Opinion Manipulation and Disagreement in Social Networks

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Abstract

I study a bounded rationality model of opinion formation in which there are two different types of agents: naive agents and sophisticated agents. All agents update opinions by taking weighted averages of neighbors’ opinions. Naive agents truthfully report their opinions, but sophisticated agents can strategically report opinions to manipulate naive agents. I show that the limiting opinions are completely determined by sophisticated agents’ bias and the structure of the network; and generically, there is no consensus. I analyze how disagreement is affected by the lying cost, diverging interests and the spectral gap of the social network. I also show that naive agents don’t have any social influence and sophisticated agents’ social influence can be decomposed into two separate factors: direct influence and indirect influence. (JEL D83, D85, Z13)

Keywords: Learning, Social networks, Persistent disagreement

Field: Network Economics

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1. Introduction

The 2016 election exposed an America of deep divides over politics. A survey conducted by Gallup after the election found 77 percent of Americans believe the nation is more divided than united on its fundamental values. (Jones 2016) In view of this, it is important to understand why disagreement among individuals in a society can persist. In this paper, I try to provide one explanation: individuals with diverging interests misrepresent opinions for their own benefit.

I base my study on the model of DeGroot (1974). In this classical framework, there are \( N \) agents in the society. Agent \( i \) has an initial opinion \( x_0^i \in R \). A weighted directed network describes the social structure of the society in which the weight represents how much an agent trust another agent. At each date, the agents receive information from their friends/neighbors in the social network and update their opinions. An agent’s new opinion is simply the weighted average of the information he/she receives. In the model of DeGroot (1974), the agents report their opinions truthfully. If the network is strongly connected and aperiodic, agents will eventually reach consensus. As some researchers (Acemoglu et al. 2013, Bindel et al. 2015) have argued, this emphasis on consensus can only describe a particular type of opinion dynamics and unable to explain the persistent disagreement among individuals in the society.

I study a model of opinion dynamics in which, generically, there is no consensus. I analyze how the degree of disagreement is affected by the network structure and derive a new measure of social influence. The crucial difference between this model and DeGroot (1974) is that some agents can lie. More specifically, there are two types of agents in the society: sophisticated and naive. Sophisticated agents have different personal bias and are willing to persuade others to believe in their bias. The bias is exogenous. They can lie to naive agents, but there is a cost of lying, scaled by the parameter \( \alpha > 0 \). To ensure tractability, I assume that the sophisticated agents are myopic and the utility is quadratic. Naive agents behave the same as the agents in DeGroot (1974), and they can not infer accuracy of the signals.

Since the sophisticated agents are myopic, the equilibrium can be calculated for each period separately. First, I show that, in each period, there exists a unique Nash equilibrium in which the signals sent by sophisticated agents can be represented by linear combinations of the opinions of all agents in the previous period and the bias of the sophisticated agents. When lying cost (\( \alpha \)) is large, the results are more clear and intuitive, the signals are perturbation of the real opinions of the sophisticated agents. The perturbation has the order of \( \frac{1}{\alpha} \). It is positively correlated with the opinions of other agents in the previous period and negatively associated with his/her bias.

Second, I investigate the opinion dynamics. I show that, when the network is strongly connected and aperiodic, there is a threshold of lying cost (\( \alpha \)) such that when lying cost is larger than that threshold, opinions converge but generically, the opinions do not converge to a consensus. Then I provide a more detailed analysis of the asymptotic opinions. I find that the asymptotic opinion vector equals to a vector of same value \( \hat{c} \) plus the vector representing disagreement. The asymptotic opinions are proved to be determined by the sophisticated agents’ bias and the structure of the network. They are not affected by the initial opinions. I then discuss how lying cost and the structure of the network influence the disagreement. Since the signals are perturbation of the actual opinions with the order of \( \frac{1}{\alpha} \), the disagreement also has the order of \( \frac{1}{\alpha} \). Disagreement is also positively related to the inverse of spectral gap (the difference between the two largest eigenvalues) of the network. There is a large literature of probability theory devoted to studying the relation between this difference and other properties of the social network. Details can be found in Jackson (2008), Levin et al. (2009), Golub and Sadler (2016). The basic take-away is that the spectral gap is small if the society is segregated.
Third, I study the social influence. When the lying cost is large, disagreement vanishes and the asymptotic opinions are all close to $c$ which is a weighted average of the bias of the sophisticated agents. The weight can be interpreted as a measure of social influence. It can be decomposed into two separate factors: direct influence and indirect influence. The direct influence is simply a summation of the weights that naive agents place on the sophisticated agent. The indirect influence is a summation of these weights multiplied by the eigenvector centrality of respective naive agents.

I next examine the opinion dynamics in a slightly different environment in which the sophisticated agents place different degrees of importance on different naive agents. The results are almost the same as before. The only significant difference is that the direct influence now is the summation of the weights that naive agents place on the sophisticated agent multiplied by the respective degree of importance.

I turn next to change the assumption that the sophisticated agents are myopic. They are now forward-looking. The model then is so complicated that I can only solve a special case in which there are only two stubborn sophisticated agents and one naive agent who place weight only on these two sophisticated agents. I show that, in comparison with the case of myopic agents, the agent who has more influence gain even more influence, the utility of both sophisticated agents are larger. I also examine comparative statics on the discount factor. I prove that these two effects become more significant when the discount factor is greater.

My paper is related to an extensive non-Bayesian learning literature. In this literature, the classical result is that if the network is strongly connected and aperiodic, agents will eventually reach consensus. (DeGroot 1974, DeMarzo et al. 2003, Golub and Jackson 2010) Besides this paper, several other models have been proposed to explain persistent disagreements. One approach is provided by models incorporating some homophily which means that opinions that are too far from one’s own are weighted little or not at all. Opinions in this kind of models eventually converge to a limit opinion profile, in which agents are partitioned into several groups and agents have the same limit opinion if and only if they are in the same group. A survey of this approach is provided by Lorenz (2007). Another method is provided by models in which some agents are stubborn which means that they always put some weight on their initial opinion. This updating rule is first proposed by Fiedlin and Johnsen (1990). Bindel et al. (2015) use this framework to quantify the inherent social cost of this lack of consensus. Acemoglu et al. (2013) also investigate a model with stubborn agents. They use an inhomogeneous stochastic gossip model and show that the presence of stubborn agents leads to persistent opinion fluctuations and disagreement.

The model most similar in structure to this paper is by Buechel et al. (2015). They also study a dynamic model of opinion formation in social networks in which agents can lie (misrepresent their opinion). We both use quadratic utility function. The difference is that in their paper, the motive to misrepresent the true opinion is conformity or counter-conformity, which means the agent states opinion close to or away from the group opinion. The characteristics of their model are also quite different. In their paper, when the opinion dynamics converge, agents reach consensus. They focus on how the long-run group opinion is affected by conformity and whether information aggregation is undermined by misrepresentation of opinions.

2. Model

2.1. Description of the Environment
A finite set of agents, indexed by \( i \in \{1, 2, \ldots, N\} \), interacts in a social network. The social network structure is captured by a directed graph, denoted by a matrix \( V \), where \( v_{ij} \geq 0 \) represents the trust that agent \( i \) places on agent \( j \). The matrix \( V \) is row stochastic, which means \( \sum_{j=1}^{N} v_{ij} = 1 \) for any \( i \).

I study a discrete time dynamic model. Each agent \( i \) starts with an initial opinion \( x_{i}^{0} \) and then they exchange information with their neighbors at discrete times \( t = 1, 2, 3, \ldots \). At time \( t \), the opinions of all agents is given by \( X^{t} = (x_{1}^{t}, x_{2}^{t}, \ldots, x_{N}^{t})' \).

There are two kinds of agents: naive agents and sophisticated agents. Without loss of generality, I assume that the first \( d \) agents are sophisticated. I denote the set of sophisticated agents by \( S(d) = \{1, 2, \ldots, d\} \) and the set of naive agents by \( A(d) = \{d+1, \ldots, N\} \). Naive agents report their opinions truthfully. On the contrary, sophisticated agents report their opinions strategically. Sophisticated agent \( i \) has a exogenous bliss point \( b_{i} \) and he try to convince naive agents to believe in \( b_{i} \), so he reports his opinion strategically. Denote the opinion that the sophisticated agent \( i \) has a exogenous bliss point \( b_{i} \).

For simplicity, I use quadratic preference and assume that agents are myopic. A sophisticated agent \( i \)'s utility function at time \( t \) is:

\[
 u_{i}(x_{i}^{t-1}, s_{i}^{t}, b_{i}) = -\sum_{j \in A(d)} (x_{j}^{t} - b_{i})^2 - \alpha (s_{i}^{t} - x_{i}^{t-1})^2 
\]  

(2.1)

\(-\alpha (s_{i}^{t} - x_{i}^{t-1})^2\) represents the cost of lying, where \( \alpha > 0 \) displays the importance of the preference for honesty. I assume that all sophisticated agents know each other’s true opinions. Thus their opinions are updated according to:

\[
 x_{i}^{t} = \sum_{j=1}^{N} v_{ij} x_{j}^{t-1} 
\]  

(2.2)

In contrast, naive agents believe every information that they get from their neighbors and update their beliefs according to:

\[
 x_{i}^{t} = \sum_{j \in S(d)} v_{ij} s_{j}^{t} + \sum_{j \in A(d)} v_{ij} x_{j}^{t-1} 
\]  

(2.3)

2.2. Equilibrium

First, let me partition matrix \( V \) into 4 blocks as follows:

\[
 V = \begin{pmatrix}
  V_{1} & \vdots & V_{N-d} \\
  \vdots & \ddots & \vdots \\
  V_{d} & \vdots & V_{2} \\
 V_{N-d} & \vdots & V_{3} \\
 \end{pmatrix}
\]  

(2.4)

in which, \( V_{1} \) and \( V_{2} \) represent how sophisticated agents get information from sophisticated agents and naive agents, \( V_{3} \) and \( V_{4} \) represent how naive agents get information from sophisticated agents and naive agents.

Given other sophisticated agents’ strategies, agent \( i \in S(d) \) solve:

\[
 \max_{s_{i}^{t}} u_{i}(x_{i}^{t-1}, s_{i}^{t}, b_{i}) = -\sum_{j \in A(d)} (x_{j}^{t} - b_{i})^2 - \alpha (s_{i}^{t} - x_{i}^{t-1})^2 
\]  

(2.5)

in which \( x_{j}^{t} = \sum_{k \in S(d)} v_{jk} s_{k}^{t} + \sum_{k \in A(d)} v_{jk} x_{k}^{t-1} \). The FOC is:

\[
 -2 \sum_{j \in A(d)} v_{ji} (x_{j}^{t} - b_{i}) - 2\alpha (s_{i}^{t} - x_{i}^{t-1}) = 0
\]

\[
 \iff \sum_{j \in A(d)} v_{ji} (\sum_{k \in S(d)} v_{jk} s_{k}^{t}) + \alpha s_{i}^{t} - \sum_{j \in A(d)} v_{ji} (\sum_{j \in A(d)} v_{jk} x_{k}^{t-1}) + \sum_{j \in A(d)} v_{ji} b_{i} = 0
\]  

(2.6)
Then the equilibrium strategies of all sophisticated agents are the solution of the following simultaneous equations:

\[
\sum_{j \in A(d)} v_j (\sum_{k \in S(d)} v_{jk}s_k^t) + \alpha s_1^t = \alpha x_1^{t-1} - \sum_{j \in A(d)} v_j (\sum_{k \in A(d)} v_{jk}x_k^{t-1}) + \sum_{j \in A(d)} v_j b_1 \\
\sum_{j \in A(d)} v_j (\sum_{k \in S(d)} v_{jk}s_k^t) + \alpha s_2^t = \alpha x_2^{t-1} - \sum_{j \in A(d)} v_j (\sum_{k \in A(d)} v_{jk}x_k^{t-1}) + \sum_{j \in A(d)} v_j b_2 \\
\vdots \\
\sum_{j \in A(d)} v_j (\sum_{k \in S(d)} v_{jk}s_k^t) + \alpha s_d^t = \alpha x_d^{t-1} - \sum_{j \in A(d)} v_j (\sum_{k \in A(d)} v_{jk}x_k^{t-1}) + \sum_{j \in A(d)} v_j b_d
\]

This system of equations can be written as a matrix equation:

\[
\alpha \begin{pmatrix} s_1^t \\ s_2^t \\ \vdots \\ s_d^t \end{pmatrix} + V_3^TV_3 = \alpha \begin{pmatrix} x_1^{t-1} \\ x_2^{t-1} \\ \vdots \\ x_d^{t-1} \end{pmatrix} - V_3^TV_4 + \begin{pmatrix} \sum_{j \in A(d)} v_j b_1 \\ \sum_{j \in A(d)} v_j b_2 \\ \vdots \\ \sum_{j \in A(d)} v_j b_d \end{pmatrix}
\]

Let

\[
B = \begin{pmatrix} \sum_{j \in A(d)} v_j b_1 \\ \sum_{j \in A(d)} v_j b_2 \\ \vdots \\ \sum_{j \in A(d)} v_j b_d \end{pmatrix}, \quad S^t = \begin{pmatrix} s_1^t \\ s_2^t \\ \vdots \\ s_d^t \end{pmatrix}, \quad X_1^{t-1} = \begin{pmatrix} x_1^{t-1} \\ x_2^{t-1} \\ \vdots \\ x_d^{t-1} \end{pmatrix} \text{ and } X_2^{t-1} = \begin{pmatrix} x_1^{t-1} \\ x_2^{t-1} \\ \vdots \\ x_d^{t-1} \end{pmatrix} \] Then the matrix equation is equivalent to:

\[
(\alpha I_d + V_3^TV_3)S^t = \alpha X_1^{t-1} - V_3^TV_4X_2^{t-1} + B
\]

I first show that for any \(\alpha > 0\) and any social network, there exists a unique Nash equilibrium in each period.

