The Coalitional Nash Bargaining Solution with Simultaneous Payoff Demands

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Abstract We consider a standard coalitional bargaining game where once a coalition forms it exits, however, instead of alternating offers, we consider simultaneous payoff demands. Each player is selected with equal probability. If that is the case, she can choose any coalition she belongs to. A coalition can form if and only if payoff demands are feasible, as in the Nash demand game. In the limit, for almost all sharing rules (used for refining purposes), if there exists a grand coalition stationary subgame perfect equilibrium, then the expected payoffs are in the core. If the expected payoffs are in the interior of the core, then such an equilibrium exists. If the Nash bargaining solution is the sharing rule, such an equilibrium exists regardless of the discount factor if and only if the per capita worth of the grand coalition is greater than or equal to that of any coalition; when this rule is applied to Shapley and Shubik’s production economy with identical workers, the coalitional Nash bargaining solution obtains; this is also the unique stationary subgame perfect equilibrium outcome if we don’t use a sharing rule but we add uncertainty, the noise vanishes, and the discount factor is close to 1. Preliminary work shows that these results can be extended to the general case. However, when the core is empty our results differ from those in the Rubinstein framework.

Keywords Coalitional Bargaining · Nash Program · Simultaneous Payoff Demands · Uncertainty

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1 Introduction

Compte and Jehiel (2010), and Okada (2011) find that the coalitional Nash bargaining solution, a cooperative (ad hoc) solution for situations where coalitions can form, is the unique stationary subgame perfect equilibrium outcome as the discount factor goes to 1 in different noncooperative coalitional bargaining games. The games they study consider alternating-offers bargaining, as in Rubinstein (1982). The main focus of our paper is to show that in the production economy that Okada (2011) studies using the random proposer model, the coalitional Nash bargaining solution is the unique stationary subgame perfect H-essential equilibrium (an extension of Van Damme’s (1991) static H-essential equilibrium) outcome of a coalitional bargaining game as that in Okada (2011) but with simultaneous payoff demands and uncertainty. Hence, this paper belongs to the Nash Program, the literature that looks for noncooperative games that have cooperative solutions as equilibrium outcomes. The secondary objective is to study the conditions under which the grand coalition forms in our framework without uncertainty if we consider different sharing rules which are needed to refine the equilibria.

In each period of our game, players are chosen with equal probability from the set of active players. If selected, a player, the initiator, can choose any coalition to belong to. Once she chooses a coalition, a simultaneous Nash demand game (as in Nash 1953) is played by the members of that coalition. This coalition forms if and only if the payoff demands are feasible given the worth generated by the coalition. If a coalition forms, its members are deleted from the set of active players, and consumption in terms of transferable utility is realized. If the coalition does not form, one period elapses and the game repeats itself with the same set of active players. The number of periods is infinite, and perpetual disagreement yields a payoff of zero. There are no externalities, and the underlying cooperative game is superadditive. Players discount future payoffs.

As an equilibrium notion for our main objective, we use a refinement of subgame perfect equilibrium (SPE). As is well known, repeated Nash demand like games have a plethora of Nash equilibria and SPE (as shown by Stahl 1990 for the two player case). So we use a refinement of the concept of stationary subgame perfect equilibrium (SSPE). In this paper, we define the notion of Nash group stationary subgame perfect equilibrium (Nash GSSPE), to further refine the equilibrium concept. In a Nash GSSPE, the expected discounted payoffs each time a coalition plays the Nash demand game are the Nash bargaining solution (NBS) of a bargaining problem where the total surplus is the worth of this coalition and the disagreement payoffs are the expected Nash GSSPE payoffs if the payoff demands are not feasible. The NBS can be thought of as an equal sharing rule of the Nash surplus, the difference between the total surplus and the disagreement payoffs.

In Section 3, we prove the existence of Nash GSSPE using Kakutani’s fixed point theorem and characterize the grand coalition Nash GSSPE in general. In contrast with Okada (2011), the grand coalition Nash GSSPE exists if and only if the per capita worth of the grand coalition is in the core regardless of the value of the discount factor. We then study the case that considers a large class of positive sharing rules of the Nash surplus and obtain the following results when the discount factor is close to 1:

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1 This maximizes the product of players utility levels over the set of points in the core of a cooperative game.

2 This refinement is an extension of Nash GSPE, a refinement for games with finite horizon as in Nieva (2005, 2008).
if the grand coalition GSSPE exists, then the expected payoffs are in the core; if the expected payoffs are in the interior of the core, then such an equilibrium exists.

Our model with different sharing rules has implications which are close to those found by Okada (2011) but for different reasons. In both models, the expected payoffs depend on the players’ bargaining power. In our case, the bargaining power of a player is determined by that player’s individual share, and in Okada’s model, it is determined by the probability of being an initiator. The existence of the grand coalition GSSPE is related to the concept of the core. However, in Okada (2011), a sufficient and necessary condition for the existence of the grand coalition equilibrium in the limit is that the expected payoffs are in the core. In the present paper, in contrast, when the discount factor is not close to 1, we cannot tell whether the expected payoffs have to be in a bigger set than the core. Lastly, the expected payoffs in the grand coalition GSSPE are, in the limit, equal to the generalized Nash Bargaining solution, as in Okada (2011). Our results are surprising as the Nash demand game had been abandoned: since its large number of equilibria becomes even larger in dynamic games, there seemed to be no hope of expecting a tractable efficiency analysis. Our results are thus encouraging. Moreover, our treatment seems to be more general, as the Rubinstein framework could be seen as corresponding to one sharing rule.

Next, we use previous results to prove, in Section 4, that the Nash GSSPE outcome coincides with the coalitional Nash bargaining solution in our game applied to Shapley and Shubik’s (1967) production economy with identical workers when the discount factor is close to 1, as in Okada (2011). The fact that in the limit we reproduce his results also means that in our game in the limit the market outcome obtains, that is, the employer acts as if maximizing profit even though we use the Nash bargaining solution as a sharing rule (note that in an alternating-offers set up, the employer maximizes profit regardless of the value of the discount factor). For the sake of comparison, our paper is organized as in Okada (2011).

Lastly, in the Appendix, we give an outline of the straightforward extension to the $n$-player case of Van Damme’s (1991) smoothing technique (inspired by Nash 1953) for the two player Nash demand game. At the same time, we show how this extension is used in our game. It is not hard to see then that the Nash GSSPE outcome is the unique SSPE outcome (or, equivalently, the unique SSPE H-essential equilibrium outcome) when the discount factor is close to 1 and the noise vanishes.3

2 Preliminaries

Let $(N, v)$ be an $n$-person game in coalitional form where utility is transferable. The set of players is $N = \{1, 2, \ldots, n\}$. A nonempty subset $S$ of $N$ is a coalition of players. The set of all coalitions of $N$ is $\mathcal{C}(N)$. The characteristic function of this game is $v$, a real valued function defined on $\mathcal{C}(N)$, that is normalized, that is, $v(\{i\}) = 0$ for each $i \in N$. We also assume that it is super-additive, that is, $v(S \cup T) \geq v(S) + v(T)$ for all pairs of disjoint coalitions $S$ and $T$. Finally, it is essential, i.e., $v(N) > 0$. For

3 We have been aware of the work of Abreu and Pierce (2015) where the smooth Nash demand is studied in an infinitely repeated game in a stochastic framework to single out as the noise vanishes the unique equilibrium (stationary) that turns out to be the variable threats Nash bargaining solution for a two player game. Besides the obvious differences, we want to point out that Nieva (2005) was the first paper to suggest the use of the smooth game after the simultaneous approach had been neglected because of the discouraging results in Stahl (1990).
each coalition $S$, $v(S)$ is the total utility that members in $S$ can distribute among themselves in any way they agree to, the *worth* of a coalition.

A payoff allocation for coalition $S$ is a vector $x^S = \left(x^S_i\right)_{i \in S}$ of real numbers where $x^S_i$ is the payoff to player $i \in S$. A payoff allocation $x^S$ is *feasible* if $\sum_{i \in S} x^S_i \leq v(S)$. Let $X^S$ denote the set of all feasible payoff allocations for $S$ and let $X^S_2$ denote the set of all non-negative elements in $X^S$. If $T$ is a finite set, $\Delta(T)$ denotes the set of all probability distributions on $T$.

