Evidence and Skepticism in Verifiable Disclosure Games

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Abstract

A shared feature of communication games with verifiable evidence is that the receiver will be skeptical following any non-disclosure: he will tend to believe that the message comes from an informed sender who is withholding unfavorable evidence. It then follows that when the receiver is more skeptical he will choose a less preferable action for the sender. This paper seeks to characterize when a change in the distribution of evidence induces any receiver to be more skeptical. We introduce the “more evidence” relation between type distributions: a distribution has more evidence than another if types with larger available sets are more probable in a monotone likelihood ratio sense. We show that when the sender has more evidence, the equilibrium action following any message is less favorable for the sender, i.e. the receiver becomes more skeptical following any message. We also show that the more evidence relation is necessary for this kind of increased skepticism in the receiver: if the sender does not have more evidence, there exists a receiver who treats the sender (strictly) more favorably following some message. Our approach also admits a full characterization of receiver optimal equilibria in a general class of verifiable disclosure games.

Keywords: Verifiable Disclosure, Hard Information, Monotone Likelihood Ratio Property, Comparative Statics

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1. Introduction

In many economic contexts, “hard evidence” is the only viable means of informative communication: a politician will say anything to get elected, a car salesman will never admit to selling a lemon, and a job applicant will never say that he is untalented. In these situations the bias of the informed party is so extreme that any “cheap talk” communication would be fraught with fabrication. Fortunately, certain types of communication—hard evidence—are immune to such fabrication: the upstanding politician can release non-scandalous tax returns, the car salesman can offer a test drive, and the good job applicant can present her $A^+$ transcript. In each case, hard evidence is informative because its availability in communication is directly related to (or is correlated with) the decision maker’s payoff.

More generally, verifiable disclosure games are communication games between an uninformed receiver and an informed sender whose set of available messages depends on his private information or “type”. These games have received extensive attention since they were introduced to the literature in Milgrom & Roberts (1986), Grossman (1981), and Dye (1985). A shared feature is that non-disclosures are met with skepticism: the receiver will believe that these omissions originate from more informed senders that are strategically withholding unfavorable information. Extending this intuition - the degree of receiver skepticism is related to his perceptions concerning the amount of available evidence: e.g. the fact that a job applicant has no references is more likely to be overlooked if he is fresh out of college than if he has held a number of different positions. Many comparative statistics results concerning seemingly different changes in the distribution of evidence echo this intuition (e.g. Acharya et al. (2011), Guttman et al. (2014), and Dziuda (2011)). A natural question is whether there exists a comparison over distributions that unifies these seemingly disparate findings and generalizes the above intuition to more complex disclosure environments.

The goal of this paper is to understand which distributions make any receiver more skeptical and lead to a less favorable equilibrium action following any message. We show that the key concept is the “more evidence” relation which compares evidence distributions: one distribution has more evidence than another if types with larger message sets are more probable in a monotone likelihood ratio (MLR) sense. Our main result-Theorem 1, shows that increased skepticism in the receiver is characterized by the sender having more evidence. That is, (i) if the sender has more evidence then any receiver takes a lower action for every type (and message) and (ii) if the sender does not have more evidence then there exists a receiver that will strictly increase his action following some message. In obtaining
this conclusion we also derive a tractable solution for equilibrium payoffs.

Our model encompasses a large class of verifiable disclosure games. An informed sender communicates with an uninformed receiver in order to influence his action choice. While the receiver’s preferences over actions depend on the private information or “type” of the sender, the sender independently prefers higher actions. Following Hart et al. (2017) (Henceforth HKP), and Ben-Porath et al. (2017), we model the structure of hard evidence as a partial order: type \( t \) dominates type \( s \) according to the “disclosure order”, or \( t \succeq_d s \), if type \( t \) has all the evidence necessary to masquerade as \( s \). For example the job applicant with 1 reference dominates the job applicant with 0 references. Importantly, there is no assumed relationship between whether a type is “high value” (commands a favorable best response from the receiver) and whether that type is dominant according to the disclosure order (has a large feasible message set). We focus on the receiver optimal equilibrium which is selected by the refinement in HKP.\(^1\)

Solving for equilibrium in general disclosure games can be difficult. We exploit the observation that any equilibrium induces a partition of sender types into payoff equivalence classes. Our first task is to characterize when any set of types can form such a payoff equivalence class, or poolable set, in the receiver optimal equilibrium. Intuitively, poolable sets involve low value types that are more dominant in the disclosure order pretending to be high value types that are less dominant in the disclosure order. For example, the set of job applicants that present no references include experienced applicants with bad references and (relatively) high quality fresh applicants who actually have no references. Proposition 2 formalizes this intuition: Poolable sets are those over which the receiver’s best response is “downward biased”: all lower contour subsets according to the disclosure order are higher value than the set as a whole.\(^2\) Thus any partition such that the receiver’s best response is downward biased on its parts constitutes an equilibrium partition.\(^3\) This observation underlies two novel solution methods for receiver optimal equilibria: (i) an algorithm to find the equilibrium partition; and (ii) an explicit expression for the equilibrium action for each type. Crucially, our analysis bypasses the issue of which messages the sender uses in equilibrium and the need to focus on pure strategies.

Armed with this characterization, we move to our main comparative statics result. First

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\(^1\) A growing literature (Sher (2011), Glazer & Rubinstein (2004), and most recently Ben-Porath et al. (2017)) has shown that the optimal mechanism solution in which the receiver can commit ex-ante to his best response can be implemented as an equilibrium without receiver commitment. HKP identify the truth leaning refinement that selects this equilibrium.

\(^2\) A lower contour subset \( S \) of a partially ordered set \( (X, \succeq) \) is all the elements dominated by elements in \( S \), i.e. \( \{s : \exists s' \in S, s \succeq s'\} \).

\(^3\) Some additional constraints on the partition are required and made precise in Proposition 1.
we define our more evidence relation that compares prior distributions over types: \( f \) has more evidence than \( g \) if whenever \( t \preceq_d t' \), the likelihood ratio \( \frac{f}{g} \) is greater at \( t \) than at \( t' \). The more evidence relation is unrestrictive as it only imposes the likelihood ratio ordering on comparable pairs of types according to the disclosure order. Consequently, many comparisons from the literature are specific examples of the more evidence relation: Grubb (2011) and Acharya et al. (2011) consider uniformly increasing the probability that the sender obtains verifiable evidence, Guttman et al. (2014) considers introducing additional "evidence types", and Dziuda (2011) considers decreasing the probability of honest types that can only reveal truthfully; all of these studies find that the equilibrium action decreases for all types. Theorem 1 generalizes these results showing that the more evidence relation characterizes when any receiver takes a lower equilibrium action for all types.

Due to our characterization of poolable sets, the key insight is that the downward biased property ensures that the receiver’s best response decreases when the sender has more evidence. This is not implied by existing comparative statics results concerning monotone likelihood ratio shifts in the distribution. To illustrate this, consider that the type space is \([0,1]\) and the disclosure order is the usual order. Let the receiver’s best response to each type be given by the measurable function \( v : [0,1] \rightarrow [0,1] \) and the best response to subsets be given by the associated conditional expectation function \( E^v_f(S) \equiv \mathbb{E}[v(x)|x \in S, x \sim f] \) for any full support probability density \( f \) on \([0,1]\) and subset \( S \subset [0,1] \).

The receiver’s best response is downward biased on \([0,1]\) if \( E^v_f([0,y]) \geq E^v_f([0,1]) \) for all \( y \in [0,1] \), i.e. all lower contour subsets have higher value than the set itself. If this holds, our characterization results entail that all types will pool together in equilibrium and thereby obtain an action given by \( E^v_f([0,1]) \). Also \( f \) has more evidence than \( g \) if \( f \) MLR dominates \( g \), i.e. if \( \frac{f(x)}{g(x)} \) is increasing in \( x \). Thus, verifying Theorem 1 requires showing that the expectation of \( v \) under \( f \) is lower than that under \( g \), i.e. \( E^v_f([0,1]) \leq E^v_g([0,1]) \).

Notice that \( E^v_f \) being downward biased is weaker than the condition that \( v \) is decreasing in \( x \). This means that we cannot use the known result that \( E^v_f([0,1]) \leq E^v_g([0,1]) \) for any pair of distributions \( f \) and \( g \) such that \( f \) MLR dominates \( g \) if and only if \( v \) is decreasing on \([0,1]\). Instead consider the slightly modified problem of characterizing the joint properties

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4 These two comparisons are made in the specific framework of Dye (1985) in which types are either evidence types or uninformed with evidence types being able to truthfully reveal or pretend to be uninformed. More details are given in Subsection 2.3.

5 This is true if the receiver’s utility is quadratic loss i.e. if \( U^R(a,x) = -(a - v(x))^2 \), where \( v(t) \) represents the optimal action for each type.

6 In the paper we consider a partial order and only finite sets, but the continuum provides a more convenient illustration here and it is straightforward to show that our results extend to this case. In addition we allow for distributions without full support.
on $\varepsilon$ and $f$ such that for any $g$ with $f$ MLR dominates $g$ we have that $E_f^\varepsilon([0, 1]) \le E_f^\varepsilon([0, 1])$. We show in Corollary 1 that the answer to this problem is that $E_f^\varepsilon([0, y]) \ge E_f^\varepsilon([0, 1])$ for all $y \in [0, 1]$ or $E_f^\varepsilon$ is downward biased. This says that the apparently disconnected characterization of poolable sets and the joint characterization problem above are resolved by the same property. This identification is the reason the value of poolable sets decreases when the sender has more evidence and Theorem 1 holds. Our methodology proves a more general result—Proposition 7 which implies the solution to the above joint characterization problem but also implies other results from the literature such as that in Athey (2002).

Finally, we show how our results apply and generalize examples from the literature. First, we use our characterization results to solve the classic “Dye evidence” model from Dye (1985) and Jung & Kwon (1988). Next we generalize the comparative statics results by Acharya et al. (2011) and Guttman et al. (2014) in the Dye Evidence Model. Then we move to the more complicated multidimensional framework of Dziuda (2011) and again solve for equilibrium using our characterization results. Finally we again generalize a key result in her model to all verifiable disclosure games: if we increase the probability of honest types - types that must declare truthfully - we increase the equilibrium utility of every type.

The paper proceeds as follows. Subsection 1.1 previews our model, characterization approach, and comparative statics result in a simple economic example. Subsection 1.2 discusses the related literature. Section 2 lays out the model and equilibrium concept, as well as listing well known examples that fit our framework. Section 3 presents our characterization results. Section 4 introduces the more evidence relation, presents our main comparative statics result, and details the novel methodology behind it. Section 5 presents some applications of our results to the literature and generalizations of their results. Finally Section 6 concludes. Unless noted otherwise, all proofs are in the appendix.

1.1. Preview

Consider an entrepreneur (the sender) who instructs his engineers to run a beta test for a new software before its launch. The test can result in four different outcomes. The software could possibly outperform expectations garnering rave reviews from its users. Alternatively the test could uncover an unknown fatal flaw in the software. For example, the users may find it inaccessible to non-engineers. Having realized the fatal flaw, another possible outcome is that the engineers could partially salvage the problem by adding a useful tutorial. Lastly, the beta test could yield no usable evidence, perhaps because the users were not a representative sample or because faulty instructions were distributed.

The entrepreneur reports to his investor (the receiver), and attempts to induce the most favorable impression for his product, and thereby induce the highest “action” choice (the
action could proxy for amount of future funding, or advertising effort). However, communication is not “cheap talk” and some of the above outcomes can be certified. If the product performs above expectations, or if the product has a fatal flaw, reviews can be presented that suggest as much. In addition if the fatal flaw is realized but partially salvaged the entrepreneur can credibly present the new tutorial. Although, in this case it will be apparent that the software was flawed to begin with. Finally independent of the test result, the entrepreneur can always claim that the test results were unusable.

There are four “types” of sender - no evidence (NE), above expectations (AE), fatal flaw (FF), and Partially Salvaged (PS). A convenient representation of this problem is illustrated in the left panel of Figure 1. The directed graph illustrates the disclosure order in that each vertex represents a type, and the available messages to each type are the set of vertices accessible via a directed path. For example, PS can declare \{PS, FF, NE\} but not \{AE\}. The investor’s type dependent value for the product is displayed above each vertex. Suppose that the prior over sender types is uniform and the receiver chooses an action equal to the expected value of the product.\footnote{Formally, let the receiver’s utility be given by $U_R(a, t) = -(a - v(t))^2$ where $v(t)$ is the value above each vertex.}

\footnote{There are many ways to set the actions for the off path declarations FF and PS. In the main text we do}

\begin{figure}
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\begin{tikzpicture}[->,shorten >=1pt,auto,node distance=1.5cm,thick,main node/.style={circle,draw}]
\node[main node] (1) {No Evidence};
\node[main node] (2) [right of=1] {Above Expectations};
\node[main node] (3) [below of=1] {Fatal Flaw};
\node[main node] (4) [below of=3] {Partially Salvaged};
\path
(1) edge node [above] {$v = 0$} (2)
(2) edge node [above] {$v = 5$} (4)
(3) edge node [above] {$v = -6$} (4)
(4) edge node [above] {$v = -4$} (1);
\end{tikzpicture}
\caption{Disclosure Order and Best Response}
\end{subfigure}
\begin{subfigure}{0.45\textwidth}
\centering
\begin{tikzpicture}[->,shorten >=1pt,auto,node distance=1.5cm,thick,main node/.style={circle,draw}]
\node[main node] (1) {No Evidence};
\node[main node] (2) [right of=1] {Above Expectations};
\node[main node] (3) [below of=1] {Fatal Flaw};
\node[main node] (4) [below of=3] {Partially Salvaged};
\node[main node] (5) [above of=2] {AE};
\node[main node] (6) [left of=5] {NE};
\node[main node] (7) [below of=5] {FF};
\node[main node] (8) [right of=7] {PS};
\path
(1) edge node [above] {$v = 0$} (6)
(6) edge node [above] {$v = 5$} (8)
(2) edge node [above] {$v = -6$} (5)
(5) edge node [above] {$v = -4$} (1)
(7) edge node [above] {$v = -6$} (8);
\end{tikzpicture}
\caption{Equilibrium Strategies}
\end{subfigure}
\caption{Entrepreneur with Uniform Prior}
\end{figure}

What is the receiver optimal equilibrium in this simple example? It turns out that the equilibrium involves the pure strategies represented by the dotted arrows in the right panel of Figure 1. Types in \{PE, FF, NE\} all claim to have no evidence, and induce the receiver to take $a(NE) = -\frac{10}{3}$. The AE type truthfully reveals and obtains an action $a(AE) = 5$.\footnote{There are many ways to set the actions for the off path declarations FF and PS. In the main text we do}
The interpretation is that an entrepreneur will only reveal that the test produced results if said results are positive and will claim the test was faulty otherwise. The receiver anticipates this, and is skeptical upon receiving $NE$, i.e. he forms a lower expectation of the value of the product than if he were certain that the test were faulty.

Notice that the above equilibrium can also be seen as a partition into sender payoff equivalence classes. This partition is $(P_1, P_2)$ where $P_1 = \{NE, FF, PS\}$ and $P_2 = \{AE\}$. The most important feature of this partition is that the receiver’s expected value is downward biased on each part: the lower contour subsets of $P_1$, i.e. subsets with relatively less evidence, induce a higher receiver expectation than $P_1$ as a whole. To verify this, note that the expected values of $NE$ and $\{NE, FF\}$ (the two (strict) lower contour subsets) are $0$ and $-3$ respectively which are greater than $-\frac{10}{3}$. In Proposition 2 we show that the downward biased property characterizes these poolable sets in the general model.

Now suppose that the beta test and engineers are of higher quality and provide the entrepreneur with more evidence about his product: He is both more likely to get a test result and more likely to partially salvage the fatal flaw. Specifically, assume the test now has only a $\frac{1}{12}$ probability of being faulty, the probability of a fatal flaw is $\frac{1}{6}$, the probability of partially salvaging the fatal flaw increases to $\frac{1}{3}$, while the probability of performing above expectations remains unchanged at $\frac{1}{4}$. This distribution and the original uniform distribution are compared by the more evidence relation: the likelihood ratio between any type with more evidence and some other type he can masquerade as has increased. For example, the $PS$ type can pretend to be the $FF$ type, and the likelihood ratio between $PS$ and $FF$ has increased from $1$ to $2$. The receiver optimal equilibrium is given by the same partition as with the uniform prior. However, the equilibrium actions for types in $P_1$ and $P_2$ are now $-4$ and $5$ respectively. This means that the equilibrium utility has (weakly) decreased for all types ($-4 < -\frac{10}{3}$). Theorem 1 shows that this comparative statics result holds for any two distributions comparable by the more evidence relation and for any evidence structure.

