Multimarket Contact under Imperfect Observability and Impatience*

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Abstract

We study a model of infinitely repeated games where two or more identical prisoners’ dilemmas with imperfect public monitoring, whose monitoring structures are mutually independent, are simultaneously played every period. Our central question is whether the most cooperative public strategy equilibrium per-game payoff is greater than that of an individual repeated game. This question translates into a debate in industrial organization as to whether multimarket contact facilitates collusion. While existing results are concerned with limit results on either the number of markets or patience, we allow any number of games and any level of discounting.

We show that adding one more game never reduces the most cooperative equilibrium per-game payoff. Further, except the case where the players cannot cooperate at all in any equilibrium, adding one more game almost always increases the most cooperative equilibrium per-game payoff, and adding two or more games always increases it. Finally, we ask to what extent an added game can have an impact on the most cooperative equilibrium payoff and show the following “critical mass result.” Namely, for any given number of games $m$, there exist a stage game and a discount factor such that (i) if the number of games is $m$ or less, the only equilibrium is repeated play of the static equilibrium, and (ii) if the number of games is $m + 1$, this forms a critical mass and the most cooperative equilibrium payoff is arbitrarily close to the payoff of full cooperation in all games. This result is a caution to antitrust authorities.

JEL Classification: C72, C73, D43, L13.

Keywords: multimarket contact, collusion, repeated games, imperfect monitoring, impatience

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1 Introduction

The present paper studies a model where two or more repeated prisoners’ dilemmas with imperfect public monitoring are simultaneously played. Those repeated prisoners’ dilemmas are identical in the sense that they share the same stage payoffs, the same discount factor, and the same information structure. Further, those games are mutually independent in the sense that the payoffs and signals each player receives in one game are independent of play and outcomes of the other games. This mutual independence may lead the reader to think that it makes much sense to play those games separately; namely, playing an equilibrium of one prisoners’ dilemma in all games simultaneously and independently. Instead, the present paper asks whether a new strategic possibility arises by playing those games interdependently; namely, making future play of one prisoners’ dilemma dependent on outcomes of the other games. Particularly, we ask whether the most cooperative public strategy equilibrium per-game payoff is greater than that of an individual repeated game.

This question translates into a debate in industrial organization as to whether multimarket contact facilitates collusion. Our model is a most simplified one of multimarket contact; the prisoners’ dilemma is interpreted as a model of price competition in each local market played by nationwide firms. Starting from Bernheim and Whinston [2], several papers study the effect of multimarket contact on collusion via a model of multiple repeated games. The present paper differs from the literature because of the following two reasons. First, we do not assume perfect monitoring, so that the players receive only a noisy public signal of their actions. Second, while existing papers which assume imperfect monitoring are concerned with limit results on either the number of games or patience, we allow any number of games and any level of discounting.

In our model, two players play \( m \) infinitely repeated prisoners’ dilemmas simultaneously. Each repeated prisoners’ dilemma is a version of the model by Radner, Myerson and Maskin [9]. That is, the players choose from cooperation and defection and receive a binary signal, good or bad, every period. The signal is stochastic, and its probability distribution depends on their actions. The assumption of binary signals causes failure of pairwise distinguishability (Fudenberg, Levine and Maskin [3]), so that the folk theorem by public strategy equilibria fails. We assume that the payoffs and the information structure are symmetric, and we limit attention to symmetric public strategy equilibria. We are interested in the equilibrium attaining the greatest payoff, which we call the most cooperative equilibrium.

We first formulate the problem of finding the most cooperative equilibrium payoff as a linear programming problem. Based on the analysis of the problem, we show that the most cooperative equilibrium has a cutoff level of bad signals so that (i) if the number of bad signals is less than the cutoff, no punishment occurs, and (ii) if the number of bad signals exceeds the cutoff, they are punished by the Nash reversion in all games. Moreover,
the most cooperative equilibrium has a feature that the most profitable deviation is either to defect in all games or to defect in a single game. This corner-solution property greatly simplifies our analysis of finding the most cooperative equilibrium payoff.

Our results are summarized as follows. First, we show that given a stage game and a number of games, more patience weakly increases the most cooperative equilibrium payoff and strictly increases it if the most cooperative equilibrium corresponds to neither full cooperation nor no cooperation. The rest of the results are on the effect of multimarket contact. We show that adding one game never reduces the most cooperative equilibrium per-game payoff. In order to obtain a stronger result, let us now assume that the most cooperative equilibrium sustains a nonzero level of cooperation. Then, adding one game strictly increases the most cooperative equilibrium per-game payoff, unless a very stringent condition, which we call a no-sunspot condition, holds. The no-sunspot condition states that play of the most cooperative equilibrium does not need availability of a public randomization device, and holds only for finitely many discount factors given the stage game. The most cooperative equilibrium per-game payoff remains the same only when that rare condition holds.

We also show that adding two or more games always increases the most cooperative equilibrium per-game payoff. This result is due to the fact that, given a stage game and a discount factor, the no-sunspot condition never holds for two consecutive number of games. Therefore, whenever adding a first game does not improve the most cooperative equilibrium per-game payoff, adding a second (and more) game does improve it. We thus conclude from those results that multimarket contact generally facilitates collusion.

Finally, we ask to what extent an added game can have an impact on the most cooperative equilibrium payoff. In the context of multimarket contact, this is an attempt to explore a worst case scenario to an antitrust authority about allowing one more market to an oligopoly. Our “critical mass result” is a sharp answer to that question. Namely, for any given number of games \( m \), there exist a stage game and a discount factor under which (i) if the number of games is \( m \) or less, the only equilibrium is repeated play of the static equilibrium, and (ii) if the number of games is \( m + 1 \), the most cooperative equilibrium payoff is arbitrarily close to the payoff of full cooperation in all games. Here \( m + 1 \) games form a critical mass, and the players who have reached it suddenly attain a very high level of cooperation, despite their inactiveness before reaching to the critical mass. This result serves as a caution to antitrust authorities.

Our model and solution concepts are basically the same as Matsushima [8] and Kobayashi and Ohta [5]. [8] derives a limit result on the number of games, stating that if the number of games goes to infinity, the range of discount factors which approximately sustain the per-game payoff of full collusion is the same as the range under perfect monitoring. [5] rather derives the most cooperative equilibrium payoff for any number of markets, assuming that the players are sufficiently patient. We instead consider any number of markets and any level of patience.