**Proposition 1.** For any \(\alpha > 0\), and any social network, there exists a unique Nash equilibrium in each period such that

\[
S^t = \alpha(\alpha I_d + V_3^TV_3)^{-1}X_1^{t-1} - (\alpha I_d + V_3^TV_3)^{-1}V_3^TV_4X_2^{t-1} + (\alpha I_d + V_3^TV_3)^{-1}B
\]

As shown in proposition 1, in each period there exists a unique Nash equilibrium. The sophisticated agents report the opinions as a linear combination of the true opinion, the opinion of the naive agents and their bias. When \(\alpha\) is large, I can rewrite \(S^t\) as follows:

\[
S^t = \alpha(\alpha I_d + V_3^TV_3)^{-1}X_1^{t-1} - (\alpha I_d + V_3^TV_3)^{-1}V_3^TV_4X_2^{t-1} + (\alpha I_d + V_3^TV_3)^{-1}B \\
= (I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}X_1^{t-1} - \frac{1}{\alpha}(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^TV_4X_2^{t-1} + \frac{1}{\alpha}(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}B \\
= X_1^{t-1} - \frac{1}{\alpha}V_3^TV_3X_1^{t-1} - \frac{1}{\alpha}V_3^TV_4X_2^{t-1} + \frac{1}{\alpha}B + O(\frac{1}{\alpha^2})
\]

The opinions that sophisticated agents report are perturbation of the real opinions. The perturbation has the order of \(\frac{1}{\alpha}\). It is positively correlated with the opinions of other agents in the previous period and negatively associated with his/her bias.

2.3. Opinion Dynamics
The opinion updating rule (3.2) and (3.3) can also be written as two matrix equations:

\[ X_t^1 = V_1 X_{t-1}^1 + V_2 X_{t-1}^2 \]  
\[ X_t^2 = V_3 S_t^1 + V_4 X_{t-1}^2 \]  

(2.11)

(2.12)

Substitute equation (3.10) into equation (3.12), then

\[ X_t^2 = \alpha V_3 (\alpha I_d + V_3^T V_3)^{-1} X_{t-1}^1 - V_3 (\alpha I_d + V_3^T V_3)^{-1} V_3^T V_4 X_{t-1}^1 + V_3 (\alpha I_d + V_3^T V_3)^{-1} B + V_4 X_{t-1}^2 \]

\[ = \alpha V_3 (\alpha I_d + V_3^T V_3)^{-1} X_{t-1}^1 + (I_{N-d} - V_3 (\alpha I_d + V_3^T V_3)^{-1} V_3^T) V_4 X_{t-1}^1 + V_3 (\alpha I_d + V_3^T V_3)^{-1} B \]

Let \( U_1 = V_1, U_2 = V_2, U_3 = \alpha V_3 (\alpha I_d + V_3^T V_3)^{-1}, U_4 = (I_{N-d} - V_3 (\alpha I_d + V_3^T V_3)^{-1} V_3^T) V_4, \)

\[ \hat{B} = \begin{pmatrix} 0_d \\ V_3 (\alpha I_d + V_3^T V_3)^{-1} B \end{pmatrix} \]

and \( X^t = \begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix}, \) then the opinion dynamics can be represented by the following matrix equation:

\[ X^t = UX^{t-1} + \hat{B} \]  

(2.13)

3. Main results

In this section, I will first derive sufficient conditions for convergence, and then discuss the condition under which there is no consensus. I then analyze how the disagreement is affected by the lying cost and the structure of the network.

3.1. Convergence

First, let me assume that all sophisticated agents have the same bliss point:

\[ b_1 = b_2 = \ldots = b_d = b \]  

(3.1)

Lemma 1 shows that without loss of generality I can assume that \( b = 0. \)

Lemma 1. When all sophisticated agents have the same bliss point \( b, \) let \( Z^t = \begin{pmatrix} x_t^1 - b \\ x_t^2 - b \\ \vdots \\ x_t^N - b \end{pmatrix}, \) then

\[ Z^t = U Z^{t-1}. \]

Then the opinion dynamics can be characterized by:

\[ X^t = UX^{t-1} \]  

(3.2)

In order to describe the first main result, I need to introduce some standard graph-theoretic definitions. In this paper, I use the same definitions as Golub and Jackson (2010).

Definition 1. A walk in \( V \) is a sequence of nodes \( i_1, i_2, \ldots, i_K, \) not necessarily distinct, such that \( V_{i_k i_{k+1}} > 0 \) for each \( k \in 1, \ldots, K - 1. \) The length of the walk is defined to be \( K - 1. \)

Definition 2. A path in \( V \) is a walk consisting of distinct nodes.
Definition 3. The matrix $V$ is strongly connected if there is path in $V$ from any node to any other node.

Definition 4. A cycle is a walk $i_1, i_2, ..., i_K$ such that $i_1 = i_K$. The length of a cycle with $K$ (not necessarily distinct) entries is defined to be $K - 1$. A cycle is simple if the only node appearing twice in the sequence is the starting (and ending) node.

Definition 5. The matrix $V$ is aperiodic if the greatest common divisor of the lengths of its simple cycles is 1.

In DeGroot model, if $V$ is aperiodic and strongly connected, then for any $X^0$, $\lim_{t \to \infty} V^t X^0 = l X^0$, where $l$ is the unique left eigenvector of $V$ corresponding to eigenvalue 1. In my model, since I introduce sophisticated agents, the result is different.

My first result shows that under a little more assumptions than DeGroot model ($\alpha$ need to be large), if all sophisticated agents have the same bliss point, sophisticated agents can perfectly manipulate public opinion. The proof is provided in Appendix.

Proposition 2. If $V$ is strongly connected and aperiodic, then there exist a $\hat{\alpha} > 0$ such that for any $\alpha > \hat{\alpha}$ and any vector $X^0 \in \mathbb{R}^N$, $\lim_{t \to \infty} U^t X^0 = 0$.

This proposition states that, if all sophisticated agents have the same bliss point, sophisticated agents determine the limiting opinions. The sophisticated agents don’t need to be at special locations of the network or have large social influence. I need $\alpha$ to be large because the sophisticated agents are myopic. If $\alpha$ is small, then they may send some extreme signals, which leads to divergence of opinions. A special case of proposition 2 is that there is only one sophisticated agent. Then all agents asymptotically believe what he wants.

3.2. Disagreement

I now turn to a more general setting in which sophisticated agents have different bliss points. Proposition 3 shows that in a strongly connected and aperiodic network, when sophisticated agents’ lying costs are high, the opinions of all agents converge; the limiting opinions are determined by the sophisticated agents’ bias and the structure of the network.

Proposition 3. If $V$ is strongly connected and aperiodic, then there exist a $\hat{\alpha} > 0$ such that for any $\alpha > \hat{\alpha}$ and any vector $X^0 \in \mathbb{R}^N$, $\lim_{t \to \infty} X^t = (I - U)^{-1} \tilde{B}$.

The proof of proposition 3 is basically an application of proposition 2, the details are provided in Appendix. One of the features of proposition 3 is that the initial beliefs of all agents don’t affect the limiting beliefs. This feature is different from all other non-Bayesian social learning models.

The next step is analyzing the distribution of the limiting opinions. Since there are two inverse in $(I - U)^{-1}$, It is not easy to analyze what it indicates. One thing I can show is that under some mild conditions, there is no consensus.

Proposition 4. If all beliefs converge, $\text{rank}(V_{\tilde{B}}) = d$ and sophisticated agents don’t have the same bliss point, then there is no consensus.

The condition $\text{rank}(V_{\tilde{B}}) = d$ in proposition 3 means that the weights that naive agents assign to sophisticated are not quite linearly corellated. It also indicates that all sophisticated agents can get
attention from some naive agents. This condition should be true for most networks, so in this model agents are highly likely to have different limiting opinions.

I now try to measure the disagreement. Generically $V$ is diagonalizable, suppose $V = GAG^{-1}$, where

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix}$$

$$G^{-1} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_N \end{pmatrix}$$

and

$$G = \begin{pmatrix} r_1 & r_2 & \cdots & r_N \end{pmatrix}$$

$\{\lambda_i\}$ are the eigenvalues listed in descending order by modulus, $l_i$ and $r_i$ are the respective left and right eigenvectors. Since $V$ is a row stochastic matrix, $\lambda_1 = 1, l_1 = l, r_1 = 1_N$.

I can now analyze how large the distance between opinions can be.

**Proposition 5.** If $V$ is strongly connected and aperiodic, then there exist a $\hat{\alpha} > 0$ such that for any $\alpha > \hat{\alpha}$, generically, there exists a constant $\tilde{c}$ such that

$$\hat{X} = \tilde{c}1_N + \frac{1}{\alpha} \left( \sum_{i=2}^{N} \frac{s_{r_i}s_{l_i}}{1 - \lambda_i} \right) \hat{B} + O\left(\frac{1}{\alpha^2}\right)$$

in which

$$\hat{B} = \begin{pmatrix} 0_d \\ \sum_{i \in S(d)} \left( \sum_{j \in A(d)} v_{ji} \right) v_{d+1,i}(b_i - \hat{c}) \\ \vdots \\ \sum_{i \in S(d)} \left( \sum_{j \in A(d)} v_{ji} \right) v_{N,i}(b_i - \hat{c}) \end{pmatrix}$$

Now there is no inverse of matrix, but the result is still hard to interpret. It is obvious that disagreement is positively associated to the difference between $b_i$ and negatively related to $\alpha$. It is also true that when spectral gap $(1 - \lambda_2)$ is small the disagreement is large. As discussed in Jackson (2008), Levin et al. (2009), Golub and Sadler (2016), this statement means that when the society is segregated, the disagreement is large.

### 3.3. Social Influence

The last part of this section is to ascertain how each agent in social network influences the limiting opinions. In DeGroot model, the social influence is represented by the left-hand unit eigenvector of $V$. Denote this vector by $l$. As discussed in section 3.2, the initial opinions of all agents don’t affect the limiting opinions, so naive agents should not have any social influence. If no naive agents pays attention to sophisticated agent $i$, since other sophisticated agents know his true belief, no agents listen to his lies, so his best response is to tell the truth. Then agent $i$ behave the same as naive agents, so
he also has no social influence. In this section, I assume that for any \( i \in S(d) \), \( \sum_{j \in A(d)} v_{ji} > 0 \), which means that for any sophisticated agent, there is at least one naive agent pays attention to him.

For the same reason I mentioned above (two inverse in the formula), it is not easy to calculate a general form for social influence. Here I derive the approximate numbers of social influence of sophisticated agents when \( \alpha \) is large. Denote \( \lim_{t \to \infty} X^t \) by \( \hat{X} = \left( \begin{array}{c} \hat{x}_1 \\ \vdots \\ \hat{x}_N \end{array} \right) \). Let

\[
\hat{l}_i = \frac{\left( \sum_{j \in A(d)} v_{ji} \right) \left( \sum_{j \in A(d)} v_{ji} \right)}{\sum_{k \in S(d)} \left( \sum_{j \in A(d)} v_{jk} \right) \left( \sum_{j \in A(d)} v_{jk} \right)}
\]

for any \( i \in S(d) \).

**Proposition 6.** \( \lim_{\alpha \to \infty} \hat{X} = 1_N \hat{c} \), in which \( \hat{c} = \sum_{i \in S(d)} \hat{l}_i b_i \).

