Multiproduct trading of indivisible goods with many seller and buyers\textsuperscript{*}

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Abstract

This paper analyzes oligopolistic markets in which indivisible goods are sold by multiproduct firms to a finite set of heterogeneous buyers, extending the analysis of Arribas and Urbano (2017, a) and Arribas and Urbano (2017,b). We show the existence of efficient subgame perfect equilibrium by formulating the problem as the linear programming relaxation of the standard Package Assignment model. We prove that a set of modified versions of the dual programming problem characterizes the efficient (non-linear) equilibrium prices. We study the conditions for the existence of efficient equilibrium in terms of the consumers value functions.

Keywords: Multiproduct price competition, Indivisibility, Mixed bundling prices, Subgame perfect Nash equilibrium,

1. Introduction

In many common situations, buyers have complementary preferences for objects in the marketplace. Consider a buyer trying to construct a computer system by purchasing components. Among other things, the buyer needs to buy a CPU, a keyboard and a monitor, and may have a choice over several models for each component. The buyer’s valuation of a package depends on the components in any particular combination, involving products of either only one firm or several firms. This example is a general instance of allocation problems characterized by

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heterogeneous, discrete resources and complementarities in consumers’ preferences. In addition, preferences need not be monotonic. These kinds of models are probably close to many circumstances in real world markets, but they are also more difficult to analyze. On the one hand, with indivisibilities, it is well-known that many familiar properties of the profit functions may fail to ensure the existence of pure strategy Nash equilibrium prices. On the other, without monotonicity of the buyer’s preferences equilibrium efficiency may require additional restrictions.

This paper focuses on oligopolistic markets in which indivisible goods are sold by multiproduct firms to a finite set of heterogeneous buyers, who may have non-monotonic preferences over bundles of products. In these settings, neither non-linear pricing not anonymous pricing guarantee the existence of efficient subgame perfect Nash-equilibrium outcomes and, even worse, sometimes equilibrium -either efficient or inefficient- fails to exist (see Bikhchandani and Ostroy, 2002; Arribas and Urbano, 2017, a).

We extend the analysis of Arribas and Urbano (2017, a) and Arribas and Urbano (2017,b), and show the existence of efficient subgame perfect equilibrium by formulating the problem as the linear programming relaxation of the standard Package Assignment model. We prove that a set of modified versions of the dual programming problem characterizes the efficient (non-linear) equilibrium prices and study the conditions for the existence of efficient equilibrium in terms of the consumers value functions.

Arribas and Urbano (2017, a) analyze a kind of non-linear subadditive prices –mixed bundling prices– and analytically show that efficient pure strategy subgame perfect Nash equilibria always exist in such settings. Mixed bundle refers to the practice of offering a consumer the option of buying goods separately or else packages of them at a special price. The authors also show that may also exist inefficient equilibria in which the representative buyer chooses a suboptimal bundle and no firm has a profitable deviation inducing the buyer to buy the surplus-maximizing bundle because of a coordination problem among the firms. Inefficient equilibria can be ruled out by either assuming that all firms are pricing their unsold bundles at the same profit margin as the bundle sold at equilibrium, or refining the equilibrium correspondence using the solution concept of Strong equilibrium (Aumann, 1959), which requires the absence of profitable deviations by any subset of firms and the buyer. Furthermore, they prove that the set of the firms’ Strong equilibrium profits is some projection of the core of such games. The above result is important because even theoretically, there are few general results for bundles of more than two goods in strategic settings. In fact, McAdams (McAdams, 1997) found that the analytical machinery for analyzing mixed-bundling could not be easily generalized to even three goods,
because of the interaction among sub-bundles.

In spite of the above theoretical results, when the number of firms is big and/or the number of products of each firm is huge, it is not easy to compute all the efficient subgame perfect equilibria of such games. In fact, even the solution of a simple example of three firms, producing three products each, turns out to be an heroic task, because of the sub-bundles interaction and the firms’ possible deviations. To solve this problem, Arribas and Urbano (2017,b) apply the machinery of the integer programming package problem, or more precisely, the dual problem of its linear relaxation, to identify the Pareto-efficient frontier of such games and hence to find all price vectors satisfying efficient subgame perfection in a huge set. Specifically, they show that the optimal solutions of the linear relaxation of the above integer programming package assignment problem and its dual problem give some particular subgame perfect Nash equilibrium consumption set and profits. It is interesting to notice that the optimal solutions of any linear programming problem shape a polyhedron, and so is the projection of the dual problem solutions on firms’ profit vectors. This polyhedron is completely determined by its vertices. The Pareto frontier of the above projection has to be identified in order to characterize the set of all subgame perfect Nash equilibrium profits vectors belonging to some equivalence class. These payoff-equivalence classes come from subgame perfect Strong Nash equilibrium outcomes.

Although integer programming package problems have been used to find Walrasian equilibrium prices (see the related literature section below), their use has not been generalized to identify efficient subgame perfect Nash equilibria. One exception is Arribas and Urbano (2005), who study, as an assignment game, the market interaction of a finite number of single-product firms and a representative buyer, where the buyer consumes bundles of goods. They show that the Nash equilibrium outcomes are solutions of the linear relaxation of an integer programming assignment problem. Another one is Arribas and Urbano (2017,b), who, as already mentioned, extend the analysis to a package assignment game.

