Confounded Observational Learning with Common Values

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Abstract

We modify the standard herding model so that a fraction of players are naive and rely exclusively on private information. The rest players are rational and uncertain about proportion of naive players. We find that learning could be confounded in the long run, despite private signal strength could be unbounded.

This paper examines the standard model of observational learning by Bayesian rational players with common values, pioneered by Banerjee (1992) and Bikhchandani et al. (1992). Nature chooses the state of the world once and for all, and every player wants to choose the action that matches the state. We modify this model so that a fraction of the players are naive or “lone-wolves” and rely exclusively on their own private signals, rather than public information. The existence of players who overweight private signals are suggested in several experimental literature. See Duffy et al. (2016), Weizsäcker (2010) and Ziegelmeyer et al. (2013). When the proportion of lone wolves is commonly known, their presence ensures that social learning never stops. This permits the rational players to eventually learn the true state of the world, even when private signals are boundedly informative. What happens to social learning when rational players are uncertain about the proportion of lone wolves, and have a common prior on this proportion, as in a standard Bayesian game? Our key finding is that confounded learning is a possibility, even though players have common values, and agree on the action to be taken in each state of the world. That is, after some histories, rational players will forever choose both actions with positive probability, as a function of their private signal. Since uncertainty is two-dimensional – players are uncertain about the state as well as the proportion of lone wolves – history may fail to allow them to disentangle
the two components separately. We also show that there will either be complete learning or confounded learning – incorrect learning is impossible in our Bayesian game.

While there is an extensive literature on observational learning, our paper is most closely related to the papers by Smith and Sørensen (2000) (hereafter, simply SS) and Bohren (2016). SS allow for heterogeneous preferences and were the first to show that confounded learning is possible in herding models when players have divergent preferences. A fraction of players would like their action to match the state, while the remaining fraction prefer to mismatch action and state. While our analysis utilizes many of their techniques and insights, our underlying economic environment giving rise to confounded learning is very different – every player would like her action to match the state. Bohren allows for lone wolves, and assumes that rational players have wrong but certain beliefs about the proportion of lone wolves. She finds that if the beliefs are not too wrong, asymptotic learning is ensured, but for larger errors, beliefs may assign probability zero to the true state or fail to converge. Our results are very different (there cannot be incorrect learning, and there can be confounded learning), since rational players use history to revise their beliefs on the true proportion of lone wolves.

Our paper is also related to the literature of observational learning with mis-specified model. Eyster and Rabin (2010) allows every rational player mistakenly think other players are naive. They find incorrect herding could happen even with continuum actions and unbounded signals. Bohren and Hauser (2017) allows each player to have a general mis-specification of how other player process information. Our paper could be viewed as an attempt to answer what if players can learn and correct their mis-specified model.

1 Model

At \( t = 0 \), nature moves and chooses the economic fundamental from the set \( \{A, B\} \), and the proportion of lone wolves, \( p \), from \( p_L, p_H \) \((0 < p_L < p_H < 1)\). For simplicity, we assume each element of \( \{A, B\} \times \{p_L, p_H\} \) is chosen with equal probability. No player observes nature’s choice. There is a countable infinity of players indexed by \( t \in \mathbb{N} \), who must choose, in sequence, either action \( a \) or action \( b \). A player gets a payoff of 1 if her action matches the economic fundamental, and zero otherwise.

The player’s public information is as follows. With a constant probability \( p \), player \( t \) is a
lone wolf and does not observe the actions of any previous players. With probability $1 - p$, she observes the actions of all previous players.

Both types of players observe a private signal, whose distribution is state-dependent and i.i.d across players. Following SS, we identify one player’s private signal with one’s posterior belief after observing the signal, and use $s_t$ to denote the private posterior probability that player $t$ believes economics fundamental to be $A$. Here $\langle s_t \rangle$ is conditional i.i.d. with distribution denoted as $F^\omega(s), \omega = \{A, B\}$. We assume signal strength is unbounded, that is, $\text{supp}(F^A(s)) = \text{supp}(F^B(s)) = (0, 1)$. With unbounded signal strength, one player could be arbitrarily sure about one state after observing private signal. We also assume $f^\omega(s) > 0$ for all $\omega \in \{A, B\}, s \in (0, 1)$. This will guarantee the standard assumption that no private signal fully reveal the underlying state.