As a noncooperative bargaining procedure for a game $(N, v)$, we consider the random proposer model as in Okada (2011) but with a twist. The negotiations can take an infinite number of bargaining rounds $t = (1, 2, \ldots)$. Let $N^t$ be the set of all *active* players who have not formed a coalition yet at the beginning of period $t$. In the initial round, $N^1 = N$. At the beginning of period $t$, a player $i \in N^t$ is selected with equal probability; we will call this player the *initiator*. This initiator $i$ can choose a coalition $S$ with $i \in S \subset N^t$. Once she has chosen $S$, a Nash demand game (Nash 1953) is played between the $s$ players in $S$ as follows: all players in $S$ state simultaneously their nonnegative payoff demands. The payoff demand for player $i \in S$ is $x^S_i$. If the payoff demand profile $x^S$ is feasible, then they agree, the coalition $S$ forms, and consumption takes place. Negotiations continue in the next period, where the set of active players is $N^{t+1} = N^t - S$. If the payoff demand profile is not feasible, then negotiations continue in the next period, where $N^{t+1} = N^t$. The game ends when every player in $N$ joins some coalition.

When a coalition $S$ forms after a payoff demand profile $x^S$ is agreed upon in period $t$, the payoff to each player $i \in S$ is $\delta^{t-1}x^S_i$, where $\delta$ ($0 \leq \delta < 1$) is the discount factor for future payoffs. When bargaining does not stop, all players who fail to join any coalition obtain zero payoffs.

The game is denoted by $\Gamma(N, \delta)$, where $N$ is the initial set of active players and $\delta$ is the discount factor. This is a multistage game with observed actions, chance moves, and with an infinite horizon.

A (behavior) strategy for player $i$ in $\Gamma(N, \delta)$ is defined in the standard manner. A history $h^t_i$ in period $t$ is a sequence of all past actions including the selection of the initiators. A strategy $\sigma_i$ for player $i$ is a function that assigns to every possible history $h^t_i$ the (random) action $\sigma_i(h^t_i)$ that will be taken by player $i$. If player $i$ is an initiator in period $t$, $\sigma_i(h^t_i)$ is a probability distribution over all possible coalitions with $i \in S \subset N^t$. If a coalition $S \subset N^t$ is chosen in period $t$, $\sigma_i(h^t_i)$ is a payoff demand $x^S_i$ with $i \in S$. Given a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$, the expected discounted payoff to player $i$ in $\Gamma(N, \delta)$ is defined in the usual way. A strategy $\sigma_i$ induces a strategy, a *restriction* $\sigma_i|\Gamma(N^t, \delta)$, in each subgame $\Gamma(N^t, \delta)$. In each period $t$, it will be useful to denote the subgame after a coalition $S$ has been chosen by some initiator by $\Gamma(N^t, S, \delta)$.

A strategy $\sigma_i$ for player $i$ in $\Gamma(N, \delta)$ is called stationary if it depends on a small set of state variables. In our model, a strategy depends on the set $N^t$ of active players when the initiator has been selected; it depends on the coalition $S$ that has been chosen and the initiator who chose it when a player plays the Nash demand game in subgame $\Gamma(N^t, S, \delta)$.

It is known that repeated Nash demand games have many subgame perfect equilibrium (SPE) outcomes, see (Stahl 1991); take the divide dollar game; any division of the dollar, including $(0, 0)$, is a possible SPE equilibrium outcome. It is not hard to see that these repeated games also have infinitely many stationary subgame perfect equilibria.
(SSPE). Hence, as an equilibrium notion for \( \Gamma(N, \delta) \), we use in most of this paper a refinement of the stationary subgame perfect equilibrium concept consistent with the Nash bargaining solution, as in (Nieva 2005, 2008). In the Appendix, we explain that introducing uncertainty, as in Van Damme (1991), is equivalent to using our refinement of SSPE. Formally, we have the following definition.

**Definition 1** A strategy profile \( \sigma = (\sigma_1, \ldots, \sigma_n) \) of \( \Gamma(N, \delta) \) is a stationary subgame perfect equilibrium if \( \sigma \) is a subgame perfect equilibrium of \( \Gamma(N, \delta) \) and every strategy \( \sigma_i \) is stationary for each \( i \in N \).

For \( \sigma \) a stationary subgame perfect equilibrium of \( \Gamma(N, \delta) \), let \( v^S_i \) denote the expected payoff to player \( i \) in subgame \( \Gamma(S, \delta) \).

**Definition 2** A strategy profile \( \sigma \) of \( \Gamma(N, \delta) \) is a Nash group stationary subgame perfect equilibrium (Nash GSSPE) if \( \sigma \) is a stationary subgame perfect equilibrium of \( \Gamma(N, \delta) \) such that for each \( S \subseteq N^t \) and each \( N^t \), the expected discounted payoffs associated to the restriction \( \sigma \) to the subgame \( \Gamma(N^t, S, \delta) \) is the solution to a Nash bargaining problem (whenever it is well defined) where the total transferable utility is \( v(S) \) and the disagreement payoffs are the discounted payoffs if the demands are not feasible, that is, \( \delta v^N_i \) for each \( i \in S \).

As utilities are transferable, the Nash GSSPE discounted payoff to player \( i \) in subgame \( \Gamma(N^t, S, \delta) \) is equal to \( \frac{v(S) - \sum_{j \in S} \delta v^N_j}{\sum_{j \in S} \delta v^N_j} \) plus the disagreement payoff to player \( i \).

It is clear that if \( v(S) > \sum_{j \in S} \delta v^N_j \), the Nash GSSPE profile \( \sigma(N^t, S, \delta) \) could consist of each player in \( S \) demanding the payoff \( x^S_i = \frac{v(S) - \sum_{j \in S} \delta v^N_j}{\sum_{j \in S} \delta v^N_j} + \delta v^N_i \). Note, also, that Definition 2 implies that if the Nash bargaining solution is not well defined, say, if \( v(S) \leq \sum_{j \in S} \delta v^N_j \), any SSPE restriction \( \sigma \) to the subgame \( \Gamma(N^t, S, \delta) \) can be chosen. In particular, if \( v(S) < \sum_{j \in S} \delta v^N_j \), a Nash GSSPE predicts a disagreement at \( \Gamma(N^t, S, \delta) \); the Nash GSSPE profile \( \sigma(N^t, S, \delta) \) could consist of all players in \( S \) demanding unfeasible payoffs.

The next two key lemmas make use of the following concepts. For \( \sigma \) a Nash GSSPE of \( \Gamma(N, \delta) \), let \( q^S_i \in \Delta(\{T | i \in T \subset S\}) \) denote the random choice by initiator \( i \) of coalitions \( T \) (of \( S \)) in \( \Gamma(S, \delta) \). Recall that \( v^S_i \) is the expected payoff to player \( i \) in subgame \( \Gamma(S, \delta) \) (in a Nash GSSPE). We refer to a collection \( (v^S, q^S)_{S \subseteq C(N)} \), where \( v^S = (v^S_i)_{i \in S} \) and \( q^S = (q^S_i)_{i \in S} \), as the configuration of \( \sigma \).

**Lemma 1** In every Nash GSSPE \( \sigma = (\sigma_1, \ldots, \sigma_n) \) of \( \Gamma(N, \delta) \), given an initiator, some coalition with more than one member forms with positive probability in the initial round, and each member \( j \) receives more than \( \delta v^N_j \).

**Proof** For each \( i \in N \), let \( v_i \) be player \( i \)'s expected payoff for \( \sigma \) in \( \Gamma(N, \delta) \) in a Nash GSSPE. Because of super-additivity, any average over the discounted payoffs to players that may be obtained in each coalition structure that occurs with nonnegative probability is less than or equal to \( v(N) \); this implies \( \sum_{j \in N} v_j \leq v(N) \). Because the game is zero normalized, we have \( v_i \geq 0 \) for all \( i \in N \). It follows that each initiator \( i \) can choose

\[ v_i \geq \delta v^N_i \]
the grand coalition $N$ and obtain strictly more than $\delta v_i$ by the definition of the Nash bargaining solution. Hence, the claim follows, as then the Nash surplus in the coalition that she ends up choosing optimally, say $S$, is positive, that is, $v(S) - \sum_{j \in S} \delta v_j > 0$; so $S$ has more than one member and member $j$ gets strictly more than $\delta v_j$ for each $j \in S$.