Our paper extends the above results to a package assignment problem and illustrates how to modify the integer programming problem and its dual to find such a set. More specifically, since we are interested in the full characterization of the solutions where the firms’ profits are Pareto-undominated, we want to characterize the vertex of the polyhedron of optimal solutions of the dual problem whose corresponding coordinates are non Pareto-dominated.

If the optimal solution of the linear relaxation of the integer programming problem is not integer, then equilibrium fails to exist. Therefore, we give some sufficient conditions to guarantee that both the integer programming assignment problem and its linear relaxation have the same
optimal solution. Some of those conditions are related with the unimodularity of the matrix of constraints of the primal problem.

To sum up, we formulate the efficient subgame perfect Nash equilibrium outcomes of a multiproduct trading with a representative buyer as a modified extension of the standard package assignment model. We give some general conditions to guarantee that the solutions of the linear relaxation of the integer programming problem are integral; and we prove the equivalence of integer programming solutions and efficient (subgame perfect) Nash equilibrium outcomes.

1.1. Related literature

A large literature (Crawford and Knoer, 1981; Quinzii, 1984; Zhao, 1992; among others) on markets with indivisibilities has grown following Shapley and Shubik’s housing market (Shapley and Shubik, 1972). All these works assume price-taking behavior and study the equivalence between core outcomes and those of competitive equilibria. Linear programming has been applied to the standard assignment model, with remarkable results. In particular, there is a linear programming characterization of the standard assignment model yielding optimal solutions to the underlying integer programming problem. The primal and dual solutions of the linear program coincide with price-taking Walrasian equilibrium and with the core. Later authors have addressed variations of the assignment game (Bikhchandani and Mamer, 1997; Ma, 1998; Bikhchandani and Ostroy, 2002). Bikhchandani and Mamer gave a linear programming characterization of the package assignment model. These papers made the usual assumption that the prices of packages are the sum of the prices of the objects contained in it, i.e., linear prices. In an excellent paper, Bikhchandani and Ostroy study assignment problems where individuals trade packages consisting of several, rather than single, objects. Efficient assignments can be formulated as a linear programming problem in which the pricing function expressing duality may be nonlinear in the objects constituting the packages, thus extending the price characterization of the standard assignment problem to the package assignment model. The package assignment model was first investigated by Kelso and Crawford (1982), who obtained sufficient conditions for existence of Walrasian equilibrium. Gul and Stacchetti (1999) obtained equivalent sufficient conditions for existence of equilibrium and showed that under this condition the core has the lattice property. The point of departure in Bikhchandani and Ostroy is to consider pricing functions which are non-additive over objects and also possibly non-anonymous.

More recently Sun and Yang (2006), examine an exchange economy with heterogeneous indivisible objects that can be substitutable or complementary. They show that a competitive equilibrium exists provided that all the objects can be partitioned into two groups, and from
the viewpoint of each agent, objects in the same group are substitutes and objects across the
two groups are complements. This condition generalized the Kelso-Crawford gross substitutes
condition. A related literature is Vries and Vohra (2003), in combinatorial auctions, the set
Packing problem and the solvable instances in terms of unimodularity, balanced matrices, etc.
Finally, Baldwin and Kemplerer (2016) propose a new techniques for understanding agents’
valuations. Their classification into “demand types”, incorporates existing definitions (sub-
stitutes, complements, strong substitutes, etc) and permits new ones. Their Unimodularity
Theorem generalizes previous results about when competitive equilibrium exists for any set of
agents whose valuations are all of a demand type for indivisible goods.

The above models consider an economy, with many sellers and many buyers and where all
agents are price takers. Thus, prices have to clear the market. Our model also deals with
many multiproduct firms and many agents. The firms post prices for all their feasible bundles
and then buyers choose the priced consumption set that maximizes their surplus. It is then a
sequential game dealing with strategic equilibrium which can also be formulated as a package
assignment problem, with the dual problem providing the non-linear prices (when needed).

The paper is organized as follows. Section 2 sets up the game framework and Section 3
characterizes both efficient and inefficient subgame perfect Nash equilibrium in terms of the
incentive compatibility constraints. Section 4 deals with the Package Assignment problem and
Section 5 characterizes the subgame perfect equilibrium in terms of linear programming and
offer some results on the consumers’ value function guaranteeing integer solutions.

2. The Game

2.1. Framework

Consider a finite set of firms \( N = \{1, 2, ..., n\} \) and buyers \( M = \{1, 2, ..., m\} \). Each firm \( i \in N \)
produces a finite set of heterogeneous and indivisible goods \( \Omega_i \), where firm’s products can be
different from or identical to those of any other firm. Let \( c_i(w) \) be the unit cost of production
of firm \( i \) for good \( w \in \Omega_i \), where costs are additive, i.e. \( c_i(T) = \sum_{w \in T} c_i(w), T \subseteq \Omega_i \). Let
\( \Omega = \Omega_1 \times \ldots \times \Omega_n \) and let \( 2^\Omega = 2^{\Omega_1} \times \ldots \times 2^{\Omega_n} \). Each buyer \( j \in M \) buys \( n \) bundles, one for each
firm, where the null bundle is allowed. We have a market game that take place in two steps:
first firms move simultaneously announcing a price for each of their bundles; second buyers
move simultaneously choosing their preferred bundles.