## 2 Main Result

There are four potential states in total. For notation abbreviation, denote $z_{AL} = \{A, p_L\}$, $z_{AH} = \{A, p_H\}$, $z_{BL} = \{B, p_L\}$, $z_{BH} = \{B, p_H\}$. Without loss of generality, we assume the true state to be $z_{AL}$. All arguments apply to other cases directly.

For a rational player who observes history up to period $t - 1$, her public belief can be summarized by a triple of likelihood ratios $\pi_t = (\lambda_t^{AH}, \lambda_t^{BL}, \lambda_t^{BH})$. Here $\lambda_t^{AH} = \frac{\text{Pr}(z_{AH}|h_{t-1})}{\text{Pr}(z_{AL}|h_{t-1})}$ is the conditional likelihood ratio of state $z_{AH}$ over true state $z_{AL}$, similarly for $\lambda_t^{BL}, \lambda_t^{BH}$.

Furthermore, the public likelihood ratio of economic fundamentals $B$ over $A$, after observing $h_{t-1}$, is given by $\lambda_t = \frac{\text{Pr}(B|h_{t-1})}{\text{Pr}(A|h_{t-1})} = \frac{\lambda_t^{BL} + \lambda_t^{BH}}{1 + \lambda_t^{AH}}$.

If player $t$ is rational, then she will choose action $a$ if and only if $\frac{\text{Pr}(B|s_t, h_{t-1})}{\text{Pr}(A|s_t, h_{t-1})} = \frac{1 - s_t}{s_t} \lambda_t < 1$, which is equivalent to $s_t > \frac{\lambda_t}{\lambda_{t+1}}$.

If player $t$ is a lone wolf, then she doesn’t observe public history $h_{t-1}$. She will choose action $a$ if and only if $\frac{\text{Pr}(B|s_t)}{\text{Pr}(A|s_t)} = \frac{1 - s_t}{s_t} < 1$, which is equivalent to $s_t > \frac{1}{2}$.

Let $\psi(\alpha|z_t, \pi_t)$ be the probability of observing action $\alpha \in \{a, b\}$ under state $z_t$ and public belief $\pi_t$. We can explicitly write out $\psi(\alpha|z_t, \pi_t)$ for each $i \in \{AH, BL, BH\}$. For example,

$$\psi(b|Z_{AH}, \pi_t) = p_H F^A(\frac{1}{2}) + (1 - p_H) F^A(\frac{\lambda_t}{\lambda_{t+1}}).$$

We observe that such probability only depends on the likelihood ratio of econ fundamentals $\lambda_t$, hence $\psi(\alpha|z_t, \pi_t) = \psi(\alpha|z_t, \lambda_t)$.

Therefore, after observing player $t$’s action $\alpha \in \{l, h\}$, likelihood ratios are bayesian
updated to

$$\lambda_{i+1}^t(\alpha) = \frac{\psi(\alpha|z_i, \pi_t)}{\psi(\alpha|z_{AL}, \pi_t)}.$$  \hspace{1cm} (1)$$

As standard in the literature, we say learning is complete if public belief assigns all weight to true state $z_{AL}$ in the long run, that is, $\langle \pi_t \rangle$ converges to $(0, 0, 0)$. Otherwise, learning is incomplete.

As standard in the literature, in defining the likelihood ratio, we intentionally put probability associated with true state AL in the denominator. It is immediate that for $i \in \{AH, BL, BH\}$

$$E[\lambda_{i+1}^t | \pi_t, Z_{AL}] = \lambda_i^t.$$  \hspace{1cm} (2)$$

Therefore, $\lambda_i^t$ is a conditional martingale. As in standard argument, $\lambda_i^t$ converges almost surely to a finite r.v. $\lambda_i^\infty$ due to martingale convergence theorem.