Proposition 6 shows that when the cost of lying is large enough, the limiting opinions should be close to weighted average of the biases of sophisticated agents. The weight can be interpreted as a new kind of social influence. The value of \( \left( \sum_{j \in A(d)} v_{ji} \right) \left( \sum_{j \in A(d)} v_{ji} \right) \) determine the value of the new social influence \( \hat{l}_i \). The first part \( \left( \sum_{j \in A(d)} v_{ji} \right) \) is the total trust that all naive agents assign to \( i \), thus can be viewed as the direct influence. On the other hand, the second part \( \left( \sum_{j \in A(d)} v_{ji} \right) \) is the summation of the product of how much a naive agent trust \( i \) and how much influence this naive agent has on other agents. The second part can be viewed as the indirect influence.

4. Extension

4.1. Different Degrees of Importance

In this section, I try to extend the model to the case where the sophisticated agents place different degrees of importance on different naive agents. A sophisticated agent \( i \)'s utility function in time \( t \) is:

\[
u_i(x^{t-1}, s^t_i, b_i) = -\sum_{j \in A(d)} \beta_{ji}(x^t_j - b_i)^2 - \alpha(s^t_i - x^{t-1}_i)^2 \quad (4.1)
\]

in which \( \beta_{ji} \) represents how important naive agent \( j \) is to sophisticated agent \( i \). \( \beta_{ji} \) satisfies that for any \( i \), \( \sum_{j \in A(d)} \beta_{ji} = 1 \). The FOCs now are

\[
\sum_{j \in A(d)} \beta_{ji} v_{ji} \left( \sum_{k \in S(d)} v_{jk} s^t_k \right) + \alpha s^t_i = \alpha x^{t-1}_i - \sum_{j \in A(d)} \beta_{ji} v_{ji} \left( \sum_{j \in A(d)} v_{jk} x^{t-1}_k \right) + \sum_{j \in A(d)} \beta_{ji} v_{ji} b_i \quad (4.2)
\]

let

\[
\bar{V}_3 = \begin{pmatrix} 
\beta_{d+1,1} v_{d+1,1} & \cdots & \beta_{d+1,d} v_{d+1,d} \\
\vdots & & \ddots \\
\beta_{N,1} v_{N,1} & \cdots & \beta_{N,d} v_{N,d} 
\end{pmatrix}
\]

and

\[
\bar{B} = \begin{pmatrix} 
\sum_{j \in A(d)} \beta_{j1} v_{j1} b_1 \\
\sum_{j \in A(d)} \beta_{j2} v_{j2} b_2 \\
\vdots \\
\sum_{j \in A(d)} \beta_{jd} v_{jd} b_d 
\end{pmatrix}
\]
then system of FOCs can be rewritten as

\[(\alpha I_d + \tilde{V}_3^T V_3)S^{t} = \alpha X_1^{t-1} - \tilde{V}_3^T V_4 X_2^{t-1} + \tilde{B}\] (4.3)

It is easy to check that there exists a \(\alpha^*\), such that for any \(\alpha > \alpha^*\), \(\alpha I_d + \tilde{V}_3^T V_3\) is invertible. Then the opinion updating rule can also be written as two matrix equations:

\[X_1^t = V_1 X_1^{t-1} + V_2 X_2^{t-1}\] (4.4)

\[X_2^t = V_3 S^{t} + V_4 X_2^{t-1}\] (4.5)

then

\[X_2^t = \alpha V_3 (\alpha I_d + \tilde{V}_3^T V_3)^{-1} X_1^{t-1} - V_3 (\alpha I_d + \tilde{V}_3^T V_3)^{-1} \tilde{V}_3^T V_4 X_2^{t-1} + V_3 (\alpha I_d + \tilde{V}_3^T V_3)^{-1} \tilde{B} + V_4 X_2^{t-1}\]

\[= \alpha V_3 (\alpha I_d + \tilde{V}_3^T V_3)^{-1} X_1^{t-1} + \left[I_{N \times d} - V_3 (\alpha I_d + \tilde{V}_3^T V_3)^{-1} \tilde{V}_3^T \right] V_4 X_2^{t-1} + V_3 (\alpha I_d + \tilde{V}_3^T V_3)^{-1} \tilde{B}\]

Let \(\tilde{U}_3 = \alpha V_3 (\alpha I_d + \tilde{V}_3^T V_3)^{-1}\), \(\tilde{U}_4 = \left[I_{N \times d} - V_3 (\alpha I_d + \tilde{V}_3^T V_3)^{-1} \tilde{V}_3^T \right] V_4\), \(\tilde{U} = \begin{pmatrix} U_1 \\ \tilde{U}_3 \\ \tilde{U}_4 \end{pmatrix}\), \(\tilde{B}_1 = \begin{pmatrix} 0_d \\ V_3 (\alpha I_d + \tilde{V}_3^T V_3)^{-1} \tilde{B} \end{pmatrix}\) and \(X^{t} = \begin{pmatrix} X_1^{t} \\ X_2^{t} \end{pmatrix}\), then the opinion dynamics can be represented by the following matrix equation:

\[X^{t} = \tilde{U} X^{t-1} + \tilde{B}_1\] (4.6)

Proposition 1-4 all still hold and are almost the same as before, so I only show how social influence changes in this different environment. Let

\[
\hat{l}_i = \frac{\left(\sum_{j \in A(d)} \beta_{ji} v_{ji}\right) \left(\sum_{j \in A(d)} v_{ji} s_j\right)}{\sum_{k \in S(d)} \left(\sum_{j \in A(d)} \beta_{jk} v_{jk}\right) \left(\sum_{j \in A(d)} v_{jk} s_j\right)}
\]

for any \(i \in S(d)\).

**Proposition 7.** \(\lim_{\alpha \to \infty} \lim_{t \to \infty} X^t = 1_N \hat{c}\), in which \(\hat{c} = \sum_{i \in S(d)} \hat{l}_i b_i\).

Proposition 7 shows that when the sophisticated agents place different degrees of importance on different naive agent, the limiting beliefs are different. It is still the weighted average of the ideal points of sophisticated agents, but the weight is different. The indirect influence is still the same as before but the direct influence now is \(\sum_{j \in A(d)} \beta_{ji} v_{ji}\). The trust that all naive agents assign to \(i\) is weighted according to the degrees of importance.

4.2. Farsighted Sophisticated Agents

So far I have examined the case where the sophisticated agents are myopic. In this section, I am trying to extend the model to the case where the sophisticated agents are forward looking. I only consider the simplest case due to tractability considerations. Suppose there are two sophisticated agents and one naive agent. The sophisticated agents are stubborn which means they never change their minds (\(v_{11} = v_{22} = 1\)). Without loss of generality, I assume that the initial belief of the sophisticated agents are -1 (agent 1) and 1 (agent 2). The sophisticated agents try to convince the naive agent to believe what they believe. (\(b_1 = -1, b_2 = 1\)). I also assume that \(v_{33} = 0\). Then the stage payoffs are given by

\[\Pi_1 = -(v_{31} x_t + v_{32} y_t + 1)^2 - \alpha(x_t + 1)^2\]
\[ \Pi_2 = -(v_{31}x + v_{32}y - 1)^2 - \alpha (y - 1)^2 \]

in which \( x_t \) and \( y_t \) are signals sent by sophisticated agents and \( z_{t-1} \) is the belief of naive agent.

I follow the method in Maskin and Tirole (1987) to solve this model. I also assume that in odd period \( 2t + 1 \), agent 1 choose \( x_{2t+1} \) which will remain unchanged until period \( 2t + 3 \); and in even period \( 2t + 2 \), agent 1 choose \( y_{2t+2} \) which will remain unchanged until period \( 2t + 4 \). I also focus on Markov Perfect Equilibrium. Since \( v_{33} = 0 \), the state in each period is the action of the other sophisticated agent in last period. Denote the the pairs of dynamic reaction functions by \((R_1, R_2)\). The same as Maskin and Tirole (1987), \((R_1, R_2)\) is a MPE iff there exist valuation functions \(\{(V_1, W_1), (V_2, W_2)\}\) such that for any \(\{\hat{x}, \hat{y}\}\):

\[
V_1(\hat{y}) = \max_x \{\Pi_1(x, \hat{y}) + \delta W_1(x)\}
\]

\[
R_1(\hat{y}) \in \text{argmax}_x \{\Pi_1(x, \hat{y}) + \delta W_1(x)\}
\]

\[
W_1(\hat{x}) = \Pi_1(\hat{x}, R_2(\hat{x})) + \delta V_1(R_2(\hat{x}))
\]

The same as Maskin and Tirole (1987), the solution should satisfy that

\[
\frac{dR_2}{dx}(\hat{x}) = \frac{-\Pi_1^1(\hat{x}, R_1^{-1}(\hat{x})) - \delta \Pi_1^2(\hat{x}, R_2(\hat{x}))}{\delta \Pi_1^2(\hat{x}, R_2(\hat{x})) + \delta^2 \Pi_1^2(R_1(R_2(\hat{x))), R_2(\hat{x}))}
\]

\[
\frac{dR_1}{dy}(\hat{y}) = \frac{-\Pi_2^1(\hat{y}, \hat{y}) - \delta \Pi_1^2(R_1(\hat{y}), \hat{y})}{\delta \Pi_1^2(R_1(\hat{y}), \hat{y}) + \delta^2 \Pi_1^2(R_1(\hat{y}), R_2(R_1(\hat{y})))}
\]

I also assume that \( R_i \) are linear functions:

\[ R_1(y) = a_1 + b_1 y \]

\[ R_2(x) = a_2 + b_2 x \]

Then

\[ b_2 = -\frac{\alpha(x + 1) + v_{31}(v_{31}x + v_{32}R_1^{-1}(x) + 1) + \alpha \delta(x + 1) + \delta v_{31}(v_{31}x + v_{32}R_2(x) + 1)}{\delta v_{32}(v_{31}x + v_{32}R_2(x) + 1) + \delta^2 v_{32}(v_{31}R_1(R_2(x)) + v_{32}R_2(x) + 1)} \]

\[ b_1 = -\frac{\alpha(y - 1) + v_{32}(v_{32}R_2^{-1}(y) + v_{32}y - 1) + \alpha \delta(y - 1) + \delta v_{32}(v_{32}R_1(y) + v_{32}y - 1)}{\delta v_{31}(v_{31}R_1(y) + v_{32}y - 1) + \delta^2 v_{31}(v_{31}R_1(y) + v_{32}R_1(R_1(y)) - 1)} \]

Then it must be true that

\[
(1 + \delta)(\alpha + v_{31}^2) + \frac{1}{b_1} v_{31} v_{32} + \delta v_{31} v_{32} b_2 + \delta v_{31} v_{32} b_2 + \delta v_{32}^2 b_2 + \delta^2 v_{31} v_{32} b_1 b_2 + \delta^2 v_{32}^2 b_2 = 0 \]

\[ \iff \delta^2 v_{31} v_{32} b_1^2 b_2^2 + \delta (1 + \delta) v_{31} v_{32} b_1 b_2 + 2 \delta v_{31} v_{32} b_1 b_2 + (1 + \delta)(\alpha + v_{31}^2) b_1 + v_{31} v_{32} = 0 \quad (4.7) \]

\[ \delta^2 v_{31} v_{32} b_1^2 b_2^2 + \delta (1 + \delta) v_{31} v_{32} b_1 b_2 + 2 \delta v_{31} v_{32} b_1 b_2 + (1 + \delta)(\alpha + v_{32}^2) b_2 + v_{31} v_{32} = 0 \quad (4.8) \]

and

\[
-(\delta v_{32} b_2 + v_{31} + \alpha)(1 + \delta) + v_{31} v_{32} \frac{a_1}{b_1} = \delta v_{31} v_{32} a_2 + \delta (1 + \delta) v_{32}^2 a_2 b_2 + \delta^2 v_{31} v_{32} a_2 b_2 + \delta^2 v_{31} v_{32} a_2 b_1 b_2 \quad (4.9)
\]
(δv_{31} b_1 + v_{32} + a)(1 + δ) + v_{31} v_{32} \frac{a_2}{b_2} \\
= δv_{31} v_{32} a_1 + δ(1 + δ)v_{31}^2 a_1 b_1 + δ^2 v_{31} v_{32} a_2 b_1 + δ^2 v_{31} v_{32} a_1 b_1 b_2 \tag{4.10}

Firstly, let’s analyze the symmetric case. If v_{31} = v_{32} = 0.5, then from equations 1 and 2, it is clear that b_1 = b_2 = b, then

$$
\delta^2 b^4 + \delta(1 + \delta)b^3 + 2\delta b^2 + (1 + \delta)(4\alpha + 1)b + 1 = 0
$$

It is easy to check the following facts:

1. For any $0 < \delta \leq 1$ and $\alpha > 0$, there always exists a real root $b^* \in (-1, 0)$.
2. Given $b_1 = b_2 = b^*$, $a_1 = -a_2^* = \frac{2(\delta b^* + 1 + 2\alpha)(1 + \delta)b^*}{(1 + \delta b^*)^2(1 + \delta b^*)}$.
3. There exists $\alpha^*, \epsilon > 0$, such that for any $\alpha > \alpha^*$, $b^* \in \left(-\frac{1+\epsilon}{1+\delta(4\alpha+1)}, -\frac{1-\epsilon}{1+\delta(4\alpha+1)}\right)$

The same as Maskin and Tirole (1987), I can check that when $b^* \in (-1, 0)$, $P_1(x, \hat{y}) + \delta W_1(x)$ is concave, so there exists an equilibrium where $b = b^*$, $a_1 = -a_2^* = \frac{2(\delta b^* + 1 + 2\alpha)(1 + \delta)b^*}{(1 + \delta b^*)^2(1 + \delta b^*)}$ and the strategies are

$$
R_1(y) = a_1^* + b^* y \\
R_2(x) = a_2^* + b^* x
$$

When $\alpha$ is large, then $b^* \approx -\frac{1}{(1+\delta)(4\alpha+1)}$ and $a_1^* = -a_2^* \approx -1$. It is easy to check that the linear system is stable.