A strategy of firm \( i \in N \) is a function \( p_i : 2^{\Omega_i} \to \mathbb{R}_+ \) specifying the price of each subset of
\( \Omega_i \). Let \( p_i(T) \) be the price of \( T \subseteq \Omega_i \) and set \( p_i(\emptyset) = 0 \). Let \( \mathcal{P}_i \) be the set of firm \( i \)’s strategies
and $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$. The typical element of $\mathcal{P}$ is a tuple $p = (p_1, \ldots, p_n)$ and is called a price vector.

A strategy for buyer $j$ is a function $\gamma_j$ from $\mathcal{P}$ to $2^\Omega$ such that $\gamma_j(p) = (S_j^1, \ldots, S_j^n) \in 2^\Omega$ and $S_j^i$ is the bundle bought by buyer $j$ to firm $i$. If there is not transaction between firm $i$ and buyer $j$, then $S_j^i = \emptyset$. Let $\Gamma_j$ be the set of strategies of buyer $j$, i.e., the set of functions from $\mathcal{P}$ to $2^\Omega$. Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_m$ and $\gamma = (\gamma_1, \ldots, \gamma_m) \in \Gamma$ be a typical vector of buyers’ strategies.

A pair of strategies’ vectors $(\gamma, p) \in \Gamma \times \mathcal{P}$ defines a consumption set $S = \gamma(p) = (S_j^i)_{i \in N, j \in M}$ where $\gamma_j(p) = (S_j^1, \ldots, S_j^n)$. Therefore, it defines both buyer $j$’s purchase $S_j = \gamma_j(p)$ and firm $i$’s sale $S^i = (S_j^1, \ldots, S_j^n)$. Moreover, let $c_i(S^i) = \sum_j c_i(S_j^i)$ be the cost of the consumption set $S^i$ (where $c_i(\emptyset) = 0$), and let $F(S)$ be the set of active firms in $S$, i.e., $F(S) = \{i \in N | \cup_j S_j^i \neq \emptyset\}$.

Notice that we will reserve index $i$ and $j$ for firms and buyers respectively, so that, $\sum_i = \sum_{i \in N}$ and $\sum_j = \sum_{j \in M}$. Moreover, a consumption set $S$ can be understand as an $n$-vector where coordinates are the firms’ sales, i.e., $S = (S^1, \ldots, S^n)$; or as an $m$-vector where coordinates are the buyers’ purchases, i.e., $S = (S_1, \ldots, S_m)$.

A feasible consumption set is a consumption set $S = (S_j^i)_{i \in N, j \in M}$ such that for all $i \in N$ and $j, j' \in M$ with $j \neq j'$, $S_j^i \cap S_{j'}^i = \emptyset$. A pair of strategies’ vectors $(\gamma, p)$ is feasible if it defines a feasible consumption set.

For any given pair $(\gamma, p) \in \Gamma \times \mathcal{P}$, the payoff of each firm $i \in N$, its profit function, is

$$\pi_i(\gamma, p) = \pi_i(S^i, p) = \sum_{j \in M} p_i(S_j^i) - c_i(S^i),$$

where $S^i$ is firm $i$’s sale defined by $\gamma(p)$.

Buyers have utility over bundles, thus let $v_j : 2^\Omega \rightarrow \mathbb{R}_+$ denotes the buyer $j$’s utility function in monetary terms. Then, buyer $j$’s payoff function, her consumer surplus, is

$$cs_j(\gamma, p) = cs_j(S_j, p) = v_j(S_j) - \sum_{i \in N} p_i(S_j^i).$$

where $S_j = \gamma_j(p)$.

The extensive form game is as follows. In a first stage, each firm $i$ choses a price schedule $p_i \in \mathcal{P}_i$, independently and simultaneously to the other firms. Then, in the second stage, each buyer $j$ observes the price vector $p = (p_1, \ldots, p_n)$ and selects a consumption bundle from the firms, the sale take place and both firms and buyers obtain their payoffs.
A pair of strategies \((\gamma, p)\) defines a consumption set \(\gamma(p)\), therefore when there is not ambiguity we will use \(S\) instead of \(\gamma(p)\), such that \((S, p)\) can be interpreted as a pair of consumption set and price vector, with \(S = \gamma(p)\).

Hence, formally, we have a game with a set \(N\) of firms and \(M\) buyers. Let \(G(N, M, (P_i)_{i \in N}, (\Omega_i)_{i \in N}, (v_j)_{j \in M}, c)\) denote such a game.

2.2. Equilibrium concept

In what follows we are interested in pure-strategy equilibria. A pure-strategy equilibrium for firms and buyers is an element of \(\Gamma \times P\).

**Definition 1.** A subgame perfect Nash equilibrium (SPE) of \(G\) is a pair of strategies, \((\gamma, p)\), such that \(\gamma(p)\) is a feasible consumption set and:

\[
\gamma_j(p) \in \arg \max_{T \in 2^{\Omega_i}} cs_j(T, p) \quad \text{for all } j \in M, \quad \text{and} \quad (3)
\]

\[
p_i \in \arg \max_{p'_i \in P_i} \pi_i(\gamma(p_{-i}, p'_i), (p_{-i}, p'_i)) \quad \text{for all } i \in N \quad (4)
\]

where \((p_{-i}, p'_i) = (p_1, \ldots, p_{i-1}, p'_i, p_{i+1}, \ldots, p_n)\).