Theorem B.2 in SS characterized limit points of process $\langle \lambda_i^{AH}, \lambda_i^{BL}, \lambda_i^{BH} \rangle$. It claims if process of belief $\langle \lambda_i^{AH}, \lambda_i^{BL}, \lambda_i^{BH} \rangle$ converges to limit $\pi_\infty = \langle \pi_{AH}, \pi_{BL}, \pi_{BH} \rangle$ with positive probability, then any action that is observable under limit belief must keep the limit belief unchanged. Because lone wolves never herd, both actions are observable in the long run. Therefore, if $\pi_\infty$ could be a limit belief, then

$$\pi_i = \frac{\psi(\alpha|\pi_\infty, z_i)}{\psi(\alpha|\pi_\infty, z_{AL})}.$$  \hspace{1cm} (3)$$

for all $i \in \{AH, BL, BH\}$ and $\alpha \in \{a, b\}$. Furthermore, martingale convergence theorem requires $\pi_i < \infty$ for all $i$.

Obviously, $\pi_i = 0$ for all $i$ is a potential candidate of limit belief, which assign all weight to true state $\{A, p_L\}$. And learning is complete under such limit belief. Any other limit point must contain at least one non-zero component, and represents incomplete learning. From equation (3) if $\pi_i \neq 0$, then $\psi(\alpha|\pi_\infty, z_i) = \psi(\alpha|\pi_\infty, z_{AL})$ for $\alpha \in \{a, b\}$. That is, both actions must equally like to be observed across states $z_i$ and $z_{AL}$ under limit belief $\pi_\infty$. However, this is not always possible. For example, state $z_{BL}, z_{AL}$ specify the same lone wolf population but different economic fundamental, and action $b$ is strictly more plausible under state $z_{BL}$ for all potential limit belief. Following lemma states that only $\pi_{BH}$ could be non-zero. That is, if learning is incomplete, then positive weight can only be assigned to the true state, and the state which specifies both wrong economic fundamental and lone wolf population.
Lemma 1 If $\pi_t$ converges to $\pi_\infty = (\pi_{AH}, \pi_{BL}, \pi_{BH})$ with positive probability, then $\pi_{AH}, \pi_{BL} = 0; \pi_{BH}$ is either 0 or solves

$$(1 - p_L)F_A^A\left(\frac{\pi_{BH}}{1 + \pi_{BH}}\right) + p_LF_A^A\left(\frac{1}{2}\right) = (1 - p_H)F_B^B\left(\frac{\pi_{BH}}{1 + \pi_{BH}}\right) + p_HF_B^B\left(\frac{1}{2}\right). \quad (4)$$

Proof. If $\pi_{AH} \neq 0$, according to equation 3, then $\psi(b|z_{AH}, \pi_\infty) = \psi(b|z_{AL}, \pi_\infty)$, which is

$$p_LF_A^A\left(\frac{1}{2}\right) + (1 - p_L)F_A^A\left(\frac{\lambda_\infty}{\lambda_\infty + 1}\right) = p_HF_A^A\left(\frac{1}{2}\right) + (1 - p_H)F_A^A\left(\frac{\lambda_\infty}{\lambda_\infty + 1}\right).$$

Here $\lambda_\infty = \frac{p_{BL} + p_{BH}}{1 + p_{AH}}$. Because $p_H > p_L$, above equation holds only if $\lambda_\infty = 1$. It is direct to check that $\lambda_\infty = 1$ implies $\psi(b|z_{BL}, \pi_\infty) \neq \psi(b|z_{AL}, \pi_\infty)$ and $\psi(b|z_{BH}, \pi_\infty) \neq \psi(b|z_{AL}, \pi_\infty)$. Apply equation 3 again, $\pi_{BL}, \pi_{BH} = 0$. This contradicts $\lambda_\infty = 1$.

If $\pi_{BL} \neq 0$, then $\psi(b|z_{BL}, \pi_\infty) = \psi(b|z_{AL}, \pi_\infty)$, which is

$$p_LF_B^B\left(\frac{1}{2}\right) + (1 - p_L)F_B^B\left(\frac{\lambda_\infty}{\lambda_\infty + 1}\right) = p_LF_A^A\left(\frac{1}{2}\right) + (1 - p_L)F_A^A\left(\frac{\lambda_\infty}{\lambda_\infty + 1}\right).$$

This equation is never true because $F_B^B(c) > F_A^A(c)$ for any $c \in (0, 1)$. See SS lemma A.1 (c) for a proof. Intuitively, if a player’s decision rule is to choose action $b$ if private signal $s < c$, then the probability of choosing action $b$ under state $B$ must be bigger than the probability of choosing action $b$ under state $A$.