When it is not symmetric, the system becomes much harder to analyze, but the main results still hold. Let $p = v_{31}$ then $v_{32} = 1 - p$, I first prove the existence of real roots:

**Lemma 2.** Given $0 < p < 1$ and $0 < \delta \leq 1$, there exist $\alpha^* > 0$, such that for any $\alpha > \alpha^*$,

1. there exist real solutions to the system of equations (3.7) and (3.8), denoted by $b_1^*$ and $b_2^*$,
2. 

$$
b_1^* = \frac{-p(1 - p)}{(1 + \delta)(\alpha + p^2)} + O(\frac{1}{\alpha^3})
$$

$$
b_2^* = \frac{-p(1 - p)}{(1 + \delta)(\alpha + (1 - p)^2)} + O(\frac{1}{\alpha^3})
$$

**Lemma 2** shows that when $\alpha$ is large, there exists real solutions and the solutions can be represented by much simpler formulas. I can then start to calculate $a_1$ and $a_2$.

**Lemma 3.** Given $0 < p < 1$ and $0 < \delta \leq 1$, there exist $\alpha^* > 0$, such that for any $\alpha > \alpha^*$, if $b_i = b_i^*$, then $a_1^* = \frac{\Delta_1}{\Delta_1}$ and $a_2^* = \frac{\Delta_2}{\Delta_1}$, where

$$
\Delta_1 = 1 - \frac{3\delta^2 p^2(1 - p)^2}{(1 + \delta)^2(\alpha + p^2)(\alpha + (1 - p)^2)} + O\left(\frac{1}{\alpha^3}\right)
$$

$$
\Delta_2 = 1 - \frac{p(1 - p)}{\alpha + p^2} - \frac{\delta p(1 - p)\alpha}{(1 + \delta)(\alpha + p^2)(\alpha + (1 - p)^2)} + \frac{[-3p + 1 + \delta]}{(1 + \delta)^2(\alpha + p^2)(\alpha + (1 - p)^2)} + O\left(\frac{1}{\alpha^3}\right)
$$

12
\[ \Delta_3 = 1 + \frac{p(1-p)}{\alpha + (1-p)^2} + \frac{\delta p(1-p)\alpha}{(1+\delta)(\alpha + p^2)(\alpha + (1-p)^2)} + \frac{3(1-p) - (1+\delta)}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right) \]

By lemma 2 and 3, I can find that when \( \alpha \) is large, there are always real solutions to the system of equations such that \( b_1^*, b_2^* \) are both negative numbers close to 0 and \( a_1^* \approx -a_2^* \approx -1 \). The same as the symmetric case, these solutions induce a MPE and it is stable.

When \( \alpha \) is large, the state will converge quickly to the steady state:

\[ (x^*, y^*) = \left( \frac{a_1^* + b_1^* a_2^*}{1 - b_1^* b_2^*}, \frac{a_2^* + b_2^* a_1^*}{1 - b_1^* b_2^*} \right) \]

By lemma 2 and 3, I can analyze some properties of the steady state. I will compare the steady state with the case where the sophisticated agents are myopic. Denote the belief of the naive agent in steady state and in myopic case by \( z \) and \( \hat{z} \), respectively. Then I can prove the following proposition.

**Proposition 8.** \( z^* > \hat{z} \) if and only if \( 1 - p > p \)

Proposition 8 shows that when the sophisticated agents are forward-looking instead of myopic, the agent who has more influence when all agents are naive, gain even more influence. I can also compare the stage payoff in these two different cases. Denote the stage payoffs of the sophisticated agents in steady state and in myopic case by \( \Pi_i^* \) and \( \Pi_i \), respectively. Then I can prove the following proposition.

**Proposition 9.** Given \( 0 < v_{31}, v_{32} < 1 \) and \( 0 < \delta \leq 1 \), there exist \( \alpha^* > 0 \), such that for any \( \alpha > \alpha^* \), \( \Pi_i^* > \Pi_i \).

Proposition 9 shows that when the sophisticated agents are forward-looking instead of myopic, the stage payoffs of both sophisticated agents increase, even though one has more influence while the other has less.

The last proposition discusses some comparative statics with respect to \( \delta \).

**Proposition 10.** Given \( 0 < v_{31}, v_{32} < 1 \) and \( 0 < \delta_1 < \delta_2 \leq 1 \), there exist \( \alpha^* > 0 \), such that for any \( \alpha > \alpha^* \),

1. \( \Pi_i^*(\delta_2) > \Pi_i^*(\delta_1) \)
2. \( z*(\delta_2) > z*(\delta_1) \) if and only if \( 1 - p > p \)

Proposition 10 shows that when the sophisticated agents are more patient, the effects I discuss in proposition 6 and 7 become more significant.

5. **Conclusion**

In this paper, I study a bounded rationality model of opinion formation in which some agents can lie. I show that the opinion dynamics converge as long as the network is strongly connected and aperiodic and that the cost of lying is large enough. I also show that the limiting opinions are completely determined by sophisticated agents’ bias and the structure of the network. When the sophisticated agents’ bliss points are not the same and that the weight naive agents assign to sophisticated agents don’t collinear, there is no consensus. I analyze what influence the disagreement and show that disagreement is positively related to the inverse of spectral gap of the network and negatively related to the lying cost. I also show that naive agents don’t have any social influence and sophisticated agents’ social influence can be decomposed into two separate factors: direct influence and indirect influence.
6. Appendix: Proofs

6.1. Proof of Proposition 1

Proof. There exists a unique NE if and only if $\alpha I_d + V_3^T V_3$ is invertible. For every non-zero column vector $z$ of $d$ real numbers, denoted by $\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{pmatrix}$, $z^T (\alpha I_d + V_3^T V_3) z = \alpha z^T z + z^T V_3^T V_3 z$. It is easy to calculate that:

$$
\alpha z^T z = \alpha \left( \sum_{i=1}^{d} z_i^2 \right)
$$

and

$$
z^T V_3^T V_3 z = \sum_{i=d+1}^{N} \left( \sum_{j=1}^{d} v_{ij} z_j \right)^2
$$

Since $z$ is non-zero, $\alpha \left( \sum_{i=1}^{d} z_i^2 \right) > 0$ and $\sum_{i=d+1}^{N} \left( \sum_{j=1}^{d} v_{ij} z_j \right)^2 \geq 0$. Then $z^T (\alpha I_d + V_3^T V_3) z = \alpha z^T z + z^T V_3^T V_3 z > 0$. By definition, $\alpha I_d + V_3^T V_3$ is positive definite, thus invertible.

6.2. Proof of Lemma 1

Proof. Denote $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ of $n$ elements by $1_n$, then:

$$
\begin{pmatrix}
    x_1^t - b \\
    x_2^t - b \\
    \vdots \\
    x_N^t - b
\end{pmatrix}
- \mathbf{U}
\begin{pmatrix}
    x_1^{t-1} - b \\
    x_2^{t-1} - b \\
    \vdots \\
    x_N^{t-1} - b
\end{pmatrix} = X^t - UX^{t-1} - b1_N + bU1_N
$$

Since $V$ is row stochastic, $V_1 1_d + V_2 1_{N-d} = 1_d$ and $V_3 1_d + V_4 1_{N-d} = 1_{N-d}$

$$
-b1_N + bU1_N = b
$$

$$
-1_{N-d} + \alpha V_3 (\alpha I_d + V_3^T V_3)^{-1} 1_d + (I_{N-d} - V_3 (\alpha I_d + V_3^T V_3)^{-1} V_3^T) V_4 1_{N-d}
$$

$$
-1_{N-d} + \alpha V_3 (\alpha I_d + V_3^T V_3)^{-1} 1_d + (I_{N-d} - V_3 (\alpha I_d + V_3^T V_3)^{-1} V_3^T) (1_{N-d} - V_3 1_d)
$$

$$
\alpha V_3 (\alpha I_d + V_3^T V_3)^{-1} 1_d + V_3 (\alpha I_d + V_3^T V_3)^{-1} V_3^T V_3 1_d - V_3 (\alpha I_d + V_3^T V_3)^{-1} V_3^T 1_{N-d} - V_3 1_d
$$

$$
V_3 (\alpha I_d + V_3^T V_3)^{-1} (\alpha I_d + V_3^T V_3) 1_d - V_3 1_d - V_3 (\alpha I_d + V_3^T V_3)^{-1} V_3^T 1_{N-d}
$$

$$
- V_3 (\alpha I_d + V_3^T V_3)^{-1} V_3^T 1_{N-d}
$$
Then

\[-b_1N + bU_1N = b \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \hat{B}\]

so

\[
\begin{pmatrix} x_1^t - b \\ x_2^t - b \\ \vdots \\ x_N^t - b \end{pmatrix} - U \begin{pmatrix} x_1^{t-1} - b \\ x_2^{t-1} - b \\ \vdots \\ x_N^{t-1} - b \end{pmatrix} = X^t - UX^{t-1} - b_1N + bU_1N = X^t - UX^{t-1} - \hat{B} = 0
\]

6.3. Proof of proposition 2

Proof. As proved in DeGroot (1974), there exists a \( c \in \mathbb{R} \) such that \( \lim_{t \to \infty} V^tX^0 = 1_{NC} \), and that \( c = lX^0 \), in which \( l \) is a left-hand unit eigenvector of \( V \). By Perron–Frobenius theorem, \( l_i > 0 \) for any \( 1 \leq i \leq N \). Let \( \bar{X}^0_k = \begin{pmatrix} x_{1,k} \\ \vdots \\ x_{N,k} \end{pmatrix} \) satisfy that

\[
\begin{cases}
  x_{i,k} = 1 & \text{if } i = k \\
  x_{i,k} = 0 & \text{if } i \neq k
\end{cases}
\]

Then \( c = lX^0_k > 0 \) for any \( k \). Let me first consider the case in which \( X^0 = X^0_k \).

Denote \( V^tX^0 \) by \( \begin{pmatrix} x_1(t) \\ \vdots \\ x_N(t) \end{pmatrix} \). Since \( N \) is finite, there exists a \( H \) such that for any \( t \geq H \),

\[
\min_i \{x_i(t)\} > \frac{c}{2} > 0.
\]

Denote

\[
U = \begin{pmatrix} V_1 \\ \alpha V_3(\alpha I_d + V_3^T V_3)^{-1} - V_3(\alpha I_d + V_3^T V_3)^{-1}V_3^TV_4 \end{pmatrix}
\]

so

\[
U - V = \begin{pmatrix} V_3((I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} - I_d) \\ \alpha V_3(\alpha I_d + V_3^T V_3)^{-1} - V_3(\alpha I_d + V_3^T V_3)^{-1}V_3^TV_4 \end{pmatrix}
\]

\[
= \begin{pmatrix} V_3((I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} - (I_d + \frac{1}{\alpha} V_3^T V_3)) - \frac{1}{\alpha} V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1}V_3^TV_4 \\ -\frac{1}{\alpha} V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1}V_3^TV_4 - \frac{1}{\alpha} V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1}V_3^TV_4 \end{pmatrix}
\]

\[
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Then

\[
U = \left[ I_N - \frac{1}{\alpha} \begin{pmatrix} 0 \\ V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1}V_3^T \end{pmatrix} \right] V
\]

Since \( \lim_{\alpha \to \infty} (I_d + \frac{1}{\alpha} V_3^T V_3) = I_d \) and due to lemma 1, \( I_d + \frac{1}{\alpha} V_3^T V_3 \) is always invertible, \( \lim_{\alpha \to \infty} \)


\[ \frac{1}{\alpha} V_3^T V_3 \] 

\[ = \lim_{\alpha \to \infty} \frac{1}{\alpha} V_3^T V_3 = \frac{1}{\alpha} \frac{1}{\alpha} \frac{1}{\alpha} = I_d. \] 

Then \[ \lim U = V. \] 

Since \( H \) is finite, \[ \lim U^H X^0 = V^H X^0. \] 

Then there exists a \( a_1 > 0 \), such that for any \( \alpha > a_1, \) \[ min_i \{ x_i^H \} > min_i \{ \alpha \} - \frac{\alpha}{\alpha} > \frac{\alpha}{\alpha} > 0. \]

Since \( V \) is strongly connected and aperiodic, there exists a \( M_1 \) such that for any \( m \geq M_1 \) every element of matrix \( V^m \) is positive. Let \( M = M_1 + 1 \), then every element of both \( V^M \) and \( V^{M-1} \) is positive. Denote \[ \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3 \end{pmatrix} = P. \] 

Then

\[ U^M = [V - PV]^M = V^M - V^{M-1} PV + \cdots + (-PV)^M \]

Since \( M \) is finite and \[ \lim P = 0, \] there exists a \( a_2 > 0 \) such that for any \( \alpha > a_2, \) every element of matrix \( U^M \) is positive. Then for any \( \alpha > \max \{ a_1, a_2 \} \) and any \( l \in N_+ \), every element of vector \( U^{M+l+T} X^0 \) is positive.