1.2. Related Literature

The first verifiable disclosure models were introduced in Milgrom & Roberts (1986) and Grossman (1981), and Dye (1985) and Jung & Kwon (1988). The former two studies prevent the sender from "lying" but allow him to be "vague", i.e. the sender can declare any subset of types to which his true type belongs. These papers introduced the concept that the receiver is "skeptical" upon hearing vague messages, i.e. the receiver will believe that

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this according to the truth leaning refinement by HKP, which in this case dictates that $a(FF)$ and $a(PS)$ are $-6$ and $-4$ respectively.
any vague message comes from the lowest value type who could have possibly sent the message. The second class of models initiated by Dye (1985) and Jung & Kwon (1988) include the possibility that the sender is uninformed. More specifically types either do or do not obtain verifiable information and types with verifiable information can either truthfully reveal or pretend to be uninformed.

These original models have seen a number of extensions. Shin (2003) and Dziuda (2011) consider multidimensional versions of this model in which the agent obtains potentially multiple pieces of verifiable evidence and can disclose any subset. Dziuda (2011) also considers uncertainty over the preferences of the sender and over whether he is honest or strategic. Guttman et al. (2014) considers a dynamic model with multidimensional evidence in which the receiver is also uncertain about when the sender has obtained evidence. Acharya et al. (2011) also considers a dynamic model in which public information continuously arrives about the value of the sender’s evidence. We will refer back to these models in Section 5 when we generalize their comparative statics results. The above verifiable disclosure games (in their static form) fit the model of this paper. The last class of disclosure games initiated by Verrecchia (1983) involves disclosure costs and is not covered by our model.

Recently, a set of results have shown that the receiver’s utility in some equilibrium of the verifiable disclosure game is the same as the case in which the receiver can commit to a best response before hearing the sender’s message. This equivalence was first introduced in Glazer & Rubinstein (2004) and further generalized by Sher (2011) and Ben-Porath et al. (2017). HKP identify the equilibrium that achieves this equivalence through what they term the “truth leaning refinement”. We focus on this receiver optimal equilibrium throughout. In addition to the above equivalence results, Glazer & Rubinstein (2004) and Sher (2014) derive methods to find the receiver optimal equilibrium. However, their models involve a binary action choice and only two types of senders - acceptable and unacceptable. Our algorithm bares similarities to that in Bertomeu & Cianciaruso (2016) which also takes a characterization approach to a general class of disclosure games with a focus on pure strategy equilibria. We instead focus on the equilibrium partition allowing us to solve for equilibria in games with only mixed strategy equilibria such as that in Dziuda (2011).

Although our comparative statics result generalizes findings from the literature, to our knowledge there is no other study that examines general likelihood ratio shifts in the distribution of types according to the disclosure order. Our methodology involves iteratively applying the classic result that a monotone likelihood ratio shift in the distribution lowers the expectation of a decreasing function. This result can be found in Milgrom (1981) and
2. Model

The setting involves a single sender and a single receiver. The sender is endowed with a type \( t \in T \), which constitutes his private information. \( T \) is a finite set with \( |T| = n \). Let \( h \in \Delta T \) be the prior over types. Throughout the main text (excluding examples and applications) we assume that the prior has full support. In Appendix E we show that our results extend without modification to general prior distributions. After observing his type, the sender chooses a message from those available to his type and communicates this to the receiver. The receiver then takes an action \( a \in A \), where \( A \) is a compact convex subset of \( \mathbb{R} \).

2.1. Preferences

The receiver’s preferences depend on both the action and the sender’s private information. The sender simply wants the highest action. The sender’s utility is given by \( U^S : A \to \mathbb{R} \); it will be without loss of generality to let \( U^S(a) = a \). The receiver’s utility is given by \( U^R : A \times T \to \mathbb{R} \) which is assumed to be strictly concave and differentiable in \( a \). Call the set of all such receiver utilities \( \Upsilon \).\(^9\) Define the receiver’s unique best response to type \( t \) as \( v : T \to \mathbb{R} \) given by \( v(t) \equiv \arg \max_a U^R(a, t) \). Similarly, define \( V^h : 2^T \to \mathbb{R} \) as \( V^h(S) \equiv \arg \max_a \mathbb{E}[U^R(a, t) | t \in S, t \sim h] \) to be the receiver’s best response action conditional on the sender’s type being in the set \( S \) and distributed according to the prior.\(^10\) We will refer to sets of types with relatively high (low) optimal actions, i.e. high \( V^h \), as ”high (low) value”.

The leading example for the receiver’s utility will be quadratic loss defined by \( U^R(a, t) = -(a - v(t))^2 \) for any function \( v : T \to \mathbb{R} \). Here, \( v(t) \) specifies the optimal action for each type and \( V^h(S) = \mathbb{E}[v(t) | t \in S, t \sim h] \) takes the form of a conditional expectation.

**Lemma 1.** For any distribution over types \( q \in \Delta T \), define \( a^*(q) \equiv \arg \max_a \mathbb{E}[U^R(a, t) | t \sim q] \). Consider two distributions \( q_1, q_2 \in \Delta T \) and \( \lambda \in (0, 1) \),

\[
\min_{i \in \{1, 2\}} a^*(q_i) \leq a^*(\lambda q_1 + (1 - \lambda)q_2) \leq \max_{i \in \{1, 2\}} a^*(q_i).
\]

\(^9\)We could make the same weaker assumptions as in HKP, i.e. that the receiver’s utility over actions is single-peaked given any fixed distribution over \( T \). More specifically \( \forall q \in \Delta T \sum_i q_i U^R(a, t) \) is strictly quasiconcave in \( a \).

\(^10\)We consider a fixed prior until Section 4.
Lemma 1 says that the optimal action for the mixture of two distributions is between the optimal actions in response to each individual distribution. In the quadratic loss example the best response is linear, i.e. \( a^*(\lambda q_1 + (1 - \lambda)q_2) = \lambda a^*(q_1) + (1 - \lambda) a^*(q_2) \), so that Lemma 1 is satisfied.

2.2. Messaging Technology

The potential for informative communication in this model is driven by type dependence in the set of available messages to the sender. We make the assumption that the message space is the type space and that the set of feasible messages available to each type is given by the correspondence \( M : T \rightarrow 2^T \): type \( t \) can send any message in \( M(t) \). We make the following assumptions on this correspondence,

\[
\begin{align*}
t \in M(t), \forall t & \quad \text{(Reflexivity),} \quad (1) \\
s \in M(t) \implies M(s) \subset M(t) & \quad \text{(Transitivity).} \quad (2)
\end{align*}
\]

The motivation for these assumptions is that there is actually some underlying space of hard evidence \( C \). For example, \( C \) could correspond to all the potential witness statements, hard evidence, and expert testimony possible in a criminal trial. Each sender type \( t \) corresponds to a subset of evidence \( M(t) \subset C \) to which type \( t \) has access. In this framework, the above assumptions are tantamount to allowing each type to present any subset of evidence within his possession.\(^{11}\) (1) then refers to the feasibility of presenting all of one’s available evidence. (2) is also satisfied because if the subset of evidence available to type \( s \) is contained in that for type \( t \), type \( t \) can also present any subset of evidence available to \( s \). The verifiable disclosure literature refers to message structures satisfying (1) and (2) as “normal”.\(^{12}\) However, the assumption that any subset of available evidence may be presented may be less realistic in the presence of time constraints which may limit the amount of evidence presented by each type. These issues are beyond the scope of this paper.\(^{13}\)

The above assumptions induce a preorder on \( \succeq_d \) on \( T \) given by inclusion of feasible message sets, i.e. \( t \succeq_d s \) if \( M(s) \subset M(t) \), or in other words if \( t \) can “pretend” to be \( s \). Because sender preferences are type independent it will be without loss to consider the

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\(^{11}\)The assumptions are slightly weaker: we only require the sender be able to present any subset of evidence that can be realized with positive probability.

\(^{12}\)For further details on normality see Bull & Watson (2004), Green & Laffont (1986), or Kartik & Tercieux (2012). The definition of normality is superficially different in these papers. Message availability for each type is given by a correspondence \( E : T \rightarrow C \). A message structure is normal if for all types \( t \), there exists \( e_t \), such that for any \( t, t' \in T, e_t \in E(t') \implies E(t) \subset E(t') \).

\(^{13}\)For examples of disclosure games without normality, see Sher (2014) and Rubinstein & Glazer (2006).
quotient type space $T/\sim_d$ and the associated partial order.\footnote{Moving to the quotient space requires redefining the receivers preferences. For the representative $t$ of an equivalence class $E \equiv \{t' \in T : t' \equiv t \}$ we let $U^R(a,t) \equiv E[U^R(a,t') | t' \in E, t' \sim h]$} Throughout, we simply refer to $(T, \succsim_d)$ as this partially ordered quotient space and $\succsim_d$ as the disclosure order.\footnote{A preorder is a pair $(S, \succeq)$ where $\succeq$ is a binary relation over $S$ that is transitive and reflexive. A partially ordered set $(S, \succeq)$ is a preorder where $\succeq$ is also antisymmetric.}

The set of available messages $M(t)$ is also the lower contour set of $t$ in the disclosure order, i.e. $M(t) \equiv \{s : t \succsim s\}$. Abusing notation, for $S \subset T$ we refer to $M(S) \equiv \bigcup_{s \in S} M(s)$ as the lower contour set of $S$. This is the set of types that can be declared by some type in $S$. Similarly for any subset $S \subset T$, we will refer to the upper contour set of $S$ as $B(S) \equiv \{s \in T : \exists t \in S, s \succsim_d t\}$. $B(S)$ is the set of types that can declare some type in $S$.\footnote{For notational simplicity we omit the dependence of $M$ and $B$ on the disclosure order. When we refer to different partially ordered sets $(X, \succeq)$, the lower and upper contour correspondences will be denoted by $M_{\succeq}$ and $B_{\succeq}$ respectively.}

Notice that due to (1), the above message structure always permits agents to report truthfully, i.e. report a message with the same label as their own type.

We emphasize two key generalities of this model. First, the disclosure order is arbitrary. And second, there is no assumed relationship between the disclosure order and the receivers best response to each type given by $v$. We next list some well known examples of disclosure orders from the literature that fit our framework.

2.3. Examples

Dye Evidence Assume that $T = \{t_1, ..., t_{n-1}, t_\emptyset\}$. The disclosure order is given by $M(t_i) = \{t_i, t_\emptyset\} \ \forall i < n$ and $M(t_\emptyset) = \{t_\emptyset\}$. The interpretation is that $\{t_1, ..., t_{n-1}\}$ are “evidence” types, while $t_\emptyset$ is the “uninformed” type. The evidence types can credibly certify their type or pretend to be uninformed, while the uninformed type cannot credibly certify his lack of evidence. This model was first introduced by Dye (1985), and Jung & Kwon (1988), and has been widely used in the verifiable disclosure literature, e.g. by Grubb (2011), Acharya et al. (2011), and Bhattacharya & Mukherjee (2013). In subsubsection 5.1.2 we quickly solve this model using the methods from Section 3, and then show that existing comparative statics results are specific examples of Theorem 1.

Multidimensional Evidence The agent draws an integer $n$ from some $g \in \Delta\{0, 1, ..., k\}$. The agent then draws a sample of size $m$ from some distribution $f \in \Delta X$ where $X$ is a finite subset. Each type is some subset of size $m$, i.e. $t = \{x_1, x_2, ..., x_n\}$, and the type space is $T = \{t \in 2^X : |t| \leq k\}$. The disclosure order is given by $M(t) = \{t' : t' \subset t\} \ \forall t \in T$. The interpretation is that each type can report any combination of pieces of evidence in his

Vagueness All agents obtain a single piece of verifiable evidence $x \in X$ drawn from $h \in \Delta X$, where $X$ is some finite set. The type space is all non-empty subsets of $X$, $T = 2^X \setminus \emptyset$. The disclosure order is given by $M(t) = \{t' \in T, t \subset t'\} \forall t \in T$. The interpretation is that each positive probability type $t$ can credibly reveal himself or be “vague”. For example, if $X = \{0, 1\}$, type $\{0\}$ can either truthfully report $\{0\}$, or be vague and report $\{0, 1\}$, however she cannot “lie” and report $\{1\}$. These message structures were first introduced by Grossman (1981) and Milgrom & Roberts (1986).

Honest Types Consider a “payoff relevant” type space $T'$ endowed with disclosure order $\succeq_d$. In addition to the payoff relevant type, agents can either be strategic, $s$ with probability $q$, or honest, $h$, with probability $1 - q$. The receiver’s utility $U^R : A \times T' \rightarrow \mathbb{R}$ only depends on the payoff relevant type, and not on whether the agent is strategic or honest. The total type space and disclosure order given by $(T, \succeq_d)$ is defined as follows: $T = T' \times \{s, h\}$, and $\forall t \in T'$, $M(t, s) = M_{\succeq_d} \times \{s, h\}$ and $M(t, h) = (t, h)$. The interpretation is that honest types must report truthfully and therefore have a single available message. Strategic types can report any honest or strategic type according to some arbitrary disclosure order. For an illustration of an honest types model see Figure 4. In subsubsection 5.2.1, we use the results of Section 4 to show that the equilibrium utility of all types increases with the probability of honest types.

Complete Order and Empty Order Although less commonly used in the literature, the cases where the disclosure order is complete or empty serve as illustrative examples. A completely disclosure ordered type space is given by $(T, \succeq_d)$, with $T = \{t_1, ..., t_n\}$ and $i \geq j \iff t_i \succeq_d t_j$. That is, types with higher indices can report all types with lower indices. An empty disclosure ordered type space $(T, \succeq_d)$ is given by $t \succeq_d t' \iff t = t'$. That is, each type is forced to truthfully reveal his type.

17 Notice that we are explicitly using zero probability types to model additional messaging options for positive probability types. Any non-singleton type $t$ is zero probability, but represents a feasible message for types $t' \subset t$. In Appendix E we show that this possibility does not alter our results.
2.4. Strategies, Equilibria, and Preliminaries

A feasible strategy for the sender is $\sigma : T \rightarrow \Delta T$ where $\text{Supp}(\sigma_t) \subset M(t) \ \forall t$. A pure strategy for the receiver is $a : T \rightarrow A$ which specifies an action choice in response to each message. A Bayes Nash equilibrium is strategies for the sender and receiver such that,

$$\sigma_t(s) > 0 \implies s \in \arg \max_{s' \in M(t)} a(s') \ 	ext{and} \ a(s) = \arg \max_a \mathbb{E}[U_R(a, t)|\sigma, s] \ \forall s \in \text{Supp}(\sigma).$$

We omit the implications of perfect bayesian equilibria since we focus on a stronger refinement.

We focus on the receiver optimal equilibrium. A number of studies (Ben-Porath et al. (2017), Sher (2011), Glazer & Rubinstein (2004)) have found that commitment is of no value in this disclosure game. This means that there exists an equilibrium in which the receiver’s utility is equivalent to that when he can commit to a strategy before receiving the sender’s message. HKP show that this (receiver optimal) equilibrium is found through their truth leaning equilibrium refinement below. A pair of strategies, $\sigma$ and $a$ is truth leaning if they satisfy the following conditions,

$$t \in \arg \max_{s' \in M(t)} a(s') \implies \sigma_t(t) = 1$$

$$\sigma_t(s) = 0 \ \forall t \in T \implies a(s) = v(s)$$

Truth leaning imposes that if the sender is indifferent between truthful revelation and some other report, he truthfully reveals with probability one. On the receiver’s side, upon observing an off path message, the receiver assumes it is a truthful message. HKP prove existence, but through a fixed point argument. In contrast, we will show existence through an explicit construction. For any truth leaning equilibrium denote the payoff to the sender of type $t$ given receiver preferences $U_R$ by the function $\pi_h(t|U_R)$. Throughout the paper, we will sometimes refer to $\pi_h$ as the sender payoff vector.

3. Equilibrium Characterization

We now provide equilibrium characterization results that will be key to our comparative statics analysis in Section 4. These results lead to both an algorithm that solves for equilibrium as well as an explicit expression for the sender equilibrium payoff vector.

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18 $\sigma_t(s)$ refers to the probability that type $t$ declares type $s$.