If $\pi_{BH} \neq 0$, then $\psi(b|z_{BH}, \pi_\infty) = \psi(b|z_{AL}, \pi_\infty)$, which is equation 4.

Therefore, if equation 4 has no non-zero solution, then $\pi_t$ must converge to $(0, 0, 0)$ and learning is complete. To understand when incomplete learning could happen, we need to roughly understand when equation 4 has a solution. Following lemma states if and only if condition for equation 4 to has unique solution.

Lemma 2 Equation 4 has an unique non-zero solution if and only if

$$p_LF_A^A\left(\frac{1}{2}\right) + (1 - p_L) > p_HF_B^B\left(\frac{1}{2}\right) + (1 - p_H), \quad (5)$$

Proof. If rational players decision rule is given by cutoff $p$ (choosing action $b$ iff $s < p$), then probability of observing action $b$ under state $Z_{AL}$ is $\Pr(b|Z_{AL}, p) = p_LF_A^A\left(\frac{1}{2}\right) + (1 - p_L)F_A^A(p)$. Similarly $\Pr(b|Z_{BH}, p) = p_HF_B^B\left(\frac{1}{2}\right) + (1 - p_H)F_B^B(p)$. 

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When cutoff $p$ is 0 (rational players herd on action $a$), then
\[ \Pr(b|z_{AL}, 0) = p_L F^A(\frac{1}{2}) < \Pr(b|z_{BH}, 0) = p_H F^B(\frac{1}{2}). \]

This is not surprising because state $BH$ specifies more naive players and econ fundamental $B$ in favor of action $b$.

As cutoff $p$ increases, marginal increases of probability of action $b$ are given as
\[ \frac{d}{dp} \Pr(b|z_{AL}, p) = (1 - p_L) f^A(p) \quad \frac{d}{dp} \Pr(b|z_{BH}, p) = (1 - p_H) f^B(p). \]

Because $\frac{f^B(p)}{f^A(p)} = \frac{1 - p}{p}$ (see SS lemma A1.(a)), $\frac{d}{dp} \Pr(b|z_{BH}, p) > \frac{d}{dp} \Pr(b|z_{AL}, p)$ if and only if $p < \frac{1 - p_H}{2 - p_L - p_H}$.

Therefore, $\Pr(b|z_{AL}, p) \neq \Pr(b|z_{BH}, p)$ for any $p \in (0, \frac{1 - p_H}{2 - p_L - p_H})$. This simply because $\Pr(b|z_{BH}, p)$ is bigger at $p = 0$ and increase faster on this region.

Assume $\Pr(b|z_{AL}, 1) > \Pr(b|z_{BH}, 1)$, then $\Pr(b|z_{AL}, p) = \Pr(b|z_{BH}, p)$ for some $p \in (\frac{1 - p_H}{2 - p_L - p_H}, 1)$. These two probabilities can’t intersect twice because $\Pr(b|z_{AL}, p)$ increases strictly faster on $(\frac{1 - p_H}{2 - p_L - p_H}, 1)$.

It is relatively easy to find a set of primitives satisfy the sufficient condition in Lemma 2. For example, we can choose $p_H = 0.6, p_L = 0.4, F^B(\frac{1}{2}) = 0.5, F^A(\frac{1}{2}) = 0.4$. Also, the set of four tuple $\{p_H, p_L, F^A(\frac{1}{2}), F^B(\frac{1}{2})\}$ solve strict inequality 5 is obviously open. Hence we can roughly say incomplete learning arise with a robust set of primitives.

When inequality 5 is satisfied, then belief $\pi_t$ can converge either to $\langle 0, 0, 0 \rangle$ or $\langle 0, 0, \pi_{BH} \rangle$.

We remark that the convergence to $\langle 0, 0, \pi_{BH} \rangle$ is a revisit of confounded learning in SS. However, the underlying economic environments are quite different. In SS, confounded learning arise because players have different preferences, that is, certain fraction of players want to mismatch the state.