Now let me prove that \[ \lim_{l \to \infty} U^{M+l+T} X^0 = 0. \] First let me do some estimation.

\[ U^M 1_N = [V - PV]^M 1_N = 1_N - V^{M-1} P 1_N + \cdots + (-PV)^M 1_N = 1_N - V^{M-1} P 1_N + O\left( \frac{1}{\alpha^2} 1_N \right) \]

in which

\[ P 1_N = \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3 \end{pmatrix} 1_N = \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3 1_N \end{pmatrix} \]

Then

\[ U^M 1_N = 1_N - \frac{1}{\alpha} V^{M-1} \begin{pmatrix} 0 & 0 \\ V_3 V_3^T 1_N \end{pmatrix} + O\left( \frac{1}{\alpha^2} 1_N \right) = 1_N - \frac{1}{\alpha} \left[ V^{M-1} \begin{pmatrix} 0 & 0 \\ V_3 V_3^T 1_N \end{pmatrix} + O\left( \frac{1}{\alpha} 1_N \right) \right] \]

In which

\[ \begin{pmatrix} 0 \\ V_3 V_3^T 1_N \end{pmatrix} = \begin{pmatrix} \sum_{j \in A(d)} v_{j1} \\ \vdots \\ \sum_{j \in A(d)} v_{jd} \end{pmatrix} \]

Since \( V \) is strongly connected, \( V_3 \neq 0 \), then all elements of matrix \( V_3 V_3^T \) are nonnegative and at least one of them is positive. Then all elements of vector \( \begin{pmatrix} 0 & 0 \\ V_3 V_3^T 1_N \end{pmatrix} \) are also nonnegative and that at least one of them is positive. As proved above, every element of \( V^{M-1} \) is positive, thus every element of \( V^{M-1} \begin{pmatrix} 0 & 0 \\ V_3 V_3^T 1_N \end{pmatrix} \) is positive. Let \( r \) be the minimum of elements of \( V^{M-1} \begin{pmatrix} 0 & 0 \\ V_3 V_3^T 1_N \end{pmatrix} \). Then there exists a \( a_3 > 0 \), such that for any \( \alpha > a_3 \), every element of matrix \( V^{M-1} \begin{pmatrix} 0 & 0 \\ V_3 V_3^T 1_N \end{pmatrix} + O\left( \frac{1}{\alpha} 1_N \right) \) is larger than \( \frac{\alpha}{\alpha} \). Then for any \( \alpha > \max \{ \alpha, a_3 \} \), every element of \( U^M 1_N \) is positive and no larger than \( 1 - \frac{\alpha}{\alpha} \).

Now I can prove that \[ \lim_{l \to \infty} U^{M+l+H} X^0 = 0. \] Given a \( \alpha > \max \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \), as proved above, for any \( l \in N_+ \), every element of vector \( U^{M+l+H} X^0 \) and every element of matrix \( U^M \) is positive. Let \( R^t = \max \{ x_i^t \} \). Then for any \( l \in N_+ \), every element of vector \( U^{M+l+H} X^0 = U^M U^{M+(l-1)+H} X^0 \) is no larger than \( U^M 1_N R^{M+(l-1)+H} \). Since every element of \( U^M 1_N \) is positive and no larger than \( 1 - \frac{\alpha}{\alpha} \), \( R^{M+(l-1)+H} \leq (1 - \frac{\alpha}{\alpha}) R^{(l-1)+H} \). Then \( R^{M+l+H} \leq (1 - \frac{\alpha}{\alpha}) R^H. \) \( r \) and \( \alpha \) is given and \( R^H \) is
a constant, so \( \lim_{t \to \infty} R_i^{M*+H} = 0 \). Since for any \( 1 \leq i \leq N \), \( 0 < x_i^{M*+H} \leq R_i^{M*+H} \), \( \lim_{t \to \infty} U_i^{M*+H} X^0 = \mathbf{0} \). Since \( U \) and \( M \) are finite, for any \( 1 \leq k \leq M - 1 \), \( \lim_{t \to \infty} U_i^{M*+k+H} X^0 = \mathbf{0} \).

\( U^k \lim_{t \to \infty} U_i^{M*+H} X^0 = 0 \), therefore \( \lim_{t \to \infty} U^t X^0 = 0 \).

For other dynamics with \( X^0 = X_k^0, k \neq 1 \), the proof still works. Therefore, I have proved that there exist a \( \alpha(k) > 0 \) such that for any \( \alpha > \alpha(k) \), \( \lim_{t \to \infty} U_i^t X^0 = 0 \). Let \( \alpha = \max_{1 \leq k \leq N} \{ \alpha(k) \} \), then for any \( \alpha > \alpha(k) \),

\[
\alpha(k), \text{ and any } X^0 = \begin{pmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_N^0 \end{pmatrix} \in \mathbb{R}^N, \lim_{t \to \infty} U_i^t X^0 = \lim_{t \to \infty} \left[ U_i^t \left( \sum_{i=1}^N x_i^0 X_i^0 \right) \right] = \sum_{i=1}^N x_i^0 \left( \lim_{t \to \infty} U_i^t X_i^0 \right) = 0
\]

6.4. Proof of proposition 3

**Proof.** First, let me prove that \( I - U \) is invertible.

Since \( V \) is strongly connected and aperiodic, then, as I proved in proposition 2, there exist a \( \alpha > 0 \) and \( M \in \mathbb{N}^+ \) such that for any \( \alpha > \alpha \), every element of \( U_i 1_{N} \) is positive and no larger than \( 1 - \frac{x}{2\alpha} \), so \( \| U_i \| \_\infty < 1 \). Then the Neumann serie \( \sum_{k=1}^\infty U_i^k \) converge and \( (I - U)^{-1} = \lim_{k \to \infty} \sum_{i=1}^k U_i^i \).

Let \( Y^t = X^t - (I - U)^{-1} \Hat{B} \), then

\[
Y^t - U Y^{t-1} = X^t - (I - U)^{-1} \Hat{B} - U X^{t-1} + U (I - U)^{-1} \Hat{B} = X^t - U X^{t-1} - \Hat{B} = 0
\]

By proposition 2,

\[
\lim_{t \to \infty} Y^t = \lim_{t \to \infty} U^t Y^0 = 0
\]

so

\[
\lim_{t \to \infty} X^t = \lim_{t \to \infty} Y^t + (I - U)^{-1} \Hat{B} = (I - U)^{-1} \Hat{B}
\]

6.5. Proof of proposition 4

**Proof.** Since \( \text{rank}(V_3) = d \), I can rearrange the code name of naive agents such that \( V_3 \) can be partitioned into 2 blocks as follows:

\[
V_3 = \begin{pmatrix} V_{31} \\ V_{32} \end{pmatrix}
\]

in which \( V_{31} \) is a square matrix \((d \times d)\) and \( \text{rank}(V_{31}) = d \). This will not change the network structure, thus does not affect the result of proposition 3.
Suppose that there is consensus, denoted by \( c \). Then it must be true that

\[
\hat{B} = (I - U)1_Nc
\]

\[
= \left\{ 1_N - \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T & V_3^T \end{pmatrix} \right\} V1_N \}
\]

\[
= \left\{ 1_N - \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T \end{pmatrix} \right\} c
\]

\[
= \left[ 1_N - 1_N + \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T \end{pmatrix} \right] 1_N \}
\]

\[
= \frac{c}{\alpha} \left( V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T1_{N-d} \right)
\]

Since

\[
\hat{B} = \frac{1}{\alpha} \left( V_3(I_d + \frac{0_d}{\alpha}V_3^TV_3)^{-1}B \right)
\]

the equation above is equivalent to

\[
V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}V_3^T1_{N-d}c = V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}B
\]

\[
\iff V_3(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}(V_3^T1_{N-d}c - B) = 0
\]

\[
\iff \left( V_{31}(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}(V_3^T1_{N-d}c - B) \right) = 0
\]

Then it must be true that

\[
V_{31}(I_d + \frac{1}{\alpha}V_3^TV_3)^{-1}(1_{N-d}c - B) = 0
\]

Since \( V_{31} \) and \( (I_d + \frac{\alpha}{\alpha}V_3^TV_3)^{-1} \) are both invertible, \((V_3^T1_{N-d}c - B) = 0\).

\[
V_3^T1_{N-d}c = \begin{pmatrix} \sum_{j\in A(d)} v_{j1} \\ \sum_{j\in A(d)} v_{j2} \\ \vdots \\ \sum_{j\in A(d)} v_{jd} \end{pmatrix} c
\]

and

\[
B = \begin{pmatrix} \sum_{j\in A(d)} v_{j1}b_1 \\ \sum_{j\in A(d)} v_{j2}b_2 \\ \vdots \\ \sum_{j\in A(d)} v_{jd}b_d \end{pmatrix}
\]

since rank\((V_3) = d\), for any \( 1 \leq i \leq d \), \( \sum_{j\in A(d)} v_{ji} > 0 \).

\[
(V_3^T1_{N-d}c - B) = 0
\]

\[
\implies b_1 = b_2 = \ldots = b_d = c
\]

Contradiction.
6.6. Proof of Proposition 5

Proof.

\[ G^{-1}(I - U)G \]

\[ = G^{-1}(I - V)G + \frac{1}{\alpha} G^{-1} \begin{pmatrix} 0 & 0 \\ V_3(J_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} V G \]

\[ = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 - \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 - \lambda_N \end{pmatrix} + \frac{1}{\alpha} H - \frac{1}{\alpha^2} \hat{H} + O(\frac{1}{\alpha^3}) \]

\[ = \begin{pmatrix} 0 & 0 \\ 0 & I_{N-1} - A_{22} \end{pmatrix} + \frac{1}{\alpha} H - \frac{1}{\alpha^2} \hat{H} + O(\frac{1}{\alpha^2}) \]

in which \( h_{ij} = l_i \left( \begin{array}{cc} 0 & 0 \\ V_3 V_3^T \end{array} \right) V r_j = \lambda_j l_i \left( \begin{array}{cc} 0 & 0 \\ V_3 V_3^T \end{array} \right) r_j \) and \( \hat{h}_{ij} = \lambda_j l_i \left( \begin{array}{cc} 0 & 0 \\ V_3 V_3^T V_3 V_3^T \end{array} \right) r_j \), then

\[ G^{-1}(I - U)G = \begin{pmatrix} \frac{1}{\alpha} h_{11} - \frac{1}{\alpha^2} \hat{h}_{11} + O(\frac{1}{\alpha^3}) & \frac{1}{\alpha} H_{12} - \frac{1}{\alpha^2} \hat{H}_{12} + O(\frac{1}{\alpha^3}) \\ \frac{1}{\alpha} H_{21} - \frac{1}{\alpha^2} \hat{H}_{21} + O(\frac{1}{\alpha^3}) & I_{N-1} - A_{22} + \frac{1}{\alpha} H_{22} - \frac{1}{\alpha^2} \hat{H}_{22} + O(\frac{1}{\alpha^3}) \end{pmatrix} \]