19 Because the receiver’s utility is strictly concave, it is without loss to focus on pure strategies.
3.1. Equilibria as Partitions

We begin by noticing that an equilibrium is associated with a partition of the type space, $P = \{P_1, ..., P_m\}$, such that any type in a given part obtains the same action. More specifically, let the set of available sender payoffs in an equilibrium be $\{\pi_1 < ... < \pi_m\}$, and let $P_i \equiv \{ t : \pi_h(t|U^R) = \pi_i \}$. Because the sender’s payoff is strictly increasing in the action, all types in $P_i$ also obtain the same action with probability 1. We call this partition into payoff or action equivalence classes the equilibrium partition.

We now turn to the two necessary and sufficient conditions for any partition to constitute an equilibrium partition. First, for each type $t$, the set of types that $t$ declares with positive probability must also be in the same payoff equivalence class (or part of the equilibrium partition) as $t$. In this sense the strategy of types in each part of the equilibrium partition is "self-contained", i.e. $t \in P_i \implies \text{Supp}(\sigma_t) \subset P_i$. This means there exists a sender strategy for the types in each $P_i$, such that the best response of the receiver is to choose the same action for all on path declarations in $P_i$. Using Lemma 1 we can deduce that the value of this action must be equal to $V_h(P_i)$. That is, the receiver best responds to the prior belief conditioned on $t \in P_i$. The types in $P_i$ are "pooling" in the sense that the receiver treats each type in $P_i$ the same and "as if" he only knew that the type were in $P_i$.

If such strategies exist we call them pooling strategies and say that $P_i$ is poolable.

**Definition 1.** A subset $S \subset T$ is poolable under $h$, if there exists a feasible sender strategy for types in $S$, $\sigma_S : S \to \Delta(S)$ and a receiver best response $a : S \to \mathbb{R}$ such that $\forall s \in \text{Supp}(\sigma_S) a(s) = V_h(S)$, and $\forall s \notin \text{Supp}(\sigma_S) a(s) < V_h(S)$.

Because we focus on receiver optimal equilibrium, poolable sets have another important interpretation. If $T = S$ is poolable under $h$ if there does not exist another equilibrium (possibly not truth leaning) that separates types in $S$, i.e. an equilibrium in which different types in $S$ obtain different actions.

The second feature of any equilibrium partition ensures that no type can deviate to obtain a higher payoff. As higher payoff parts have higher indices, no type $t \in P_j$ should be able to declare any type $s \in P_k$ when $j < k$. Otherwise, $t$ would deviate to the strategy.

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20 We omit the dependence of $P$ on the prior $h$ and receiver utility $U^R$.

21 The fact that any equilibrium partitions sender types into action equivalence classes holds for all equilibria and not just those satisfying the truth leaning refinement.

22 In addition, the action for off path declarations must be less than $\pi_i$.

23 This does not mean that given a declaration in $P_i$, the receiver’s belief about the type is the same as the prior conditioned on $t \in P_i$. Nor does it mean that the belief is the same following each on path declaration in $P_i$. The only conclusion is that the set of beliefs following on path declarations in $P_i$ induce the receiver to take the same action as that of the prior conditioned on $t \in P_i$. 

of $s$ and obtain a strictly higher payoff. We call a partition that satisfies this property an *interval partition*. We formalize this concept in the context of arbitrary partial orders and then summarize the above discussion in Proposition 1

**Definition 2.** Let $(X, \geq)$ be a partially ordered set. An interval partition of $(X, \geq)$ is $P = (P_1, ..., P_m)$ such that,

$$M \geq (P_k) \cap P_j = \emptyset \ \forall j > k.$$  

**Remark 1.** We use the term interval partition, because any interval partition $P = (P_1, ..., P_m)$ of the reals $(\mathbb{R}, \geq)$, has that each part is an interval, i.e. $P_i = [a, b] \ \forall i$ with (extended) real numbers $a \leq b$. In a partial order, the set of interval partitions is larger and has less structure. For example, if the partial order is empty, then every partition is an interval partition.

**Proposition 1.** Let $\pi_h : T \rightarrow \mathbb{R}$ with equivalence classes $\bigcup s \{\pi_h(s|UR)\} = \{\pi_1 < ... < \pi_m\}$ and let $P_i \equiv \{s : \pi_h(s|UR) = \pi_i\}$. $\pi_h$ is an equilibrium sender payoff vector $\iff$

$$P_i \text{ is a poolable set } \forall i, \quad (3)$$

$$(P_1, ..., P_m) \text{ is an interval partition of } (T, \succeq_d), \quad (4)$$

$$\text{and } \pi_i = V_h(P_i) \ \forall i. \quad (5)$$

This result gives us a partial road map to finding equilibrium payoffs, however it leaves a fundamental question unanswered. Given a sender payoff vector $\pi_h$, (4) and (5) are relatively easy to check. Conversely, condition (3) is more opaque. The next section characterizes poolable sets.

**3.2. Downward Biased Functions and Poolable Sets**

A natural intuition is that poolable sets will involve types of lower value pretending to be types of higher value. This says that each poolable set is composed of relatively higher value types that are less dominant in the disclosure order and relatively lower value types that are more dominant in the disclosure order. We formalize this intuition below.

**Definition 3.** The set function $H : 2^X \rightarrow \mathbb{R}$ is downward biased on $(X, \geq)$ if

$$H(M \geq (\bar{X})) \geq H(X) \ \forall \bar{X} \subset X \quad (6)$$

---

24 Before moving on we note which parts of Proposition 1 depend on the refinement under consideration. The only aspect that is specific to truth leaning equilibria is our definition of poolable sets. This definition requires that the strategies that ensure pooling satisfy the refinement. If we were to remove these conditions from the definition of poolable sets, Proposition 1 would still characterize equilibria without change.

25 We state this definition for general partial orders and set functions because it will be useful in Section 4.
A function is downward biased when all lower contour sets have higher value than the set as a whole. If the best response function \( V_h \) is downward biased on \((S, \succeq_d)\) we have that,

\[
V_h(M(\tilde{S}) \cap S) \geq V_h(S) \quad \forall \tilde{S} \subset S.
\] (7)

Therefore \( V_h \) is downward biased on \((S, \succeq_d)\) if less dominant subsets according to the disclosure order of \( S \) command a higher optimal action than the set \( S \) as a whole. It is worth noting that as a consequence of Lemma 1, an equivalent “upper-contour version” of (7) can be stated as

\[
V_h(B(\tilde{S}) \cap S) \leq V_h(S) \quad \forall \tilde{S} \subset S.
\] (8)

That is, upper contour subsets of \( S \)-more dominant subsets of \( S \) according to the disclosure order-have lower value than the set as a whole. Downward biased functions are the key concept in characterizing poolable sets.

**Proposition 2.** A set \( S \) is poolable under \( h \) \iff \( V_h \) is downward biased on \((S, \succeq_d)\).

**Example 1.** Consider that the disclosure order is the empty order as in Subsection 2.3. In this case, any subset of \( S \) is a lower contour subset. In particular, \( V_h \) is downward biased on \((S, \succeq_d)\) means that \( v(s) \leq V_h(S) \quad \forall s \in S \). Using Lemma 1, this holds if and only if \( v(s) = V_h(S) \quad \forall s \in S \), i.e. the optimal action is constant across types.

**Example 2.** Let \((S, \succeq_d)\) be completely ordered as in Subsection 2.3. If \( V_h \) is downward biased on \((S, \succeq_d)\) we have that \( \forall i = 1, ..., m, V_h(\{s_1, ..., s_i\}) \geq V(S) \). A more specific example is illustrated in Figure 2. In this case the receiver’s utility is quadratic loss, \( S \equiv (s_1, ..., s_8) \), and \( h \) is the uniform distribution. The left panel shows the receiver’s best response to each type and the right panel shows the receiver’s best response to lower contour subsets confirming that \( V_h \) is downward biased on \((S, \succeq_d)\). This demonstrates that on a completely ordered set, \( V_h \) is downward biased on \((S, \succeq_d)\) is strictly weaker than \( v \) is decreasing on \((S, \succeq_d)\).

**Remark 2.** These two extreme examples suggest the more general property that refining the disclosure order makes the condition that \( V_h \) is downward biased on \((S, \succeq_d)\) less restrictive. Consider two disclosure orders, \( \succeq_d \) and \( \succeq'_d \) on \( S \), such that \( \succeq'_d \) is a refinement of \( \succeq_d \). \(^{26}\) Note that any lower contour subset of \((S, \succeq'_d)\) is also a lower contour subset of \((S, \succeq_d)\). This means that If \( V_h \) is downward biased on \((S, \succeq_d)\), then \( V_h \) is also downward biased on \((S, \succeq'_d)\).

\(^{26}\) \( \succeq'_d \) is a refinement of \( \succeq_d \) if \( \forall t, t' \in S \ t \succeq_d t' \implies t \succeq'_d t' \)
While the formal argument is deferred to Subsection B.2, we give some intuition for necessity. The idea is that if $V_h$ is not downward biased on $(S, \succeq_d)$, then in the receiver optimal equilibrium some more dominant types in $(S, \succeq_d)$ will separate. To see this, consider that $(T, \succeq_d)$ is completely ordered as in Subsection 2.3. Suppose that the entire type set $T$ pools together in the receiver optimal equilibrium. However, in contradiction to the above result, suppose that $V_h$ is not downward biased on $(T, \succeq_d)$, i.e. there exists $i$ such that $V_h(\{t_1, \ldots, t_i\}) < V_h(\{t_{i+1}, \ldots, t_n\})$. But then there exists another equilibrium with associated partition $(P_1, P_2) = (\{t_1, \ldots, t_i\}, \{t_{i+1}, \ldots, t_n\})$. Because $V_h(P_1) < V_h(P_2)$, there will be no profitable deviation. However, the receiver is obviously better off in the latter equilibrium contradicting that $T$ is poolable in the receiver optimal equilibrium. Proposition 2 shows that this intuition extends to general partial disclosure orders, and to when the equilibrium partition is non-trivial.

The above result is a powerful tool in abstracting from the complex equilibrium strategies that make pooling possible. Indeed, armed with this result, the remainder of this paper does not make any further reference to sender strategies.

3.3. Solving for Equilibrium

We first introduce a lemma that allows us to easily identify poolable subsets. Using this lemma iteratively, we provide an algorithm that constructs the equilibrium partition. Finally we provide an explicit expression for the sender’s equilibrium payoff vector.

**Lemma 2.** For any subset $S \subset T$, let $J \subset \operatorname{arg max}_{\tilde{S} \subset S} V_h(B(\tilde{S}) \cap S)$. $\cup_{\tilde{S} \in J} B(\tilde{S}) \cap S$ is poolable.

---

27We can sustain this (potentially non-truth leaning) equilibrium with the sender strategy- $\sigma_t(s) = 1$ if $s = t_1, t \in P_1$ or $s = t_i, t \in P_2$ and $\sigma_t(s) = 0$ otherwise.
**Algorithm 1: Partition into Poolable Sets**

**Input:** Partially ordered type space \((T, \succeq_d)\)

**Output:** Equilibrium partition

\[ i = 1; S_1 = T; \]

**while** \(S_i \neq \emptyset\) **do**

\[ P_i = \arg \max_{\tilde{S}_i \subseteq S_i} V_h(B(\tilde{S}_i) \cap S_i); \]

\[ i = i + 1; \]

\[ S_i = S_{i-1} \setminus P_{i-1}; \]

**end**

Relabel \(P_j \equiv P_{i-j} \forall 1 \leq j \leq i - 1;\)

Lemma 2 will imply that any maximal valued upper contour subset is poolable. The argument for this result is straightforward. Say that \(B(\hat{S}) \cap S \equiv \hat{B}\) were not poolable. By using the upper contour version of the definition of downward biased in (8), there exists \(R \subset \hat{B}\) such that \(V_h(B(R) \cap \hat{B}) > V_h(\hat{B})\). But this contradicts the maximality of \(\hat{B}\).

The ability to find poolable sets through maximization is useful in finding the equilibrium partition. Consider applying the above result iteratively as follows. Begin with the entire type set \(T\) and use Lemma 2 to find the highest payoff poolable set \(P_m\). Next remove \(P_m\) and apply Lemma 2 to \(T \setminus P_m\) to find another poolable set \(P_{m-1}\). Repeat this process, until every type is in some \(P_i\). This algorithm, which we call partition into poolable sets, generates the equilibrium partition described in Proposition 1.

**Proposition 3.** The output of “Partition into Poolable Sets” \((P_1, \ldots, P_m)\) is the unique equilibrium partition with sender payoff vector given by \(t \in P_i \iff \pi_h(t|U^R) = V_h(P_i)\).\(^{28}\)

Algorithm 1 produces a partition of \(T\)-\((P_1, P_2, \ldots, P_m)\). First, each \(P_i\) poolable by Lemma 2. Second, because of the iterative maximization, types in higher index parts obtain strictly higher actions. Finally, at each stage we remove an upper contour subset relative to the remaining set of elements which implies that the output is an interval partition. In summary, the requirements of Proposition 1 are satisfied and \((P_1, P_2, \ldots, P_m)\) is the equilibrium partition. We next present an explicit expression for the sender payoff vector.

**Proposition 4.** Let \(\pi_h : T \rightarrow \mathbb{R}\) be the equilibrium payoff vector.

\[
\pi_h(t|U^R) = \min_{\{S_a, t \in S_a\}} \max_{\{S_b, t \in S_b\}} V_h(M(S_a) \cap B(S_b))
\]

\(^{28}\)The indexing in the output partition is reversed from that of Proposition 1. The highest value part is actually \(P_1\) and the action is decreasing in the index. This is why we correct the indexing in the last step of the algorithm.
The expression in (9) corresponds to the equilibrium utility (and thereby obtained action) of the sender of type \( t \). The intuition for the result is as follows. Again consider that \((T, \succeq)\) is completely ordered as in Subsection 2.3. Fix any type \( t_i \). For any feasible \( S_a \) we will have \( M(S_a) = \{t_1, \ldots, t_i, \ldots, t_a\} \) for some \( a \geq i \). Similarly, for any feasible \( S_b \) we will have \( B(S_b) = \{t_b, \ldots, t_i, \ldots, t_a\} \) for some \( b \leq k \). Thus \( B(S_b) \cap M(S_a) = \{t_b, \ldots, t_i, \ldots, t_a\} \). In this case, the above problem reduces to choosing sets of types \( \{t_i, \ldots, t_a\} \) and \( \{t_b, \ldots, t_i\} \) that will pool with type \( t_i \) from "above" and "below" respectively. This means we can rewrite the problem in (9) as,

\[
\pi_h(t_i|U^R) = \min_{a \geq i} \max_{b \leq i} V_h(\{t_b, \ldots, t_a\}) \tag{10}
\]

The problem in (10) suggests that equilibrium forms through the following process. Given a set of types \( \{t_i, \ldots, t_a\} \) that pool with \( t_i \) from above, \( t_i \) will choose to pool with a set of lower types \( \{t_b, \ldots, t_i\} \) in order to maximize his value. Similarly, types in \( \{t_i, \ldots, t_a\} \) that pool from above will only do so if it increases their value. This second process serves to minimize the expression in (10).

It is also worth noting that we could rewrite the equaltion in (9) as,

\[
\pi_h(t|U^R) = \max_{S_b: t \in S_b} \min_{S_a: t \in S_a} V_h(M(S_a) \cap B(S_b))
\]

i.e. the "minmax = maxmin" equivalence holds.

In this section, we have put Proposition 1 and Proposition 2 to work in order to develop two simple ways to find the equilibrium partition and sender payoff vector. Using these results we now move to our characterization of distributions of evidence that make the receiver more skeptical.

4. Distributions with More Evidence

There is a widespread intuition in the verifiable disclosure literature that the sender benefits from the perception of a lack of evidence. The idea is that when the sender is perceived to have more evidence, the receiver becomes more skeptical that any message comes from types that are strategically withholding unfavorable information. This is consistent with our analysis of equilibrium in the previous section: poolable sets involve types with larger message sets (more evidence) and lower values declaring types with smaller message sets (less evidence) and higher values. Thus, if the receiver believes the sender to have more evidence,

\footnote{For solutions \( S_a \) and \( S_b \) to (9), \( M(S_a) \cap B(S_b) \) is the part of the equilibrium partition that contains \( t \).}
evidence, he will also believe that the sender is of lower value and correspondingly lower his best response action. This means that the sender obtains a higher action when he is perceived to have less evidence.