Let us now characterize the configuration of a Nash GSSPE.

**Lemma 2** A collection $(v^S, q^S)_{S \in C(N)}$, where $v^S = (v^S_i)_{i \in S}$ and $q^S = (q^S_i)_{i \in S}$, is the configuration of a Nash GSSPE in $\Gamma(N, \delta)$ if and only if the following conditions hold for every $S \in C(N)$ and every $i \in S$:

(i) $q^S_i(\bar{S}) > 0$ implies that the coalition $\bar{S}$ is a solution of

$$\max_{T \subset S} \frac{v(T) - \sum_{j \in T} \delta v^S_j}{t} + \delta v^S_i.$$  \hspace{1cm} (1)

(ii) $v^S_i \in R_+$ satisfies

$$v^S_i = \frac{1}{s} \max_{T \subset S} \left( \frac{v(T) - \sum_{j \in T} \delta v^S_j}{t} + \delta v^S_i \right) + \frac{1}{s} \sum_{j \in S, j \neq i} \left( \sum_{T \subset S, i \in T} q^S_j(T) \left( \frac{v(T) - \sum_{k \in T} \delta v^S_k}{t} + \delta v^S_i \right) + \sum_{T \subset S, i \notin T} q^S_j(T) \delta v^S_{T - i} \right).$$  \hspace{1cm} (2)

**Proof** First, we prove necessity. Given the configuration of the Nash GSSPE, it is clear that if initiator $i$ chooses optimally to randomize between coalitions, then the expected value that the initiator gets in each coalition that is assigned a positive probability has to be the maximum of (1); otherwise, she would gain by assigning zero probability to a coalition that is not a maximizer and increasing the probability assigned to a coalition that is a solution of (1); hence, (i) follows. Equation (2) in (ii) is just the recursive definition of the expected payoffs $v^S_i$. For sufficiency, we can use the single-period deviation property, which states that the local optimality of a strategy implies its global optimality in an infinite-length multistage game with observed actions. \hfill \Box

### 3 Existence, Grand Coalition Nash GSSPE, and Other Sharing Rules

First, we prove the existence of a Nash GSSPE in the general case. We also characterize an equilibrium where only the grand coalition forms, regardless of the initiator. Next, we do the same considering other sharing rules, that is, other ways of dividing the Nash surplus.

#### 3.1 Existence

The only difference from the proof in proposition 3.1 in Okada (2011) is that in our case the initiator chooses the coalition that maximizes (1) and not Okada’s expression 1.
Proposition 1 There exists a Nash GSSPE of the bargaining model $\Gamma(N, \delta)$.

Proof By Lemma 2, it suffices to prove that there exists a collection $(v^i, q^i)_{S \in C(N)}$ of players’ expected payoffs $v^S = (v^i)_{i \in S}$ and their random choices $q^S = (q^i)_{i \in S}$ of coalitions in all subgames $\Gamma(S, \delta)$ such that (i) and (ii) in Lemma 2 hold for each coalition $S \in C(N)$ and each $i \in S$.

We prove this claim by induction in the cardinality $s$ of coalition $S$. When $s = 1$, say $S = \{i\}$, the claim trivially holds by setting $v^i = 0$ and $q^i = 1$. For any $2 \leq s \leq n$, suppose that the claim holds for all $t = 1, \ldots, s - 1$. Let $S \in C(N)$ be any coalition with $s$ members. For all proper subsets $T$ of $S$, let $v^T = (v^j)_{j \in T}$ be the expected payoffs to the members in $T$ and $q^T = (q^j)_{j \in T}$ be their random choices of coalitions in the subgame $\Gamma(T, \delta)$ such that (1) and (2) in Lemma 2 hold. By the inductive assumption, $v^T$ and $q^T$ exist. Let $\Delta^S = \Delta(T_i \in T \subset S)$, that is, it is the set of probability distributions over the subsets of $S$ that the player $i$ belongs to. We define a multi-valued mapping $F$ from a compact and convex set $X_S \times \prod_{i \in S} \Delta^S$ to itself as follows. For $(x, q) \in X_S \times \prod_{i \in S} \Delta^S$, $F(x, q)$ is the set of all $(y, r) \in X_S \times \prod_{i \in S} \Delta^S$ that satisfy the following for all $i \in S$:

(i) $r_i \in \Delta(S) \in S \subset S$ and $\hat{S}$ is a solution of

$\max_{\hat{S} \subset S} \left( \frac{v(T) - \sum_{k \in T} \delta x_k}{t} + \delta x_i \right)$

(ii) $y_i \in H_i$ satisfies

$y_i = \frac{1}{\delta} \max_{\hat{S} \subset S} \left( \frac{v(T) - \sum_{k \in T} \delta x_k}{t} + \delta x_i \right)$

$= \frac{1}{\delta} \sum_{j \in S, j \neq i} \left( \sum_{j \in T \subset S, i \in T} q^j(T) \left( \frac{v(T) - \sum_{k \in T} \delta x_k}{t} + \delta x_i \right) + \sum_{j \in T \subset S, i \notin T} q^j(T) \delta v^{S-T}_i \right)$.

It is not hard to see that $F(x, q)$ is a non-empty convex set in $X_S \times \prod_{i \in S} \Delta^S$. We can show that $F$ is upper-hemicontinuous and compact valued using the maximum theorem. As then the assumptions for Kakutani’s fixed point theorem are satisfied, there exists a fixed point $(x^*, q^*)$ of $F$ with $(x^*, q^*) \in F(x^*, q^*)$. Set $v_i^N = x_i^N$ and $q_i^N = q_i^N$ for all $i \in S$ and the proposition follows. □

3.2 Grand Coalition Nash GSSPE

We next study the conditions under which the grand coalition $N$ is formed regardless of the initiator that is selected at random.

Definition 3 A behavior strategy $\sigma$ for $\Gamma(N, \delta)$ is called a grand coalition Nash GSSPE if it is a Nash GSSPE of $\Gamma(N, \delta)$ and the grand coalition forms, independently of the proposer.

Theorem 1 The grand-coalition Nash GSSPE of $\Gamma(N, \delta)$ is characterized as follows: the expected payoff $v_i$ is given by $v_i = \frac{v(N)}{n}$. The grand-coalition Nash GSSPE exists if and only if its expected payoff vector $(\frac{v(N)}{n}, \ldots, \frac{v(N)}{n})$ is in the core of $(N, v)$, that is,\n
$\frac{v(N)}{n} \geq \frac{v(S)}{n}$ for all $S \subset N$.\n
Proof: From Equation (2), we obtain

\[ v_i = \frac{v(N) - \sum_{j \in N} \delta v_j}{n} + \delta v_i, \]

for each \( i \in N \). After summing over all \( i \), we obtain \( \sum_{j \in N} v_j = v(N) \), and so \( \sum_{j \in N} \delta v_j = \delta v(N) \). Using this in the last equation, we obtain \( v_i = \frac{v(N)}{n} \). Next, since \( N \) forms in equilibrium, from (1), we have that

\[ \frac{v(N) - \sum_{j \in N} \delta v_j}{n} + \delta v_i \geq \frac{v(S) - \sum_{j \in S} \delta v_j}{s} + \delta v_i, \] (3)

for each \( i \in S \), for each \( S \subset N \). After substituting \( v_i = \frac{v(N)}{n} \) and cancelling out, the necessity of the second claim follows. For sufficiency, suppose \( \sum_{j \in S} v_j = \frac{v(S)}{n} \geq v(S) \), and hence \( \frac{v(N)}{n} \geq \frac{v(S)}{s} \); add \( \frac{\sum_{j \in N} \delta v_j}{n} - \frac{\sum_{j \in S} \delta v_j}{s} = 0 \) to the right hand side for each \( S \), and now (3) obtains. By Lemma 2 the claim follows. \( \square \)

Note that this result differs from that of the standard Rubinstein coalitional bargaining model (see Okada 2011) where he has instead a limiting result. In other words, in the Rubinstein paradigm, the expected payoff vector of the grand coalition \( (v(N) n, \ldots, v(N) n) \) does not need to be in the core if the discount factor \( \delta \) is not close to 1 for his result to be obtained.