The next natural question is, provided existence of $\langle 0, 0, \pi_{BH} \rangle$, will belief $\pi_t$ converges to both limits $\langle 0, 0, 0 \rangle$ and $\langle 0, 0, \pi_{BH} \rangle$ with positive probability? Following Lemma 3 states that if the belief is within a small neighborhood of $\langle 0, 0, 0 \rangle$, it will converge there with positive probability. In other words, $\langle 0, 0, 0 \rangle$ is local stable. Lemma 4 states a sufficient condition for $\langle 0, 0, \pi_{BH} \rangle$ to be local stable.

**Lemma 3** There exist $\varepsilon > 0$ such that if $\| (\lambda_t^{AH}, \lambda_t^{BL}, \lambda_t^{BH}) - (0, 0, 0) \| < \varepsilon$, then
\[ \Pr[(\lambda_t^{AH}, \lambda_t^{BL}, \lambda_t^{BH}) \rightarrow (0, 0, 0)] > 0. \]
Proof. See Appendix. ■

Lemma 4 If $\pi_{BH}$ solves equation 4 and $1 + \frac{\pi_{BH}}{(\pi_{BH} + 1)^2} G_1$ and $1 + \frac{\pi_{BH}}{(\pi_{BH} + 1)^2} G_2$ are non-negative; where

$$G_1 = \frac{(1 - p_L) f^A(\frac{\pi_{BH}}{\pi_{BH} + 1}) - (1 - p_H) f^B(\frac{\pi_{BH}}{\pi_{BH} + 1})}{p_L[1 - F^A(\frac{1}{2})] + (1 - p_L)[1 - F^A(\frac{\pi_{BH}}{\pi_{BH} + 1})]}$$

and

$$G_2 = \frac{-(1 - p_L) f^A(\frac{\pi_{BH}}{\pi_{BH} + 1}) + (1 - p_H) f^B(\frac{\pi_{BH}}{\pi_{BH} + 1})}{p_L[F^A(\frac{1}{2})] + (1 - p_L)[F^A(\frac{\pi_{BH}}{\pi_{BH} + 1})]}.$$ 

Then there exists $\varepsilon > 0$ such that if $\|((\lambda^A_t, \lambda^B_t, \lambda^H_t) - (0, 0, \pi_{BH}))\| \leq \varepsilon$, then

$$\Pr[(\lambda^A_t, \lambda^B_t, \lambda^H_t) \rightarrow (0, 0, \pi_{BH})] > 0.$$ 

Some readers may ask whether there exists primitives satisfying those sufficient condition. We claim that $f^A(p) = 2p$, $f^B(p) = 2(1 - p)$, and $(p_L, p_H)$ within a small neighborhood of $(0.2, 0.8)$ are sets of such primitives.

To summarize, we have obtained our main result.

Theorem 5 Assume underlying state to be $\{A, p_L\}$, and private signal strength to be unbounded. In the long run, public belief $\langle \pi_t \rangle$ almost surely converges to one of following two limits: (1) $\langle 0, 0, 0 \rangle$; (2) $\langle 0, 0, \pi_{BH} \rangle$ where $\pi_{BH}$ solves equation 4. Furthermore, $\langle 0, 0, 0 \rangle$ is always local stable.

3 Appendix

Here both proofs follows the arguments in Appendix C. of SS. We sketch the idea and reproduce details for reader’s convenience.

Proof. The proof primarily consists of three steps. First, construct an auxiliary process which dominate the original process in norm on a small neighborhood around limit point. Second, show that auxiliary process moves toward the limit point in expectation if restricted to potentially a smaller neighborhood. Third, using law of large numbers to conclude the auxiliary process (restricted on the smaller neighborhood) will converge to the limit point a.s, hence will stay within that smaller neighborhood with positive probability. Thus original process, if started within the smaller neighborhood, will converge to the limit point with positive probability.
For notation convenience, denote $\pi^*_1 = \langle 0,0,0 \rangle$, $\pi^*_2 = \langle 0,0,\pi_{BH} \rangle$. The original process moves from $\pi_t = \langle \lambda_t^{AH}, \lambda_t^{BL}, \lambda_t^{BH} \rangle$ to