The rest of the paper is organized as follows. Section 2 sets up our model. Section 3 formulates a linear programming problem that characterizes the most cooperative equilibrium payoff as its solution and establishes structure of the equilibrium. Section 4 presents our main results; comparative statics with respect to discounting, positive effects of mul-
timarket contacts on collusion, and the critical mass result. Some technical proofs are contained in appendix.

2 Model

Two players simultaneously play $m$ infinitely repeated prisoners’ dilemmas. The $m$ stage games are identical, in each of which they choose from cooperation ($C$) and defection ($D$). After they chose their actions, a signal of the actions is publicly observed in each game. The signal is binary, good ($G$) or bad ($B$), and stochastic. In each stage game, the probability of a bad signal depends entirely on the action pair chosen in the stage game and is denoted by $\operatorname{Prob}(B|a_1 a_2)$, where $a_1 a_2$ is the action pair of this stage game. In particular, we denote $p = \operatorname{Prob}(B|CC)$ and $q = \operatorname{Prob}(B|CD) = \operatorname{Prob}(B|DC)$. We assume $0 < \operatorname{Prob}(B|a_1 a_2) < 1$ for any $a_1 a_2$, and $p < q$. Each player $i$ receives the stage payoff of $v_i(a_i y)$ in any stage game where he played $a_i \in \{C, D\}$ and observed $y \in \{G, B\}$.

We are more interested in the expected payoffs of the players, given their actions. Let us define $u_i(a_1 a_2) = \operatorname{Prob}(B|a_1 a_2)v_i(a_i B) + \{1 - \operatorname{Prob}(B|a_1 a_2)\}v_i(a_i G)$, which denotes player $i$’s expected stage payoff of an action pair $a_1 a_2$. We assume that the expected payoffs are represented by the following payoff matrices. We further assume $g > 0$, $l > 0$ and $g - l < 1$, so that each stage game is a prisoners’ dilemma. Note that the stage games are mutually independent, since both the payoff and the information structure are independent of the outcomes of the other games.

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<td>$D$</td>
<td>$1 + g, -l$</td>
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Stage Game 1

Stage Game $m$

The entire repeated game starts at $t = 0$. The information a player obtains within one period is all his actions and all signals of the $m$ prisoners’ dilemmas, together with a sunspot realized at the beginning of the period. The sunspot is a random variable from the uniform distribution on the interval $[0, 1]$, which is publicly observable. Thus player $i$’s history at period $t$ is a sequence of his past actions and signals in all past periods, as well as the sunspots in all past and the current periods. Therefore, player $i$’s history at period $t$ is an element of $\{(A_i)^m \times Y^m\}^t \times [0, 1]^{t+1}$, where $A_i = \{C, D\}$ and $Y = \{B, G\}$. The present paper deals with only pure strategies, and a pure strategy (simply a strategy in what follows) is a function mapping each history to a stage-action, an element of $(A_i)^m$. The repeated game payoff of each player is the average, discounted sum of all stage payoffs (the sum of the payoffs in all $m$ prisoners’ dilemmas) using a common discount factor $\delta \in (0, 1)$.

A player’s strategy is public if it does not explicitly depend on his own actions. Note that a player’s own past actions are the only private information of this game. A public strategy and a pair of public strategies are regarded as a function on the set of all public
histories, \( \bigcup_{t=0}^{\infty} \{ (Y^m)^t \times [0,1]^{t+1} \} \). A public strategy pair is strongly symmetric if it prescribes either CC in all games or DD in all games at any public history. In addition, a strongly symmetric public strategy pair is configuration independent if it depends only on the sequence of the sunspots and the numbers of bad signals in all past periods. Under a configuration independent strategy pair, which games produced bad signals is irrelevant. Our solution concept is Nash equilibrium by strongly symmetric and configuration independent public strategy pairs, which is hereafter called an “equilibrium” simply.\(^4\)

Note that a trivial equilibrium exists, where each player is prescribed to defect in all games at any history. Note also that the greatest payoff among all strongly symmetric Nash equilibria is attained by a strongly symmetric and configuration independent Nash equilibrium. In other words, configuration independence is not a real constraint. However, strong symmetry might be.

3 Characterizing the Most Cooperative Equilibrium Payoff

This section formulates a linear programming problem whose value corresponds to the most cooperative equilibrium payoff.\(^5\) Fix all parameters, \( g, l, p, q, \delta, \) and \( m \). Let \( v \) be the greatest payoff among all equilibria, and suppose \( v > 0 \). Then the corresponding equilibrium prescribes CC in all games in the initial period. For each \( r \in \{0, 1, \ldots, m\} \), let \( w_r \) be the expected continuation payoff from period 1 on when \( r \) bad signals are observed in period 0, where the expectation is taken with respect to the sunspot at the beginning of period 1. We have the following value equation.

\[
v = (1 - \delta) m + \delta \sum_{r=0}^{m} f_r^m(0) w_r, \tag{1}
\]

where \( f_r^m(k) \) is the probability of \( r \) bad signals when there are \( m \) games in total, one player chooses D in \( k \) games, and the other player chooses C in all \( m \) games.\(^6\)

Since this is an equilibrium, we have

\[
v \geq (1 - \delta)(m + kg) + \delta \sum_{r=0}^{m} f_r^m(k) w_r, \quad \forall k \geq 1
\]

from which and (1) we obtain

\[
(1 - \delta) g \leq \delta \sum_{r=0}^{m} \frac{f_r^m(0) - f_r^m(k)}{k} w_r, \quad \forall k \geq 1 \tag{2}
\]

\(^4\) We do not explicitly impose sequential rationality. The equilibria we examine in the subsequent analysis are sequential equilibria with suitably defined systems of beliefs.

\(^5\) This approach follows Abreu, Milgrom, and Pearce [1].

\(^6\) Since all stage games are identical and since the other player’s actions are the same, it does not matter in which \( k \) games the player chooses D.
Further, since this attains the greatest equilibrium payoff, we have

\[ 0 \leq w_r \leq v. \quad \forall r \]  

(3)

Indeed, finding the greatest equilibrium payoff corresponds to solving the following maximization problem.

\[ \max_{v, w_0, \ldots, w_m} v \quad \text{subject to (1), (2) and (3)} \]

If the value of the problem is positive, this is the greatest equilibrium payoff. If the value is nonpositive or if the problem is infeasible, the greatest equilibrium payoff is zero. In the latter case, the trivial equilibrium where \( DD \) is played whenever and wherever is the most cooperative equilibrium. The results hold because any \( (v, w_0, \ldots, w_m) \) satisfying the constraints (1)–(3) has a corresponding equilibrium such that

- the play is either in the reward state where \( CC \) is played in all games, or in the punishment state where \( DD \) is played in all games,
- the initial state is the reward state,
- when the play is in the reward state and \( r \) bad signals are observed, the sunspot in the next period induces a shift to the punishment state with probability \( 1 - (w_r/v) \), and
- the punishment state is absorbing.