Let function \(V(S) = \sum_j v_j(S_j) - \sum_i c_i(S_i)\) be the social surplus function of the consumption set \(S\). Then,

**Definition 2.** A feasible consumption set \(S\) is socially efficient if \(S \in \arg \max_T V(T)\). In particular, let \(V^* = \max_T V(T)\).

Notice that the above game with constant costs can be transformed into an equivalent game where production costs are zero due to cost are linear. This allows us to simplify the notation and the proofs. In particular, given \(G(N, M, (P_i)_{i \in N}, (\Omega_i)_{i \in N}, (v_j)_{j \in M}, c)\), we can define a new strategic game \(G'(N, M, (P_i)_{i \in N}, (\Omega_i)_{i \in N}, (v'_j)_{j \in M}, c')\), where \(v'_j(S_j) = v_j(S_j) - \sum_i c_i(S_i^j)\) and \(c'_i(w) = 0\) for all \(i \in N, w \in \Omega_i\). Consider strategies \((\gamma, p)\) and \((\gamma, p')\), where \(p'_i(S_i^j) = (p_i - c_i)(S_i^j)\) for all \(j \in M, i \in N\) and \(S_i^j \subseteq \Omega_i\). It is straightforward to check that \((\gamma, p)\) is a SPE for game \(G\) if and only if \((\gamma, p')\) is a SPE for game \(G'\). Moreover, the firms’ profits and the buyers’ payoffs from both strategies are the same. Given this fact, in what follows we will assume without loss of generality that \(c_i(w) = 0\) for all \(i \in N\) and \(w \in \Omega_i\), i.e., the game \(G\) has zero production costs.

As shown by Arribas and Urbano (2017, a), some subgame perfect Nash equilibrium might not be efficient. To guarantee efficiency, we use the solution concept of subgame perfect Strong
Nash equilibrium introduced by Aumann (1959). **Strong equilibrium** is an equilibrium such that no subset of players has a *joint* deviation that strictly benefits *all* of them. In our game this implies that there is not a subset of firms and buyers that can benefit from a jointly deviation. Let $SPE^*$ be the set of **Strong Nash equilibrium** in our game $G$.

Unfortunately, $SPE^*$ might not exist, as the following example from Bikhchandani and Ostroy (2002) (who borrowed from Kelso and Crawford (1982)) shows.

**Example 1 (No existence of $SPE^*$):** There are three firms $N = \{1, 2, 3\}$, each one producing a single product, say $a, b$ and $c$, respectively; and there are two buyers $M = \{1, 2\}$. The buyers’ value functions, that are monotonic and subadditive, are in Table 1.

<table>
<thead>
<tr>
<th>$S$</th>
<th>${a}$</th>
<th>${b}$</th>
<th>${c}$</th>
<th>${a, b}$</th>
<th>${b, c}$</th>
<th>${a, c}$</th>
<th>${a, b, c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_1$</td>
<td>4</td>
<td>4</td>
<td>4.25</td>
<td>7.5</td>
<td>7</td>
<td>7</td>
<td>9</td>
</tr>
<tr>
<td>$v_2$</td>
<td>4.25</td>
<td>4</td>
<td>4</td>
<td>7</td>
<td>7.5</td>
<td>7</td>
<td>9</td>
</tr>
</tbody>
</table>

As we prove below, $SPE^*$ outcomes are efficient, that in our example implies that either buyer 1 buys $\{a\}$ and buyer 2 buys $\{b, c\}$, or buyer 1 buys $\{a, b\}$ and buyer 2 buys $\{c\}$. Both give a social surplus of 11.5 and are equivalent, so let us prove that the former cannot be part of an equilibrium. Suppose that $p_a, p_b$ and $p_c$ are the prices set by firms. For this to be an equilibrium outcome, the following inequalities must hold as implications of the buyer best response as in (3):

\[
\begin{align*}
4 - p_a & \geq 7.5 - p_a - p_b \\
4 - p_a & \geq 4.25 - p_c \\
7.5 - p_b - p_c & \geq 4 - p_c \\
7.5 - p_b - p_c & \geq 4.25 - p_a.
\end{align*}
\]

The first and third inequalities together imply that $p_b = 3.5$. Given this, the second and forth inequalities can be rewritten as $p_a - p_c \leq -0.25$ and $p_a - p_c \geq 0.25$, a contradiction. Thus, there is no price vector supporting the consumption set $\{S_1 = \{a\}, S_2 = \{b, c\}\}$ as an equilibrium outcome.

Now, let us suppose that there is a single firm producing the three products $\Omega = \{a, b, c\}$, so that the firm have to set a price for each one individually, and also for each possible bundle of them. Suppose that $(p_a, p_b, p_c, p_{ab}, p_{ac}, p_{bc}, p_{abc})$ is such a price vector. Let us see that
even allowing non linear prices, the consumption set \( S_1 = \{ a \}, S_2 = \{ b, c \} \) cannot be an equilibrium outcome.