$$
\phi(\alpha, \pi_t) = \langle \lambda_t^{AH} \psi(\alpha|z_{AH}, \pi_t), \lambda_t^{BL} \psi(\alpha|z_{BL}, \pi_t), \lambda_t^{BH} \psi(\alpha|z_{BH}, \pi_t) \rangle
$$

with probability $\psi(\alpha|z_{AL}, b_t)$ for $\alpha \in \{a,b\}$. For notation convenience, let $A_1 = D_{\pi}\phi(a, \pi^*_1)$, $B_1 = D_{\pi}\phi(b, \pi^*_1)$; and $A_2 = D_{\pi}\phi(a, \pi^*_2)$, $B_2 = D_{\pi}\phi(b, \pi^*_2)$.

(Step 1) In this step, we construct an auxiliary process which dominates original process in norm on a small neighborhood around limit point. Without loss of generality, we take $\pi$ if $\pi$ moves from $\pi$ if $\pi$ moves to $\pi$ or if player is lone wolf and $\pi$ moves to $\pi$ or if player is rational and $\pi$ moves to $\pi$ provided that $\pi$ stays within a small neighborhood of $\pi^*_1$ as an example. For any $\eta$ which is slightly larger than 1, we can find a small neighborhood $N(\eta)$ around $\pi^*_1$ such that

$$
\|\phi(a, \pi) - \pi^*_1\| = \|A_1\pi + \gamma(\pi)\| \leq \|\eta A_1\pi\|,
$$

and

$$
\|\phi(b, \pi) - \pi^*_1\| = \|B_1\pi + \gamma(\pi)\| \leq \|\eta B_1\pi\|,
$$

provided that $\pi \in N(\eta)$.

For $\pi_t = \langle \lambda_t^{AH}, \lambda_t^{BL}, \lambda_t^{BH} \rangle \in N(\eta)$, we must have associated $\lambda_t = \frac{\lambda_t^{BH} + \lambda_t^{BL}}{1 + \lambda_t^{BH}}$ very close to 0. Let $\sigma(\eta) = \sup_{\pi_t \in N(\eta)} \frac{\lambda_t}{\lambda_t + 1}$. We construct following auxiliary process $\tilde{\pi}^1_t = \langle \tilde{\lambda}_t^{AH}, \tilde{\lambda}_t^{BL}, \tilde{\lambda}_t^{BH} \rangle$ as

$$
\tilde{\pi}^1_{t+1} = \begin{cases} 
\eta A_1 \tilde{\pi}^1_t, & \text{if player is lone wolf and } s_t > \frac{1}{2}, \text{ or if player is rational and } s_t > \sigma(\eta); \\
\eta B_1 \tilde{\pi}^1_t, & \text{if player is lone wolf and } s_t < \frac{1}{2}; \\
\eta C_1 \tilde{\pi}^1_t, & \text{otherwise.}
\end{cases}
$$

Here $C_1$ is defined to be $\max\{A_1, B_1\}$ component-wise. For any sequence of private signals, if $\pi_t$ moves to $\phi(a, \pi_t)$, then $\tilde{\pi}^1_t$ moves to either $\eta A_1 \tilde{\pi}^1_t$ or $\eta C_1 \tilde{\pi}^1_t$; if $\pi_t$ moves to $\phi(b, \pi_t)$, then $\tilde{\pi}^1_t$ moves to either $\eta B_1 \tilde{\pi}^1_t$ or $\eta C_1 \tilde{\pi}^1_t$. If starting with $\tilde{\pi}^1_{t_0} = \pi_{t_0} \in N(\eta)$, above construction guarantees $\|\tilde{\pi}^1_t\| \geq \|\pi_t\|$, provided that $\tilde{\pi}^1_t \in N(\eta)$ for $t \geq t_0$.

Following exactly the same process, we can construct auxiliary process $\tilde{\pi}^2_t$ such that $\|\tilde{\pi}^2_t - \pi^*_2\| \geq \|\pi^2_t - \pi^*_2\|$ provided that $\tilde{\pi}^2_t$ stays within a small neighborhood of $\pi^*_2$.

(Step 2) This step involves some computation. We first directly compute $A_1, B_1, A_2, B_2$. 