We provide a series of results about structure of the most cooperative equilibrium.

**Proposition 1** Let \( \hat{r} \) be the greatest integer not exceeding \( mp \). Any solution of the above maximization problem satisfies

\[ v = w_0 = \cdots = w_{\hat{r}}. \]

(4)

**Proof.** Note first that

\[ f^{m-1}_r(0) = \frac{(m-1)!}{r!(m-1-r)!} p^r(1-p)^{m-1-r}, \]

from which we obtain

\[ \frac{f^{m-1}_r(0)}{f^{m-1}_{r-1}(0)} = \frac{(m-r)p}{r(1-p)} \]

for any \( r \geq 1 \). Therefore, \( f^{m-1}_r(0) \geq f^{m-1}_{r-1}(0) \) for any \( r \leq \hat{r} \). Combining this with Lemma 2 in Appendix A, we have

\[ \frac{f^{m-1}_r(m-1)}{f^{m-1}_{r-1}(m-1)} > \frac{f^{m-1}_r(m-2)}{f^{m-1}_{r-1}(m-2)} > \cdots > \frac{f^{m-1}_r(0)}{f^{m-1}_{r-1}(0)} \geq 1. \]

Hence \( f^{m-1}_r(k) > f^{m-1}_{r-1}(k) \) for any \( r \leq \hat{r} \) and \( k \geq 1 \).
Next, for any \( k \in \{0, \ldots, m - 1\} \) and any \( r \leq \hat{r} \), it follows that
\[
f_r^m(k + 1) = q f_{r-1}^m(k) + (1 - q) f_{r-1}^m(k), \quad f_r^m(k) = pf_{r-1}^m(k) + (1 - p) f_{r-1}^m(k),
\]
where the convention is that \( f_{m-1}^m(k) = f_{m-1}^m(k) = 0 \). These imply that
\[
f_r^m(k) - f_r^m(k + 1) = (q - p)\{f_{r-1}^m(k) - f_{r-1}^m(k)\}. \tag{5}
\]
From the previous result and from \( q > p \), \( f_r^m(k) > f_r^m(k + 1) \) for any \( k \geq 1 \) and \( f_r^m(0) \geq f_r^m(1) \) follow. As a result,
\[
f_r^m(0) \geq f_r^m(1) > f_r^m(2) > \cdots > f_r^m(m), \quad \forall r \leq \hat{r} \tag{6}
\]
Fix \((v, w_0, \ldots, w_m)\) satisfying all constraints (1)--(3), and suppose \( w_r < v \) for some \( r \leq \hat{r} \). For \( \varepsilon > 0 \), let us define \((v', w_0', \ldots, w_m')\) as follows.
\[
v' = v + \delta f_r^m(0) \varepsilon, \quad w_k' = \begin{cases} w_k & \text{if } k \neq r \\ w_r + \varepsilon & \text{if } k = r \end{cases}
\]
It is easy to verify that \((v', w_0', \ldots, w_m')\) satisfies (1). Since \( w_r < v \), (3) also holds for sufficiently small \( \varepsilon > 0 \). Since \( f_r^m(0) \geq f_r^m(k) \) for any \( k \) from (6), (2) also holds. This establishes that \((v', w_0', \ldots, w_m')\) is feasible and attains a greater value than \((v, w_0, \ldots, w_m)\). Hence at the solution, (4) must hold.

Q.E.D.

A key to this result is that \( k = 0 \) maximizes \( f_r^m(k) \) for any \( r \leq \hat{r} \). Thus, no deviation is most likely when the number of bad signals is small. In order to provide incentives effectively, one should not punish the players when there are less bad signals.

**Proposition 2** At any solution, if \( w_r < v \) for some \( r, w_{r+1} = 0 \).

**Proof.** Suppose otherwise. That is, a solution \((v, w_0, \ldots, w_m)\) exists such that \( w_r < v \) and \( w_{r+1} > 0 \) for some \( r \). From Lemma 2 of Appendix A, there exists \( \eta > 0 \) such that
\[
f_{r+1}^m(k) > f_r^m(0) \left\{ 1 + \eta - \eta \frac{f_r^m(0)}{f_r^m(k)} \right\} \tag{7}
\]
for any \( k \geq 1 \).

For \( \varepsilon > 0 \), let us define \((v', w_0', \ldots, w_m')\) as follows.
\[
v' = v + \delta \eta \varepsilon f_{r+1}^m(0) f_r^m(0), \quad w'_r = w_r, \quad \forall r' \notin \{r, r + 1\}
\]
\[
w'_r = w_r + f_{r+1}^m(0)(1 + \eta) \varepsilon, \quad w'_{r+1} = w_{r+1} - f_r^m(0) \varepsilon.
\]
Since \( w_r < v \) and \( w_{r+1} > 0 \), \((v', w_0', \ldots, w_m')\) satisfies (3) for sufficiently small \( \varepsilon > 0 \). It is
easy to see that it also satisfies (1). Further, for any \( k \geq 1 \),
\[
\{ f^m_r(0) - f^m_r(k) \} f^m_{r+1}(0)(1 + \eta) \varepsilon - \{ f^m_{r+1}(0) - f^m_r(k) \} f^m_r(0) \varepsilon
\]
\[
= \varepsilon \left[ f^m_{r+1}(k) f^m_r(0) - f^m_r(k) f^m_{r+1}(0) + \{ f^m_r(0) - f^m_r(k) \} f^m_r(0) \eta \right]
\]
\[
= \varepsilon f^m_r(k) f^m_r(0) \left[ \frac{f^m_{r+1}(k)}{f^m_r(k)} - \frac{f^m_{r+1}(0)}{f^m_r(0)} \{ 1 + \eta - \eta f^m_r(0) \} \right] > 0,
\]
where the inequality is due to (7). Hence, \( (v', w'_0, \ldots, w'_m) \) satisfies (2).

To sum up, \( (v', w'_0, \ldots, w'_m) \) satisfies all constraints and we have \( v' > v \). This is a contradiction against optimality of \( (v, w_0, \ldots, w_m) \), which completes the proof. Q.E.D.