In this section we formalize this intuition, and characterize distributions of evidence that make the receiver more skeptical. The key concept is the "more evidence" relation which compares two prior distributions over the type space.

4.1. More Evidence and Monotone Likelihood Ratio Dominance

Type $t$ has more evidence than type $t'$ if $t \succeq_d t'$. A natural extension of this comparison to distributions $f, g \in \Delta T$ is that if $t \succeq_d t'$ and $f$ has more evidence than $g$, then $t$ is relatively more likely than $t'$ under $f$ than under $g$. We repeat this definition below maintaining our restriction to full support distributions.30

**Definition 4.** Let $f, g \in \Delta T$. $f$ has more evidence than $g$ ($f \geq_{ME} g$) if,

$$\forall t, t' \in T \quad t \succeq_d t' \implies \frac{f(t)}{g(t')} \geq \frac{f(t')}{g(t)}.$$ 

If the disclosure order is connected, then the more evidence relation $\geq_{ME}$ is a partial order on distributions.31 If $\succeq_d$ were a complete order, then $f \geq_{ME} g$ would be equivalent to $f$ MLR dominates $g$ on $(T, \succeq_d)$ as in Milgrom (1981). Thus Definition 4 is a natural extension of monotone likelihood ratio dominance on a partially ordered support.

This definition is unrestrictive in the sense that it only imposes the likelihood ratio inequality for comparable pairs of types. For example, if the disclosure order is empty, then every distribution has more evidence than any other distribution. In less pathological cases the condition is still unrestrictive. For example $f \geq_{ME} g$ in the Dye Evidence model of Subsection 2.3 if and only if

$$\frac{f(t_0)}{g(t_0)} \leq \min_{i<n} \frac{f(t_i)}{g(t_i)}.$$ 

4.2. More Evidence and Sender Equilibrium Payoff

Despite being unrestrictive, the more evidence relation allows for the general comparative statics result presented below.

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30 We extend the main results without modification to distributions without full support in Appendix E.

31 A partial order is connected if its directed graph representation has one component.
Theorem 1. Let \( f, g \in \Delta T \). \( f \) has more evidence than \( g \) \iff
\[
\pi_f(t|U^R) \leq \pi_g(t|U^R) \quad \forall t \in T, \forall U^R \in \Upsilon.
\]

The strength of this result is that it compares equilibrium utility for all types and all receiver preferences.\(^{32}\) In this sense, skepticism is not in the "eye of the beholder". Instead, if \( f \) has more evidence than \( g \), any receiver is more skeptical facing \( f \), i.e. he chooses a less favorable action for all messages. Moreover, the result shows if some receiver is more skeptical facing \( f \) than \( g \), then \( f \geq_{ME} g \).

The key step in establishing Theorem 1 is showing that the value of poolable subsets decreases when the sender has more evidence. However, this is not implied by existing comparative statics results. We begin elucidating the intuition for the result by considering the simple example below. We then discuss why the argument in this example fails in general necessitating the use of novel methodology.

Example 3. Let \( U^R \) be quadratic loss, \((T, \succeq_d)\) be completely ordered as in Subsection 2.3, and \( v(t_i) \) be decreasing \( i \). First note that the equilibrium partition for any prior distribution \( h \) is the trivial partition, i.e. \( P = (T) \) and all types pool together. The sender equilibrium payoff vector is given by \( \pi_h(t_i|U^R) = V_h(T) \forall i \).

As the disclosure order is complete, we know that \( f \geq_{ME} g \) implies \( f \) MLR dominates \( g \) on \((T, \succeq_d)\). Thus, we can use the classic result that the expectation of a monotonically decreasing function is lowered by an MLR increase in the distribution.\(^{33}\) This will imply that \( V_f(T) \leq V_g(T) \) establishing Theorem 1 in this case.

This example captures the intuition mentioned at the beginning of this section. Within the poolable set (the entire type set \( T \)), more dominant types in the disclosure order have relatively lower value. Thus when we shift mass to these types, i.e. we give the sender more evidence, the receiver becomes more skeptical and chooses a lower action. The example uses the following classic result which we print below for reference.

Proposition 5. Let \((X, \succeq)\) be a completely ordered set and \( \varepsilon : X \to \mathbb{R} \). \( \mathbb{E}[\varepsilon(x)|x \sim f] \leq \mathbb{E}[\varepsilon(x)|x \sim g] \forall f, g, \) such that \( f \) MLR dominates \( g \) \iff \( \varepsilon \) is decreasing in \( x \).

\(^{32}\) One should note that this does not mean that a sender with more evidence obtains a lower ex-ante utility. Theorem 1 is an ex-post statement, in that given each type realization, the sender is worse off if he has more evidence. It is entirely plausible that despite the decrease in utility "type by type", the sender with more evidence has a higher probability of being higher valued types, and therefore a higher ex-ante utility.

\(^{33}\) Readers may be more familiar with the result in the case when \( f \) first order stochastically dominates \( g \). However, monotone likelihood ratio ratio dominance implies first order stochastic dominance as shown in Shaked & Shanthikumar (2007).
Can the above simple argument using Proposition 5 be used in general to obtain that the value of poolable sets decreases when the sender has more evidence? The answer is negative. To see this, maintain that \( U^R \) is quadratic loss and that \((S, \succeq_d)\) is completely ordered. Our goal is to establish that if \( S \) is poolable under \( f \in \Delta T \), then

\[
\mathbb{E}[v(x)|x \sim f] \leq \mathbb{E}[v(x)|x \sim g] \ \forall g \text{ such that } f \text{ MLR dominates } g.
\] (11)

Recall from Proposition 2 that \( S \) being poolable under \( f \) is equivalent to \( V_f \) being downward biased on \((S, \succeq_d)\). The problem then in applying Proposition 5 is that the downward biased property is weaker than the condition that \( v \) is decreasing on \((S, \succeq_d)\). However, the statement in (11) is also less demanding as it holds the distribution \( f \) fixed instead of requiring that the inequality hold for arbitrary pairs of distributions \( f \) and \( g \) such that \( f \) MLR dominates \( g \). Thus the relevant approach in our problem is to consider the joint properties on \( v \) and \( f \) such that (11) holds. As the next result reveals, \( V_f \) being downward biased on \((S, \succeq_d)\) (and therefore \( S \) being poolable under \( f \)) is the requisite joint property for (11). This is the key result in proving Theorem 1.34

**Proposition 6.** Fix \( f \in \Delta T \). \( V_f(S) \leq V_g(S) \ \forall g \in \Delta T, \text{ such that } f \geq_{ME} g \iff S \text{ is poolable under } f. \)

Proposition 6 is established in the next section. As shown above, fixing the distribution \( f \) allows us to make comparative statics predictions with less restrictive conditions on the underlying function. We push this observation further in the sequel by showing that for a fixed arbitrary pair of a function and a distribution, we can make comparative statics predictions under likelihood ratio shifts in the distribution.

### 4.3. General Comparative Statics Result

This section presents a comparative statics result on the conditional expectation of an arbitrary function for MLR shifts in the distribution. First, we provide some additional notation. For any two distributions \( f, g \in \Delta X \) with \( X \) finite, denote the \( f - g \) likelihood ratio preorder-\( \geq_{f/g} \), defined by,

\[
x \geq_{f/g} x' \iff \frac{f(x)}{g(x)} \geq \frac{f(x')}{g(x')}.
\]

34While Proposition 6 is necessary in proving Theorem 1, it is only sufficient in the case where the equilibrium partition under \( f \) is the same as that under \( g \). In general, the equilibrium partition can change when the sender has more evidence. In the main text we argue for Proposition 6 and leave arguments concerning the changing equilibrium partition to Appendix F.
Thus \((X, \geq_{f/g})\) is a completely ordered set in which higher elements are relatively more likely under \(f\) than under \(g\). Notice that we can restate Definition 4 as \(f \geq_{ME} g\) if \((T, \geq_{f/g})\) is an order completion of \((T, \geq_d)\).

Finally, for a function \(\varepsilon : X \to \mathbb{R}\) and distribution \(h \in \Delta X\), denote the conditional expectation function \(E^\varepsilon_h : 2^X \to \mathbb{R}\) defined by \(E^\varepsilon_h(S) \equiv \mathbb{E}[\varepsilon(x)|x \in S, x \sim h]\).

**Proposition 7.** Let \(X\) be a finite set, and fix \(f \in \Delta X\) and \(\varepsilon : X \to \mathbb{R}\). For any \(g \in \Delta X\), there exists an interval partition \(P = (P_1, ..., P_m)\) of \((X, \geq_{f/g})\) with,

\[
E^\varepsilon_f(P_1) < ... < E^\varepsilon_f(P_m), \tag{12}
\]

and

\[
E^\varepsilon_f(P_i) \leq E^\varepsilon_g(P_i) \forall i, \tag{13}
\]

Where the partition \(P\) does not depend on \(g\) given \(\geq_{f/g}\).

Our approach is to fix both the function \(\varepsilon\) and distribution \(f\) and investigate the comparative statics results that hold for any distribution \(g\) such that \(f\) MLR dominates \(g\). To interpret Proposition 7 consider its implications in two extreme cases. If \(\varepsilon\) is strictly increasing on \((X, \geq_{f/g})\), then the only interval partition satisfying (13) is the complete partition \(P = (x_1, ..., x_n)\). Conversely, if \(\varepsilon\) is decreasing on \((X, \geq_{f/g})\), then the only interval partition satisfying (12) is the trivial partition \(P = (X)\). Broadly, Proposition 7 says that some ”decomposition” into the above two cases holds for arbitrary functions \(\varepsilon\) and distributions \(f\).

At an intuitive level, every function has decreasing and increasing portions on \((X, \geq_{f/g})\). On the decreasing regions, we can apply Proposition 5 which says that the expectation of a decreasing function is lower under monotone likelihood ratio shifts. The proof of Proposition 7 repeatedly applies this result, to find an interval partition such that there are ”no more decreasing regions”. The consequences are (12) and (13). The former says that the conditional expectation is strictly increasing in the part’s index, or the function is ”increasing across parts”. The latter says that for each part the conditional expectation is lower under \(f\) than under \(g\), or the function is ”decreasing within each part”.

**4.3.1. Implications for Downward Biased Functions**

One way to apply Proposition 7 is as follows: if the only interval partition satisfying (12) is the trivial partition \(P = (X)\), then (13) holds as \(E^\varepsilon_f(X) \leq E^\varepsilon_g(X)\), i.e. the expectation of \(\varepsilon\)

\[
\forall t, t' \in T \quad t \geq_d t' \implies t \geq_{f/g} t'.
\]

\[
\text{The last part of the result says that } P \text{ only depends on } g \text{ through } \geq_{f/g}; \text{i.e. if } P \text{ satisfies (12) and (13) for } g' \in \Delta X, \text{ then } P \text{ will also satisfy these conditions for any } g'' \text{ such that } \geq_{f/g'} = \geq_{f/g''}.\]
is lower under \( f \) than under \( g \). For example, under every distribution \( f \), this condition is satisfied when \( \varepsilon \) is decreasing on \((X, \geq f/g)\). This is because for any interval partition \( P = (P_1, \ldots, P_m) \) of \((X, \geq f/g)\), \( E_f(P_i) \) is decreasing in \( i \), contradicting (12) unless \( P \) is trivial. Thus (13) gives that \( E_f^\varepsilon(X) \leq E_g^\varepsilon(X) \) demonstrating that Proposition 5 is an implication of Proposition 7.

However with \( f \) fixed, \( \varepsilon \) being decreasing is not the only condition that permits the above argument. We can also obtain sufficiency in Proposition 6 - if \( f \) has more evidence than \( g \), and \( S \) is poolable under \( f \), then \( V_f(S) \leq V_g(S) \).\(^{37}\) We show this by using Proposition 2 which characterizes poolable sets as admitting downward biased best responses. It turns out that the downward bias property will also prevent any non-trivial interval partition from satisfying (12).

Consider \( \varepsilon : X \to \mathbb{R} \) and \( f \in \Delta X \) such that \( E_f^\varepsilon \) is downward biased on \((X, \geq f/g)\). In search of a contradiction, let (12) be satisfied by a non-trivial interval partition of \((X, \geq f/g)\), \((P_1, \ldots, P_m)\). By iterated expectations, \( E_f^\varepsilon(P_1) < E_f^\varepsilon(X) \). But because \( E_f^\varepsilon \) is downward biased on \((X, \geq f/g)\) and \( P_1 \) is a lower contour set, we have that \( E_f^\varepsilon(P_1) \leq E_f^\varepsilon(X) \) contradicting the above inequality. So the only interval partition satisfying Proposition 7 must be the trivial partition which means that (13) holds as \( E_f^\varepsilon(X) \leq E_g^\varepsilon(X) \). We restate this conclusion below.

**Corollary 1.** Let \( X \) be a finite set, \( f, g \in \Delta X \), and \( \varepsilon : X \to \mathbb{R} \). If \( E_f^\varepsilon \) or \( E_g^\varepsilon \) is downward biased on \((X, \geq f/g)\), then

\[ E_f^\varepsilon(X) \leq E_g^\varepsilon(X) \]

If \( U^R \) is quadratic loss so that \( V_f \) takes the form of a conditional expectation it is straightforward to see that Corollary 1 implies Proposition 6. First, \( V_f \) is downward biased on \((S, \geq d)\) by Proposition 2. But \( f \geq ME \) \( g \) implies that \((S, \geq f/g)\) is a completion of \((S, \geq d)\), and so \( V_f \) is also downward biased on \((S, \geq f/g)\), and we can directly apply Corollary 1 to obtain that \( V_f(S) \leq V_g(S) \).\(^{38}\) In Subsection C.2 we extend this logic to general receiver utilities.\(^{39}\)

Proposition 7 has other implications beyond verifiable disclosure games. In Appendix G we show that the result in Athey (2002) that the expectation of a single crossing functions is single crossing in monotone likelihood ratio shifts in the distribution.

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\(^{37}\) Necessity in Proposition 6 is more straightforward. Suppose \( V_f \) is not downward biased on \((S, \geq d)\). This means there exists a lower contour subset \( L = M(L) \subset S \) such that \( V_f(L) < V_f(S) \implies V_f(L) < V_f(L') \). Consider defining \( g(s) = \frac{f(s)}{\int f(s)} \) if \( s \in L \) and \( g(s) = 0 \) otherwise (where \( F \) is the corresponding measure of \( f \)). Naturally \( f \geq ME \) \( g \) but \( V_f(S) > V_g(S) \).

\(^{38}\) Recall from Remark 2 that the downward biased property is preserved by refinements of the partial order. A preordered set \((S, \geq)\) refines a partial ordered set \((S, \geq)\) if \( t' > t'' \implies t' \geq t'' \forall t', t'' \in S \).

\(^{39}\) The key is to define \( m(t) \equiv U^R(V_f(S), t) \). Then \( V_f(S) \leq V_g(S) \iff E_f^m(S) \leq E_g^m(S) \) and \( E_f^m \) is downward biased on \((S, \geq f/g)\) \iff \( V_f \) is downward biased on \((S, \geq f/g)\), and we can apply the above argument to obtain the result.
Algorithm 2: Monotonic Coarsening

**Input:** Completely ordered set \((S, \geq) = \{t_1, ..., t_m\}\) and conditional expectation function \(E_f^\varepsilon : 2^S \to \mathbb{R}\)

**Output:** Interval Partition of \((S, \geq)\) satisfying (12) and (13)

\[
i = 1; m(1) = m; P^1 = (\{t_1\}, ..., \{t_m\});
\]

**while** \(i = 1\) or \(P^i \neq P^{i-1}\) **do**

\[
I_0 = 0;
\]

\[
k = 0;
\]

**while** \(I_k < m(i)\) **do**

\[
k = k + 1;
\]

\[
I_k = \max\{j > I_{k-1} : E_f^\varepsilon(P^i_{I_{k-1}+1}) \geq ... \geq E_f^\varepsilon(P^i_j)\}
\]

**end**

\[
i = i + 1;
\]

\[
P^i = (\bigcup_{I_0 < j \leq I_1} P^{i-1}_j, ..., \bigcup_{I_{k-1} < j \leq I_k} P^{i-1}_j);
\]

\[
m(i) = k;
\]

**end**

4.3.2. Methodology

While we have shown that Proposition 7 is the correct tool, we have yet to show why it holds. In this section we show how to explicitly construct the interval partition in Proposition 7 using the algorithm described below.