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We have
\[
A_1 = \begin{bmatrix}
p_H[1-F^A(\frac{1}{2})]+(1-p_H) & 0 & 0 \\
p_L[1-F^A(\frac{1}{2})]+(1-p_L) & 0 & 0 \\
0 & p_L[1-F^B(\frac{1}{2})]+(1-p_L) & 0 \\
0 & 0 & p_H[1-F^B(\frac{1}{2})]+(1-p_H)
\end{bmatrix}
\]
\[
B_1 = \begin{bmatrix}
p_H[F^A(\frac{1}{2})] & 0 & 0 \\
p_L[F^A(\frac{1}{2})] & 0 & 0 \\
0 & p_L[F^B(\frac{1}{2})] & 0 \\
0 & 0 & p_H[F^B(\frac{1}{2})]
\end{bmatrix}
\]

and
\[
A_2 = \begin{bmatrix}
p_H[1-F^A(\frac{1}{2})]+(1-p_H)\frac{1-F^A(\frac{1}{2})}{\pi_{BH}+1} & 0 & 0 \\
p_L[1-F^A(\frac{1}{2})]+(1-p_L)\frac{1-F^A(\frac{1}{2})}{\pi_{BH}+1} & 0 & 0 \\
0 & p_L[1-F^B(\frac{1}{2})]+(1-p_L)\frac{1-F^B(\frac{1}{2})}{\pi_{BH}+1} & 0 \\
-(\frac{\pi_{BH}}{\pi_{BH}+1})^2 G_1 & \frac{\pi_{BH}}{(\pi_{BH}+1)^2} G_1 & 1 + \frac{\pi_{BH}}{(\pi_{BH}+1)^2} G_1
\end{bmatrix}
\]
\[
B_2 = \begin{bmatrix}
p_H[F^A(\frac{1}{2})]+(1-p_H)\frac{F^A(\frac{1}{2})}{\pi_{BH}+1} & 0 & 0 \\
p_L[F^A(\frac{1}{2})]+(1-p_L)\frac{F^A(\frac{1}{2})}{\pi_{BH}+1} & 0 & 0 \\
0 & p_L[F^B(\frac{1}{2})]+(1-p_L)\frac{F^B(\frac{1}{2})}{\pi_{BH}+1} & 0 \\
-(\frac{\pi_{BH}}{\pi_{BH}+1})^2 G_2 & \frac{\pi_{BH}}{(\pi_{BH}+1)^2} G_2 & 1 + \frac{\pi_{BH}}{(\pi_{BH}+1)^2} G_2
\end{bmatrix}
\]

Here
\[
G_1 = \frac{(1-p_L)\frac{F^A(\frac{1}{2})}{\pi_{BH}+1} - (1-p_H)\frac{F^B(\frac{1}{2})}{\pi_{BH}+1}}{p_L[1-F^A(\frac{1}{2})]+(1-p_L)[1-F^A(\frac{1}{2})]},
\]
and
\[
G_2 = \frac{-(1-p_L)\frac{F^A(\frac{1}{2})}{\pi_{BH}+1} + (1-p_H)\frac{F^B(\frac{1}{2})}{\pi_{BH}+1}}{p_L[F^A(\frac{1}{2})] + (1-p_L)[F^A(\frac{1}{2})]}.
\]

The big terms look scary but the logic is relatively simple. For example, we can compute the expected change of \(\log \tilde{\lambda}_t^{1,BH}\), which is the third component of \(\tilde{\pi}_t^1\). We have
\[
E[\log \tilde{\lambda}_{t+1}^{1,BH} | \log \tilde{\lambda}_t^{1,BH}] \log \tilde{\lambda}_t^{1,BH} = \{p_L[1-F^A(\frac{1}{2})] + (1-p_L)[1-F^A(\sigma)]\} \max(\frac{p_H[1-F^B(\frac{1}{2})]}{p_L[1-F^A(\frac{1}{2})] + (1-p_L)} + (1-p_H), \frac{p_H[F^B(\frac{1}{2})]}{p_L[F^A(\frac{1}{2})] + (1-p_L)}).
\]
It is immediate that above formula is continuous in $\eta, \sigma$. By let $\eta \to 1$ (which forces $\sigma \to 0$), the limit goes to

$$
\{p_L[1 - F^A(\frac{1}{2})] + (1 - p_L)\} \frac{p_H[1 - F^B(\frac{1}{2})] + (1 - p_H)}{p_L[1 - F^A(\frac{1}{2})] + (1 - p_L)} + [p_L F^A(\frac{1}{2})] \frac{p_H[F^B(\frac{1}{2})]}{p_L[F^A(\frac{1}{2})]} < \log\{p_H[1 - F^B(\frac{1}{2})] + (1 - p_H) + p_H[F^B(\frac{1}{2})]\} = 0.
$$