Propositions 1 and 2 reveal that the solution of the maximization problem has simple structure. First, we have \( v = w_0 \) from Proposition 1, because \( \hat{r} \geq 0 \). Second, \( v = w_0 = \cdots = w_m \) is not a solution because it violates (2). Combining these observations with Proposition 2, we see that there exists \( r^* \geq 1 \) such that the solution satisfies
\[
v = w_0 = \cdots = w_{r^*-1} > w_{r^*} = w_{r^*+1} = \cdots = w_m = 0. \tag{8}
\]

It is possible that \( r^* = m \). It is also possible that \( w_{r^*} = 0 \).

One important implication of (8) is that the solution of the maximization problem is, if it exists, unique. This is because any two distinct vectors satisfying (8) must have different \( v \)'s, in order to be consistent with (1).

From (8), we see that the most cooperative equilibrium has simple structure with a cutoff \( r^* \). That is, (i) the players are not punished at all if the number of bad signals is less than the cutoff, and (ii) they are maximally punished if there are more than \( r^* \) bad signals; namely, they revert to repeated play of the static equilibrium. When there are exactly \( r^* \) bad signals, they are in general punished partially. The sunspot adjusts size of the punishment.

**Proposition 3** Fix \( (v, w_0, \ldots, w_m) \) such that (8) holds for \( r^* \geq 1 \). If it satisfies the “first” and the “last” incentive conditions, namely,
\[
(1 - \delta) g \leq \delta \sum_{r=0}^{m} \{ f^m_r(0) - f^m_r(1) \} w_r, \tag{9}
\]
\[
(1 - \delta) g \leq \delta \sum_{r=0}^{m} \frac{f^m_r(0) - f^m_r(m)}{m} w_r, \tag{10}
\]
then \( (v, w_0, \ldots, w_m) \) satisfies (2) for any \( k \).

**Proof.** Since there is nothing to prove in case of \( m = 2 \), let us assume \( m \geq 3 \).

Fix \( (v, w_0, \ldots, w_m) \) such that (8) holds for \( r^* \geq 1 \). For \( k \in \{0, 1, \ldots, m\} \), define \( w(k) \) as
\[
w(k) = (1 - \delta)(m + kg) + \delta \sum_{r=0}^{m} f^m_r(k) w_r.
\]
holds. From (11), we therefore obtain

\[ w(k) - w(k-1) - (1 - \delta)g \]

by the definition of \( ^{\hat{}} \), where the second equality follows from (5).

Any \( k > 1 \), it follows that

\[ w(k + 1) - w(k) = \delta \sum_{r=0}^{m-1} \{ f_r^m(k) - f_r^m(k-1) \} w_r \]

implies that for any \( w \in \mathbb{R} \), where the second equality is due to (8) and the third one is due to (5). This in turn

implies that for any \( k \in \{1, \ldots, m-1\} \),

\[ w(k + 1) - w(k) = \delta \sum_{r=0}^{m-1} \{ f_r^m(k) - f_r^m(k-1) \} w_r \]

where the second equality is due to (8) and the third one is due to (5). This in turn implies that for any \( k \in \{1, \ldots, m-1\} \),

\[ w(k + 1) - w(k) = \delta \sum_{r=0}^{m-1} \{ f_r^m(k) - f_r^m(k-1) \} w_r \]

where the second equality follows from (5).

Let \( \hat{k} \) be the smallest \( k \) such that \( w(k+1) > w(k) \). If no such \( k \) exists, define \( \hat{k} = m \).

By the definition of \( \hat{k} \), we have \( w(\hat{k}) \leq w(\hat{k}-1) \). Hence, \( w(k+1) - w(\hat{k}) > w(\hat{k}) - w(\hat{k}-1) \)

holds. From (11), we therefore obtain

\[ \left\{ 1 - \frac{f_r^{m-2}(\hat{k} - 1)}{f_r^{m-2}(\hat{k} - 1)} \right\} (v - w_{r\cdot}) + \left\{ \frac{f_r^{m-2}(\hat{k} - 1)}{f_r^{m-2}(\hat{k} - 1)} - 1 \right\} w_{r\cdot} > 0. \]

From Lemma 2 in Appendix A, any \( k > \hat{k} \) satisfies

\[ \left\{ 1 - \frac{f_r^{m-2}(k - 1)}{f_r^{m-2}(k - 1)} \right\} (v - w_{r\cdot}) + \left\{ \frac{f_r^{m-2}(k - 1)}{f_r^{m-2}(k - 1)} - 1 \right\} w_{r\cdot} > 0, \]

which implies

\[ w(k + 1) - w(k) > w(k) - w(k-1). \]

Applying (12) iteratively, we have \( w(k+1) > w(k) \). Therefore, the necessary and sufficient condition for \( w(k+1) > w(k) \) is \( k \geq \hat{k} \).
For any $k < \hat{k}$, we have

$$w(k) - w(0) = \sum_{k' = 1}^{k} \{w(k') - w(k'-1)\} \leq w(1) - w(0),$$

where the inequality follows because $w(k') \leq w(k'-1)$ for any $k' \leq k < \hat{k}$. Consequently, if (9) holds, $w(k) \leq w(0)$ for any $k < \hat{k}$.

Next, for any $k \geq \hat{k}$, we have

$$w(k) - w(0) = \sum_{k' = 1}^{k} \{w(k') - w(k'-1)\}$$

$$\leq \sum_{k' = 1}^{k} \{w(k') - w(k'-1)\} + \sum_{k' = k+1}^{m} \{w(k') - w(k'-1)\}$$

$$= w(m) - w(0),$$

where the inequality follows because $w(k') > w(k'-1)$ for any $k' \geq k + 1 > \hat{k}$. Consequently, if (10) holds, $w(k) \leq w(0)$ for any $k \geq \hat{k}$. Hence all incentive conditions are satisfied. Q.E.D.

In any most cooperative equilibrium, the binding incentive condition is either for a deviation that defects in a single game, or for a deviation that defects in all games. Any deviation that chooses defection in an intermediate number of games is not more profitable than one of the above two deviations. Proposition 3 generalizes existing results in two senses. First, Kobayashi and Ohta [5] explicitly derives the most cooperative equilibrium for any number of games, assuming that the players are sufficiently patient. Under the equilibrium, the most profitable deviation is the one that defects in a single market. Our result generalizes their result to the case of less patient players. Second, Matsushima [8] shows that even if the players are quite impatient, there exists an equilibrium whose per-game payoff is arbitrarily close to the payoff of mutual cooperation (namely, 1), if the number of markets goes to infinity. [8] constructs a review strategy equilibrium, a special instance of the strategies with a cutoff, and shows that the most profitable deviation is either a deviation in all games or a deviation in a single market. Proposition 3 generalizes this observation to any strategy with a cutoff.