For the purposes of Proposition 7 consider inputting \((X, \geq_{f/g})\) for some \(f, g \in \Delta X\). The algorithm begins with the complete interval partition of \((X, \geq_{f/g}), P^1 = (\{t_1\}, \{t_2\}, ..., \{t_m\})\). We then form the largest sequence of elements such that \(E_f^\varepsilon(t_j)\) (which is simply \(\varepsilon(t_j)\)) is decreasing in \(j\), i.e. \(\{t_1, t_2, ..., t_{I_1}\}\) such that \(E_f^\varepsilon(t_1) \geq ... \geq E_f^\varepsilon(t_{I_1})\). We repeat this process forming another maximal decreasing sequence beginning with \(t_{I_1+1}\). We continue forming maximal decreasing sequences until we exhaust all the elements of \(X\). Using these sequences we form a new coarser interval partition \(P^2\) by collapsing all the elements of each decreasing sequence into an associated single part. More specifically, \(P^2_1 \equiv \{t_1, ..., t_{I_1}\}, P^2_2 = \{t_{I_1+1}, ..., t_{I_2}\}\), and so on. This coarsening is then repeated with \(P^2\) and so on: at each stage, \(P^i\) is coarsened into \(P^{i+1}\) where each part of \(P^{i+1}\) is composed of a consecutive sequence of \(P^i_j\) over which \(E_f^\varepsilon(P^i_j)\) is decreasing in \(j\).

The algorithm concludes when \(P^T = P^{T+1}\), which implies that \(E_f^\varepsilon(P^T_1) < ... < E_f^\varepsilon(P^T_m)\). Thus \(P^T\) is a candidate interval partition for Proposition 7 as it satisfies (12). It turns out that \(P^T\) also satisfies (13). The idea is that because each part of \(P^i\) is composed of a sequence
of parts from $P^{i-1}$ over which $E^*_j$ is decreasing, we can apply Proposition 5 to every $P^i_j$ and obtain

$$\frac{1}{G(P^i_j)} \sum_{l=k(j)} E^*_j(P^{i-1}_l)G(P^{i-1}_l) \geq \frac{1}{F(P^i_j)} \sum_{l=k(j)} E^*_j(P^{i-1}_l)F(P^{i-1}_l) = V_f(P^i_j)$$

Using a sequence of these inequalities we establish that $P^T$ satisfies (13). While the details are left to the Appendix, we illustrate the application of the algorithm in the following example.

**Example 4.** Recall the example in Figure 2.\(^{40}\) Let $g \in \Delta S$ such that $f \succeq_{ME} g$, and define associated probability measures $F, G$. Recall that $V_f$ is downward biased on $(S, \succeq_d)$. Our goal is to show that $V_f(S) \leq V_g(S)$ establishing Proposition 6 in this case. However as $v$ is not decreasing, we cannot simply apply Proposition 5 to obtain the result. Instead we will use algorithm 2.

We begin with the complete interval partition $P^1 \equiv (s_1, \ldots, s_8)$, and form the longest consecutive sequences such that $v(s_i)$ is decreasing in $i$. We then collapse these sequences into parts of a new coarser partition-$P^2$. These are $P^1_1 \equiv \{s_1, s_2\}$, $P^1_2 \equiv \{s_3, s_4\}$, $P^1_3 \equiv \{s_5, s_6\}$, and $P^1_4 \equiv \{s_7, s_8\}$. Because $v(s_i)$ is decreasing on each subset, we can apply Proposition 5 as in (14) to obtain that $\forall j = 1, \ldots, 4$ $V_g(P^2_j) \geq V_f(P^2_j)$. Then, taking the expectation over $j$ with respect to $G$ gives $V_g(S) \geq \sum_{j=1}^4 V_f(P^2_j)G(P^2_j)$.

We next repeat the process on $P^2$ by finding the largest sequences of $P^2_j$ such that $V_f(P^2_j)$ is decreasing in $j$. Notice that $(V_f(P^2_1), V_f(P^2_2), V_f(P^2_3), V_f(P^2_4)) = (5, 4, 3, 2)$ is the single maximal such sequence, i.e. $P^3 = (S)$ is the trivial partition. As before, we apply Proposition 5 to get that $\sum_{j=1}^4 V_f(P^2_j)G(P^2_j) \geq \sum_{j=1}^4 V_f(P^2_j)F(P^2_j) = V_f(S)$ Combining the above two inequalities we obtain that $V_g(S) \geq V_f(S)$.

This construction is illustrated in Figure 3 below. The formation of $P^2$ from $P^1$ is shown in the left panel and the formation of $P^3$ from $P^2$ is shown in the right panel. The fact that the process ends with the trivial partition is a general feature of all downward biased conditional expectation functions.

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\(^{40}\)Specifically, $U^R$ is quadratic loss, the disclosure order is complete on $S = (s_1, \ldots, s_8)$, $f$ is the uniform distribution on $S$, $(v(s_1), \ldots, v(s_8)) = (6, 4, 5, 3, 4, 2, 3, 1)$. 

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5. Applications

5.1. Dye Evidence Model

In this section we use our methods to fully characterize the receiver optimal equilibrium in the Dye Evidence model of Subsection 2.3. In addition we show that a number of comparative statics results known in the literature are specific examples of Theorem 1.

5.1.1. Equilibrium characterization

Recall the Dye evidence model from Subsection 2.3. Without loss of generality let $v(t_i)$ be increasing in $i$. Due to the particular disclosure order, evidence types either pool with the uninformed types, or truthfully reveal. This means that in order to characterize the equilibrium partition, we only need to characterize the part containing $t_\emptyset$. Let us apply Proposition 4,

$$\pi_h(t_\emptyset | U^R) = \min_{t_\emptyset \in S_a} \max_{t_\emptyset \in S_b} V_h(M(S_a) \cap B(S_b))$$

For any feasible $S_b$, we have that $T = B(t_\emptyset) \subset B(S_b)$. Also, for any feasible $S_a$ note that $M(S_a) = S_a$. Thus the objective reduces to $V(M(S_a) \cap T) = V(S_a)$. We can therefore rewrite the above problem as

$$\pi_h(t_\emptyset | U^R) = \min_{S \subseteq T} V_h(S \cup \{t_\emptyset\})$$

This is the minimum principle presented in Acharya et al. (2011), i.e. the action given to $t_\emptyset$ is the minimum possible action over any feasible strategy.

Note that solving this problem is equivalent to finding some threshold $i$ and associated
minimizer $S_i = \{t_1, ..., t_i\}$ such that $v(t_i) \leq V(S_i \cup t_0)$ and $v(t_{i+1}) > V(S_i \cup t_0)$\footnote{To see this note that if $v(t_i) > V(S_i \cup t_0)$ or if $v(t_{i+1}) \leq V(S_i \cup t_0)$, we could lower the objective by removing $t_i$ from $S_i$ or including $t_{i+1}$ in $S_i$ respectively.}. All other evidence types $\{t_{i+1}, ..., t_{n-1}\}$ truthfully reveal. We summarize these conclusions below.

**Corollary 2.** In the Dye Evidence model $\exists i < n$ such that,

$$\pi(t_j) = \begin{cases} V(\{t_1, ..., t_i, t_0\}) & j \leq i \text{ or } j = \emptyset \\ v(t_j) & j > i \end{cases}$$

and

$$v(t_i) \leq V(\{t_1, ..., t_i, t_0\}) < v(t_{i+1})$$

### 5.1.2. Comparative Statics with Dye Evidence

We now show that two comparative statics results in the Dye evidence model are examples of our Theorem 1. Recall that in this model, for two distributions $f, g \in \Delta T$, $f \succeq_{ME} g$ $\iff$

$$\frac{f(t_0)}{g(t_0)} \leq \min_{i<n} \frac{f(t_i)}{g(t_i)}$$

**Changing the probability of Evidence Types** Let the prior distribution over types be given by the following two step process. With probability $p \in [0, 1]$ each agent is an evidence type, then conditional on being an evidence type, $t_i$ is drawn from $f \in \Delta \{t_1, ..., t_{n-1}\}$. Call the resulting prior distribution $h_p \in \Delta T$. Grubb (2011) and Acharya et al. (2011) both conclude that the equilibrium action for the uninformed type is decreasing in $p$. It is straightforward to verify that for $p_2 > p_1$, $h_{p_2}$ has more evidence than $h_{p_1}$\footnote{The RHS objective in (15) is constant in $t_i$ at $\frac{p_2}{p_1}$, which is greater than the LHS which is $\frac{1-p_2}{1-p_1}$.}. We summarize below in the following corollary.

**Corollary 3.** If $p_2 > p_1$, $h_{p_2}$ has more evidence than $h_{p_1}$ and so $\pi_{h_{p_2}}(t|U^R) \leq \pi_{h_{p_1}}(t|U^R) \forall t \in T$, and $U^R \in \Upsilon$.

**Introducing New Evidence Types** Consider some subset of evidence types $S \subset \{t_1, ..., t_{n-1}\}$. For an original prior $h \in \Delta T$, define the restricted prior $h_S$ as

$$h_S(t) = \begin{cases} \frac{h(t)}{H(S \cup \{t_0\})} & t \in S \cup \{t_0\} \\ 0 & \text{otherwise} \end{cases}$$
For $S' \subset S''$, $h_{S'}$ excludes evidence types from $h_{S''}$. Despite the fact that these excluded evidence types may be of high value, Guttman et al. (2014) finds that the equilibrium utility of $t_0$ is lower under $h_{S''}$ than under $h_{S'}$. However, this result is also a specific example of Theorem 1, as $h_{S''}$ has more evidence that $h_{S'}$.

**Corollary 4.** If $S' \subset S'' \subset \{t_1, \ldots, t_{n-1}\}$, $h_{S''}$ has more evidence than $h_{S'}$ and so $\pi_{h_{S''}}(t) \leq \pi_{h_{S'}}(t)$ $\forall t \in T$ and $U^R \in \Upsilon$.

### 5.2. Strategic Argumentation and Honest Types

Dziuda (2011) considers a multidimensional verifiable disclosure model in which a seller tries to persuade a decision maker to purchase a good. The seller is endowed with potentially multiple pieces of verifiable evidence that are informative about the quality of the good. In Appendix H we use our characterization results to solve for the receiver optimal equilibrium in a generalized version of her model. In the main text we generalize her comparative statics result concerning honest types.

#### 5.2.1. Changing the Probability of Honest Types

In Strategic Argumentation, Dziuda shows that the equilibrium utility for each type is increasing in the probability of the honest type. It turns that this is a general property of verifiable disclosure games.

Recall the general model of honest types in Subsection 2.3, with prior distribution $h_q \in \Delta T$, where $q$ is the probability of the strategic type. Note that if $q = 1$, i.e. there are no honest types, then $\succeq_d = \succeq'_d$. Also if $q = 0$, i.e. there are no strategic types, the disclosure order is empty. Thus, in a sense, increasing the probability of strategic types is providing the sender with a larger message set. It is straightforward to verify that if $q_2 \geq q_1$, then the distribution $h_{q_2}$ has more evidence than $h_{q_1}$. This means that Theorem 1 applies, and the utility for each type is lower in equilibrium under a distribution with less honest types.

**Corollary 5.** Let $q_2 \geq q_1$, then $h_{q_2}(x) \geq ME h_{q_1}(x)$ and $\pi_{q_2}(x|U^R) \leq \pi_{q_1}(x|U^R)$ $\forall x \in T \forall U^R \in \Upsilon$.

### 6. Conclusion

This paper has two main contributions: (i) it characterizes the receiver optimal equilibrium in a large class of verifiable disclosure games and (ii) it shows that distributions which

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43 As $h_{S'}$ and $h_{S''}$ have different supports, we use the version of more evidence defined in Appendix E.

44 For any pair of strategic types or pair of honest types the likelihood ratio is constant. For $(t, s)$ and $(t', h)$, $\frac{h_{q_1}((t, s))}{h_{q_2}((t, s))} = \frac{q_1}{q_2} \leq 1-\frac{q_2}{1-q_2} = \frac{h_{q_1}((t', h))}{h_{q_2}((t', h))}$.
induce greater skepticism in the receiver are characterized by the more evidence relation. While quite general, our disclosure model does not incorporate all related examples from the literature. A prime example is that we do not allow for message dependent disclosure costs such as that of Verrecchia (1983). With disclosure costs, the equilibrium action for any type is not pinned down by the set of types with which he pools which impedes the equilibrium partition approach. Another open generalization is allowing the sender to have type dependent preferences.

References


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A. Preliminary Proofs

A.1. Proof of Lemma 1

Proof of (1)

Proof. Let \( a_1 \equiv \min_i a^*(q_i) \) and \( a_2 \equiv \max_i a^*(q_i) \). Note that because \( U^R \) is strictly concave, \( a^*(q) \geq a \iff \sum_t U^R_a(a, t)q(t) > 0 \). This means that \( \sum_t U^R_1(a_1, t)q(t) > 0 \) and \( \sum_t U^R_2(a_2, t)q(t) > 0 \). This implies that

\[
\sum_t U^R(a_2, t)(\lambda q_1(t) + (1 - \lambda)q_2(t)) = \lambda \sum_t U^R(a_2, t)q_1(t) + (1 - \lambda) \sum_t U^R(a_2, t)q_2(t) < 0
\]

This implies that \( a^*(\lambda q_1 + (1 - \lambda)q_2) < a_2 \). The argument is symmetric for \( a_1 < a^*(\lambda q_1 + (1 - \lambda)q_2) \).

Q.E.D.

A.2. Implications of Truth leaning

We recall one result from HKP concerning truth leaning equilibria. For any truth leaning equilibrium denote the payoff to the sender of type \( t \) by the function \( \pi_h : T \rightarrow \mathbb{R} \). Throughout the paper, we will sometimes refer to \( \pi_h \) as the sender payoff vector.

Lemma 3. If \((\sigma, a)\) constitute a truth leaning equilibrium, then for every \( t \in T \) exactly one of the following holds,

\[
\sigma_t(t) = 1, \text{ and } \pi_h(t|U^R) = a(t) \leq v(t), \text{ or } \tag{16}
\]

\[
\sigma_s(t) = 0 \forall s, \text{ and } \pi_h(t|U^R) > v(t) = a(t) \tag{17}
\]

Proof. See HKP Q.E.D.

B. Proofs from Section 3

B.1. Proof of Proposition 1

Proof. \( \Rightarrow \) Take equilibrium strategies \( \sigma^* : T \rightarrow \Delta T \) and \( a^* : T \rightarrow \mathbb{R} \) that generate the sender payoff vector \( \pi_h \). Due to sender optimality, it must be that \( \forall s \in P_i \ \text{Supp}(\sigma^*_s) \subset P_i \),
otherwise the type \( s \) sender would be mixing across alternatives that provide different payoffs. Also note that \( \forall t \in \bigcup_{s \in P_i} \text{Supp}(\sigma^*_s) \Delta a^*(t) = \pi_t, \) so \( \pi_t = V_h(P_i) \) by Lemma 1. Thus each \( P_i \) is a poolable set. Lastly, Since \( \pi_t \) is increasing in \( i \), no profitable sender deviation implies that \( M(P_k) \cap P_j = \emptyset \ \forall j > k \). Otherwise a type from \( P_k \) would deviate to play the strategy of some type in \( M(P_k) \cap P_j \) thereby obtaining \( \pi_j > \pi_k \). Thus \( (P_1, ..., P_m) \) is an interval partition of \( (T, \succeq_d) \).

" \( \iff \) " Let \( \sigma^i : P_i \to \Delta P_i \) and \( a^i : P_i \to \mathbb{R} \) be pooling strategies on each \( P_i \). Take candidate equilibrium strategies \( \sigma^* \) and \( a^* \) defined as follows. Let \( i(t) \equiv \{ i : t \in P_i \} \), with \( \sigma^*_i = \sigma^{i(t)}_i \) and \( a^*(t) = a^{i(t)}(t) \) for all \( t \in T \). Because \( a^i \) is pooling, \( a^*(t) = V_h(P_i(t)) = \pi_i(t) \). Now say that there is a profitable deviation for type \( t \), i.e. there exists a declaration \( s \in M(t) \), such that \( a^*(s) > V_h(P_i(t)) \). We know that \( a^*(s) \leq V_h(P_i(s)) \). Thus, because the deviation is profitable, \( V_h(P_i(t)) < V_h(P_i(s)) \). Since \( V_h(P_j) = \pi_j \) is increasing in \( j \), this implies that \( i(t) < i(s) \). But because \( (P_1, ..., P_m) \) is an interval partition of \( (T, \succeq_d) \), \( M(P_i(t)) \cap P_i(s) = \emptyset \) which means \( s \notin M(t) \), a contradiction.