3.3 Grand Coalition GSSPE under other Sharing Rules

Consider the following extension of our game. Let \( a \) be a function that assigns to every coalition \( S \subset N \) a positive vector \( a^S > 0 \), where \( \sum_{i \in S} a^S_i = s \). In addition, we assume that there exist \( S \) and \( i \) such that \( a^S_i \neq 1 \). As the number \( \frac{a^S_i}{s} \) is to be interpreted as the share of the Nash surplus that goes to player \( i \), the last assumption rules out that the Nash bargaining rule is always used. When we use the Nash bargaining rule, \( a^S_i = 1 \) for each \( i \) and for each \( S \subset N \).

It is straightforward to restate the previous definitions and prove the previous Lemmas and Propositions. However, we prove the analogue of Theorem 1 for this extension.

**Theorem 2** The grand coalition GSSPE of \( \Gamma(N, a, \delta) \) is characterized as follows:

(i) The expected payoff \( v_i \) is given by \( v_i = a^N_i \frac{v(N)}{n} \).
(ii) The grand coalition GSSPE exists if and only if for each \( i \in S \), for each \( S \subset N \)

\[ \frac{v(N)}{n} \left( a^N_i \frac{1 - \delta}{a^S_i} + \frac{\delta \sum_{j \in S} a^N_j}{s} \right) \geq \frac{v(S)}{s}, \] (4)

(iii) If the grand-coalition GSSPE exists when \( \delta \) is close to 1, then its expected payoff vector \( \left( \frac{a^N_1 v(N)}{n}, \ldots, \frac{a^N_n v(N)}{n} \right) \) is in the core of the game. If the expected payoff vector is in the interior of the core, that is, if \( \frac{v(N)}{n} \sum_{j \in S} a^N_j > v(S) \) for each \( S \subset N \), then the grand-coalition GSSPE exists when \( \delta \) is close to 1.
Proof (i) An extension of Equation (2) if we use the general sharing rule implied by \( a \) yields
\[ v_i = a_i^N v(N) - \frac{\sum_{j \in N} \delta v_j}{n} + \delta v_i, \]
for all \( i \in N \). After summing over all \( i \), we obtain \( \sum_{j \in N} v_j = v(N) \) or, equivalently, \( \sum_{j \in N} \delta v_j = \delta v(N) \). After using this result in the last equation, we obtain
\[ v_i = a_i^N v(N). \]

(ii) For necessity, since the grand coalition GSSPE exists, from a generalization of the expression in (1), we have that
\[ a_i^N v(N) - \frac{\sum_{j \in N} \delta v_j}{n} + \delta v_i \geq a_i^S v(S) - \frac{\sum_{j \in S} \delta v_j}{s} + \delta v_i, \]  \tag{5}
for each \( i \in S \), and for each \( S \subset N \). After substituting \( v_i = a_i^N v(N) \) in (5) and rearranging, we get (4). For sufficiency, apply the generalization of Lemma 2 to (4) that is equivalent to (5).

(iii) The necessity of the condition follows from taking the limit of (4). Suppose \( \frac{v(N)}{n} \sum_{j \in S} a_j^N > v(S) \) for each \( S \subset N \). Then \( \frac{v(N)}{n} \sum_{j \in S} a_j^N > v(S) \) for each \( S \subset N \) when \( \delta \) is close to 1. Dividing by \( s \) and then adding \( \frac{v(N)}{n} \frac{\gamma^N(1-\delta)}{a_j^N} > 0 \) to the left hand side of the previous inequality (given that \( \delta \) is close to 1) for each \( S \subset N \) yields (4). Next, we apply the generalization of Lemma 2 to (4) that is equivalent to (5). \( \Box \)

Our model’s implications are close to those in Okada (2011) but for different reasons. The expected payoffs depend on the individual shares. In Okada’s model, these depend on the probability of being an initiator. The existence of the grand coalition GSSPE is related to the core. However in their work, a sufficient and necessary condition for the existence of a grand coalition equilibrium in the limit is that the expected payoffs are in the core. When \( \delta \) is not close to 1, then in contrast, we cannot tell whether the expected payoffs are in a bigger set than the core for a grand coalition GSSPE to exist. Our results are surprising, as the Nash demand game had been abandoned: because of the fact that its large number of equilibria get even larger in dynamic games, there seemed to be no hope of obtaining a tractable efficiency analysis. The results in Theorem 2 are thus encouraging.

Finally, the expected payoffs in the grand coalition GSSPE are, as in Okada (2011), equal to the generalized Nash bargaining solution that maximizes \( \prod_{i \in N} \frac{x_i^N}{x_i^N} \) subject to \( \sum_{i \in N} x_i = V(N) \).

4 The Coalitional Nash Bargaining Solution in a Production Economy

The uniqueness of the Nash GSSPE in the general case for the coalitional bargaining game we study is an open question (as it is the uniqueness of SSPE in the alternating-offers coalitional bargaining model). So the coalitional Nash bargaining solution for cooperative situations in general may not be an appropriate solution concept. However, it has been singled out as the unique noncooperative prediction in the alternating-offers coalitional bargaining framework used by Okada (2011) in a particular situation when the discount factor is close to 1; hence, it is appropriate in this case. We focus on the same situation and show that we obtain the same limiting results but with a
simultaneous approach. We follow that paper’s structure as closely as possible, so as to emphasize the great similarity. The idea is that in the limit, Nash GSSPE that are different from the grand coalition Nash GSSPE occur, except for one degenerate case in circumstances under which the grand coalition Nash GSSPE can not form. In any case, the expected payoffs are unique and are those in the coalitional Nash bargaining solution. That is the main message of Theorem 3.

Consider a production economy $\xi$ with an employer (player 1) and $n-1$ identical workers $i (= 2, ..., n)$, as in Shapley and Shubik (1967). A coalition of the employer and $s-1$ ($s \geq 1$) workers yields the benefit $f(s)$, which is monotonically increasing in $s$ with $f(1) = 0$. The benefit of any other coalition is zero. The core of the economy is non-empty since the allocation with the employer exploiting the total benefit $f(n)$ is in the core. To analyze the outcome of surplus bargaining between the employer and workers, we apply our coalitional bargaining game with random initiators and simultaneous payoff demands. The grand coalition Nash GSSPE will be called the full-employment equilibrium, and any other Nash GSSPE a partial-employment equilibrium. Let $v_i$ be the expected payoff to player $i (= 1, ..., n)$ in a Nash GSSPE.

We start by showing that in a Nash GSSPE, all workers have identical expected payoffs because of competition between them.

**Lemma 3** For all workers $i$ and $j$, $v_i = v_j$ in every Nash GSSPE.

**Proof** By contradiction. First, denote the per capita Nash surplus generated by coalition $T$ by $W(T)$, where $W(T) = v(T) - \sum_{j \in T} \delta e_j$ (unless otherwise specified, this depends on the set of active players $N$). Note that the coalitions that maximize the per capita Nash surplus over all coalitions that include the employer solve expression (1) for the employer. Denote the set of coalitions that are assigned a positive probability in equilibrium by the employer by $C_1$, with generic element $S_1$. The set over which each worker solves expression (1) consists of a subset of the coalitions that include the producer and coalitions that don’t include the producer which yield zero worth. As the expected payoffs when choosing a coalition have to be positive by Lemma 1, if $S_1$ is such that $i \in S_1$, we have $S_1 \subset C_i$, and so $\frac{W(S_i)}{s_i} = \frac{W(S_1)}{s_1}$.