Suppose that \( p_a < 4 \) and \( p_{bc} < 7.5 \). We know that the following inequalities must hold as implications of (3):

\[
\begin{align*}
4 - p_a & \geq 7 - p_{bc} \\
7.5 - p_{bc} & \geq 4.25 - p_a
\end{align*}
\] (5) (6)

Let us see that the firm has incentives to deviate profitable by setting the following prices:

\( p'_a = p'_b = p'_{ac} = p'_{bc} = 10, \ p'_c = p_a + \epsilon \) and \( p'_{ab} = p_{bc} + \epsilon \), where \( 0 < \epsilon < \frac{1}{4} \min\{4 - p_a, 7.5 - p_{bc}\} \). First, these prices induce buyer 1 to choose \( \{ a, b \} \) and buyer 2 to choose \( \{ c \} \). To see that we have to check that \( 7.5 - p'_{ab} \geq 4.25 - p'_c \) and \( 4 - p'_c \geq 7 - p'_{ab} \), but these inequalities are equivalent to (5) and (6). Second, the firm’s profit with the new prices is greater than with the old ones: \( p'_c + p'_{ab} = p_a + \epsilon + p_{bc} + \epsilon > p_a + p_{bc} \). Therefore either \( p_a = 4 \) or \( p_{bc} = 7.5 \).

However, if \( p_{bc} = 7.5 \), then the inequality (6) does not hold, therefore it must be verified that \( p_a = 4 \) and \( p_{bc} \leq 7.5 \). But then, inequalities (5) and (6) imply that \( 7 \leq p_{bc} \leq 7.25 \). Again, we can see that the firm has incentives to deviate profitable by setting the prices \( p'_a = p'_b = p'_{ac} = p'_{bc} = p'_{abc} = 10, \ p'_c = p_a = 4 \) and \( p'_{ab} = p_{bc} + \epsilon \), where \( 0 < \epsilon < .25 \), which induces buyer 1 to choose \( \{ a, b \} \) and buyer 2 to choose \( \{ c \} \).

In conclusion, there is not such a non linear price vector that supports the consumption set \( \{ S_1 = \{ a \}, S_2 = \{ b, c \} \} \) as an equilibrium outcome.

3. Equilibrium analysis

In what follows, we characterize both sets \( SPE \) and \( SPE^* \) of game \( G \).

The subgame perfect Nash equilibrium conditions preclude unilateral deviations from buyers and firms. Namely, we need conditions to guarantee that:

1. Each buyer maximizes her surplus. This condition is Buyer Maximization (BM) below.

2. Each non-active firm cannot benefit from price reductions to marginal cost. This is condition Firm Rationality (FR).

3. Each firm does not have an incentive to increase the equilibrium prices of its sold bundles to increase his profit. Firm Maximization (FM) condition

4. Each firm does not have an incentive to modify prices in order to increase his profit by
selling a different vector of bundles. Incentive Compatibility (IC) condition.

More precisely, let \((\gamma, \mathbf{p})\) be a SPE and let \(\mathbf{S} = \gamma(\mathbf{p})\), then each buyer should verify the buyer maximization property which states that no bundle gives her a higher surplus than \(\mathbf{S}_j\):

\[
v_j(\mathbf{S}_j) - \sum_{i \in \mathcal{N}} p_i(S_j^i) \geq v_j(\mathbf{T}) - \sum_{i \in \mathcal{N}} p_i(T_i) \quad \text{for all } j \in \mathcal{M}, \ T \in 2^\Omega. \tag{BM}
\]

In an equilibrium outcome, no non-active firm, which obtains a profit equal to zero, benefits from price reductions. There is no way in which a non-active firm can change buyers’ decisions by reducing his equilibrium prices (firm rationality condition):

\[
v_j(\mathbf{S}_j) - \sum_{k \in \mathcal{N}} p_k(S_j^k) \geq v_j(\mathbf{T}) - \sum_{k \in \mathcal{N} \setminus i} p_k(T_i) \quad \text{for all } j \in \mathcal{M}, i \notin \mathcal{F}(\mathbf{S}), \ T \in 2^\Omega. \tag{FR}
\]

Each firm sets its prices so that the profit of implementing \(\mathbf{S}\) is maximal. There is no way in which an active firm can increase the equilibrium prices for \(\mathbf{S}_i\) without losing its sale (firm maximization condition):

\[
v_j(\mathbf{S}_j) - \sum_{k \in \mathcal{N}} p_k(S_j^k) = \max_{\bigcup_{\mathbf{T}_i \neq \emptyset}} \left\{ v_j(\mathbf{T}) - \sum_{k \in \mathcal{N}} p_k(T_k) \right\} \quad \text{for all } j \in \mathcal{M}, i \in \mathcal{F}(\mathbf{S}). \tag{FM}
\]

The maximum price for firm \(i\) to convince buyer \(j\) to move from \(\mathbf{S}_j\) to \(\mathbf{T}\) is \((v_j(\mathbf{T}) - \sum_{k \neq i} p_k(T_k)) - (v_j(\mathbf{S}_j) - \sum_k p_k(S_j^k))\). If \(\mathbf{S}\) is an equilibrium consumption set, then the loss of a deviation must be greater than the benefit from that deviation, for all firm \(i\) and bundle \(\mathbf{S}'\) (incentive compatibility condition):

\[
\sum_{j \in \mathcal{M}} \left( v_j(\mathbf{S}_j) - \sum_{k \in \mathcal{N} \setminus i} p_k(S_j^k) \right) \geq \sum_{j \in \mathcal{M}} \left( v_j(\mathbf{S}_j) - \sum_{k \in \mathcal{N} \setminus i} p_k(S_j^k) \right) \quad \text{for all } i \in \mathcal{F}(\mathbf{S}) \text{ and } \mathbf{S}'. \tag{IC}
\]

**Proposition 1.** \((\gamma, \mathbf{p})\) is an SPE-outcome of \(G\) if and only if buyer maximization, firm rationality, firm maximization and incentive compatibility conditions hold.