Thanks to Proposition 3, we can simplify the maximization problem which derives the most cooperative equilibrium payoff, by omitting irrelevant incentive constraints. More concretely, it suffices to solve

$$\max_{v,w_0,\ldots,w_m} v \quad \text{subject to (1), (3), (9) and (10)}.$$ 

In the next section, we work with this version.
4 Main Results

This section provides three main results of this paper, all of which are an application of the methodology provided in the previous section. The following three subsections correspond to the results respectively; comparative statics with respect to discounting, positive effects of multimarket contacts on collusion, and the critical mass result.

4.1 Patience

This subsection fixes all parameters on the payoffs and monitoring of the stage games, \( g, l, p, \) and \( q, \) together with the number of prisoners’ dilemmas \( m. \) The following preliminary result is useful, because it gives an upper bound on the value of the maximization problem we delineated in the previous section which uniformly applies to any level of discounting. If the upper bound is nonpositive, the maximization problem is always infeasible. If the upper bound is positive, it is indeed a tight bound in the sense that it coincides with the maximum when \( \delta \) sufficiently close to 1. Further, we provide a condition under which the maximization problem is bang-bang. Namely, a threshold discount factor exists such that any maximization problem with \( \delta \) clearing the threshold attains the upper bound and any maximization problem with \( \delta \) below the threshold is infeasible.

**Proposition 4** Let us define

\[
 v^* = m - \frac{pg}{q-p}.
\]

Then it follows that

(i) the value of the maximization problem does not exceed \( v^* \) for any \( \delta, \)

(ii) if \( v^* \leq 0, \) the maximization problem is infeasible for any \( \delta, \) and

(iii) if \( v^* > 0, \) the value of the maximization problem is \( v^* \) for any \( \delta \geq \delta, \)

where

\[
 \delta = \frac{g}{p^{m-1}(q-p)v^* + g} = \frac{g}{p^{m-1}(q-p)m - pg + g}.
\]

Furthermore, if \((1-p)v^* \leq 1\) additionally holds, then the maximization problem is infeasible for any \( \delta < \delta. \)

**Proof.** Fix \( \delta, \) and let \((v, w_0, w_1, \ldots, w_m)\) be a solution of the maximization problem. It must satisfy (8) for some \( r^* \geq 1. \) Define \( \rho = 1 - (w_{r^*}/v). \) Note that \( \rho \in (0, 1]. \) Now (1) and (9) reduce to

\[
 v = (1 - \delta)m + \delta \left\{ 1 - \sum_{r=r^*+1}^{m} f_r^m(0) - \rho f_r^{r^*}(0) \right\} v, \tag{13}
\]

\[
 (1 - \delta)g \leq \delta \left[ \sum_{r=r^*+1}^{m} \left\{ f_r^m(1) - f_r^m(0) \right\} + \rho \left\{ f_r^{r^*}(1) - f_r^{r^*}(0) \right\} \right] v. \tag{14}
\]
Using (14), we can rearrange (13) as follows.

\[ v = m - \frac{\delta}{1 - \delta} \left\{ \sum_{r=r^*+1}^{m} f_r^m(0) + \rho f_r^m(0) \right\} v \leq m - \frac{\sum_{r=r^*+1}^{m} f_r^m(0) + \rho f_r^m(0)}{\sum_{r=r^*+1}^{m} \{ f_r^m(1) - f_r^m(0) \} + \rho \{ f_r^m(1) - f_r^m(0) \}} g. \] (15)

From Lemma 2 in Appendix A,

\[ \frac{f_r^m(0)}{f_r^m(1) - f_r^m(0)} > \frac{f_m^m(0)}{f_m^m(1) - f_m^m(0)} \]

for any \( r < m \). Substituting this into (15), we obtain

\[ v \leq m - \frac{f_m^m(0)}{f_m^m(1) - f_m^m(0)} g = m - \frac{p}{q - p} g = v^*, \]

which proves part (i).

To prove part (ii), note that the maximization problem either has a positive value or is infeasible. This is because \( v = 0 \) requires \( w_r = 0 \) for any \( r \) by (3), which violates the incentive conditions. Hence, if \( v^* \leq 0 \), it is infeasible.

Finally, in order to prove part (iii), let us first fix \( \delta \geq \delta \) (note that \( \delta \in (0, 1) \) because \( v^* > 0 \)). Define \( (v, w_0, \ldots, w_m) \) as follows.

\[ v = w_0 = \cdots = w_{m-1} = v^*, \quad w_m = v^* - \frac{(1 - \delta)g}{\delta(q - p)p^{m-1}}. \]

Part (i) implies that the value of the maximization problem with \( \delta \) is \( v^* \) if \( (v, w_0, \ldots, w_m) \) satisfies all constraints.

It is easy to verify that (1) and (9) hold. Since \( \delta \geq \delta \),

\[ \frac{(1 - \delta)g}{\delta(q - p)p^{m-1}} \leq \frac{p^{m-1}(q - p)v^* g}{g(q - p)p^{m-1}} = v^*. \]

This verifies (3). As for (10), \( q > p \) implies

\[ \frac{f_m^m(m) - f_m^m(0)}{m} (v - w_m) = \frac{q^m - p^m}{m} (v - w_m) \geq (q - p)p^{m-1}(v - w_m) = \frac{1 - \delta}{\delta} g, \]

where the last equality follows from (9). This establishes (10), as desired.

Next, fix \( \delta < \delta \) and assume \( (1 - p)v^* \leq 1 \). Suppose that the maximization problem with \( \delta \) has a value \( v > 0 \), so that there exist \( r^* \geq 1 \) and \( \rho \in (0, 1) \) such that (13) and (14) hold. Solving (13) for \( v \), substituting it into (14), and rearranging, we obtain

\[ \delta \geq \frac{g}{\{ \Phi(1) - \Phi(0) \} m + \{ 1 - \Phi(0) \} g}, \] (16)
where
\[ \Phi(k) = \sum_{r=r^*+1}^{m} f^m_r(k) + \rho f^m_r(0), \quad k \in \{0, 1\}. \]

Note that
\[ \{\Phi(1) - \Phi(0)\} m - \Phi(0)g = \sum_{r=r^*+1}^{m} \left[ \{f^m_r(1) - f^m_r(0)\} m - f^m_r(0)g \right] \]
\[ + \rho \left[ \{f^m_r(1) - f^m_r(0)\} m - f^m_r(0)g \right]. \tag{17} \]