Q.E.D.

**B.2. Proof of Proposition 2**

**Proof.** " \( \implies \) "

Say \( S \) is poolable under \( h \). This means there exist feasible truth leaning strategies \( \sigma : S \to \Delta(S) \) and \( a : S \to \mathbb{R} \) with corresponding sender payoff vector \( \pi_h \) satisfying that \( \forall s \in S \pi_h(s|U^R) = V_h(S) \), or more specifically \( \forall s \in (\emptyset) \cup_t \text{Supp}(\sigma_t)a(s) = (\prec) V_h(S) \).

Now say that, \( V_h \) is not downward biased on \( S \), i.e. there exists \( \tilde{S} \subset S \) such that \( V_h(M(\tilde{S}) \cap S) < V_h(S) \). Let \( \tilde{M} \equiv M(\tilde{S}) \cap S \). Take the set of declared types in \( \tilde{M} \) to be \( \tilde{M}_d \equiv \cup_t \text{Supp}(\sigma_t) \cap \tilde{M} \). Because \( M(\tilde{M}) \cap S = \tilde{M} \), we know that \( \cup_{t \in \tilde{M}} \text{Supp}(\sigma_t) \subset \tilde{M}_d \). Since \( a \) is a pooling strategy, \( a(t) = V_h(S) \forall t \in \tilde{M}_d \). This means that \( \forall t \in \tilde{M}_d \)

\[
\sum_{s} U^R_a(V_h(S), s)\sigma_s(t)h(s) = 0
\]

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If we sum over \( t \in \bar{M}_d \),

\[
0 = \sum_{t \in \bar{M}_d} \sum_s U^R_a(V_h(S), s) \sigma_s(t) h(s)
\]

\[
= \sum_{t \in \bar{M}_d, s \notin \bar{M}} U^R_a(V_h(S), s) \sigma_s(t) h(s) + \sum_{s \in \bar{M}} U^R_a(V_h(S), s) h(s) \sum_{t \in \bar{M}_d} \sigma_s(t)
\]

\[
= \sum_{t \in \bar{M}_d, s \notin \bar{M}} U^R_a(V_h(S), s) \sigma_s(t) h(s) + \sum_{s \in \bar{M}} U^R_a(V_h(S), s) h(s)
\]  \quad (18)

The last equality follows from the fact that \( \cup_{t \in \bar{M}} \text{Supp}(\sigma_t) \subset \bar{M}_d \) and so \( \sum_{t \in \bar{M}_d} \sigma_s(t) = 1 \forall s \in \bar{M} \). By assumption \( V_h(\bar{M}) < V_h(S) \) and so the second term in (18) is negative i.e.,

\[
\sum_{s \in \bar{M}} U^R_a(V_h(S), s) f(s) < 0
\]

now consider the first term. We know from Lemma 3, that every type \( s \) that declares some other type \( t \) must have lower value than the action \( a(t) \) type \( s \) obtains. This means that for \( s \notin \bar{M} \) such that \( \sigma_s(t) > 0 \), we have that \( v(s) < a(t) = V_h(S) \). This means that every summand in the first term in (18), \( U^R_a(V_h(S), s) < 0 \). But these two facts imply that the expression in (18) is negative, a contradiction.

" \( \Leftarrow \) "

Now say that \( V_h \) is downward biased on \( S \). Bipartition \( S \) into \( U \equiv \{ t : v(t) \geq V_h(S) \} \), \( D \equiv U^c \). Now consider any feasible strategy \( \eta : D \to DU \) and define \( \sigma : S \to DS \) as \( \sigma_t(s) = \eta_t(s) \) for \( t \in D \) and \( \sigma_t(t) = 1 \) for \( t \in U \). call the set of all such \( \sigma \) strategies \( Z \). First, note that\( Z \) is non-empty. \( Z \) is empty only if \( \exists s, \text{ s.t. } M(s) \cap U = \emptyset \), i.e. \( M(s) \cap S \subset D \) and \( \forall t \in M(s) \cap S v(t) < V_h(S) \). By Lemma 1 we have \( V_h(M(s) \cap S) < V_h(S) \), contradicting the fact that \( V_h \) is downward biased on \( S \). Thus the restricted set of strategies \( Z \) is nonempty. Define the receiver’s unique truth-leaning best response to \( \sigma \in Z \) as \( a^\sigma : S \to \mathbb{R} \). Now consider the problem,

\[
\min_{\sigma \in Z} f(\sigma) \equiv \sum_{t \in U} (a^\sigma(t) - V_h(S))^2
\]  \quad (19)

\( \sum_{t} U^R(a, t) \sigma_t(s) f(t) \) is continuous in \( a \) and \( \sigma \). The receiver can be taken to maximize over a compact set \( ([\min_t v(t), \max_t v(t)]) \) so the theorem of the maximum holds. Moreover the maximum is unique, so for each \( t \in U \) \( a^\sigma(t) \) is continuous in \( \sigma \). Since \( Z \) is a compact set, the problem has a solution \( \sigma^* \) by Weierstrass theorem. Let the corresponding receiver best response be \( a^{\sigma^*} \).
Now say $f(\sigma^*) > 0$, we will find a contradiction. $f(\sigma^*) > 0 \implies \exists t \in U : a^\sigma(t) \neq V_h(S)$. Notice that $\cup_s \text{Supp}(\sigma_s) = U$, so by Lemma 1 $U^B \equiv \{s \in U : a^\sigma(s) \geq V_h(S)\}$ and $U^W \equiv \{s \in U : a^\sigma(s) < V_h(S)\}$ are both nonempty. Let

$$X \equiv \{s : \exists t \in \text{Supp}(\sigma^*_t), V_h(S) > a^\sigma(t) > v(s)\}$$

$$Y \equiv \{s : U^B \cap M(s) \neq \emptyset\}$$

$X$ is the set of types that obtain an action with positive probability that is less than $V_h(S)$, but greater than its own value. $Y$ is the set of types that have value less than $V_h(S)$, but have the ability to obtain an action greater than $V_h(S)$. We will show that $X \cap Y \neq \emptyset$.

First note that $X \neq \emptyset$. This is because $U^W$ is non-empty, meaning there exists a type $t \in U$ with $a(t) < V_h(S)$. But by Lemma 1, the best response action must be in between the values of the types that declare it. Since $t \in U$, $v(t) \geq V_h(S)$ and by the definition of $\sigma^*$, we have that $\sigma^*_t(t) = 1$. This in turn means there must exist an $s$ that declares $t$ such that $v(s) < a(t)$, or in other words $X$ is non-empty. Let $\tilde{M} \equiv M(X) \cap S$

Now assume in search of a contradiction that $X \cap Y = \emptyset$. Consider the sender strategy of only types in $\tilde{M}$, and recompute receiver best responses $a^\sigma_{\tilde{M}} : \tilde{M} \rightarrow \mathbb{R}$. That is, let

$$a^\sigma_{\tilde{M}}(s) \equiv \arg \max_{a} \sum_{t \in \tilde{M}} U^R(a, t)\sigma^*_t(s)f(t)$$

We claim that $a^\sigma_{\tilde{M}}(s) < V_h(S) \forall s \in \tilde{M}$. To see this, first note that $a^\sigma(s) < V_h(S) \forall s \in \tilde{M}$, because by assumption $X \cap Y = \emptyset$, i.e. types in $X$ cannot obtain a higher action than $V_h(S)$. Now inspect the types that declare some $s \in \tilde{M}$ under $\sigma^*$, but are excluded by the restriction to types in $\tilde{M}$. More formally, call this set

$$W \equiv \{t \notin \tilde{M} : \exists s \in \tilde{M} \cap \text{Supp}(\sigma^*_t)\}$$

Now take $t \in W$. By definition $t \notin X$ because $t \notin \tilde{M}$ and $X \subset \tilde{M}$. Now consider $s \in \tilde{M} \cap \text{Supp}(\sigma^*_t)$. Because, $s \in \tilde{M}$, $a^\sigma(s) < V_h(S)$. Then, because $t \notin X$ and $s \in \text{Supp}(\sigma^*_t)$, $v(t) \geq a^\sigma(s)$. Now when we restrict $\sigma^*$ to the types in $\tilde{M}$, we decrease $\sigma_t(s)$. But by Lemma 1, this decreases the best response action of the receiver. Thus since $a^\sigma(s) < V_h(S) \forall s \in \tilde{M}$ we also have that $a^\sigma_{\tilde{M}}(s) < V_h(S) \forall s \in \tilde{M}$. Also by Lemma 1, the best response action to the entire set $\tilde{M}$ must be in between the best response actions to each type for any strategy, i.e.

$$\min_{s \in \tilde{M}} a^\sigma_{\tilde{M}}(s) \leq V_h(\tilde{M}) \leq \max_{s \in \tilde{M}} a^\sigma_{\tilde{M}}(s)$$
This implies that $V_h(\bar{M}) < V_h(S)$. But this is a contradiction, because the fact that $V_h$ is downward biased on $S$ means that $V_h(\bar{M}) \geq V_h(S)$.

Thus $X \cap Y \neq \emptyset$. Let $s \in X \cap Y$. This means $\exists t', t'' \in S$ such that $t' \in \text{Supp}(\sigma^*_s)$ with $v(s) < a^{\sigma^*_s}(t) < V_h(S)$ and $t'' \in \bar{M}$ with $a^{\sigma^*_s}(t'') \geq V_h(S)$. Now construct a new strategy $\tilde{\sigma} \in Z$ as follows.

$$
\tilde{\sigma}(\tilde{t}) = \begin{cases}
  \sigma^*_s(\tilde{t}) & \tilde{s} \neq s, \tilde{t} \neq t', t'' \\
  \sigma^*_s(\tilde{t}) - \varepsilon & \tilde{s} = s, \tilde{t} = t' \\
  \sigma^*_s(\tilde{t}) + \varepsilon & \tilde{s} = s, \tilde{t} = t''
\end{cases}
$$

with $\varepsilon > 0$ small enough such that this is a feasible strategy. Because $V_h(S) > a^{\sigma^*_s}(t') > v(s)$, and we are decreasing the probability that $s$ declares $t'$, this change increases the best response $a^{\sigma^*_s}(t)$ by Lemma 1. This decreases $(a^{\sigma^*_s}(t') - V_h(S))^2$. Similarly because $a(t'') \geq V_h(S) > v(s)$ and we are increasing the probability that $s$ declares $t''$, this change decreases the best response $a^{\sigma^*_s}(t'')$ by Lemma 1. This decreases $(a^{\sigma^*_s}(t'') - V_h(S))^2$. All other best responses remain unchanged. This change thereby decreases the expression in (19) so that $f(\tilde{\sigma}) < f(\sigma^*_s)$, contradicting the minimality of $\sigma^*_s$. So at the minimum we must have $f(\sigma^*_s) = 0 \implies a^{\sigma^*_s}(t) = V_h(S) \forall t \in U$. Q.E.D.

**B.3. Proof of Lemma 2**

**Proof.** Let $S^* \in \arg \max_{S \subset S} V_h(B(\bar{S}) \cap S)$ with value $\bar{V}$, let $\bar{B} \equiv B(S^*) \cap S$. Using the upper contour version of downward biased in (8), we must prove that,

$$
V_h(B(\bar{S}) \cap \bar{B}) \leq V_h(\bar{B}) \quad \forall \bar{S} \subset \bar{B}
$$

Suppose not, and take $\bar{B}' \equiv B(\bar{S}) \cap \bar{B}$ such that $V_h(\bar{B}') > V_h(\bar{B})$. Note that $B(\bar{B}') \cap S = \bar{B}'$, which contradicts the maximality of $\bar{B}$ in the above problem. Thus each maximizer of the above problem is poolable.

Now take $J \subset \arg \max_{\bar{S} \subset S} V_h(B(\bar{S}) \cap S)$ with $J \equiv (S_1, \ldots, S_c)$ and $\bar{B}_i \equiv B(S_i) \cap S$ and $\bar{B} \equiv \bigcup_{k=1}^c \bar{B}_k$. Note that because each $\bar{B}_i$ is poolable, for each $i$ we have $V_h(B_i \cup \bigcup_{k=1}^{i-1} \bar{B}_i) \geq V_h(B_i) = \bar{V}$. Since $\bar{B}$ is the disjoint union of these sets, i.e. $\bar{B} = \bigcup_{i=1}^c (B_i \setminus \bigcup_{k=i-1}^{i-1} \bar{B}_i)$, we have by Lemma 1 that $V_h(\bar{B}) = \bar{V}$. Thus $\bar{B} \in \arg \max_{\bar{S} \subset S} V_h(B(\bar{S}) \cap S)$, and so by the previous argument $\bar{B}$ is poolable. Q.E.D.

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45 If $V_h(S) = a^{\sigma^*_s}(t'')$ then this change actually increases $(a^{\sigma^*_s}(t'') - V_h(S))^2$. However, for small enough $\varepsilon$ this increase will be second order.
B.4. Proof of Proposition 3

Proof. Algorithm 1 produces a partition of $T$ into disjoint sets $(P_1, P_2, ..., P_m)$. Now we argue that this partition satisfies the requirements of Proposition 1, and thereby constitutes an equilibrium partition. We know from Lemma 2 that each $P_i$ is a poolable set. We must therefore only check that $i > j \implies V_h(P_i) > V_h(P_j)$ and that $i > j \implies M(P_j) \cap P_i = \emptyset$. First, suppose that $V_h(P_i) \geq V_h(P_{i+1})$. Note that $B(P_i \cup P_{i+1}) \cap S_i = P_i \cup P_{i+1}$. This is because, $B(P_{i+1}) \cap S_{i+1} = P_{i+1}$ by definition, and $B(P_i) \cap S_{i+1} \subset P_i \cup P_{i+1}$ by the construction in algorithm 1. Thus, we have $V_h((P_i \cup P_{i+1}) \cap S_i) \geq V_h(P_{i+1})$ by Lemma 1. But this means that $P_{i+1}$ could not have been the output of the algorithm at as we have found a larger (in the sense of set inclusion) maximizing set than $P_{i+1}$. Second, notice that for $i < j$, $P_j \subset S_i$. But $B(P_i) \cap S_i = P_i$. Since $P_i \cap P_j = \emptyset$, this means that $B(P_i) \cap P_j = \emptyset$, or using lower contour sets as in Definition 2, $M(P_j) \cap P_i = \emptyset$. This means that the partition $(P_1, ..., P_m)$ satisfies the conditions of Proposition 1, and is therefore an equilibrium partition. Q.E.D.

B.5. Proof of Proposition 4

Proof. Take the equilibrium partition $(P_1, ..., P_m)$ where $t \in P_i$. Recall that from Proposition 1 for every $i > j$, we have that (i) $V_h(P_i) > V_h(P_j)$, and (ii) $M(P_j) \cap P_i = \emptyset$. In addition, from Proposition 2 we have that $V_h$ is downward biased on $P_i \; \forall i$. We know that $V_h(P_k) = \pi_h(t|U^R)$, so we must prove that the solution to the problem on the right hand side of (9) is $V_h(P_k)$. Consider setting $S_a^* \equiv \cup_{k=1}^i P_k$ and $S_b^* \equiv \cup_{k=1}^m P_k$. In words, $S_a^*$ is the union of parts of the equilibrium partition that have value less than or equal to $P_i$, and $S_b^*$ is the union of parts of the equilibrium partition that have value greater than or equal to $P_i$. Note that because of (ii) $M(S_a^*) = S_a^*$ and $B(S_b^*) = S_b^*$. Using these choices in equation (9) gives,

$$V_h(M(S_a^*) \cap B(S_b^*)) = V_h(S_a^* \cap S_b^*) = V_h(P_i)$$

We prove the result in two steps. First, we show that given $S_a^*, S_b^*$ solves the partial problem $\max_{\{s_i : t \in S_i\}} V_h(M(S_a^*) \cap B(S_b^*))$. This shows that $V_h(P_i)$ is achievable in the problem in (9). Second, we show that for any feasible $S_a$, choosing $S_b = S_b^*$ gives $V_h(S_a \cap S_b^*) \geq V_h(P_i)$. This means that $\forall \{S_a : t \in S_a\}$ the value to the partial problem $\max_{\{S_b : t \in S_b\}} V_h(M(S_a) \cap B(S_b)) \geq V_h(P_i)$. This means that choosing $S_a = S_a^*$ achieves the minimum value, establishing the result.