Second, let $q^k_i$ be the probability that player $i$ receives an offer when player $k (\neq i)$ is a coalition chooser, an initiator. Using the recursive definition in Equation (2) for worker $i$, and noting that worker $i$ gets zero in expectation if this worker is not offered a coalition that ends up forming, we have

$$v_i = \frac{1}{n} \left( \frac{W(S_i)}{s_i} + \delta v_i \right) + \frac{1}{n} \sum_{k \neq i} q^k_i \left( \frac{W(S_k)}{s_k} + \delta v_i \right).$$

After solving for $v_i$, we get for each worker $i$

$$v_i = \frac{W(S_i)}{s_i} + \frac{\sum_{k \neq i} q^k_i W(S_k)}{s_i} \frac{n}{n - \delta \left( 1 + \sum_{k \neq i} q^k_i \right)}.$$

Without loss of generality, we assume that $v_i > v_j$ where $i \neq j$; in addition, $v_i$ and $v_j$ are the highest and lowest value respectively among workers. We prove two claims and then find a contradiction.

We first claim that $j$ must be included in any coalition that is chosen with positive probability by the employer, $S_1$, and so $\frac{W(S_j)}{s_j} = \frac{W(S_1)}{s_1}$. Suppose not. Then the
producer could gain by proposing \( (S_1 - \{i'\}) \cup \{j\} \) for some worker \( i' \in S_1 \). So the claim follows.

Second, we claim that \( i \) is not included in each \( S_1 \in C_1 \). Suppose that \( i \) is included in some \( S_1 \). Then it is not hard to see that each worker \( l \in N \) is included in some \( S_1 \), even if \( v_l = v_l \) for some \( l \). But then \( \frac{W(S_l)}{s_1} = \frac{W(S_1)}{s_1} \) for each worker \( l \in N \). Hence, Equation (6) becomes for each worker \( i \)

\[
v_i = \frac{W(S_1)}{s_1} \frac{1 + \sum_{k \neq i} q^k_i}{n - \delta \left( 1 + \sum_{k \neq i} q^k_i \right)}.
\]

We want to show that \( \sum_{k \neq j} q^k_j \geq \sum_{k \neq i} q^k_i \), and hence, in view of the last equation, \( v_i \leq v_j \), thereby obtaining a contradiction (the following proofs of (a) and (b) are based on Okada’s (2011) proof of Lemma 4.1).

Consider (a) \( q^1_j \geq q^1_i \) for any \( k \neq i, j \), and (b) \( q^1_j \geq q^1_i \). (a) follows from the fact that for any \( S \in C_k \), \( i \in S \) implies \( j \in S \) because \( \delta v_i > \delta v_j \). To prove (b), it suffices to show that \( q^1_j < 1 \) implies \( q^1_i = 0 \), that is, if there exists some \( S_i \in C_i \) with \( j \notin S_i \), and \( \exists \delta > 0 \) such that \( \delta v_i > \delta v_j \). Suppose not. Then, there exists some \( S_i \in C_i \) with \( j \not\in S_i \) and \( \exists \delta > 0 \) such that \( \delta v_i > \delta v_j \). Because \( S_i \in C_i \) and \( i \in S_i \), we have \( \frac{W(S_i)}{s_i} \geq \frac{W(S_i)}{s_i} + \delta v_i \delta v_j \). In contrast, since \( S_i \in C_j \), we have \( \frac{W(S_i)}{s_j} \geq \frac{W(S_i)}{s_i} + \delta v_j \delta v_j \), a contradiction, and our second claim follows.

From the first and second claims, we have that \( \frac{W(S_1)}{s_1} > \frac{W(S_1)}{s_i} \). After using (a) and (b) again, \( q^1_j \geq q^1_i \) for any \( k \neq i, j \). But then \( j \)'s expected payoff when \( k \) proposes is greater than or equal to that of \( i \), that is, \( q^1_j \frac{W(S_1)}{s_k} \geq q^1_i \frac{W(S_1)}{s_k} \). Also \( q^1_j \geq q^1_i \), because \( q^1_i = 0 \), hence \( q^1_j \frac{W(S_1)}{s_k} \geq q^1_i \frac{W(S_1)}{s_k} = 0 \). After plugging these results into equation (6) for workers \( j \) and \( i \), we obtain \( v_j \geq v_i \), a contradiction.\( \square \)

Now we study all possible equilibria in the producer game and its expected payoffs in the limit. First, note that the full-employment equilibrium is characterized by Theorem 1.

Next, we characterize the partial-employment equilibria. We emphasize two types. For \( 2 \leq s < n \), a Nash GSSPE is called an \( s \)-equilibrium if only coalitions with \( s \) members form with positive probability. For \( 2 \leq s < t \leq n \), a Nash GSSPE is called an \( (s, t) \)-equilibrium if only coalitions with \( s \) and \( t \) members form with positive probability (equilibria with three or more coalitions have payoffs, in the limit, that are identical to those for the latter type, so we disregard these for now).

The central idea here is that in the limit, if a partial-employment equilibrium occurs, then the allocation \( \left( \frac{f(n)}{n}, \frac{f(n)}{n}, \ldots, \frac{f(n)}{n} \right) \) is not in the core except possibly for a degenerate case.

For example, in Proposition 2, the maximum output level is reached at \( S \) in an \( s \)-equilibrium in the limit. We also show that in the limit the employer gets all and workers nothing in expectation; this is the unique point in the core.

**Proposition 2** For \( 2 \leq s < n \), an \( s \)-equilibrium of the production economy \( \xi \) is characterized as follows.

(i) The employer and each worker receive non-negative expected payoffs

\[
v_1 = f(s) - (n - 1) v_2,
\]
\[ v_2 = \frac{f(s)(s-1)(\delta - 1)}{n\delta(s-1) - s(n-1)} \]  

respectively. Every worker receives an offer with probability \( \frac{n(s-2)+1}{n(n-1)} \).

(ii) An \( s \)-equilibrium exists for any \( \delta \) close to 1 if and only if \( f(s) > f(t) \) for all \( t < s \) and \( f(s) = f(t) \) for all \( t > s \). As \( \delta \) goes to 1, the equilibrium allocation converges to the unique core allocation of the economy \( \xi \) for which the employer exploits the total payoff \( f(n) \).

**Proof** (i). As only \( S \) coalitions are chosen with positive probability, it follows from the recursive definition of expected payoffs for the case of the employer in (2) that

\[ v_1 = \frac{f(s) - \sum_{j \in S} \delta v_j}{s} + \delta v_1. \]

After using Lemma 3, we have

\[ v_1 = f(s) - (s-1)\delta v_2 - \delta v_1 + \delta v_1 \]

and

\[ f(s) = v_1 + (n-1)v_2. \]  

The unique solution of the last two equations is (7) and (8). Let \( q \) be the probability that every worker receives an offer. Again, from the recursive definition for the worker in (2), we have

\[ v_2 = \frac{1}{n} \left( f(s) - \frac{\sum_{j \in S} \delta v_j}{s} + \delta v_2 \right) + q \left( f(s) - \sum_{j \in S} \delta v_j + \delta v_2 \right). \]

After using Lemma 3 and then solving for \( q \), we have

\[ q = \frac{sv_2}{f(S) - \delta(v_1 - v_2) - \frac{1}{n}}. \]  

Use Equation (9) to substitute for \( v_1 - v_2 \), then equation (8) to substitute for \( v_2 \). Work first with the first expression in the right hand side in (10), and then get in the numerator \( sf(s)(s-1)(\delta - 1) \) and in the denominator, after some algebra, \( f(s)s(n-1)(\delta - 1) \); after cancelling out \( f(s)s(\delta - 1) \), we obtain \( \frac{s-1}{n-2} \); we substract \( \frac{1}{n} \), and the last part of (i) follows.

(ii) Using the maximization problem for the initiator in (1), an \( s \)-equilibrium exists if and only if the per capita Nash product is maximized at \( S \), that is, \( \frac{W(S)}{s} \geq \frac{W(T)}{t} \) for all \( t \neq s \) (recall \( W(T) = f(t) - \sum_{j \in T} \delta v_2 \)). After using Lemma 3, we have, equivalently, for all \( t \neq s \),

\[ \frac{f(s) - (s-1)\delta v_2 - \delta v_1}{n} \geq \frac{f(t) - (t-1)\delta v_2 - \delta v_1}{t}. \]  

In view of (7) and (8), \( v_2 \) converges to zero and \( v_1 \) to \( f(s) \) as \( \delta \) goes to 1. Noting this, we can show that the above inequality holds for any \( \delta \) close to 1 if and only if \( f(s) \geq f(t) \) for all \( t \neq s \). Since \( f \) is a monotonically increasing function, we get that \( f(s) = f(n) \).

