rho$ satisfies ETE, $\rho(\succ_I)$ and $\rho(\succ'_J, \succ_{I \setminus J})$ can be denoted as follows:

\footnote{For example, in the bargaining theory literature bilateral bargaining is often more tractable than multilateral bargaining.}

\footnote{Of course, there may exist smaller manipulating groups. In my proofs I choose large groups to make the proofs easier.}
Since $\rho$ satisfies TA, if $x_1 + y_1 > 0$, then $a > c$; if $(x - x_1) + (y - y_1) > 0$, then $d > b$; if $(x - x_1) + y_1 > 0$, then $a' > c'$; if $x_1 + (y - y_1) > 0$, then $d' > b'$. Since $J$ strongly group manipulates $\rho$, if $x_1 > 0$, then $c' \geq a$; if $x - x_1 > 0$, then $b' \geq d$.

There are two cases to consider: $(x - x_1) + y_1 > 0$, or $x_1 + (y - y_1) > 0$. Since the two cases are symmetric, I only prove the case of $(x - x_1) + y_1 > 0$. In this case we know that $a' > c'$. There are further two cases to consider.

**Case 1:** $x_1 > 0$. It implies that $x_1 + y_1 > 0$, which further implies that $a > c$ and $c' \geq a$.

Then the total probability of $o_1$ that assigned in $\rho(\succ_J, \succ_{I \setminus J})$ is

$$
[(x - x_1) + y_1]a' + [x_1 + (y - y_1)]c' > [(x - x_1) + y_1]c' + [x_1 + (y - y_1)]c' \\
= (x + y)c' \\
\geq (x + y)a \\
\geq (x_1 + y_1)a + [(x - x_1) + (y - y_1)]c.
$$

Note that $(x_1 + y_1)a + [(x - x_1) + (y - y_1)]c$ is the total probability of $o_1$ that assigned in $\rho(\succ_J)$. Since $\rho$ is ExPE, $o_1$ must be exhausted in both $\rho(\succ_J)$ and $\rho(\succ_J', \succ_{I \setminus J})$. So the above strict inequality is a contradiction.

**Case 2:** $x_1 = 0$. It implies that $x - x_1 = x > 0$, which further implies that $d > b$ and $b' \geq d$. There are further two subcases to consider.

- If $x_1 + (y - y_1) > 0$, then $d' > b'$. Then the total probability of $o_2$ that assigned in
\( \rho(\succ_J, \succ_{I \setminus J}) \) is

\[
[(x - x_1) + y_1]b' + [x_1 + (y - y_1)]d' > [(x - x_1) + y_1]b' + [x_1 + (y - y_1)]b'
\]
\[
= (x + y)b' 
\]
\[
\geq (x + y)d 
\]
\[
\geq (x_1 + y_1)b + [(x - x_1) + (y - y_1)]d, 
\]

which, as before, is a contradiction.

- If \( x_1 + (y - y_1) = 0 \), then \( y_1 = y \). So all agents in \( J \) prefer \( o_2 \) to \( o_1 \), while all agents in \( I \setminus J \) prefer \( o_1 \) to \( o_2 \). Therefore, in the misreported preference profile \((\succ'_J, \succ_{I \setminus J})\) all agents prefer \( o_1 \) to \( o_2 \). ETE implies that \( b' = \frac{1}{x + y} \), while TA implies that \( d > \frac{1}{x + y} \), which contradicts \( b' \geq d \).

**Proof of Theorem 5**

As before, suppose at \( \succ_I \), a group \( J \subseteq I \) weakly group manipulates a mechanism \( \rho \) by reporting \( \succ'_J \). Besides the properties in Theorem 4, if \( \rho \) further satisfies UTAUT, then in the proof of Theorem 4 we have \( a + b = c + d = a' + b' = d' + c' = \frac{2}{n} \). There are two cases to consider.