Step 1: Take any other feasible $S_b$. We can write $B(S_b) \cap M(S_a^*) = \cup_{k=1}^i (B(S_b) \cap P_k)$. By the fact that $V_h$ is downward biased on each part $P_k$, we have that for $k \leq i$ (whenever
non-empty) \( V_h((B(S_b) \cap P_k) \leq V_h(P_k) \leq V_h(P_i) \). But this means that by Lemma 1 that \( V_h(\bigcup_{k=1}^m (B(S_b) \cap P_k)) \leq V_h(P_i) \). Since \( S_b^* \) achieves \( V_h(P_i) \), we have that \( S_b^* \) solves the partial problem \( \max_{\{S_i:t \in S_b\}} V_h(M(S_b^*) \cap B(S_b)) \).

**Step 2:** Take any feasible \( S_a \). Notice that \( B(S_b^*) \cap M(S_a) = \bigcup_{k=1}^m (M(S_a) \cap P_k) \). By the fact that \( V_h \) is downward biased on each part \( P_k \), we have that for \( k \geq i \) (whenever non-empty) \( V_h((M(S_a) \cap P_k) \geq V_h(P_k) \geq V_h(P_i) \). But this means that by Lemma 1 that \( V_h(\bigcup_{k=1}^m (M(S_a) \cap P_k)) \geq V_h(P_i) \). Thus, since \( S_b^* \) is feasible, we have that the solution to the partial problem \( \max_{\{S_i:t \in S_b\}} V_h(M(S_a) \cap B(S_b)) \geq V_h(P_i) \). Since choosing \( S_a = S_a^* \) achieves \( V_h(P_i) \), this choice achieves the minimal value.

Q.E.D.

**C. Proofs from Section 4**

**C.1. Proof of Corollary 1**

**Proof.** Using \( E_f^\epsilon \) in Proposition 7, we obtain an interval partition \( P = (P_1, ..., P_m) \) of \((X, \geq_{f/g})\) such that (12) and (13) hold. If \( P \) is not the trivial partition take \( W \equiv P_1 \) and \( U \equiv \bigcup_{i=2}^m P_i \). Because of (12) and the iterated expectation property \( E_f^\epsilon(W) < E_f^\epsilon(X) < E_f^\epsilon(U) \). But since \( E_f^\epsilon \) is downward biased on \((X, \geq_{f/g})\) and \( W = M_{\geq_{f/g}}(W) \cap X \), we have that \( E_f^\epsilon(W) \geq E_f^\epsilon(S) \). This contradiction implies that \( P \) must be the trivial partition, in which case (13) gives \( E_f^\epsilon(S) \leq E_g^\epsilon(S) \). The proof is analogous when \( E_g^\epsilon \) is downward biased on \((X, \geq_{f/g})\). Q.E.D.

**C.2. Proof of Proposition 6**

**Proof.** Let \( m : S \rightarrow \mathbb{R} \) be defined as \( m(s) \equiv U_a^R(V_f(S), s) \). If \( S \) is poolable under \( f \), then by Proposition 2, \( V_f \) is downward biased on \((S, \geq_d)\). Notice that because \( U_a^R \) is strictly concave,

\[
V_g(\tilde{S}) > V_f(S) \iff E_g^m(\tilde{S}) > E_f^m(S) = 0 \quad \forall \tilde{S} \subset S, g \in \Delta S
\]  

(20)

Thus because \( V_f \) is downward biased on \((S, \geq_d)\), \( E_f^m \) is downward biased on \((S, \geq_d)\). Because \( f \) has more evidence than \( g \), \((S, \geq_{f/g})\) is a refinement of \((S, \geq_d)\), and so \( E_f^m \) is also downward biased on \((S, \geq_{f/g})\). Thus we can apply Corollary 1 to obtain that \( E_f^m(S) \leq E_g^m(S) \). Finally by (20), we have that \( V_f(S) \leq V_g(S) \).

The proof is symmetric when \( S \) is poolable under \( g \). Q.E.D.
C.3. Proof of Proposition 7

Proof. We prove the existence of a partition satisfying (12) and (13) by construction. Specifically the output of algorithm 2 satisfies (12) and (13).

Input \( (X, \geq_{f/g}) \) for some \( f, g \in \Delta X \), and the conditional expectation function \( E^\varepsilon_f \) associated with \( \varepsilon : X \rightarrow \mathbb{R} \). Because \( X \) is a finite set, and the algorithm repeatedly returns coarser and coarser partitions, the process must terminate at some stage \( T \). At this point \( P^T = P^{T+1} \) which means that each decreasing sequence in \( P^T \) is a singleton. But this implies that \( E^\varepsilon_f(P^T_T) < \ldots < E^\varepsilon_f(P^T_n) \). Thus \( P^T \) satisfies (12). We will show that \( P^T \) also satisfies (13), thereby proving the result.

Consider the partition \( P_i = (P^i_1, \ldots, P^i_m) \) generated at stage \( i > 1 \). Each part \( P^i_j \) is the union of an "interval" of parts from the previous partition \( P^{i-1} \). More specifically, for each \( j \) there exists \( \bar{k}(j) \leq \tilde{k}(j) \) such that \( P^i_j = \bigcup_{l=\bar{k}(j)}^{\tilde{k}(j)} P^{i-1}_l \). Because \( P^{i-1} \) is an interval partition of \( (X, \geq_{f/g}) \), and \( E^\varepsilon_f(P^{i-1}_l) \) is decreasing for \( \bar{k}(j) \leq l \leq \tilde{k}(j) \) we can use Proposition 5 on the set \( P^i_j \), to obtain,

\[
\frac{1}{G(P^i_j)} \sum_{l=\bar{k}(j)}^{\tilde{k}(j)} E^\varepsilon_f(P^{i-1}_l)G(P^{i-1}_l) \geq \frac{1}{F(P^i_j)} \sum_{l=\bar{k}(j)}^{\tilde{k}(j)} E^\varepsilon_f(P^{i-1}_l)F(P^{i-1}_l)
\]

Consider a given part \( P^T_k \) of the final partition \( P^T \). We can iteratively use the above inequality for every part of the interval partition at every stage of the algorithm to obtain
the following string of inequalities,

\[
E_g(P^T_k) = \frac{1}{G(P^T_k)} \sum_{t_i \in P^T_k} \varepsilon(t_i) g(t_i) = \frac{1}{G(P^T_k)} \sum_{P^2_j \subseteq P^T_k} \left( \frac{1}{G(P^T_j)} \sum_{t_i \in P^2_j} \varepsilon(t_i) g(t_i) \right) G(P^2_j) \\
\geq \frac{1}{G(P^T_k)} \sum_j \left( \frac{1}{F(P^2_j)} \sum_{t_i \in P^2_j} \varepsilon(t_i) f(t_i) \right) G(P^2_j) = \frac{1}{G(P^T_k)} \sum_j E_f(P^2_j) G(P^2_j) \\
\cdots \\
\geq \frac{1}{G(P^T_k)} \left( \frac{1}{F(P^T_k)} \sum_{P^T_{l-1} \subseteq P^T_k} E_f(P^T_{l-1}) G(P^T_{l-1}) \right) G(P^T_k) \\
= E_f(P^T_k)
\]

Combining these inequalities we have that \( E_g(P^T_k) \geq E_f(P^T_k) \) \( \forall k \) establishing (13) and thereby Proposition 7.

\[ Q.E.D. \]

D. Uniqueness

**Corollary 6.** The equilibrium partition and thereby sender payoff vector is unique.

**Proof.** Take two equilibrium partitions \( P = (P_1, ..., P_m) \) and \( Q = (Q_1, ..., Q_l) \) such that \( P \neq Q \). Let \( P_k = Q_k \) for \( k > r \), and with WLOG \( P_r \not\subset Q_k \) \( \forall k \). We assume that \( w \) is the largest index such that \( Q_w \cap P_r \neq \emptyset \). Note that \( B(P_r \cap Q_k) \cap Q_k = P_r \cap Q_k \) so whenever non-empty, Proposition 2 implies \( V_h(P_r \cap Q_k) \leq V_h(Q_k) \). This means that by Lemma Lemma 1 \( V_h(P_r) = V_h(\cup_{k=1}^w P_r \cap Q_k) \leq V_h(Q_w) \). By the symmetric logic we obtain that \( V_h(P_r) = V_h(Q_w) \). Now, \( V_h(P_r \cup \cup_{k=1}^{w-1} P_r \cap Q_k) \geq V_h(P_r) = V_h(Q_w) \) and \( V_h(P_r \cup \cup_{k=1}^{w-1} P_r \cap Q_k) \leq V_h(Q_l) \) for some \( l < w \). But this is a contradiction as \( V_h(Q_w) > V_h(Q_l) \).

\[ Q.E.D. \]

E. Generalization to Non-Full Support Distributions

E.1. Results from Section 3

It turns out that all our construction results go through without further refinement of the equilibrium concept. Adding zero probability types is like order embedding \((\text{Supp}(h), \succeq_d)\)
into some larger ordered type set \((T, \succ')\). This change enlarges the message sets of positive probability types. One might think that we may have to deter positive probability types from taking advantage of these new messages by appropriately refining the off path best response to these zero probability types. Also the set of equilibrium payoffs changes by adding these zero probability types.

However the receiver optimal equilibrium does not change and does not involve declaration of zero measure types with positive probability. This is because the truth leaning refinement which identifies the commitment solution ensures that zero probability types will be off path. Consider some \(t \in T \setminus \text{Supp}(h)\). Suppose \(t \in \bigcup_{t \in \text{Supp}(h)} \text{Supp}(\sigma_t)\), with best response \(a(t)\), for truth leaning \(a\) and \(\sigma\). By Lemma 1 there exists some type \(s \in \text{Supp}(h)\) such that \(\sigma_s(t) > 0\) and \(v(s) \geq a(t)\). But since \(s \neq t\), and \(\pi_h(s|U^R) = a(t)\), this is a contradiction to Lemma 3 which says that \(\pi(s|U^R) \leq v(t) \implies \sigma_s(s) = 1\). This allows us to simply ignore zero probability types in our construction of equilibrium. We do have to slightly adjust the truth leaning refinement to allow for more flexibility for off path declarations of zero probability types,

\[
\text{arg max}_{s' \in \text{M}(t)} a(s') \implies \sigma_t = 1
\]

\[
s \notin \bigcup_t \text{Supp}(\sigma_t) \implies \begin{cases} a(s) = v(s) & \forall s \in \text{Supp}(h) \\ a(s) = \min_{s' \in B(s)} v(s) & \forall s \notin \text{Supp}(h) \end{cases}
\]

Without this modification, the equilibrium may not exist. Notice that the modification still selects a PBNE. With the above refinement we will obtain the same equilibrium payoff vector for types in \(\text{Supp}(h)\) as when we restrict the \(T\) to \(\text{Supp}(h)\). Thus we can simply consider \(\tilde{T} \equiv \text{Supp}(h)\) and apply the results of Section 3.

### E.2. Results from Section 4

In this section we compare equilibrium utilities for distributions without full support. Because the supports may differ across the distributions we cannot simply discard the zero probability types as we did in the previous subsection. The main problem that arises is that the best response function \(V_h\) will not be defined on all sets if \(h\) does not have full support. Broadly, we must make this restriction explicit in our preliminary results. However, we show that our main theorem-Theorem 1 goes through with little modification.

First we make some notions robust to general distributions. Consider some finite set \(X\), and \(f, g \in \Delta X\) such that without loss \(X = \text{Supp}(f) \cup \text{Supp}(g)\).^46 Define the \(f - g\) likelihood

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^46 We can disregard elements not in \(\text{Supp}(f) \cup \text{Supp}(g)\) by the same logic presented above.
ratio order $\geq_{f/g}$ as
\[ x \geq x' \implies f(x)g(x') \geq_{f/g} f(x')g(x). \]

This definition reduces to the one in the main text for full support distributions. It is straightforward to verify that $(X, \geq_{f/g})$ is a completely preordered set.

For $(T, \succeq_d)$ and two distributions $f, g \in \Delta T$ where $T = \text{Supp}(f) \cup \text{Supp}(g)$, we say that $f$ has more evidence than $g$ if,
\[ \forall t, t' \in T \ t \succeq_d t' \implies f(t)g(t') \geq f(t')g(t). \]

We now state the revised versions of Proposition 7, Corollary 1, Proposition 6, and Theorem 1.

**Proposition 8.** Let $X$ be a finite set, and $\varepsilon : X \to \mathbb{R}$ and $f \in \Delta X$ with full support. For any distribution $g \in \Delta X$ with corresponding measure $G : 2^X \to \mathbb{R}_+$, there exists an interval partition $P = (P_1, ..., P_m)$ of $(X, \geq_{f/g})$ with,
\[ E^\varepsilon_f(P_1) < ... < E^\varepsilon_f(P_m), \text{ and } E^\varepsilon_f(P_i) \leq E^\varepsilon_g(P_i) \forall i : G(P_i) > 0. \] (21)

Moreover, $P$ is independent of $g$ given $\geq_{f/g}$.

**Lemma 4.** Let $X$ be a finite set, $f \in \Delta X$ where $f$ has full support and $\varepsilon : X \to \mathbb{R}$. If $E^\varepsilon_f$ is downward biased on $(X, \geq_{f/g})$, then $E^\varepsilon_f(X) \leq E^\varepsilon_g(X)$.

**Proposition 9.** Let $f, g \in \Delta T$ have more evidence than $g \in \Delta T$. If $S \subset \text{Supp}(f)$ is poolable under $f$ and $S \cap \text{Supp}(g) \neq \emptyset$, then $V_f(S) \leq V_g(S)$.

**Theorem 2.** Let $f, g \in \Delta T$, where $T = \text{Supp}(f) \cup \text{Supp}(g)$. $f \geq_{ME} g \implies$
\[ \pi_f(t|U^R) \leq \pi_g(t|U^R) \forall t \in \text{Supp}(f) \cap \text{Supp}(g), \forall U^R \in \Upsilon. \]

Moreover, if $\frac{f(t)}{g(t)} < \frac{f(t')}{g(t')}$ for some $t \succeq_d t'$ then $\exists U^R \in \Upsilon$ such that $\pi_f(t'|U^R) > \pi_g(t'|U^R)$.

The only real amendment in the above results is in Theorem 2. The reason for this is that we cannot compare the utility of types that are not in $\text{Supp}(f) \cap \text{Supp}(g)$ simply because $\pi_f(t|U^R)$ does not exist if $f(t) = 0$.

The proofs of the first three results are identical to those of their original versions. The proof of Theorem 2 is slightly modified.
F. Proof of Theorem 1 and Theorem 2

Proof. “$$\implies$$”

The key observation in proving the theorem is that if $$f \geq_{ME} g$$ then $$Z^g \equiv \{ t : g(t) = 0 \}$$ is an upper contour set of $$(T, \succeq_d)$$ and $$Z^f \equiv \{ t : f(t) = 0 \}$$ is a lower contour set of $$(T, \succeq_d)$$, i.e. $$B(Z_g) = Z_g$$ and $$M(Z_f) = Z_f$$.

Let $$P^f = (Z^f, P^f_1, \ldots, P^f_m)$$ and $$P^g = (P^g_1, \ldots, P^g_i, Z^g)$$ be the equilibrium partitions under $$f$$ and $$g$$ respectively. For all $$t \in \text{Supp}(f) \cap \text{Supp}(g)$$ we have that $$t \in P^f_i \cap P^g_j$$ for some $$i, j$$. We will show that $$\pi_f(t|U^R) = V_f(P^f_i) \leq V_g(P^g_l) = \pi_g(t|U^R)$$ proving the result. Now let $$D^g \equiv \cup_{k=1}^i P^g_k$$ and let $$U^f \equiv \cup_{k=1}^m P^f_k$$.