Next we show that \( f(s) > f(t) \) for all \( t < s \). We multiply (11) by \( ts \) and get

\[ tf(s) - sf(t) \geq (s-t)\delta(v_2 - v_1). \]  

(12)
for all \( t \neq s \). Set \( f(t) = f(s) \) in (12) and cancel out \((s - t)\). Use (9) to solve for \(v_1\) and plug in (12). Factor out \(\delta v_2\). Next, we use Equation (8) and cancel out \( f(s)\). We take common denominators, cancel out terms, multiply by this denominator \(n\delta(s - 1) - s(n - 1) < 0\), and, after some algebra, obtain \(1 \leq \delta\), a contradiction.

For sufficiency, after using (9), set \( v_1 = f(s) - (n - 1)v_2 \) in (12). After rearranging, we obtain

\[
(t(1 - \delta) + \delta s)f(s) - \delta n(s - t)v_2 \geq sf(t),
\]

for all \( t \neq s \). Suppose that \( f(s) > f(t) \) for all \( t < s \) and \( f(s) = f(t) \) for all \( t > s \). Case 1: \( t > s \). It follows that \( t(1 - \delta) + \delta s > s \), and so \( (t(1 - \delta) + \delta s)f(s) > sf(t) \). As \(-\delta n(s - t)v_2 \geq 0\), (13) follows. Case 2: \( t < s \). It follows that \( t(1 - \delta) + \delta s < s \). As \( f(s) > f(t) \), if \( \delta \) is close to 1, \( (t(1 - \delta) + \delta s)f(s) > sf(t) \); after adding \(-\delta n(s - t)v_2 \leq 0\) to the left hand side of the last inequality and noting that \(v_2 \to 0\) whenever \(\delta\) gets even closer to 1, (13) follows.

By (7) and (8), we have that in the limit \( v_1 = f(n) \) and \( v_2 = 0 \). When \( f(s) = f(t) \) for all \( t \geq s \), the core of the production economy \( \xi \) consists of a unique allocation \((f(n), 0, \ldots, 0)\). This proves the second part. \( \square \)

Because \( S \) is the coalition the maximizes the Nash surplus for the employer and he is always in any coalition that forms, from the recursive definition in (2), his expected payoff is \( v_1 = f(s) - \frac{\delta f(s)}{s(1 - \delta)} + \delta v_1 \). As all workers have identical expected payoffs and \( S \) forms with probability 1, we have \( f(s) = v_1 + (n - 1)v_2 \). These two equations yield \( v_1 \) and \( v_2 \). Part (i) follows. In the limit, the recursive definition of expected payoffs to the employer implies \( f(s) = v_1 + (s - 1)v_2 \), where \( v_1 \) is the limit of the employer’s expected payoff. Hence, in the limit \( v_2 = 0 \) and thus \( v_1 = f(s) \). It follows that coalition \( S \) yields the maximum output and hence \( f(s) \geq f(n) \); for otherwise, given that the outside option for the producer tends to \( f(s) \), \( S \) would not maximize the Nash surplus. As then \( f(s) = f(n) \). \((f(n), 0, \ldots, 0)\) is the unique point in the core. \( \square \) Note that the fact that \( f(s) > f(t) \) for all \( t < s \) rules out another \( s' \)-equilibrium.

Now we study partial equilibria with two coalitions, \((s, t)\)-equilibria, in the limit. In the first case, if the grand coalition does not form \((t < n)\), then \( f(s) = f(n) \) and the expected payoffs are \((f(n), 0, \ldots, 0)\), the unique point in the core (as in an \( s \)-equilibrium). The same occurs in the second case where the grand coalition can form \((t = n) \) whenever \( f(s) = f(n) \); if instead \( f(s) < f(n) \), the employer gets the minimum payoff in the core and \( \frac{f(n)}{n} \leq \frac{f(s)}{f(s)} \). In the latter subcase, we also show, based on Okada (2007), that if there is another \((s', n)\)-equilibrium, the expected payoffs are identical to those in an \((s, n)\)-equilibrium.

**Proposition 3** For \( 2 \leq s < t \leq n \), an \((s, t)\)-equilibrium of the production economy \( \xi \) is characterized as follows.

(i) The employer receives the expected payoff

\[
v_1 = f(s) - \frac{(s - 1)\delta v_2}{s(1 - \delta) + \delta}, \quad (14)
\]

where \( v_2 \), the expected payoff to the worker, is

\[
v_2 = \frac{f(t) - f(s)}{t - s} + \frac{(1 - \delta)(sf(t) - tf(s))}{\delta(t - s)}. \quad (15)
\]

\(5\) In the alternating offers model, \(\delta v_2\) is the wage for the worker and “equals” the marginal product of labor. As this wage tends to zero when \(\delta\) is close to 1, the production function reaches its maximum at \(S\) (See Okada 2007)
(ii) Assume that \( t < n \). If an \((s,t)\)-equilibrium exists for any \( \delta \) close to 1, then
\[
f(k) = f(n)
\]
for all \( s \leq k \leq n \). Moreover, \( v_1 \) and \( v_2 \) converge to (the unique core allocation) \( f(n) \) and 0, respectively, as \( \delta \) goes to 1.

(iii) Assume that \( t = n \). As \( \delta \) goes to 1, \( v_1 \) and \( v_2 \) converge to \( v_1^* = \frac{(n-1)f(s)-(s-1)f(n)}{(n-s)} \) and \( v_2^* = \frac{f(n)-f(s)}{(n-s)} \) respectively. The probability of full employment converges to 1 when \( f(s) < f(n) \). If an \((s,n)\)-equilibrium exists as \( \delta \) goes to 1, then \( f(n) \leq f(s) \). The expected payoff vector \((v_1^*, v_2^*, ..., v_n^*)\) is in the core of the production economy \( \xi \), and the employer receives the minimum payoff \( v_1^* \) in the core.

Proof (i) Using the maximization problem for the initiator in (1), if an \((s,t)\)-equilibrium exists, then the per capita Nash surplus of \( S \frac{W(s)}{s} \geq \frac{W(K_t)}{t} \) for all \( k \neq s \) and that of \( T \frac{W(T)}{t} \geq \frac{W(K_t)}{t} \) for all \( k \neq t \). This implies, after using Lemma 3, that
\[
\frac{f(s) - (s-1) \delta v_2 - \delta v_1}{s} \geq \frac{f(k) - (k-1) \delta v_2 - \delta v_1}{k}
\]
for all \( k \neq s \) and
\[
\frac{f(t) - (t-1) \delta v_2 - \delta v_1}{t} \geq \frac{f(k) - (k-1) \delta v_2 - \delta v_1}{k}
\]
for all \( k \neq t \). By (16) and (17), we have
\[
\frac{f(s) - (s-1) \delta v_2 - \delta v_1}{s} = \frac{f(t) - (t-1) \delta v_2 - \delta v_1}{t}.
\]
Equation (18) implies that the recursive definition in (2) for the case of the employer becomes (as the employer obtains the same after an equilibrium coalition forms regardless of whether it is \( S \) or \( T \))
\[
v_1 = \frac{f(s)}{s} - \sum_{j \in S} \frac{\delta v_j}{s} + \delta v_1.
\]
First, we solve for \( v_1 \) in (19) after using Lemma 3 and get Equation (14). Next, we use (14) in (18) to solve for \( v_2 \) and get (15) after tedious algebra manipulations.