If some agent in \( J \) with the true preference ordering \( o_1 \succ o_2 \) obtains a different assignment than truth-telling by misreporting preferences, then it implies that \( x_1 > 0 \). TA requires that \( a > c \), and weak group manipulation requires that \( c' > a \). So the lottery \((c' \cdot o_1, d' \cdot o_2)\) strictly first-order stochastically dominates the lottery \((a \cdot o_1, b \cdot o_2)\) for the \( x_1 \) agents in \( J \). Therefore, if \( x - x_1 = 0 \), then \( J \) strongly group manipulates \( \rho \), which contradicts the fact that \( \rho \) is minimally group strategy-proof. If \( x - x_1 > 0 \), TA requires that \( a' > c' \). Thus, \( a' > c' > a > c \). So the lottery \((b' \cdot o_2, a' \cdot o_1)\) is strictly first-order stochastically dominated by the lottery \((d \cdot o_2, c \cdot o_1)\) for the \( x - x_1 \) agents in \( J \). But this implies that \( J \) cannot weakly group manipulate \( \rho \), which is a contradiction.

If some agent in \( J \) with the true preference ordering \( o_2 \succ o_1 \) obtains a different assignment than truth-telling by misreporting preferences, then I can repeat the above proof to obtain a contradiction.

**Proof of Theorem 6**

Let \( I \equiv \{i_1, i_2, \ldots, i_n, j\} \) and \( O \equiv \{o_1, o_2\} \) with \( n \geq 2 \). Consider the following two preference profiles.
\[ i_1 \succ^* i_2 \cdots \succ^* i_n \succ^* j \]

\[ o_1 \succ o_1 \cdots \succ o_1 \succ o_2 \]

\[ o_2 \succ o_2 \cdots \succ o_2 \succ \emptyset \]

\[ \emptyset \succ \emptyset \cdots \succ \emptyset \succ o_1 \]

(a) \( \succ^*_j \)

\[ o_2 \succ o_2 \cdots \succ o_2 \succ o_2 \]

\[ o_1 \succ o_1 \cdots \succ o_1 \succ \emptyset \]

\[ \emptyset \succ \emptyset \cdots \succ \emptyset \succ o_1 \]

(b) \( \succ^*_j \)

As before, at \( \succ^*_j \) each \( i \in J = \{i_1, \ldots, i_n\} \) obtains \( 1/n \) of \( o_1 \) and less than \( 1/(n+1) \) of \( o_2 \), while at \( \succ^*_j \) each \( i \in J \) obtains \( 1/n \) of \( o_1 \) and \( 1/(n+1) \) of \( o_2 \). So \( J \) can strongly group manipulate the mechanism at \( \succ^*_j \) by reporting \( \succ^*_j \).

**Proof of Proposition 1**

Since PS is envy-free, I only need to prove TA. But it is obvious. So in the following I only prove the proposition for RSD.

- **ETAET**: Suppose any distinct \( i, j \) most prefer the same object \( o \). For any drawn ordering of agents in which \( i \) obtains \( o \), there is a symmetric ordering in which \( i, j \) switch their positions. In the symmetric ordering \( j \) must obtain \( o \). Since each ordering is drawn with equal probability, \( i, j \) obtain equal probability of \( o \).

- **TA**: Suppose \( i \) most prefers \( o \) while \( j \) most prefers a different object. For any ordering of agents in which \( j \) obtains \( o \), there is a symmetric ordering in which \( i, j \) switch their positions. In the symmetric ordering \( i \) must obtain \( o \). However, for all orderings of agents in which \( i \) is ranked first, \( i \) obtains \( o \). But for all orderings of agents in which \( j \) is ranked first, \( j \) does not obtain \( o \). Since each ordering is drawn with equal probability, \( i \) obtains higher probability of \( o \) than \( j \).

- **UTAUT**: Suppose there exists some \( o \in \hat{O} \) such that \( SU(\succ_i, o) = SU(\succ_j, o) \) and \( \succ^*_i \mid \hat{O} \setminus SU(\succ_i, o) = \succ^*_j \mid \hat{O} \setminus SU(\succ_j, o) \) for all distinct \( i, j \). For any ordering of agents, any object in \( SU(\succ_i, o) \) must be chosen before any object in \( \hat{O} \setminus SU(\succ_i, o) \) is chosen. So the outcome of RSD can be equivalently found by first running RSD to assign \( SU(\succ_i, o) \), then running RSD to assign \( \hat{O} \setminus SU(\succ_i, o) \). Since all agents have equal preferences on \( \hat{O} \setminus SU(\succ_i, o) \), they must obtain equal probability of each object in \( \hat{O} \setminus SU(\succ_i, o) \).
References


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