Now consider the set $$R \equiv U^f \cap D^g$$. This set is the union of disjoint subsets, $$R = \cup_{k=1}^m (P^f_k \cap D^g)$$. Also whenever non-empty $$M(P^f_k \cap D^g) \cap P^f_k = P^f_k \cap D^g$$, because $$M(D^g) = D^g$$ as $$P^g$$ is an equilibrium partition. Now since $$P^f_k$$ is poolable under $$f$$, Proposition 2 tells us that $$V_f$$ is downward biased on $$(P^f_k, \succeq_d)$$. This means that, whenever non-empty, $$V_f(P^f_k \cap D^g) \geq V_f(P^f_k)$$. Also because $$P^f$$ is an equilibrium partition, Proposition 1 tells us that $$V_f(P^f_k) \geq V_f(P^f_k) \forall k \geq i$$. Putting these together we get that $$V_f(P^f_k \cap D^g) \geq V_f(P^f_k) \forall k \geq i$$. Since $$R$$ is the union over these disjoint sets, by using Lemma 1 we have that $$V_f(R) \geq V_f(P^f_i)$$. Now consider the problem,

$$\max_{\tilde{S} \subset D^g \setminus Z^f} V_f(B(\tilde{S}) \cap (D^g \setminus Z^f))$$  (23)

with corresponding solution $$\tilde{S}$$ with $$\tilde{R} \equiv B(\tilde{S}) \cap (D^g \setminus Z^f)$$. Because $$B(R) \cap (D^g \setminus Z^f) = R$$, we have that $$V_f(\tilde{R}) \geq V_f(R)$$. Moreover $$\tilde{R} \subset \text{Supp}(f) \cap \text{Supp}(g)$$ and so because of Lemma 2, we know that $$\tilde{R}$$ is poolable under $$f$$. Using Proposition 9, this means that $$V_g(\tilde{R}) \geq V_f(\tilde{R})$$. Now notice that by Proposition 3,

$$V_g(P^g_j) = \max_{\tilde{S} \subset D^g} V_g(B(\tilde{S}) \cap D^g)$$

Thus since $$\tilde{R}$$ is feasible in this problem, we have by optimality that $$V_g(P^g_j) \geq V_g(\tilde{R})$$. Putting this string of inequalities together we have that

$$V_g(P^g_j) \geq V_g(\tilde{R}) \geq V_f(\tilde{R}) \geq V_f(R) \geq V_f(P^f_i)$$

proving the result.

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47 Since we do not use any refinement beyond truth leaning, we can set the off path actions for declarations in $$Z^g$$ and $$Z_f$$ arbitrarily to guarantee that this is an equilibrium.
Let $A = [a, \bar{a}]$. Define $S \equiv M(t) \cap B(t')$, and $\tilde{S} \equiv S \setminus \{t, t'\}$. By assumption, $\{t, t'\} \subset \text{Supp}(f) \cap \text{Supp}(g)$. We prove the case in which $F(\tilde{S}) \geq G(\tilde{S})$; the opposite case is symmetric. Let $U^R$ be quadratic loss, with $v(s) = \bar{a} \forall s \notin M(t)$, $v(s) = a \forall s \in M(t) \setminus S$, $v(s) = a \forall s \in \tilde{S} \cup \{t', \}$, and $v(t) = A$. The equilibrium partition is clearly $(M(t) \setminus S, S, M(t)^c)$ and prior independent. For any $h \in \Delta T$ with $H(S) > 0$, $V_f(S) = (H(\tilde{S} \cup \{t'\})a + h(t)A)/H(S)$. Thus $V_f(S) > V_g(S)$ if $\frac{f(t)}{F(S \cup \{t'\})} < \frac{g(t)}{G(S \cup \{t'\})}$ which holds by assumption. Thus $\pi_f(t'|U^R) > \pi_g(t'|U^R)$. Q.E.D.

G. Athey’s Single Crossing Result

We say that a function $\varepsilon : X \to \mathbb{R}$ is single crossing from below on $(X, \geq)$ if $\varepsilon(x) \geq 0 \implies \varepsilon(x') \geq \forall x' \geq x$\(^{48}\). For two distributions $f, g \in \Delta X$, Athey’s result is that if $\varepsilon$ is single crossing, then the expectation of $\varepsilon$ is single crossing as we “increase” the distribution in a monotone likelihood ratio dominance sense. This result, which we formalize below, is used in her paper to show monotone comparative statics in a variety of applications.

**Corollary 7.** (Athey (2002))

If $\varepsilon : X \to \mathbb{R}$ is single crossing from below on $(X, \geq_{f/g})$, then $E^\varepsilon_g(X) \geq 0 \implies E^\varepsilon_f(X) \geq 0$.

In order to show this result we note that Proposition 7 treats $f$ and $g$ asymmetrically. Using our methodology we can show that there also exists an interval partition $P = (P_1, ..., P_m)$ of $(X, \geq_{f/g})$ such that,

$$E^\varepsilon_g(P_1) > ... > E^\varepsilon_g(P_m)$$ \hspace{1cm} (24)

$$E^\varepsilon_f(P_i) \geq E^\varepsilon_g(P_i) \forall i$$ \hspace{1cm} (25)

**Proof.** We can derive this result simply from the reversed version of Proposition 7 in (24) and (25). Start with $E^\varepsilon_g(X) \geq 0$. and consider an interval partition $P = (P_1, ..., P_m)$ of $(X, \geq_{f/g})$ satisfying (24) and (25). Because $E^\varepsilon_g(P_i)$ is decreasing in $i$, it is single crossing from above in $i$. But because $\varepsilon$ on $(X, \geq_{f/g})$, is single crossing from below, so is $E^\varepsilon_g(P_i)$ in $i$. These are only not in contradiction with each other if $E^\varepsilon_g(P_i)$ is constant sign. Because we assumed $E^\varepsilon_g(X) \geq 0$ this sign must be positive. We will show that (24) means that $\forall i \ E^\varepsilon_f(P_i) \geq 0$ which combined with (25) which says that $\forall i \ E^\varepsilon_f(P_i) \geq E^\varepsilon_g(P_i)$ implies that $\forall i \ E^\varepsilon_f(X) \geq 0$. Q.E.D.

\(^{48}\)We use a notion of single crossing usually referred to as weak single crossing. Analogous notions of single crossing from above could be used to derive analogous results. Other stronger notions are sometimes employed which only permit the function to take zero values on a singleton set (strict single crossing) or on an interval (single crossing). We could also use our methodology to derive analogous results with these related definitions.
H. Equilibrium construction in Dziuda’s Strategic Argumentation

Each piece of evidence is either persuasive or dissuasive, and the quality of the good only depends on the numbers of each type of evidence. Agents are either strategic, and can present any subset of evidence in their possession, or honest, and are restricted to truthfully revealing all evidence in their possession. This is one of the few models to address partial disclosures and honest types. The paper provides a full equilibrium characterization and some comparative statics results. We will use our results to show a simple derivation as well as some generalizations of these results.

H.1. Model

We can embed this model in our framework using the honest types example from Subsection 2.3. For some \((L, R) \in \mathbb{Z}^2_+\), The payoff relevant type space is given by \(T' = \{(a, b) \in \mathbb{Z}^2_+ : a \leq L, b \leq R\}\). The total type space is given by \(T = T' \times \{s, h\}\), with the interpretation that \(t = (l, r, h(s))\) is an honest (strategic) agent with \(l\) dissuasive and \(r\) persuasive arguments. We generalize the original model and assume there is some fixed prior \(h \in \Delta T\).

Let the set of honest types be \(H\) and the set of strategic types be \(S\).

The receiver (or decision maker) has utility \(U^R : A \times T' \to \mathbb{R}\) which is strictly concave over \(a \in A\) which is a compact subset of \(\mathbb{R}\). Let \(v : T' \to \mathbb{R}\) defined by \(v(l, r) \equiv \arg \max_{a \in A} U^R(a, l, r)\) denote the best response function, and \(V_h : 2^T \to \mathbb{R}\) be the best response to subsets. The optimal action function \(v(l, r)\) is increasing in the number of persuasive arguments \(r\), decreasing in the number of dissuasive arguments \(l\), and is independent of whether the agent is honest or strategic.

Some additional notation will be useful. For \(l \leq L\), let \(\bar{l} \equiv [l, L] \cap \mathbb{Z}\) and \(\underline{l} \equiv [0, l] \cap \mathbb{Z}\). Define the sets \(\tau\) and \(\rho\) analogously. For any two sets \(A, B\), let \((A, B) \equiv A \times B \times \{s, h\}\), i.e. \((\bar{l}, \rho)\) is the set of honest and strategic types with more dissuasive arguments than \(l\) and fewer persuasive arguments than \(r\).

---

49 Strategic Argumentation also considers an extension in which there are two types of strategic agents. One prefers high actions while the other prefers low actions.

50 Dziuda considers a model with a continuum of types in order to make the solution more tractable. It turns out that our methodology renders the finite type case just as tractable.

51 Dziuda considers a continuum of decision makers, each with a binary decision, and different thresholds for making one decision over the other. These decision makers act like one decision maker with a probability of choosing the favored option that is increasing and decreasing in the number of persuasive and dissuasive arguments respectively.

52 We will also abuse notation and equivocate singleton sets with the corresponding element, i.e. \((\{l\}, B) \equiv (l, B) \equiv \{l\} \times B \times \{h, s\}\).
We assume \((T, \succeq_d)\) is endowed with a disclosure order defined as follows: \(\forall (l, r) \in T'\)

\[
M(l, r, s) = (l, r) \quad \text{(26)}
\]

\[
M(l, r, h) = \{(l, r, h)\} \quad \text{(27)}
\]

That is, strategic types can declare any honest or strategic type with fewer persuasive or dissuasive arguments, while honest types must truthfully reveal. An example of such a disclosure order with \(l + r \leq 2\) is given in Figure 4 below. Notice that honest types can only declare themselves.

![Figure 4: Disclosure Order in Strategic Argumentation](image)

**H.2. Equilibrium Characterization**

The receiver optimal equilibrium of this game can be analyzed using the results in Section 3. By Proposition 1, the equilibrium is an interval partition of the type space \((P_1, ..., P_m)\), with \(V_h(P_i)\) increasing in \(i\), and \(V_h\) downward biased on each \((P_i, \succeq_d)\). First note that because of the disclosure order, the sender payoff vector, \(\pi(l, r, s)\), must be weakly increasing in \(l, r\). We will characterize equilibrium through the use of three claims.

**Claim 1.** The equilibrium payoff to the strategic agent is constant in his number of dissuasive arguments.

**Proof.** Say that \(\exists l' < l''\) and \(r\) such that \(\pi(l', r, s) < \pi(l'', r, s)\). Without loss of generality let \(l'' = l' + 1\) with \((l', r, s) \in P_i\) and \((l'', r, s) \in P_j\) for \(i < j\). i.e. \(V_h(P_i) \leq V_h(P_j)\).

Notice that \(M(l'', r, s) \cap P_j \subset (l'', r)\). This means that \(V_h(M(l'', r, s) \cap P_j) \leq v(l'', r)\). But since \(V_h\) is downward biased on \(P_j\), \(V_h(M(l'', r, s) \cap P_j) \geq V_h(P_j)\). Thus \(v(l'', r) \geq V_h(P_j)\).

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53 If not, and \((\tilde{l}, \tilde{r}) \cap P_j\) for some \(\tilde{l} < l'', \tilde{r} < r\), then because \((\tilde{l}, \tilde{r}) \cap M(l', r, s)\) this contradicts the fact that \(P\) is an interval partition.
Similarly $B(l', r, S) \cap P_i \subset (l', r)$. This means that $V_h(B(l', r, s) \cap P_i) \geq v(l', r)$. But since $V_h$ is downward biased on $P_i$, $V_h(B(l''', r, s) \cap P_j) \leq V_h(P_i)$. Thus $v(l'', r) \leq V_h(P_i)$. Because $v(l', r) > v(l'', r)$, we have that $V_h(P_i) \geq V_h(P_j)$ which is a contradiction. \(\text{Q.E.D.}\)

Thus from Claim 1, we know that for each $i$, $\exists r_i < r_i''$ such that $P_i \cap S = (L, [r_i', r_i'')] \cap S$. That is, strategic types that pool together are those with persuasive arguments between $r_i'$ and $r_i''$ and all dissuasive arguments. Let $R_i \equiv (L, [r_i', r_i''])$. The next claim characterizes the equilibrium pooling of honest types.

**Claim 2.** $\exists \lambda : R \rightarrow L$, such that

$$P_i \cap S \neq \emptyset \implies P_i \cap H = (\cup_{r' \leq r \leq r''} (\lambda(r), r)) \cap H$$

$$P_i \cap S = \emptyset \implies P_i = (l, r, h) \text{ for some } l, r$$

With $\lambda$ defined by,

$$v(\lambda(r), r) \geq V_h(P_i), \quad v(\lambda(r) + 1, r) < V_h(P_i) \forall r' \leq r \leq r_i''$$

(28)

**Proof.** Using Corollary 8 and the structure of the current disclosure order gives,

$$V_h(P_i) = \max_{\bar{H} \subset R_i \cap H} (V_h((R_i \cap S) \cup B(\bar{H})))$$

Because $v(l, r)$ is decreasing in $l$ for each $r$, we have that $\bar{H} = \cup_{r' \leq r \leq r''} (\lambda(r), r)$, for some function $\lambda : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$. The reason why $\lambda$ is pinned down by (28) is as follows. First consider that $v(\lambda(r) + 1, r) \geq V_h(P_i)$. The type $(\lambda(r) + 1, r, h)$ would be in a lower part $P_j$ of the equilibrium partition than $(\lambda(r) + 1, r, s) \in P_i$, i.e. $V_h(P_j) < V_h(P_i)$. Since $(\lambda(r) + 1, r, h)$ is an honest type, he must be alone in $P_j$, meaning that $V_h(P_j) = v(\lambda(r) + 1, r)$ which is a contradiction. Second consider that $v(\lambda(r), r) < V_h(P_i)$. By Lemma Lemma 1 we could exclude $(\lambda(r) + 1, r, h)$ from $\bar{H}$ and strictly increase the value of the above maximization. \(\text{Q.E.D.}\)

With these claims we have fully characterized the equilibrium partition. Each part of the equilibrium partition is either strategic or honest. Honest parts are composed of a single honest type. Strategic parts are composed of two groups of types: (i) strategic agents with an interval of persuasive arguments and all dissuasive arguments, and (ii) honest types with the same interval of persuasive arguments and fewer dissuasive arguments than some
threshold\textsuperscript{54}. Also note that because of (28) and the properties of \(v(l, r)\) that \(\lambda(r)\) is strictly increasing in \(r\).

However, Dziuda goes further by making assumptions on the distribution of types. She assumes that an agent is strategic with probability \(q\), and both honest and strategic types draw a payoff relevant type from the common CDF \(F : T' + \rightarrow \mathbb{R}_+\) with conditional CDFs \(F(l|r)\) and \(F(r|l)\). She also makes a "good news is not bad news" assumption: for \(r' < r''\), \(F(\cdot| r')\) first order stochastically dominates \(F(\cdot| r'')\). In words, having more persuasive arguments implies fewer dissuasive arguments in an FOSD sense. With these assumptions we can obtain that the equilibrium action is strictly increasing in the number of persuasive arguments.

Claim 3. Let \(P_r = ((L, r) \cap S) \cup ((\lambda(r), r) \cap H)\). The equilibrium sender payoff vector is given as follows.

\[
\pi(l, r, s) = V_h(P_r) \forall (l, r)
\]

\[
\pi(l, r, h) = \begin{cases} 
V_h(P_r) & l \leq \lambda(r) \\
v(l, r) & l > \lambda(r)
\end{cases}
\]

Proof. In order to prove this result we point to a corollary of the proof of Proposition 4 that will be useful in our applications.

Corollary 8. Let the equilibrium partition be \(P = (P_1, ..., P_m)\). The following hold \(\forall P_i\).

\[
V_h(P_i) = \max_{S \subset P_i} V_h(M(P_i) \cap B(S)) 
\]  \hspace{1cm} (30)

\[
V_h(P_i) = \min_{S \subset P_i} V_h(M(S) \cap B(P_i)) 
\]  \hspace{1cm} (31)

All we must prove is that \(\forall i, r''_i = r'_i\). say that \(\exists i : r''_i > r'_i\) with \((l, r'', s) \in P_i\). For \(l^* \equiv \arg\max_i V_h(((L, r''_i) \cap S) \cup ((l, r''_i) \cap H)), \) let the maximizer be \(U \equiv ((\mathbb{Z}_+, r''_i) \cap S) \cup ((l^*, r''_i) \cap H)\). By Corollary 8, \(B(U) \cap P_i = U\). Thus \(V_h(U) \leq V_h(P_i)\) by Proposition 2. But this is a contradiction because by the "good news is not bad news" assumption \(V_h(U) > V_h(P_i)\). Q.E.D.

\textsuperscript{54}Formally, each part is given by

\[
P_i = \begin{cases} 
(R_i \cap S) \cup (\bigcup_{r'_i \leq r \leq r''_i} (\lambda(r), r)) \cap H & P_i \cap S \neq \emptyset \\
(l, r, h) & P_i \cap S = \emptyset
\end{cases} 
\]  \hspace{1cm} (29)