In our model with multiproduct firms and where buyers’ utility function over bundles need not be strictly concave, the Nash concept of stability (or bilateral efficiency) may not be enough to guarantee efficiency. This is particularly true when the equilibrium bundle pertains to several firms so that additional coordination between them may be required. Here, subgame perfection does not rule out inefficient equilibria and therefore the conditions of Proposition 1 characterize both efficient and inefficient subgame perfect Nash-equilibrium as illustrated in example 2 below.
Example 2 (The market for systems). Let the set of firms be $N = \{1, 2\}$, producing $\Omega_1 = \{a, b\}$ and $\Omega_2 = \{c, d\}$, respectively; and there is a single buyer whose value function is,

$$v(S) = \begin{cases} 
4 & S = \{a, b\} \\
9 & S = \{a, d\} \\
5 & S = \{b, c\} \\
8 & S = \{c, d\} \\
0 & \text{otherwise}
\end{cases}$$

It can be checked that the efficient bundle $S = \{a, d\}$ is a $SPE$-consumption set, supported by equilibrium prices verifying:

$$0 \leq p_a \leq 1, 4 \leq p_d \leq 5, 5 \leq p_a + p_d,$$

$$p_{ab} = p_a + p_d - 5, p_{cd} = p_a + p_d - 1,$$

$$p_b \geq 0, p_c \geq p_d - 4, p_b + p_c \geq p_a + p_d - 4.$$ 

Nevertheless, it is also easy to show that the inefficient exclusive dealing outcome $\{c, d\}$, supported by the price vector: $p_{cd} = 4; p_c \geq 1; p_d \geq 5; p_{ab} = 0; p_a \geq 1; \text{ and } p_b \geq 0$, is also a subgame perfect equilibrium. There are also multiple subgame perfect equilibrium prices sustaining the inefficient consumption set.

With this idea in mind, we would like to consider only the subset of subgame perfect equilibrium outcomes which remains as equilibrium outcomes, even if any set of non-active firms set unit cost prices and any set of active firms set prices to obtain a higher profit. In other words, we want FR to be satisfied for all subsets of non-active firms and IC to be satisfied for all subsets of active firms. Thus, the conditions of firm rationality and incentive compatibility have to be extended to a notion of stability against joint deviations that are mutually profitable to any subset of firms. We denote these conditions as strong efficiency, meaning that the equilibrium joint payoff of any subset of buyers and any subset of firms selling at equilibrium is the maximum joint payoff, given all possible strategies for all other subsets of firms, even for those subsets not selling at equilibrium. These conditions remove the set of $SPE$ in which some firms charge unreasonably high prices so that no individual firm can benefit from a price reduction of its products only.

Definition 3. The subset $SPE^*$ of SPE of $G$ is defined as the set of equilibrium outcomes satisfying BM, FM and FE (instead of FR and IC), where FE is stated as:
For all $A \subseteq N \setminus F(S)$, $B \subseteq F(S)$ and $S'$, and for all $p_i' \in P_i$, $i \in B$, such that $\sum_j p_i'(S''_j) = \sum_j p_i(S''_j)$, there exists $j \in M$ verifying:

$$v_j(S_j) - \sum_{k \in N} p_k(S''_j) \geq v_j(S'_j) - \sum_{k \in N \setminus (A \cup B)} p_k(S''_j) - \sum_{k \in B} p_k(S''_j).$$  \hfill (FE)

Thus, we restrict the analysis to a certain subset $SPE^*$ of $SPE$-outcomes.

4. The associated package assignment problem

Our main result shows that the set of $SPE^*$-outcomes is equivalent to integer-valued solutions of the linear relaxation of a package assignment problem.

Since our purpose is to find a feasible assignment of a set of products from firms to the buyers (a package assignment), for any $S_j \in 2^\Omega$, define $z(S_j, j)$ as equal to 1 if the buyer $j$ chooses consumption set $S_j$, and zero otherwise; and for any $S^i \in (\Omega_i)^m$, defines $y(S^i, i)$ as equal to 1 if the firm $i$ sells $S_i$, and zero otherwise. The integer programming defining the package assignment problem, denoted ILP is,

$$V_{ILP}(N) = \text{Max} \sum_j \sum_{S_j \in 2^\Omega} v_j(S_j)z(S_j, j)$$

s.t. 

$$\sum_{S_j \in 2^\Omega} z(S_j, j) \leq 1 \quad \forall j \in M$$  \hfill (7)

$$\sum_{S^i \in (\Omega_i)^m} y(S^i, i) \leq 1 \quad \forall i \in N$$  \hfill (8)

$$\sum_j \sum_{S_j: S_j^T = T} z(S_j, j) \leq \sum_j \sum_{S^i: S_j^T = T} y(S^i, i) \quad \forall i \in N, \forall T \subseteq \Omega_i$$  \hfill (9)

Constraint (7) ensures that only one consumption set is selected by each buyer. Constraints in (8) guarantee that each firm only sells one consumption set, and constraints (9) ensure that the bundle chosen by buyer $j$ from firm $i$ is the one sold by firm $i$ to buyer $j$.