(ii) This part follows almost without change the proof of proposition 4.2 (ii) in Okada (2011). Let \( p_s \) be the probability that an \( s \)-member coalition forms. Then
\[
v_1 + (n-1)v_2 = p_s f(s) + (1 - p_s) f(t).
\]
From (14) and (15), we can see that \( v_1 \) and \( v_2 \) converge to
\[
v_1^* = \frac{(t-1) f(s) - (s-1) f(t)}{t - s} \quad \text{and} \quad v_2^* = \frac{f(t) - f(s)}{t - s}
\]
respectively as \( \delta \) goes to 1. Let \( p_s^* \) be any accumulation point of \( \{p_s\} \). Taking the limit in Equation (20), we obtain \( v_1^* + (n-1)v_2^* = p_s^* f(s) + (1 - p_s^*) f(t) \). After substituting (21) into this last equation, we obtain
\[
\left( p_s^* + \frac{n-t}{t-s} \right) f(s) = \left( p_s^* + \frac{n-t}{t-s} \right) f(t).
\]
Because \( t < n \), \( p_s^* + \frac{n-t}{t-s} > 0 \) must hold. Thus, \( f(s) = f(t) \). Then, \( v_1^* = f(s) \) and \( v_2^* = 0 \) from (21). Finally, \( f(s) = f(n) \) is obtained by letting \( \delta \) go to 1 in (16) with \( k = n \).
Note that if there is another \((s', t')\)-equilibrium, the expected payoffs in the limit are identical to those in an \((s, t)\)-equilibrium.

(iii) Setting \(t = n\) in (21) yields \(v^n_1 = \frac{(n-1)f(s) - (s-1)f(n)}{(n-s)}\), \(v^n_2 = \frac{f(n) - f(s)}{n-s}\). Because \(t = n\), (22) implies that \(p_+ f(s) = p_- f(n)\) for any limit point \(p_\pm\) of \(\{p_\pm\}\). If \(f(s) < f(n)\), then \(p_+ = 0\). Hence, the sequence \(\{p_\pm\}\) converges to 0 as \(\delta\) goes to 1. Therefore, regardless of whether \(f(s) < f(n)\) or \(f(s) = f(n)\), we obtain \(v^n_1 = (n-1)v^n_2 = f(n)\). Because \(f(s) \leq f(n)\), we have \(v_1 + (n-1)v_2 \leq f(n)\) from (20). We substitute (14) and (15) into this inequality, then collect terms with \(f(s)\) and \(f(n)\) on different sides of the inequality; factor out, and divide by the \(f(n)\) coefficient, which is negative and hence reverses the inequality; after tedious algebra we get

\[
\frac{-(n-1)(1-\delta)(s^{(1-\delta)+\delta})}{s(n-1)(1-\delta)(s^{(1-\delta)+\delta})} f(s) \geq f(n).
\]

Thus \(\frac{f(n)}{n} \leq \frac{f(s)}{s}\) for all \(\delta < 1\), in contrast to Okada (2011) that has a limiting result.

Note that if there is another \((s', n)\)-equilibrium in the limit, \(v^n_1 = v^s_1\) and \(v^n_2 = v^s_2\). To see this, in an \((s, n)\)-equilibrium, Equation (17) with \(t = n\) yields \(k f(n) - n f(k) \geq \delta (n - k) (v_2 - v_1)\) for all \(k \neq n\). We use (14) to substitute for \(v_1\) and get for all \(k \neq n\)

\[
(k (1 - \delta) + \delta) f(n) - (1 - \delta) n f(k) - \delta f(k) \geq (n - k) \delta v_2.
\]

When \(\delta\) goes to 1 this inequality becomes \(f(n) - f(k) \geq (n - k) v_2\), which is equivalent to \(f(n) - n v_2^s \geq f(k) - k v_2^s\) for all \(k \neq n\). So, in the limit, this economy behaves as if the employer were maximizing profit. After substituting for \(v_2\), we get for all \(k \neq n\)

\[
f(n) - f(k) \geq (n - k) \frac{f(n) - f(s)}{(n-s)}.
\]

This is identical to equation (41) in the working paper version of Okada (2011), where profit maximization happens regardless of the value of \(\delta\). Similarly for \(s\) we have for all \(k \neq s\)

\[
f(s) - f(k) \geq (s - k) \frac{f(n) - f(s)}{(n-s)}.
\]

Analogous inequalities obtain for an \((s', n)\)-equilibrium. Following Okada (2007), in proposition 5.3 (iii), it is not hard to show first that \(v^n_1 = v^s_1\). Then, without loss of generality, we may suppose \(s > s'\). Now, set \(k = s'\) in (25), and get \(\frac{f(s) - f(s')}{s - s'} \geq v^s_2\).

The analogue of (25) for an \((s', n)\)-equilibrium with \(k = s\) yields \(\frac{f(s) - f(s')}{s - s'} \geq v^s_2\). So \(\frac{f(s) - f(s')}{s - s'} = v^s_2 = v^n_2\). This implies that the profit at \(s\) in an \((s, n)\)-equilibrium, \(f(s) - (s - 1)v^n_2\), is the same as the profit at \(s'\) in an \((s', n)\)-equilibrium. In the limit, the recursive definition of payoffs to the employer in an \((s, n)\)-equilibrium\(f(s) = v^n_1 + (s-1)v^n_2\) and, in the case of an \((s', n)\)-equilibrium, \(f(s') = v^n_1 + (s'-1)v^n_2\). Hence \(v^n_1 = v^n_1\).

To show that \((v^n_1, v^n_2, \ldots, v^n_2)\) is in the core of the production economy \(\xi\), following Okada (2007), Equation (21) implies for all \(k \geq 2\)

\[
v^n_1 + (k - 1)v^n_2 = \frac{(n-k)}{(n-s)} f(s) - f(n).
\]

As the right hand side of (26) equals \(f(n) - (n-k) \frac{f(n) - f(s)}{(n-s)}\), we have after using (24) that \(v^n_1 + (k-1)v^n_2 \geq f(k)\) for all \(k \geq 2\). This, together with \(v^n_1 + (n-1)v^n_2 = f(n)\), show that \((v^n_1, v^n_2, \ldots, v^n_2)\) is in the core of the production economy \(\xi\). Finally, as
\(v_1^* + (s - 1)v_2^* = f(s)\) (set \(k = s\) in (26)), the employer receives the minimum payoff \(v_1^*\) in the core. Suppose that there is a point in the core such that \(v_1 < v_1^*\). Then at that point, the workers in \(S\) have a higher average expected payoff than those not in \(S\) (this is easy to grasp if we see such a point as a deviation from \((v_1^*, v_2^*, ..., v_2^*)\)). But then there exists a player not in \(S\) and another one in \(S\) such that if the former belongs to \(S\) rather than the latter, there is a reassignment of payoffs such that this new \(S'\) can block the point in the core. \(\square\)

When \(t < n\), an \((s, t)\)-equilibrium has the same asymptotic property as an \(s\)-equilibrium. First, we show that in the limit \(f(s) = f(t)\). So (15) implies \(v_2^* = 0\). As then the maximum output is reached at \(s\), \(f(s) = f(n)\). So \(v_1^* + v_2^*\) converge to the unique core allocation \((f(n), 0, ..., 0)\). In (iii), we have in the limit \(v_1^* = f(n) - (n - 1)v_2^*\). If \(v_1^* = f(n) - (n - 1)v_2^*\) is efficient, \(v_1^* + (n - 1)v_2^* = f(n)\), it cannot be blocked, \(v_1^* + (n - 1)v_2^* \geq f(k)\) for all \(k\) (where \(2 \leq k \leq n\)). The first two equalities follow in the limit from the recursive definition of the expected payoffs to the employer. The weak inequality follows, as in the limit we show that the employer acts as a profit maximizer. Thus, the equilibrium allocation \((v_1^*, v_2^*, ..., v_2^*)\) is efficient, \(v_1^* + (n - 1)v_2^* = f(n)\), it cannot be blocked, \(v_1^* + (k - 1)v_2^* \geq f(k)\) for all \(k\), and \(v_1^* + (s - 1)v_2^* = f(s)\). So, the employer receives the minimum payoff in the core.

We have the analogue to Theorem 4.1 in Okada (2011) that shows that if there are more than two coalitions in equilibrium, in the limit, the employer receives expected payoffs as in an \((s, t)\)-equilibrium. Hence, using our previous results, the expected payoffs are unique in our game. As the employer receives the minimum payoff in the core, these expected payoffs coincide with the coalitional Nash bargaining solution.