We also have
\[ \left\{ f^m_m(1) - f^m_m(0) \right\} m - f^m_m(0)g = p^{m-1}(q - p)m - p^m g = p^{m-1}(q - p)v^* > 0. \tag{18} \]

Further, since we have
\[ f^m_{m-1}(0) = mp^{m-1}(1 - p), \quad f^m_{m-1}(1) = (1 - q)p^{m-1} + (m - 1)qp^{m-2}(1 - p), \]
it follows that
\[ \left\{ f^m_{m-1}(1) - f^m_{m-1}(0) \right\} m - f^m_{m-1}(0)g = mp^{m-2} \left[ (q - p)\{m(1 - p) - 1\} - p(1 - p)g \right] \]
\[ = mp^{m-2}(q - p)\{(1 - p)v^* - 1\} \leq 0, \tag{19} \]
where the inequality follows from \((1 - p)v^* \leq 1\). It also follows that for any \(r < m - 1\),
\[ \left\{ f^m_r(1) - f^m_r(0) \right\} m - f^m_r(0)g < f^m_r(0) \left\{ \frac{f^m_{m-1}(1) - f^m_{m-1}(0)}{f^m_{m-1}(0)} \right\} m - g \leq 0, \tag{20} \]
where the first inequality follows from Lemma 2 and the second one follows from (19).

Applying (18)–(20) to (17), we obtain
\[ \{\Phi(1) - \Phi(0)\} m - \Phi(0)g \leq \{f^m_m(1) - f^m_m(0)\} m - f^m_m(0)g = p^{m-1}(q - p)v^*. \]

Substituting this into (16) establishes that \(\delta \geq \delta\), a contradiction. This proves part (iii), and the proof is complete. \(\Box\)

It is helpful to restate Proposition 4 in terms of the most cooperative equilibrium.

**Proposition 5** Let \(v^*\) be as defined in Proposition 4. Then it follows that

(i) no equilibrium payoff exceeds \(v^*\) for any \(\delta\),

(ii) if \(v^* \leq 0\), the most cooperative equilibrium payoff is zero for any \(\delta\), and

(iii) if \(v^* > 0\), the most cooperative equilibrium payoff is \(v^*\) for any \(\delta \geq \delta\). Furthermore, if \((1 - p)v^* \leq 1\) additionally holds, then the most cooperative equilibrium payoff is zero for any \(\delta < \delta\).
The value $v^*$ is first identified by Abreu, Milgrom, and Pearce [1] for the case $m = 1$ and then by Kobayashi and Ohta [5] for any $m$. Indeed, the first part of Proposition 5(iii) just restates a main result of [5]. A novel result is its second part, a bang-bang property under some condition. Note that $(1 - p)v^* < 1$ always follows if $m = 1$. Therefore, any repeated single prisoners’ dilemma exhibits the bang-bang property.

We proceed to a comparative statics result with respect to the discount factor.

**Proposition 6** Suppose that the maximization problem has a value $v \in (0, v^*)$ at $\delta$. Then for any $\delta' > \delta$, the value of the maximization problem at $\delta'$ is greater than $v$.

**Proof.** Let $(v, w_0, \ldots, w_m)$ be the solution of the maximization problem at $\delta$. Consider the maximization problem at $\delta'$, and define $(v', w'_0, \ldots, w'_m)$ as follows.

$$v' = v, \quad w'_r = \frac{\delta(1 - \delta')}{(1 - \delta)\delta'} w_r + \frac{\delta' - \delta}{(1 - \delta)\delta'} v \quad \forall r$$

It is easy to verify that $(v', w'_0, \ldots, w'_m)$ satisfies all constraints. Moreover, since $v' = v < v^*$, $w_r < v$ for some $r \leq m - 1$. This implies that $w'_r < v$, and from the definition of $w'_m$, $w'_m > 0$. By Proposition 2, $(v', w'_0, \ldots, w'_m)$ is, though feasible, not a solution of the maximization problem at $\delta'$. Hence the value of the maximization problem at $\delta'$ is greater than $v$. Q.E.D.

From continuity and Proposition 6, we see that there exists a minimum discount factor $\delta'$ such that the maximization problem is feasible, and the value is strictly increasing in $\delta$ on the range $[\delta'; \delta]$. In other words, when the players can sustain a nonzero level of cooperation but cannot attain the level when they are sufficiently patient (namely, $v^*$), more patience allows more cooperation.

### 4.2 Multimarket Contact

This subsection fixes all parameters on the payoffs and monitoring of the stage games, $g$, $l$, $p$, and $q$, together with the discount factor $\delta$. The following result first shows that the most cooperative equilibrium per-game payoff weakly increases if the number of games increases by one. It also shows that the per-game payoff remains unchanged only if a very restrictive condition holds. In what follows, we fix the stage game, and let $v(m, \delta)$ be the most cooperative equilibrium payoff under $m$ games and the discount factor $\delta$.

**Proposition 7** For any $m$ and any $\delta$, we have $\frac{v(m+1, \delta)}{m+1} \geq \frac{v(m, \delta)}{m}$. Furthermore, if the inequality holds with equality and if $v(m, \delta) > 0$, the maximization problem under $m$ and $\delta$ has a solution such that for some $r^*$,$$
\begin{align*}
v = w_0 = \cdots = w_{r^*-1} > w_{r^*} = w_{r^*+1} = \cdots = w_m = 0.
\end{align*}
$$

---

<sup>7</sup>Precisely speaking, [5] assumes that $v^* > 0$ for $m = 1$ holds, whereas we do not need such a stronger assumption.

<sup>8</sup>This observation for $m = 1$ is already shown by [6].
Proof. There is nothing to prove if \( v(m, \delta) = 0 \), so suppose \( v(m, \delta) > 0 \). Let \( r^* \) be such that the solution of the maximization problem under \( m \) and \( \delta \) satisfies

\[
v(m, \delta) = v = w_0 = \cdots = w_{r^*-1} > w_{r^*} \geq w_{r^*+1} = \cdots = w_m = 0.
\]

(22)

Consider the maximization problem under \( m+1 \) and \( \delta \), and define \((\hat{v}, \hat{w}_0, \ldots, \hat{w}_{m+1})\) as follows.

\[
\hat{v} = \frac{m+1}{m} v(m, \delta), \quad \hat{w}_r = \frac{r}{m} w_{r-1} + \frac{m-r+1}{m} w_r, \quad \forall r \in \{0, 1, \ldots, m+1\}
\]

where \( w_{-1} = w_{m+1} = 0 \). Some calculations verify that \((\hat{v}, \hat{w}_0, \ldots, \hat{w}_{m+1})\) satisfies all constraints. This proves that \( v(m+1, \delta) \geq \hat{v} = \frac{m+1}{m} v(m, \delta) \), which in turn proves the first part.