Therefore, by properly shrinking neighborhood $N(\eta)$, we conclude $\tilde{\lambda}^{1,BH}_t$ moves to 0 in expectation. We can similarly show this for $\tilde{\lambda}^{2,AH}_t, \tilde{\lambda}^{2,BL}_t$.

If we consider $\tilde{\pi}^2_t = \langle \tilde{\lambda}^{2,AH}_t, \tilde{\lambda}^{2,BL}_t, \tilde{\lambda}^{2,BH}_t \rangle$, a slight attention is needed at computing expected movement of $\tilde{\lambda}^{2,BH}_t - \pi_{BH}$. To use the trick of taking log, we need to consider $|\tilde{\lambda}^{2,BH}_t - \pi_{BH}|$ instead of $\tilde{\lambda}^{2,BH}_t - \pi_{BH}$. Directly computation shows that

$$
E_t[|\tilde{\lambda}^{2,BH}_{t+1} - \pi_{BH}|] - \log|\tilde{\lambda}^{2,BH}_t - \pi_{BH}|
$$

is continuous in $\eta, \sigma, \lambda^{2,AH}_t, \lambda^{2,BL}_t$. Let $\eta \to 1$ (which forces $\sigma, \lambda^{2,AH}_t, \lambda^{2,BL}_t \to 0$), the limit goes to

$$
\{p_L[1 - F^A(\frac{1}{2})] + (1 - p_L)[1 - F^A(\frac{\pi_{BH}}{\pi_{BH} + 1})]\} \log|1 + \frac{\pi_{BH}}{(\pi_{BH} + 1)^2} G_1| + \{p_L[F^A(\frac{1}{2})] + (1 - p_L)[F^A(\frac{\pi_{BH}}{\pi_{BH} + 1})]\} \log|1 + \frac{\pi_{BH}}{(\pi_{BH} + 1)^2} G_2|
$$

If $1 + \frac{\pi_{BH}}{(\pi_{BH} + 1)^2} G_1$ and $1 + \frac{\pi_{BH}}{(\pi_{BH} + 1)} G_2$ are non-negative, then above limit is strictly negative, following Jensen’s inequality.

(Step 3) Finally, we prove a real-valued linear stochastic process $x_t$ converge to 0 almost surely if it moves to 0 in expectation. Consider a sequence of signal realization $\omega$ such that $\lim_{t \to \infty} x_t(\omega) \neq 0$. Then for any $t$, $\exists' t'$ such that

$$
\lim_{j \to \infty} \sup_{t'} [x_{t'+j}(\omega) - x_{t'}(\omega)] \geq 0.
$$

However, $x_{t'+j} - x_{t'} = \frac{1}{2} \sum_{i=1}^j (x_{t'+i} - x_{t'+i-1})$, which is strictly negative for large $j$.

Therefore we have shown that each component of $\tilde{\pi}^1_t$ converges to 0 almost surely. Hence $\Pr((\cup_k \cap_{n \geq k} (\|\tilde{\pi}^1_t\| \leq \varepsilon)) = 1$. By Borel-Cantelli lemma, with positive probability $\tilde{\pi}^1_t$ stay within a smaller $\varepsilon$-neighborhood of 0. From above construction, $\tilde{\pi}^1_t \to \langle 0, 0, 0 \rangle$ if the sequence stays within a $\varepsilon$-neighborhood of 0. Same argument works with $\tilde{\pi}^2_t$. 

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(Conclusion) We have $\pi_1^*$ is always local stable, and $\pi_2^*$ is local stable if $1 + \frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_1$ and $1 + \frac{\pi_{BH}}{(\pi_{BH}+1)^2}G_2$ are non-negative. ■

References