Since \( h_{11} > 0 \) and for any \( k \neq 1, 1 - \lambda_k \neq 0 \), there exists a \( \alpha > 0 \) such that for any \( \alpha > \alpha \), \( \frac{1}{\alpha} h_{11} + O(\frac{1}{\alpha}) > 0 \) and

\[ \Sigma = I_{N-1} - A_{22} + \frac{1}{\alpha} H_{22} - \frac{1}{\alpha^2} \hat{H}_{22} + O(\frac{1}{\alpha^3}) \]

\[ = I_{N-1} - A_{22} + \frac{1}{\alpha} H_{22} - \frac{1}{\alpha h_{11}} H_{21} H_{12} + O(\frac{1}{\alpha^2}) \]

is invertible. Then

\[ \frac{1}{\alpha} (I - U)^{-1} = \frac{1}{\alpha} G \left( G^{-1}(I - U)G \right)^{-1} G^{-1} \]

in which

\[ \frac{1}{\alpha} G^{-1}(I - U)G^{-1} \]

\[ = \frac{1}{\alpha} \left( \begin{array}{ccc} \frac{1}{\alpha} h_{11} - \frac{1}{\alpha^2} \hat{h}_{11} + O(\frac{1}{\alpha^3}) & \frac{1}{\alpha} H_{12} - \frac{1}{\alpha^2} \hat{H}_{12} + O(\frac{1}{\alpha^3}) \\ \frac{1}{\alpha} H_{21} - \frac{1}{\alpha^2} \hat{H}_{21} + O(\frac{1}{\alpha^3}) & I_{N-1} - A_{22} + \frac{1}{\alpha} H_{22} - \frac{1}{\alpha^2} \hat{H}_{22} + O(\frac{1}{\alpha^3}) \end{array} \right)^{-1} \]

\[ = \left( \begin{array}{ccc} \frac{1}{\alpha h_{11} - h_{11}} + \frac{\alpha}{\alpha h_{11} - h_{11}} + \Sigma^{-1} H_{21} + O(\frac{1}{\alpha^3}) & -\frac{1}{\alpha h_{11} - h_{11}} H_{12} \Sigma^{-1} + O(\frac{1}{\alpha^3}) \\ -\frac{1}{\alpha h_{11} - h_{11}} \Sigma^{-1} H_{21} + O(\frac{1}{\alpha^3}) & \frac{1}{\alpha \Sigma^{-1}} \end{array} \right) \]

Since

\[ \frac{1}{\alpha h_{11} - h_{11}} = \frac{1}{h_{11}} + \frac{1}{\alpha h_{11}^2} + O(\frac{1}{\alpha^2}) \]
\[ \left( \frac{1}{h_{11} - \frac{1}{\alpha} h_{11}} \right)^2 = \frac{1}{\alpha^2} + \frac{2}{\alpha^2} + O\left( \frac{1}{\alpha^2} \right) \]

\[ \frac{1}{\alpha} (G^{-1}(I - U)G)^{-1} \]

\[ = \left( \begin{array}{ccc} \frac{1}{h_{11}} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) + \frac{1}{\alpha} \left( \begin{array}{ccc} \frac{h_{11}}{h_{11}} + \frac{1}{h_{11}} H_{12} (I_{N-1} - A_{22})^{-1} H_{21} & -\frac{1}{h_{11}} H_{12} (I_{N-1} - A_{22})^{-1} \\ -\frac{1}{h_{11}} (I_{N-1} - A_{22})^{-1} H_{21} & (I_{N-1} - A_{22})^{-1} \end{array} \right) + O\left( \frac{1}{\alpha^2} \right) \]

Since \[ \Sigma^{-1} = (I_{N-1} - A_{22})^{-1} + O\left( \frac{1}{\alpha} \right) \]

\[ \frac{1}{\alpha} (G^{-1}(I - U)G)^{-1} \]

\[ = \left( \begin{array}{ccc} \frac{1}{h_{11}} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) + \frac{1}{\alpha} \left( \begin{array}{ccc} \frac{h_{11}}{h_{11}} + \frac{1}{h_{11}} H_{12} (I_{N-1} - A_{22})^{-1} H_{21} & -\frac{1}{h_{11}} H_{12} (I_{N-1} - A_{22})^{-1} \\ -\frac{1}{h_{11}} (I_{N-1} - A_{22})^{-1} H_{21} & (I_{N-1} - A_{22})^{-1} \end{array} \right) + O\left( \frac{1}{\alpha^2} \right) \]

\[ \frac{1}{\alpha} (I - U)^{-1} \]

\[ = \frac{1}{\alpha} G (G^{-1}(I - U)G)^{-1} G^{-1} \]

\[ = G \left( \begin{array}{ccc} \frac{1}{h_{11}} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) G^{-1} \]

\[ + \frac{1}{\alpha} G \left( \begin{array}{ccc} \frac{h_{11}}{h_{11}} + \frac{1}{h_{11}} H_{12} (I_{N-1} - A_{22})^{-1} H_{21} & -\frac{1}{h_{11}} H_{12} (I_{N-1} - A_{22})^{-1} \\ -\frac{1}{h_{11}} (I_{N-1} - A_{22})^{-1} H_{21} & (I_{N-1} - A_{22})^{-1} \end{array} \right) G^{-1} \]

\[ + O\left( \frac{1}{\alpha^2} \right) \]

Since \[ G = \left( \begin{array}{ccc} 1_N & r_2 & \cdots & r_N \end{array} \right) \]

there exists a vector \( \tilde{c}_1 \), such that

\[ G \left( \begin{array}{ccc} \frac{h_{11}}{h_{11}} + \frac{1}{h_{11}} H_{12} (I_{N-1} - A_{22})^{-1} H_{21} & -\frac{1}{h_{11}} H_{12} (I_{N-1} - A_{22})^{-1} \\ 0 & 0 \end{array} \right) G^{-1} = 1_N \tilde{c}_1 \]
so
\[
\frac{1}{\alpha} (I - U)^{-1}
= \frac{1}{h_{11}} N l + \frac{1}{\alpha} 1_N \hat{c}_1
+ \frac{1}{\alpha} G \left( -\frac{1}{h_{11}} (I_{N-1} - A_{22})^{-1} H_{21} \left( I_{N-1} - A_{22} \right)^{-1} \right) G^{-1} + O\left( \frac{1}{\alpha^2} \right)
\]

\[
G \left( -\frac{1}{h_{11}} (I_{N-1} - A_{22})^{-1} H_{21} 0 \right) G^{-1}
= G \left( -\frac{1}{h_{11}} (I_{N-1} - A_{22})^{-1} H_{21} 0 \right) \left( \begin{array}{c} l_1 \\ l_2 \\ \vdots \\ l_N \end{array} \right)
\]

\[
= G \left( -\frac{1}{h_{11}} (I_{N-1} - A_{22})^{-1} H_{21} l_1 \right)
= -\frac{1}{h_{11}} \begin{pmatrix} r_2 & \cdots & r_N \end{pmatrix} (I_{N-1} - A_{22})^{-1} H_{21} l_1
\]

\[
= -\frac{1}{h_{11}} \sum_{i=2}^{N} \frac{r_i l_i}{1 - \lambda_i} \begin{pmatrix} 0 \\ 0 \\ V_3 V_3^T \end{pmatrix} r_1 l_1
\]

and
\[
G \left( 0 \left( I_{N-1} - A_{22} \right)^{-1} \right) G^{-1}
= \sum_{i=2}^{N} \frac{r_i l_i}{1 - \lambda_i}
\]

so
\[
\frac{1}{\alpha} (I - U)^{-1}
= \frac{1}{h_{11}} N l + \frac{1}{\alpha} 1_N \hat{c}_1
+ \frac{1}{\alpha} G \left( -\frac{1}{h_{11}} (I_{N-1} - A_{22})^{-1} H_{21} \left( I_{N-1} - A_{22} \right)^{-1} \right) G^{-1} + O\left( \frac{1}{\alpha^2} \right)
\]

\[
= \frac{1}{h_{11}} N l + \frac{1}{\alpha} 1_N \hat{c}_1 + \frac{1}{\alpha} \sum_{i=2}^{N} \frac{r_i l_i}{1 - \lambda_i} \left[ I - \frac{1}{h_{11}} \begin{pmatrix} 0 \\ 0 \\ V_3 V_3^T \end{pmatrix} r_1 l_1 \right] + O\left( \frac{1}{\alpha^2} \right)
\]
then
\[
\dot{X} = (I - U)^{-1} \dot{B}
\]
\[
= \left[ \frac{1}{h_{11}} N l + \frac{1}{\alpha} N c_1 + \frac{1}{\alpha} \sum_{i=2}^{N} \frac{r_i l_i}{1 - \lambda_i} \right] \left[ I - \frac{1}{h_{11}} \begin{pmatrix} 0 & 0 \\ V_3 V_3^T \end{pmatrix} \right] \dot{r}_1 l_1 + O \left( \frac{1}{\alpha^2} \right)
\]
\[
= \frac{1}{\alpha} \sum_{i=2}^{N} \frac{r_i l_i}{1 - \lambda_i} \left[ I - \frac{1}{h_{11}} \begin{pmatrix} 0 & 0 \\ V_3 V_3^T \end{pmatrix} \right] \dot{r}_1 l_1 + O \left( \frac{1}{\alpha^2} \right)
\]
\[
= \frac{1}{\alpha} \sum_{i=2}^{N} \frac{r_i l_i}{1 - \lambda_i} \left[ \begin{pmatrix} 0 \\ V_3 \end{pmatrix} \begin{pmatrix} b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_d \end{pmatrix} - \begin{pmatrix} 0 \\ V_3 V_3^T \end{pmatrix} \right] \dot{r}_1 l_1 + O \left( \frac{1}{\alpha^2} \right)
\]
\[
= \frac{1}{\alpha} \sum_{i=2}^{N} \frac{r_i l_i}{1 - \lambda_i} \left[ \begin{pmatrix} 0 \\ V_3 \end{pmatrix} \begin{pmatrix} b_1 - \dot{c} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_d - \dot{c} \end{pmatrix} \right] + O \left( \frac{1}{\alpha^2} \right)
\]

6.7. Proof of proposition 6

**Proof.** First let me prove that when \( \alpha \) is large enough, all limiting beliefs are close to each other. I have proved that

\[
\dot{X} = \lim_{t \to \infty} X^t = (I - U)^{-1} \dot{B}
\]

Then as I proved in proposition 1 there exists a \( \hat{\alpha} > 0 \) such that for any \( \alpha > \hat{\alpha} \), there exists \( M > 0 \), such that \( U^M \) is a positive matrix and \( \| U^M \|_\infty < 1 \).

\[
\dot{X} = U \dot{X} + \dot{B}
\]
\[
= U^M \dot{X} + \dot{B} + \sum_{k=1}^{M-1} U^k \dot{B}
\]
\[
= U^M \dot{X} + O \left( \frac{1}{\alpha} \right)
\]

According to Lemma 1, without loss of generality, I can assume that \( b_i > 0 \). then

\[
l \begin{pmatrix} V_3 (I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} B \\ 0 \end{pmatrix} = l \begin{pmatrix} 0 \\ V_3 (I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} V \dot{X}
\]

Denote \( l \begin{pmatrix} 0 \\ V_3 (I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} V \) by \( w = (w_1, \ldots, w_N) \). Since

\[
\lim_{\alpha \to \infty} l \begin{pmatrix} 0 \\ V_3 (I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} B \end{pmatrix} = l \begin{pmatrix} 0 \\ V_3 B \end{pmatrix} > 0
\]
and
\[ \lim_{\alpha \to \infty} \sum_{i=1}^{N} w_i = \lim_{\alpha \to \infty} l \begin{pmatrix} 0 & 0 \\ V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} V_1 N = l \begin{pmatrix} 0_d \\ V_3 V_3^T \end{pmatrix} 1_N > 0 \]

so there exists a \( \tilde{\alpha} > 0 \) such that for any \( \alpha > \tilde{\alpha} \), \( \sum_{i=1}^{N} w_i > 0 \) and
\[ l \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} B \end{pmatrix} > 0. \]

Let \( \bar{x} = \max_i \{ \hat{x}_i \} \) and \( \underline{x} = \min_i \{ \hat{x}_i \} \), then for any \( \alpha > \tilde{\alpha} \)
\[ \sum_{i=1}^{N} w_i \bar{x} \geq l \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} B \end{pmatrix} \]
and
\[ \sum_{i=1}^{N} w_i \underline{x} \leq l \begin{pmatrix} 0_d \\ V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} B \end{pmatrix} \]

so \( \bar{x} > 0 \) and there exists a constant \( e > 0 \) such that \( \underline{x} < e \). Let \( U^M = (u_{ij}^M) \) and \( V^M = (v_{ij}^M) \), then
\[ \bar{x} = \sum_j u_{ij}^M \hat{x}_j + O(\frac{1}{\alpha}) \]
so
\[ O(\frac{1}{\alpha}) = \bar{x} - \sum_j u_{ij}^M \hat{x}_j \]
\[ = \left( 1 - \sum_j u_{ij}^M \right) \bar{x} + \sum_j u_{ij}^M (\bar{x} - \hat{x}_j) \]
\[ > \sum_j u_{ij}^M (\bar{x} - \hat{x}_j) = \left( \sum_j v_{ij}^M + O(\frac{1}{\alpha}) \right) (\bar{x} - \hat{x}_j) > 0 \]