Let us consider the linear relaxation LP of ILP in which we change the integrity constraints $z(S_j, j)$, $y(S^i, i) \in \{0, 1\}$ in ILP to $z(S_j, j) \geq 0$, $y(S^i, i) \geq 0$. Let $V_{ILP}$ and $V_{LP}$ denote the optimal value of ILP and LP respectively, thus $V_{ILP} \leq V_{LP}$.

Notice that the set of optimal solutions of LP does not need to be equal to the set of optimal solutions of ILP. In example 1 for the case of a single seller, the optimal solution of ILP has
a value of $V_{ILP} = 11.5$ and an assignment is $z(\{a\}, 1) = z(\{b,c\}, 2) = 1$, zero otherwise; and $y(\{a\}, \{b,c\}, 1) = 1$, zero otherwise. However, the optimal solution of LP has a value of $V_{LP} = 11.75$, and an assignment is $z(\{c\}, 1) = z(\{a\}, 2) = z(\{b,c\}, 2) = 0.5$, zero otherwise; and $y(\{c\}, \{a,b\}, 1) = y(\{a\}, \{b,c\}, 1) = 0.5$, zero otherwise.

Let DLP be the dual problem of LP. The interest of this formulation is that the dual variables associated with constraints (7) can be interpreted as buyers’ surplus, the dual variables associated with constraints (8) can be interpreted as firms’ profits, and each dual variable associated with each constraint in (9) as the price that firm $i$ sets for each $T \subseteq \Omega_i$. Let $\pi_j$ –the consumer $j$’s surplus– be the variables associated to constraints (7); let $\pi_i$ –firm $i$’s profit– be the ones associated to constraints (8) and finally, let $\pi^i_T$ –price of bundle $T$– be those associated with constraints (9). Then the dual problem DLP is,

\[
\text{Min} \quad \sum_j \pi_j + \sum_i \pi^i \\
\text{s.t.} \quad \pi_j + \sum_i \pi^i_{S^i_j} \geq v_j(S_j) \quad \forall j \in M, \, S_j \in 2^\Omega \\
\pi^i - \sum_j \pi^i_{S^i_j} \geq 0 \quad \forall i \in N, \, \forall S^i \in (\Omega_i)^m \\
\pi_j, \pi^i, \pi^i_{S^i_j} \geq 0.
\]

The feasible region of LP is nonempty, which implies that DLP also has an optimal solution, say $V_{DLP}$. Among the optimal solutions of DLP let us consider those whose coordinates ($\pi^i$), are not Pareto-dominated by any other optimal solution. A way to obtain some of these solutions is to consider the following restricted dual problem RDLP:

\[
\text{Max} \quad \sum_i \pi^i \\
\text{s.t.} \quad \pi_j + \sum_i \pi^i_{S^i_j} \geq v_j(S_j) \quad \forall j \in M, \, S_j \in 2^\Omega \\
\pi^i - \sum_j \pi^i_{S^i_j} \geq 0 \quad \forall i \in N, \, \forall S^i \in (\Omega_i)^m \\
\sum_j \pi_j + \sum_i \pi^i = V_{DLP} \quad (12) \\
\pi_j, \pi^i, \pi^i_{S^i_j} \geq 0.
\]

By the duality theorem of linear programming we have that $V_{DRLP} = V_{DLP} = V_{LP} \geq V_{ILP}$. It can be proven that if $V_{LP} = V_{ILP}$, then an optimal solution of LP together with a non
Pareto-dominated solution of DLP in coordinates \((\pi^i)\) yield an \(SPE^*\)-outcome of the game. The equilibrium price vectors are mixed bundling prices. Thus, the existence of \(SPE^*\)-outcomes depends on the existence of an integer optimal solution of LP. Trivially, if all buyers have additive value functions \(v_j(S) + v_j(T) = v_j(S \cup T)\) for all \(j \in M\) and \(S, T \in 2^\Omega\) then, equilibrium prices will also be linear, with \(p_i(w) = \max_j \{v_j(w)\}\), for all firm \(i \in N\), \(w \in \Omega_i\) and existence of (degenerate) \(SPE^*\)-outcomes is always guaranteed. The next section give the main result and analyze conditions on buyers value functions which ensure existence of \(SPE^*\)-outcomes.

5. Subgame perfect Nash equilibrium via Linear programming

Our main result establishes that an optimal solution of LP and RDLP is an \(SPE^*\)-outcome (Proposition 2) of G. Moreover, the \(SPE^*\)-consumption set is always efficient (Corollary 1). First, we start with a general property which states that optimal solutions of LP and DLP, set the prices of nonactive firms equal to marginal costs and hence their profits are zero (property ii) and the profits of any active firm are bigger than or equal to their selling prices (property i).