To prove the second part, suppose \( v(m+1, \delta) = \frac{m+1}{m} v(m, \delta) \). Then \((\hat{v}, \hat{w}_0, \ldots, \hat{w}_{m+1})\) solves the maximization problem under \( m+1 \) and \( \delta \). From (22) and the definition of \( \hat{w}_{r^*} \), we have

\[
\hat{w}_{r^*} < \frac{r^*}{m} v(m, \delta) + \frac{m-r^*+1}{m} v(m, \delta) = \hat{v}.
\]

Hence, \( \hat{w}_{r^*+1} = 0 \) follows from Proposition 2. This and the definition of \( \hat{w}_{r^*} \) imply \( w_{r^*} = 0 \), as desired.

Q.E.D.

We call the condition (21) the no-sunspot condition. If (21) holds, then the most cooperative equilibrium under \( m \) and \( \delta \) never punishes less than \( r^* \) bad signals and surely punishes \( r^* \) or more bad signals. Since there is no randomness, the players can play this equilibrium even in the absence of sunspots. Proposition 7 implies that unless the no-sunspot condition holds, adding one game strictly improves the most cooperative equilibrium per-game payoff.

Note that the no-sunspot condition rarely holds, because \( w_{r^*} > 0 \) is a generic property with respect to \( \delta \). As a result, given the stage game and the number of games, the no-sunspot condition holds only for finitely many discount factors. Proposition 7 thus implies that adding one game almost always improves the most cooperative equilibrium per-game payoff, which establishes that multimarket contact facilitates collusion.

To see the crux of Proposition 7, suppose the most cooperative equilibrium under \( m \) and \( \delta \) has a payoff \( v > 0 \), which never punishes less than \( r^* \) bad signals, surely punishes \( r^* \) or more bad signals, and punish \( r^* \) bad signals with probability \( (v - w_{r^*})/v \). Now consider the following trigger-type strategy pair under \( m+1 \) and \( \delta \). When \( r \) bad signals are observed, pick any signal with equal probability. Then, count the number of bad signals among the remaining \( m \) signals. If the number is less than \( r^* \), the players are not punished. If it exceeds \( r^* \), they are surely punished. If the number is \( r^* \), they are punished with probability \( (v - w_{r^*})/v \). Then, \((\hat{v}, \hat{w}_0, \ldots, \hat{w}_{m+1})\) in the proof are exactly the payoff and the continuation payoffs of this strategy pair. The new strategy pair translates the play of the original equilibrium under \( m \) games to the case of \( m+1 \) games, and for the same reason that the original strategy pair is an equilibrium, the new pair is an equilibrium with the same per-game payoff.

If the no-sunspot condition happens to hold, having one more game may not improve
the most cooperative equilibrium per-game payoff. In this case, \((\hat{v}, \hat{w}_0, \ldots, \hat{w}_{m+1})\) solves the maximization problem under \(m + 1\) and \(\delta\), and therefore the above strategy pair is the most cooperative equilibrium. Since the solution does not satisfy the no-sunspot condition (note that the proof implies that \(0 < \hat{w}_r < \hat{v}\)), having another game increases the most cooperative equilibrium per-game payoff. In other words, adding two or more games always improves the most cooperative equilibrium per-game payoff, unless it is zero. The following result summarizes this observation.

**Proposition 8** Suppose \(v(m, \delta) > 0\). Then for any \(m' \geq 2\), we have \(\frac{v(m+m', \delta)}{m+m'} > \frac{v(m, \delta)}{m}\).

Proposition 7 also implies that the range of discount factors under which a positive payoff can be sustained never shrinks if \(m\) increases. In fact, in many cases, the range strictly increases (in the sense of set inclusion) when one game is added.

**4.3 A Critical Mass Result**

This subsection fixes only the number of games \(m\), which is interpreted as the status quo of an oligopolistic industry. Now we ask the following question: to what extent adding one game can have an impact on the most cooperative equilibrium payoff? This corresponds to exploring a worst case scenario to an antitrust authority about allowing one more market to an oligopoly. The following critical mass result is a sharp answer to that question.

**Proposition 9** Fix \(m\) and \(\varepsilon > 0\) arbitrarily. Then there exist a prisoners’ dilemma game (namely, parameters \(p, q, g,\) and \(l\)) and a discount factor \(\delta\) such that

(i) if the number of games is \(m\) or less, any equilibrium payoff is zero, and

(ii) if the number of games is \(m + 1\), an equilibrium payoff exceeds \(m + 1 - \varepsilon\).

**Proof.** See Appendix B. Q.E.D.

Proposition 9 implies that, for any given number of markets \(m\), there exists a scenario where \(m + 1\) markets form a critical mass. Hence, the firms who have reached it would suddenly attain a very high level of collusion, despite that they were quite hopeless for collusion so far. This result thus serves as a caution to antitrust authorities.

Let us compare the critical mass result with the result by Kobayashi and Ohta [5]. Fix \(m\) and \(\varepsilon \in (0, 1)\), and fix a stage game under which the critical mass result holds at some \(\delta\). Since \(\varepsilon < 1\), [5] shows that for any \(m' \leq m\), the most cooperative equilibrium payoff is greater than \(m' - \varepsilon\) if the players are sufficiently patient. Therefore, the critical mass result applies to players who are not excessively patient.

It is interesting that when the critical mass result applies, the minimum discount factor which sustains \(v'\), the payoff identified by Kobayashi and Ohta [5], is decreasing in the number of games until it reaches the critical mass. This is an environment where adding one game is doubly beneficial in the sense that it improves both the most cooperative per-game payoff and the required level of patience. This “win-win” phenomenon is valid.
only when \( m \) is small. In fact, \( \hat{\delta} \) in Proposition 7 converges to one if \( m \) goes to infinity.

A key to the critical mass result is that at the minimum discount factor which sustains the payoff by [5] with \( m+1 \) games, no cooperation is sustainable with \( m \) or less games because of the win-win property.

A Appendix: Some Lemmas and Their Proofs

Recall that \( f^m_r(k) \) is the probability of \( r \) bad signals when one player chooses \( D \) in \( k \) games out of \( m \) games while the other player chooses \( C \) in all \( m \) games.