Then it must be true that for any \( 1 \leq k \leq N \), \( \hat{x}_j = \bar{x} + O(\frac{1}{\alpha}) \). Since \( \underline{x} \) is bounded above, all \( \hat{x}_j \) are bounded. Then
\[ l \begin{pmatrix} 0_d \\ V_3 B \end{pmatrix} + O(\frac{1}{\alpha}) = l \begin{pmatrix} 0_d \\ 0 \end{pmatrix} V_3 V_3^T V_1 N \underline{x} + O(\frac{1}{\alpha}) \]

Take limits of both sides of this equation, then \( \lim_{\alpha \to \infty} \hat{X} \) exist and can be denoted by \( 1_N \hat{c} \) in which
\[ \hat{c} = \frac{l \begin{pmatrix} 0_d \\ V_3 B \end{pmatrix} + \sum_{j \in S(d)} b_j \left[ \left( \sum_{j \in A(d)} v_{ji} \right) \left( \sum_{j \in A(d)} v_{ji l_j} \right) \right]} {l \begin{pmatrix} 0_d \\ V_3 V_3^T 1_N - d \end{pmatrix} \sum_{j \in S(d)} \left[ \left( \sum_{j \in A(d)} v_{ji} \right) \left( \sum_{j \in A(d)} v_{ji l_j} \right) \right]} \]

\[ 6.8. \text{Proof of proposition 7} \]

\textbf{Proof.} The proof is almost the same as the proof of proposition 6. The only two differences are
\[ \bar{U} = \begin{pmatrix} I_N - \frac{1}{\alpha} & 0 \\ 0 & V_3(I_d + \frac{1}{\alpha} V_3^T V_3)^{-1} V_3^T \end{pmatrix} V \]
and
\[ \bar{B}_1 = \begin{pmatrix} 0_d \\ V_3(\alpha I_d + V_3^T V_3)^{-1} B \end{pmatrix} \]

23
Denote the left-hand side by $g$ so

\[
\begin{pmatrix}
0_d \\ V_3(I_d + \frac{1}{\alpha}V_3^T V_3)^{-1} \tilde{B} = l \begin{pmatrix}
0 \\ 0 \\ V_3(I_d + \frac{1}{\alpha}V_3^T V_3)^{-1} \tilde{V}_3^T V_3
\end{pmatrix} V\tilde{X}
\]

so

\[
l\begin{pmatrix}
0_d \\ V_3 \tilde{B}
\end{pmatrix} + O\left(\frac{1}{\alpha}\right) = l \begin{pmatrix}
0 \\ 0 \\ V_3 \tilde{V}_3^T
\end{pmatrix} V1_N \Xi + O\left(\frac{1}{\alpha}\right)
\]

Take limits of both sides of this equation, then $\lim_{\alpha \to \infty} \tilde{X}$ exist and can be denoted by $1_N \hat{c}$ in which

\[
\hat{c} = \frac{l\begin{pmatrix}
0_d \\ V_3 \tilde{B}
\end{pmatrix}}{l\begin{pmatrix}
0 \\ V_3 \tilde{V}_3^T 1_{N-d}
\end{pmatrix}} = \frac{\sum_{i \in S(d)} b_i \left[ \left( \sum_{j \in A(d)} b_j v_{ji} \right) \left( \sum_{j \in A(d)} v_{ji} s_j \right) \right]}{\sum_{i \in S(d)} \left[ \left( \sum_{j \in A(d)} b_j v_{ji} \right) \left( \sum_{j \in A(d)} v_{ji} s_j \right) \right]}
\]

6.9. Proof of lemma 2

Proof. I will first prove that there exist $\alpha^*, c > 0$, such that for any $\alpha > \alpha^*$, there always exist real roots of the system of equations above and $0 > b_1^*, b_2^* > -\frac{c}{\alpha}$

Denote $v_{31}$ by $p$, then $v_{32} = 1 - p$. Then

\[
\delta^2 v_{31} v_{32} b_1^2 b_2^2 + \delta(1 + \delta)v_{31}^2 b_1^2 + 2\delta v_{31} v_{32} b_1 b_2 + (1 + \delta)(\alpha + v_{32}^2)b_2 + v_{31} v_{32} = 0
\]

\[\iff \delta^2 p(1-p)b_1^2 b_2^2 + \delta(1 + \delta)p^2 b_1^2 b_2 + 2\delta p(1-p)b_1 b_2 + (1 + \delta)(\alpha + (1-p)^2)b_2 + p(1-p) = 0\]

Suppose this equation is a quadratic function of $b_2$, then the roots are

\[
b_2^* = \frac{-\mu \pm \sqrt{\mu^2 - 8\delta^2 p^2(1-p)^2 b_1^2}}{2\delta^2 p(1-p)b_1}
\]

where

\[
\mu = \delta(1 + \delta)p^2 b_1^2 + 2\delta p(1-p)b_1 + (1 + \delta)(\alpha + (1-p)^2)
\]

let $f(b_1) = \frac{2p(1-p)}{-\mu - \sqrt{\mu^2 - 8\delta^2 p^2(1-p)^2 b_1^2}}$, substitute $b_2$ for $f(b_1)$ in the other equation:

\[
\delta^2 p(1-p)b_1 f^2(b_1) + \delta(1 + \delta)(1-p) b_1 f^2(b_1) + 2\delta p(1-p)b_1 f(b_1) + (1 + \delta)(\alpha + p^2)b_1 + p(1-p) = 0
\]

If there exists a real root of this equation, then there also exists real roots of the system of these two equations. Denote the left-hand side by $g(b_1)$. It is easy to check that

\[g(0) > 0\]

Then the only thing that I need to check is that there exist $c, \alpha^* > 0$ such that for any $\alpha > \alpha^*$, $g(\frac{\alpha}{\alpha^*}) < 0$. When $\alpha$ is large enough,
\[
\begin{align*}
f\left(\frac{-c}{\alpha}\right) &= \frac{2p(1-p)}{-(1+\delta)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^2}\right) - \sqrt{((1+\delta)(\alpha + (1-p)^2) + O\left(\frac{1}{\alpha^2}\right))^2 - O\left(\frac{1}{\alpha^3}\right)} \\
&= -\frac{p(1-p)}{(1+\delta)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^3}\right)
\end{align*}
\]

Then
\[
g\left(\frac{-c}{\alpha}\right) = O\left(\frac{1}{\alpha^2}\right) + (1+\delta)(\alpha + p^2)\frac{-c}{\alpha} + p(1-p) \\
= O\left(\frac{1}{\alpha}\right) - c(1+\delta) + p(1-p)
\]

let \(c = \frac{p(1-p)+1}{1+\delta}\), then \(g\left(\frac{-c}{\alpha}\right) < 0\). Since \(g(\cdot)\) is a continuous function, there exists a real root \(0 > b_1^* > \frac{-c}{\alpha}\).

Since \(0 > b_1^* > \frac{-c}{\alpha}\), \(b_2 = f(b_1^*) = \frac{-2p(1-p)}{-\mu - \sqrt{\mu^2 + 8^2p^2(1-p)^2}b_1^2} = -\frac{p(1-p)}{(1+\delta)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^2}\right)\). Following the same logic, I can also prove that
\[
b_1^* = -\frac{p(1-p)}{(1+\delta)(\alpha + p^2)} + O\left(\frac{1}{\alpha^3}\right)
\]

**Proof of lemma 3**

**Proof.** By lemma 2

\[
b_1^* = -\frac{p(1-p)}{(1+\delta)(\alpha + p^2)} + O\left(\frac{1}{\alpha^3}\right)
\]

and

\[
(1+\delta)(\alpha + p^2)b_1 = O\left(\frac{1}{\alpha^3}\right) - p(1-p) - 2\delta p(1-p)b_1b_2
\]

\[
= O\left(\frac{1}{\alpha^3}\right) - p(1-p) - 2\frac{\delta p^3(1-p)^3}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)}
\]
\[-(\delta v_{32} b_2 + v_{31} + \alpha)(1 + \delta) + v_{31} v_{32} \frac{a_1}{b_1} \]

\[= \delta v_{31} v_{32} a_2 + \delta(1 + \delta) v_{32}^2 a_2 b_2 + \delta^2 v_{31} v_{32} a_1 b_2 + \delta^2 v_{31} v_{32} a_2 b_1 b_2 \]

\[\iff - b_1 (\delta (1 - p) b_2 + p + \alpha)(1 + \delta) + p(1 - p) a_1 \]

\[= \delta p(1 - p) b_1 a_2 + \delta(1 + \delta)(1 - p) b_1 b_2 a_2 + \delta^2 p(1 - p) b_1 b_2 a_1 + \delta^2 p(1 - p) b_1^2 b_2 \]

\[\iff - b_1 (\delta (1 - p) b_2 + p + \alpha)(1 + \delta) \]

\[= \left[ \delta p(1 - p) b_1 + \delta(1 + \delta)(1 - p) b_1^2 b_2 + \delta^2 p(1 - p) b_1^2 b_2 \right] a_2 - (1 - \delta^2 b_1 b_2) p(1 - p) a_1 \]

\[\iff - \delta(1 + \delta)(1 - p) b_1 b_2 - (1 + \delta) p(1 - p) b_1 - (1 + \delta)(\alpha + p^2) b_1 \]

\[= - \frac{\delta p^2 (1 - p)^2 \alpha}{(1 + \delta)(\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) a_2 \]

\[\quad - \left[ -1 + \delta^2 \frac{p^2 (1 - p)^2}{(1 + \delta)^2 (\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) \right] a_1 \]

\[\iff 1 + \frac{p(1 - p)}{\alpha + p^2} + [2p - (1 + \delta)] \frac{\delta p(1 - p)^2}{(1 + \delta)(\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) \]

\[= - \frac{\delta p(1 - p) \alpha}{(1 + \delta)(\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) a_1 \]

\[\quad + \left[ -1 + \delta^2 \frac{p^2 (1 - p)^2}{(1 + \delta)^2 (\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) \right] a_2 \]

\[\iff -1 - \frac{p(1 - p)}{\alpha + (1 - p)^2} - [2(1 - p) - (1 + \delta)] \frac{\delta p^2 (1 - p)}{(1 + \delta)^2 (\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) \]

\[= - \frac{\delta p(1 - p) \alpha}{(1 + \delta)(\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) a_1 \]

\[\quad + \left[ -1 + \delta^2 \frac{p^2 (1 - p)^2}{(1 + \delta)^2 (\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) \right] a_2 \]

It is a system of linear equations, so it is easy to calculate that \( a_1^* = \frac{\Delta_1}{\Delta_1} \) and \( a_2^* = \frac{\Delta_2}{\Delta_1} \), where

\[ \Delta_1 = \left[ -1 + \delta^2 \frac{p^2 (1 - p)^2}{(1 + \delta)^2 (\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) \right]^2 \]

\[ - \left[ - \frac{\delta p(1 - p) \alpha}{(1 + \delta)(\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) \right]^2 \]

\[= 1 - \frac{3\delta^2 p^2 (1 - p)^2}{(1 + \delta)^2 (\alpha + p^2)(\alpha + (1 - p)^2)} + O \left( \frac{1}{\alpha^3} \right) \]
\[ \Delta_2 = \left[ -1 + \delta^2 \frac{p^2(1-p)^2}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left( \frac{1}{\alpha^3} \right) \right] \\
+ \left[ 1 + \frac{p(1-p)}{\alpha + p^2} + [2p - (1+\delta)] \frac{\delta p(1-p)^2}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left( \frac{1}{\alpha^3} \right) \right] \\
- \left[ \frac{\delta p(1-p)\alpha}{(1+\delta)(\alpha + p^2)(\alpha + (1-p)^2)} + O\left( \frac{1}{\alpha^3} \right) \right] \\
- \left[ -1 - \frac{p(1-p)}{\alpha + (1-p)^2} - [2(1-p) - (1+\delta)] \frac{\delta p^2(1-p)}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left( \frac{1}{\alpha^3} \right) \right] \\
= -1 - \frac{p(1-p)}{\alpha + p^2} - [2p - (1+\delta)] \frac{\delta p^2(1-p)}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} \\
+ \delta^2 \frac{p^2(1-p)^2}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} - \frac{\delta p^2(1-p)^2 \alpha}{(1+\delta)(\alpha + p^2)(\alpha + (1-p)^2)^2} \\
- \frac{\delta p(1-p)\alpha}{(1+\delta)(\alpha + p^2)(\alpha + (1-p)^2)} + O\left( \frac{1}{\alpha^3} \right) \\
= -1 - \frac{p(1-p)}{\alpha + p^2} - \frac{\delta p(1-p)\alpha}{(1+\delta)(\alpha + p^2)(\alpha + (1-p)^2)} \\
+ [-3p + 1 + \delta] \frac{\delta p^2(1-p)^2}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left( \frac{1}{\alpha^3} \right) \\
\]