**Theorem 3** The asymptotic values of the Nash GSSPE payoffs \(v_1^*\) and \(v_2^*\) for \(\delta\) close to 1 are uniquely characterized as follows.

(i) If the allocation \(\left(\frac{f(n)}{n}, \frac{f(n)}{n}, ..., \frac{f(n)}{n}\right)\) is in the core, then \(v_1^* = \frac{f(n)}{n}\) and \(v_2^* = f(n)\).

(ii) Otherwise, the workers receive \(v_2^* = \frac{f(n) - f(s)}{n - s}\), where \(s\) is the solution of \(\min_{1 < k < n - 1} \frac{f(n) - f(k)}{n - k}\). The employer receives the smallest payoff in the core.

(iii) The asymptotic Nash GSSPE allocation \((v_1^*, v_2^*, ..., v_2^*)\) maximizes the generalized Nash product \(x_1x_2...x_n\) within the core, that is, it is the coalitional Nash bargaining solution.

**Proof** First, we argue that if there are other Nash GSSPEs with more than two different sizes of coalitions, these have the same expected payoffs as an \((s, t)\)-equilibrium when the discount factor goes to 1. Equation (20) becomes with, say, three possible coalition sizes, \(s < t < t' < n\),

\[v_1 + (n - 1)v_2 = p_s f(s) + p_t f(t) + (1 - p_s - p_t) f(t').\]  

(27)

As \(f\) is monotonic, it follows from (27) that \(v_1 + (n - 1)v_2 \geq p_s f(s) + (1 - p_s) f(t)\). After substituting (21) into the last inequality, we imply \(f(s) \geq f(t)\). Thus, from the monotonicity of \(f\), \(f(s) = f(t)\). Then the proof proceeds as in the proof of Proposition 3(ii) but with (27) modified to be

\[v_1 + (n - 1)v_2 = (p_s + p_t) f(s) + (1 - p_s - p_t) f(t').\]  

(28)

It follows that \(f(t') = f(s)\) and \(f(s) = f(n)\). If \(s < t < t' = n\), \(f(s) = f(t)\) follows similarly. Next, the proof in Proposition 3(iii) can be applied to (28) with \(t' = n\).
Second, from Propositions 2 and 3 and Theorem 1, the uniqueness for our game follows when \( \delta \) is close to 1. Part (i) follows from Theorem 1. In part (ii), the claims that workers receive \( v_2 = \frac{f(n)-f(s)}{n} \) and the employer receives the smallest payoff in the core follow from Propositions 2 and 3. That is the solution of \( \min_{1 \leq k \leq n-1} \frac{f(n)-f(k)}{n-k} \) follows trivially for equilibria that imply in the limit \( f(n) = f(s) \). If we have an \((s, n)\)-equilibrium in the limit where \( f(s) < f(n) \), inequality (24) is strict when \( k \neq s \), but if \( k = s \), then it holds with equality. To prove (iii), note that in an \((s, t)\)-equilibrium where \( t = n \), the expected payoffs in the limit with the minimum payoff to the employer in the core maximize the Nash product. This follows because \( \frac{f(n)}{n} \leq \frac{f(s)}{s} \) implies that the employer’s expected payoff is greater than or equal to \( \frac{f(n)}{n} \) and the worker’s expected payoff after using (21); other core allocations with different payoffs to workers would have lower Nash product. In the other cases, the core is a singleton. □

5 Conclusions

We have considered the standard coalitional bargaining game without renegotiation and externalities, but with simultaneous payoffs demands. In the general model, the grand coalition forms in a Nash GSSPE (a refinement of the stationary subgame perfect equilibrium consistent with the Nash bargaining solution) if, and only if, the per capita worth of the grand coalition is greater than or equal to the per capita worth of any other coalition, regardless of the discount factor. A close result in terms of GSSPE is obtained for other sharing rules, but in the limit. In the production game as in Okada (2011), we get the author’s results if we use the Nash GSSPE as a solution concept when the discount factor goes to 1, or if we look at the SSPE if we smooth the game as in Van Damme (1991), the noise vanishes, and the discount factor goes to 1. Hence, a simultaneous approach also predict the coalitional Nash bargaining solution. A reasonable conjecture is that our results should extend to the situation where only one coalition can form, as in Compte and Jehiel (2010). More importantly, our framework should also lead to predictions in models with renegotiation and externalities where it may be possible to define extensions of the coalitional Nash bargaining solution based on our noncooperative approach. We leave that for future research. □

A Appendix: The \( n \)-player Smooth Nash Demand Game

We give an outline of how to extend the result, to any \( n \), that the Nash bargaining solution is the unique \( H \)-essential equilibrium outcome (as in Van Damme’s (1991) result for \( n = 2 \)) after smoothing the Nash demand game and letting the noise vanish. At the same time, we show how to fit this result into our coalitional bargaining model. We show the equivalence between the Nash GSSPE and the SSPE of the perturbed game when the noise vanishes. Our results in Section 4 imply that the SSPE outcome is unique in this context.

First, we extend Van Damme’s (1991) result to any \( n \).

Let \( S \) be the set of active players and suppose that coalition \( T \subset S \) has been chosen, that is, we are in subgame \( \Gamma(S, T, \delta) \). Following Van Damme (1991), we propose to smooth the Nash demand game for players in \( T \) and then look at the SSPE when the amount of smoothing approaches zero. Consider the function \( h(x^T) \) that gives the probability that the payoff demand profile \( x^T \) is feasible. More precisely, we restrict attention to perturbations in the class \( H = \cup_{\omega > 0} H^\omega \), where \( H^\omega \) is the set of functions that satisfy

\[
h : \mathbb{R}_+^T \to (0, 1], \text{ h continuous, } h(x^T) = 1 \text{ for } x^T \in X^T_+, and
\]
\[
\max \left\{ h(x^T), h(x^T) \prod_{j \in T} v_j \right\} < \epsilon \text{ if } g(x^T, X^T) > \epsilon,
\]

where \( g(x^T, X^T) \) is the Euclidean distance from \( x^T \) to \( X^T \). The latter means that \( h \) decreases to zero sufficiently fast when \( x^T \) moves away from \( X^T \). The smooth Nash demand game in sub-game \( \Gamma(S, T, \delta) \) is then \( \Psi^{S,T}(h) = \left( T, (R^h_{i})_{i \in T}, (R_{i}^{h})_{i \in T} \right) \) when the uncertainty is described by \( h \) and where the continuous payoff function is \( R^h_i (x^T) = x^T h(x^T) + (1 - h(x^T)) \delta v^S_i \).

In this setup, a (Nash) equilibrium \( x^T \) of \( \Psi^{S,T} \) is an \( H \)-essential equilibrium if to every sequence \( \{h^N\}_{N} \), with \( h^N \in H^N \), there may be associated a sequence \( \{x^{T,N}\}_{N} \), such that \( x^{T,N} \) converges to \( x^T \) as \( N \) approaches zero.

Van Damme (1991) shows, following Nash (1953), that the NBS is an \( H \)-essential equilibrium outcome of the two player standard smooth Nash demand game (Theorem 7.5.4). However, there is no assurance that it is the unique \( H \)-essential equilibrium outcome. Hence he gives an example of a “reasonable” function for which this is the case (Theorem 7.5.5).

The proof for the \( n \)-player case is a straightforward extension following his same steps using the consistency property of the NBS. It follows that \( \left( x^{T,N}, \sum_{j \in T} \delta v^S_j + \delta v^S_i \right)_{i \in T} \) is the unique \( H \)-essential equilibrium outcome of \( \Psi^{S,T} \). It follows that Lemma 2 holds if we replace Nash GSSPE by SSPE and the smoothing technique (assuming independent perturbations) has been introduced and the noise vanishes in \( \Psi^{S,T} \) for each \( T \subset S \) and for each \( S \subset C(N) \). The uniqueness of the SSPE outcome when the noise vanishes in the producer game follows from the uniqueness of the NBS GSSPE outcome when \( \delta \) goes to 1, as shown in Section 4.

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References