**Lemma 1** For any \( m \), any \( k \in \{0, 1, \ldots, m\} \) and any \( r \in \{1, \ldots, m-1\} \), we have

\[
\frac{f^m_{r+1}(k)}{f^m_r(k)} < \frac{f^m_r(k)}{f^m_{r-1}(k)}. \tag{23}
\]

**Proof.** The proof is by induction for \( m \). If \( m = 1 \), there is nothing to prove and the result holds.

Suppose that for some \( m \geq 2 \), we have

\[
\frac{f^{m-1}_{r+1}(k)}{f^{m-1}_r(k)} < \frac{f^{m-1}_r(k)}{f^{m-1}_{r-1}(k)} \tag{24}
\]

for any \( k \in \{0, 1, \ldots, m-1\} \) and any \( r \in \{0, \ldots, m-1\} \). Here, the convention is that \( f^m_0(k) = f^m_{m-1}(k) = 0 \). For such \( k \) and \( r \), we also have

\[
f^m_r(k) = pf^m_{r-1}(k) + (1-p)f^m_{r-1}(k), \quad f^m_{r+1}(k) = pf^m_{r-1}(k) + (1-p)f^m_{r+1}(k). \]

It thus follows that

\[
\frac{f^m_{r+1}(k)}{f^m_r(k)} = \frac{pf^m_{r-1}(k) + (1-p)f^m_{r+1}(k)}{pf^m_{r-1}(k) + (1-p)f^m_{r+1}(k)}.
\]

From (24) and \( p \in (0, 1) \),

\[
\frac{f^{m-1}_{r+1}(k)}{f^{m-1}_r(k)} < \frac{f^{m-1}_r(k)}{f^{m-1}_{r-1}(k)} < \frac{f^{m-1}_r(k)}{f^{m-1}_{r-1}(k)} \tag{25}
\]

holds. In case of \( r \geq 1 \), we further have

\[
\frac{f^{m-1}_r(k)}{f^{m-1}_{r-1}(k)} < \frac{f^m_r(k)}{f^m_{r-1}(k)} < \frac{f^{m-1}_r(k)}{f^{m-1}_{r-2}(k)} \tag{26}
\]

From (25) and (26), we obtain (23).

The remaining case is \( k = m \). For any \( r \in \{0, \ldots, m-1\} \),

\[
f^m_r(m) = qf^m_{r-1}(m-1) + (1-q)f^m_{r}(m-1),
\]

\[
f^m_{r+1}(m) = qf^m_{r-1}(m-1) + (1-q)f^m_{r+1}(m-1)
\]
hold, which implies that

\[
\frac{f_{r+1}^m(m)}{f_r^m(m)} = \frac{q f_r^{m-1}(m-1) + (1 - q) f_{r+1}^{m-1}(m-1)}{q f_{r-1}^{m-1}(m-1) + (1 - q) f_r^{m-1}(m-1)}.
\]

From (24) and \( q \in (0, 1) \), we obtain (25). In case of \( r \geq 1 \), we also have (26). Combining these, we obtain (23). Since all cases are considered, the proof is complete. Q.E.D.

**Lemma 2** For any \( m \), any \( k \in \{0, 1, \ldots, m - 1\} \) and any \( r \in \{0, \ldots, m - 1\} \), we have

\[
\frac{f_{r+1}^m(k+1)}{f_r^m(k+1)} > \frac{f_{r+1}^m(k)}{f_r^m(k)}.
\]

**Proof.** For any \( m \), any \( k \in \{0, 1, \ldots, m - 1\} \) and any \( r \in \{0, \ldots, m - 1\} \), we have

\[
\frac{f_{r+1}^m(k)}{f_r^m(k)} = \frac{pf_r^{m-1}(k) + (1 - p) f_{r+1}^{m-1}(k)}{pf_{r-1}^{m-1}(k) + (1 - p) f_r^{m-1}(k)},
\]

\[
\frac{f_{r+1}^m(k+1)}{f_r^m(k+1)} = \frac{q f_r^{m-1}(k) + (1 - q) f_{r+1}^{m-1}(k)}{q f_{r-1}^{m-1}(k) + (1 - q) f_r^{m-1}(k)}.
\]

Since \( \frac{f_{r-1}^m(k)}{f_{r-1}^{m-1}(k)} > \frac{f_{r+1}^m(k)}{f_{r+1}^{m-1}(k)} \) from Lemma 1, we have (27) from \( q > p \). Q.E.D.

## B Appendix: Proof of Proposition 9

Fix \( m \geq 1 \) and \( \varepsilon > 0 \). It suffices to consider the case \( \varepsilon < 1 \). There exist \( p \in (0, 1) \) and \( g > 0 \) such that

\[
\frac{2pg}{1-p} < \varepsilon, \quad m < \frac{2g + 1}{1-p}. \tag{28, 29}
\]

Given these \( p \) and \( g \), let us set \( g = (1 + p)/2 \) and \( l = g \). Then (28) and (29) reduce to

\[
\frac{pg}{q-p} < \varepsilon < 1, \quad m < \frac{pg}{q-p} + \frac{p}{1-p}. \tag{30, 31}
\]

Finally, we set

\[
\delta = \frac{g}{p^m \{(q-p)(m+1)-pg\} + g}.
\]

Note that the definition of \( \delta \) implies

\[
\delta < \frac{g}{p^{m-1} \{(q-p)m-pg\} + g}. \tag{32}
\]
because
\[
p^m\{(q - p)(m + 1) - pg\} - p^{m-1}\{(q - p)m - pg\} \\
= p^{m-1}(q - p)(1 - p)\left(\frac{pg}{q - p} + \frac{p}{1 - p} - m\right) > 0,
\]
where the inequality holds due to (31).

Let us consider repeated games with the above stage game and $\delta$. With $m$ games, we have $v^* = m - \frac{pm}{q-p} > 0$ from (30). Further, we have
\[
1 - (1 - p)v^* = (q - p)(1 - p)\left(\frac{1}{1 - p} + \frac{pg}{q - p} - m\right) > 0,
\]
where the inequality holds due to (31). Hence Proposition 5(iii) applies, and (32) implies that the most cooperative equilibrium payoff is zero. By Proposition 7, the most cooperative equilibrium payoff is zero with any $m' < m$ games.

With $m + 1$ games, we have $v^* = m + 1 - \frac{pm}{q-p} > 0$ from (30). Therefore, from Proposition 5 and the definition of $\delta$, $v^*$ is an equilibrium payoff. From (30), it is greater than $m + 1 - \varepsilon$, and the proof is complete.

**References**