Lemma 1. Let \(S \in \text{sol}(LP)\) be integral and let \(((\pi_j), (\pi^i), (\pi^i_T)) \in \text{sol}(DLP)\), then

\[
\begin{align*}
\text{i)} & \quad \pi^i = \sum_j \pi^i_{S_j} \geq \sum_j \pi^i_{T_j} \text{ for all } i \in F(\mathbf{S}), \mathbf{T} \in (\Omega_i)^m. \\
\text{ii)} & \quad \pi^i = \pi^i_{T_j} = 0 \text{ for all } i \notin F(\mathbf{S}) \text{ and } \mathbf{T} \in (\Omega_i)^m.
\end{align*}
\]

The next Proposition gives an existence result. It shows that any element of \(\text{sol}(LP) \times \text{sol}(RDLP)\) is an \(SPE^*\)-outcome, i.e., \(\text{sol}(LP)\) give the equilibrium consumption set and \(\text{sol}(RDLP)\) the consumers’ surplus, firms’ profits and price vectors.

Proposition 2. Given a game G, let \(S \in \text{sol}(LP)\) be integral and let \(((\pi_j), (\pi^i), (\pi^i_T)) \in \text{sol}(DLP)\). Then \((\mathbf{S}, \mathbf{p}) \in SPE^*\), where \(p_i(T) = \pi^i_T\), for all \(i \in N\) and \(T \subseteq \Omega_i\).

Corollary 1. Given a game G, \((\mathbf{S}, \mathbf{p}) \in SPE^*\) if and only if \(\mathbf{S}\) is socially efficient.

The above results assume that \(\mathbf{S} \in \text{sol}(LP)\) is integral. However, in Example 1 \(V_{ILP} = 11.5\) but \(V_{LP} = 11.75\) because the solutions of the the linear relaxation of an integer programming problem are not integral. Therefore, the existence of \(SPE^*\)-outcomes depends on the existence of an integer optimal solution of LP.

Trivially, if all buyers have additive value functions \(v_j(S) + v_j(T) = v_j(S \cup T)\) for all \(j \in M\) and \(S, T \in 2^\Omega\) then, equilibrium prices will also be linear, with \(p_i(w) = \max_j \{v_j(w)\}\), for all firm
i ∈ N, w ∈ Ω; and existence of (degenerate) SPE*-outcomes is always guaranteed. The next results analyze conditions on buyers value functions which ensure existence of SPE*-outcomes.

**Result 1:** Let \((\lambda_S)_{S \subseteq \Omega}\) be a balanced vector if \(\lambda_S \geq 0\) for all \(S \subseteq \Omega\) and for all \(w \in \Omega\), \(\sum_{S \ni w} \lambda_S = 1\). A value function \(v\) defined on subsets \(S \subseteq \Omega\) is balanced if for any balanced vector \((\lambda_S)_{S \subseteq \Omega}\) we have \(\sum_{S} (\lambda_S) v(S) \leq v(\Omega)\).

If all buyers have the same value function \(v\), and if \(v\) is balanced, then \(V_{LP} = V_{ILP}\) and there exists an SPE*-outcome (as solution of the primal and dual linear problems). Moreover, given that a supermodular value function is balanced we conclude that if all buyers have the same supermodular value function, then the set of SPE*-outcomes is nonempty.

**Result 2:** The existence of SPE*-outcomes can be easily extended to two types of buyers, provided they have strictly supermodular value functions i.e., there are two types of buyers such that buyers type \(j = 1, 2\) have the same supermodular value function.

The extension of this result to more than two types of buyers remains an open question.

**Result 3:** Let \(N = 1\), a single seller, and \(m\) buyers with strictly supermodular value functions. Then the set of SPE* is nonempty.

**Result 4:** Finally, consider the case in which all buyers are characterized by a value function \(v_j, j \in M\) verifying Kelso and Crawford (1982) gross substitution condition. Given a buyer \(j \in M\), with value function \(v_j\) and a price vector \(p\), let \(D_j(p)\) be:

\[
D_j(p) = \{S \in 2^\Omega | v_j(S) - \sum_i p_i(S^i) \geq v_j(T) - \sum_i p_i(T^i) \text{ for all } T \in 2^\Omega\}
\]

Now, the gross substitution condition requires: for every \(j \in M\), and two prices vectors \(p\) and \(q\) such that \(p_i(T) \leq q_i(T)\) for all \(i \in N\) and \(T \subseteq \Omega\), if \(S \in D_j(p)\), then there exists \(T \in D_j(q)\) such that \(T^i = S^i\) if \(p_i(T^i) = q_i(S^i)\).

If the gross substitution condition is verified, then the set of SPE* is nonempty.

The results in Gul and Stacchetti (1999) and Bikhchandani et al. (2002) can be extended to cover SPE*-outcomes under mixed bundling pricing. Let \(r_b(S, p) = \min_{T \in D_b(p)} |S \cap T|\) be the dual rank function of a matroid. Then by the matroid partition theorem if for all \(T \subseteq \Omega\), \(\sum_{j \in M} r_b(T, p) \leq |T|\), then there exists a partition of \(\Omega\) so that every buyer receives at most one element of \(D_b(p)\). In other case, choose \(T' \subseteq \Omega\), such that \(\sum_{j \in M} r_b(T', p) > |T'|\) and is the one verifying that property with minimal cardinality. Now, following a modification of the algorithm proposed by Gul and Stacchetti (1999), each firm \(i\) increases the price of its bundles.
$S_i \subseteq \Omega_i$ such that $S_i \cap T_i \neq \emptyset$ by $\epsilon > 0$. After a finite number of rounds we obtain a price vector and an allocation which defines an optimal solution of both LP and ILP.

6. References


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