\[ \Delta_3 = 1 + \frac{p(1-p)}{\alpha + (1-p)^2} + \frac{\delta p(1-p)\alpha}{(1+\delta)(\alpha + p^2)(\alpha + (1-p)^2)} \\
+ [3(1-p) - (1+\delta)] \frac{\delta p^2(1-p)^2}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left( \frac{1}{\alpha^3} \right) \]

\[ 1 - b_1^* b_2^* = 1 - \frac{p^2(1-p)^2}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left( \frac{1}{\alpha^3} \right) \]
\[ a_1^* + b_1^* a_2^* \]
\[ = \frac{1}{\Delta_1} [\Delta_2 + b_1^* \Delta_3] \]
\[ = \frac{1}{\Delta_1} \left[ -1 - \frac{p(1-p)}{\alpha + p^2} \right] - \frac{\delta p(1-p)\alpha}{(1+\delta)\alpha + p^2} \]
\[ + [-3p + 1 + \delta] \frac{\delta p(1-p)^2}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} \]
\[ + b_1^* + \frac{p(1-p)}{\alpha + (1-p)^2} b_1^* \]
\[ = \frac{1}{\Delta_1} \left[ -1 - \frac{p(1-p)}{\alpha + p^2} \right] - \frac{\delta p(1-p)\alpha}{(1+\delta)\alpha + p^2} \]
\[ + [-3p + 1 + \delta] \frac{\delta p(1-p)^2}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} \]
\[ - \frac{p(1-p)}{(1+\delta)(\alpha + p^2)} \]
\[ = \frac{1}{\Delta_1} \left[ -1 - \frac{2p(1-p)\alpha}{(\alpha + p^2)(\alpha + (1-p)^2)} \right] \]
\[ + \frac{p(1-p)^2(p(1-\delta)^2 - 2(1+\delta))}{(1+\delta)^2(\alpha + p^2)(\alpha + (1-p)^2)} + O\left(\frac{1}{\alpha^2}\right) \]
\[ \frac{a_2^* + b_2^* a_1^*}{1 - b_1^* b_2^*} \]
the belief of naive agent in steady state is

\[ px^* + (1 - p)y^* \]

\[ = \frac{(1 - 2p)}{(1 + \delta)^2 [\alpha^2 + (p^2 + (1 - p)^2)\alpha] + 2\delta(1 - \delta)p^2(1 - p)^2 + O\left(\frac{1}{\alpha}\right)} \]

\[ (1 + \delta)^2(\alpha^2 + \alpha) + 4\delta p^2(1 - p)^2 + O\left(\frac{1}{\alpha}\right) \]

When the sophisticated agents are myopic, the belief of the naive agent in steady state is

\[ p\hat{x} + (1 - p)\hat{y} = \frac{(1 - 2p)(\alpha^2 + \alpha)}{\alpha^2 + (p^2 + (1 - p)^2)\alpha} \]

then

\[ px^* + (1 - p)y^* > p\hat{x} + (1 - p)\hat{y} \]

\[ \iff \frac{(1 - 2p)(\alpha^2 + \alpha)}{\alpha^2 + (p^2 + (1 - p)^2)\alpha} > \frac{(1 - 2p)(\alpha^2 + \alpha)}{\alpha^2 + (p^2 + (1 - p)^2)\alpha} \]

it is easy to check that, when \( \alpha \) is large enough

\[ px^* + (1 - p)y^* \geq p\hat{x} + (1 - p)\hat{y} \iff p < \frac{1}{2} \]

\[ \square \]

**Proof of proposition 9**

**Proof.** In steady state, the stage payoff for agent 1 is

\[ \Pi_1^* = -(px^* + (1 - p)y^* + 1)^2 - \alpha(x^* + 1)^2 \]

Since

\[ px^* + (1 - p)y^* \]

\[ = \frac{(1 - 2p)}{(1 + \delta)^2 [\alpha^2 + (p^2 + (1 - p)^2)\alpha] + 2\delta(1 - \delta)p^2(1 - p)^2 + O\left(\frac{1}{\alpha}\right)} \]

\[ (1 + \delta)^2(\alpha^2 + \alpha) + 4\delta p^2(1 - p)^2 + O\left(\frac{1}{\alpha}\right) \]

and

\[ x^* = \frac{-(1 + \delta)^2(\alpha^2 + \alpha) - 4\delta p^2(1 - p)^2 - 2(1 + \delta)p(1 - p)^2 + O\left(\frac{1}{\alpha}\right)}{(1 + \delta)^2 [\alpha^2 + (p^2 + (1 - p)^2)\alpha] + 2\delta(1 - \delta)p^2(1 - p)^2 + O\left(\frac{1}{\alpha}\right)} \]

\[ \Pi_1^* - \bar{\Pi}_1 \]

\[ = -(px^* + (1 - p)y^* + 1)^2 - \alpha(x^* + 1)^2 + (p\hat{x} + (1 - p)\hat{y} + 1)^2 + \alpha(\hat{x} + 1)^2 \]

\[ = (z^* + \hat{z} + 2)(\hat{z} - z^*) + \alpha(x^* + \hat{x} + 2)(\hat{x} - x^*) \]
Let
\[
\Gamma = [\alpha^2 + (p^2 + (1-p)^2)\alpha] \left[ (1 + \delta)^2 [\alpha^2 + (p^2 + (1-p)^2)\alpha] + 2\delta(1 - \delta)p^2(1-p)^2 + O\left( \frac{1}{\alpha} \right) \right]
\]
then
\[
\hat{x} - x^* = -2\delta(1 + \delta)p(1-p)^3\alpha^2 + O(\alpha)
\]
\[
\hat{z} - z^* = -2\delta(1 + \delta)(1 - 2p)p^2(1-p)^2\alpha^2 + O(\alpha)
\]
\[
z^* + \hat{z} + 2
\]
\[
= \frac{4(1 + \delta)^2(1-p)\alpha^4 + O(\alpha^3)}{\Gamma}
\]
\[
x^* + \hat{x} + 2
\]
\[
= \frac{-4(1 + \delta)^2p(1-p)\alpha^3 + O(\alpha^2)}{\Gamma}
\]
then
\[
\Pi_1^* - \Pi_1
\]
\[
= \frac{4(1 + \delta)^2(1-p)\alpha^4 + O(\alpha^3)}{\Gamma} - \frac{2\delta(1 + \delta)(1 - 2p)p^2(1-p)^2\alpha^2 + O(\alpha)}{\Gamma}
\]
\[
+ \frac{\alpha}{\Gamma} - \frac{4(1 + \delta)^2p(1-p)\alpha^3 + O(\alpha^2)}{\Gamma} - \frac{2\delta(1 + \delta)p(1-p)^3\alpha^2 + O(\alpha)}{\Gamma}
\]
\[
= \frac{-8\delta(1 + \delta)^3(1 - 2p)p^2(1-p)^3\alpha^6 + 8\delta(1 + \delta)^3p^2(1-p)^4\alpha^6 + O(\alpha^5)}{\Gamma^2}
\]
\[
= \frac{8\delta(1 + \delta)^3p^3(1-p)^3\alpha^6 + O(\alpha^5)}{\Gamma^2} > 0
\]

On the other hand,
\[
\Pi_2^* - \Pi_2
\]
\[
= -(px^* + (1-p)y^* - 1)^2 - \alpha(y^* - 1)^2 + (p\hat{x} + (1-p)\hat{y} - 1)^2 + \alpha(\hat{y} - 1)^2
\]
\[
=(z^* + \hat{z} - 2)(\hat{z} - z^*) + \alpha(y^* + \hat{y} - 2)(\hat{y} - y^*)
\]
\[
z^* + \hat{z} - 2
\]
\[
= \frac{4(1 + \delta)^2p\alpha^4 + O(\alpha^3)}{\Gamma}
\]
\[
\hat{y} - y^*
\]
\[
= \frac{2\delta(1 + \delta)p^3(1-p)\alpha^2 + O(\alpha)}{\Gamma}
\]
\[
y^* + \hat{y} - 2
\]
\[
= \frac{4(1 + \delta)^2p(1-p)\alpha^3 + O(\alpha^2)}{\Gamma}
\]

30
then

\[
\Pi_z^* - \Pi_2 = -4(1+\delta)^2 p\alpha^4 + O(\alpha^3) - 2\delta(1+\delta)(1-2p)p^2(1-p)^2\alpha^2 + O(\alpha) \\
+ \alpha \frac{4(1+\delta)^2 p(1-p)\alpha^3 + O(\alpha^2) - 2\delta(1+\delta)p^3(1-p)\alpha^2 + O(\alpha)}{\Gamma} \\
= \frac{8\delta(1+\delta)^3(1-2p)p^3(1-p)^2\alpha^6 + 8\delta(1+\delta)^3 p^4(1-p)^2\alpha^6 + O(\alpha^5)}{\Gamma^2} > 0
\]

\[\square\]

Proof of proposition 10

Proof.

\[
z * (\delta_1) - z * (\delta_2) = \frac{(1-2p) \left[ (1+\delta_1)^2 (\alpha^2 + \alpha) + 4\delta_1 p^2 (1-p)^2 + O(\frac{1}{\alpha}) \right]}{(1+\delta_1)^2 \left[ \alpha^2 + (p^2 + (1-p)^2)\alpha + 2\delta_1 (1-\delta_1) p^2 (1-p)^2 + O(\frac{1}{\alpha}) \right]} \\
- \frac{(1+\delta_2)^2 \left[ \alpha^2 + (p^2 + (1-p)^2)\alpha + 2\delta_2 (1-\delta_2) p^2 (1-p)^2 + O(\frac{1}{\alpha}) \right]}{(1+\delta_2)^2 \left[ \alpha^2 + (p^2 + (1-p)^2)\alpha + 2\delta_2 (1-\delta_2) p^2 (1-p)^2 + O(\frac{1}{\alpha}) \right]} \\
= \frac{1-2p}{\Omega(\delta_1)\Omega(\delta_2)} \left\{ (1+\delta_1)^2 \left[ \alpha^2 + (p^2 + (1-p)^2)\alpha + 2\delta_1 (1-\delta_1) p^2 (1-p)^2 + O(\frac{1}{\alpha}) \right] \\
- (1+\delta_2)^2 \left[ \alpha^2 + (p^2 + (1-p)^2)\alpha + 2\delta_2 (1-\delta_2) p^2 (1-p)^2 + O(\frac{1}{\alpha}) \right] \right\} \\
= \frac{1-2p}{\Omega(\delta_1)\Omega(\delta_2)} \left\{ 2(1+\delta_1)^2 \delta_2 (1-\delta_2) p^2 (1-p)^2 \alpha^2 \\
+ 4\delta_1 (1+\delta_2)^2 p^2 (1-p)^2 \alpha^2 - 2\delta_1 (1-\delta_1) (1+\delta_2)^2 p^2 (1-p)^2 \alpha^2 \\
- 4(1+\delta_1)^2 \delta_2 p^2 (1-p)^2 \alpha^2 + O(\alpha) \right\} \\
= \frac{1-2p}{\Omega(\delta_1)\Omega(\delta_2)} \left\{ 4(1+\delta_1)(1+\delta_2)(\delta_1 - \delta_2) p^2 (1-p)^2 \alpha^2 + O(\alpha) \right\}
\]

Since \( \delta_1 < \delta_2, z * (\delta_1) < z * (\delta_2) \) if and only if \( 1-2p > 0 \)
\[
\Pi_1^*(\delta_1) - \Pi_1^*(\delta_2) \\
= \frac{8\delta_1(1+\delta_1)^3p^3(1-p)\alpha^6 + O(\alpha^5)}{\Gamma^2(\delta_1)} - \frac{8\delta_2(1+\delta_2)^3p^3(1-p)\alpha^6 + O(\alpha^5)}{\Gamma^2(\delta_2)} \\
= \frac{1}{\Gamma^2(\delta_1)\Gamma^2(\delta_2)} \left\{ \left[ 8\delta_1(1+\delta_1)^3p^3(1-p)\alpha^6 + O(\alpha^5) \right] \Gamma^2(\delta_2) \\
- \left[ 8\delta_2(1+\delta_2)^3p^3(1-p)\alpha^6 + O(\alpha^5) \right] \Gamma^2(\delta_1) \right\} \\
= \frac{1}{\Gamma^2(\delta_1)\Gamma^2(\delta_2)} \left[ 8\delta_1(1+\delta_1)^3(1+\delta_2)^3p^3(1-p)\alpha^{14} \\
- 8(1+\delta_1)^3\delta_2(1+\delta_2)^3p^3(1-p)\alpha^{14} + O(\alpha^{13}) \right] \\
= \frac{8(1+\delta_1)^3(1+\delta_2)^3p^3(1-p)^3(\delta_1 - \delta_2)\alpha^{14} + O(\alpha^{13})}{\Gamma^2(\delta_1)\Gamma^2(\delta_2)} < 0
\]

7. References


