Monetizing Attention on Social Media

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Abstract

Using network connecting information to discriminate prices across users or consumers is possible as social media discloses more precise individuals’ networking information. This paper concerns how the social media (Facebook) owner could use network information to do “price discrimination” across her users. In our model, users have multiple interdependent activities (creating and browsing) on friend-based social media, and social media monetizes users’ attention by sending different advertisement densities to them based on their network position. In particular, for users’ behaviors, both interpersonal local browsing externality and intrapersonal of cross-activity externalities are taken into consideration. The striking results show that the network information is usable for the monopoly to discriminate prices across users when multiple interdependent activities are introduced, even though users’ best replies are linear. This paper also tries to explain social media’s benefits from her services, such as recommending friends and events notification for users, by comparative static studies and welfare analysis. Moreover, this paper is the first to show some results of the network with mix externalities.

Keywords: Social networks, Monopoly pricing, Network externalities, Multiple activities, Cross-activity externality, Centrality measures

1 Introduction

Network externality arises when consumers care about other consumers’ consumption behavior. Earlier works consider network externality as global. As friend-based social media emerges, more precise social connecting information of consumers’ is available. Companies may use this information to perfectly discriminate prices across consumers of the good with local network externality (e.g. Bloch and Quéréou (2013), Candogan et al. (2012) ).

Consider these friend-based social media themselves. They also provide a “good” with local network externality and “price” their users. The “good” is information service provided for each user: users create content out of gratification, and they benefit from browsing their connected friends’ posting contents.

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As for “pricing”, friend-based social media monetize users’ attention by sending advertisements when they are browsing their homepage. Therefore, it is a reasonable question to ask: will service providers use network information to discriminate users and how? This paper builds a theoretical model to answer this question. Before proceeding to the model, we start from analyzing the users’ behaviors on social media.

Users have two types of different behaviors on social media: creating and browsing. Creating behavior generates content which can be seen by the creator’s friends. The behavior includes status updating, photo sharing, article/news reposting and reactions on other users’ posting, such as “likes”, comment, and repost. As for every user’s browsing behavior, it combines with browsing friends’ posting and browsing friends’ reactions to postings, which are not observable by other users.

There are two main effects of these two behaviors. One is local browsing externality arising from users’ browsing their friends’ creating behavior, which is interpersonal. The other is intrapersonal—the interaction of each user’s creating behavior and browsing behavior. Some empirical study shows the evidence of local browsing externality: Joinson (2008) stated that users spend time on social media because of the content on it. Identifying the intrapersonal effect of interaction between two behaviors of each user could be difficult. It could be substitutes due to time constraints; also, it could be complements sourcing from one behavior stimulating the other. The aggregate effect of interactions of two behavior could be cancelled out, positive or negative. The aggregate effect of interactions of two behavior is called cross-activity externality factor or externality factor in this paper for short. We consider both cases: uses have homogenous cross-activity externality factor and users have heterogeneous cross-activity externality factors in this paper.

This paper analyzes the monopoly’s discriminating “pricing strategy” across connected users in our stylized model of social media through a game-theoretical approach, in which users choose time spendings on two interdependent activities, and both interpersonal network effect and intrapersonal cross-activity effect are present. The goal is to understand if local network information is valuable to monopoly’s pricing plan or not. In particular, we want to understand the role of the three types of externalities between creating and browsing behavior. Further, we investigate the effect of network structure changes and users’ willingness to create on users’ behavior and the social media owner’s profit. Moreover, we consider the possibilities of sending advertisement whenever users are creating or browsing and the case of users having heterogeneous cross-activity externality factors.

The results of this paper are striking: When the externality factors are homogeneous, the monopoly uses network information to do price discrimination to each user in Nash equilibria. More specifically, The advertisement density sent to each user is linear in his/her weighted degree centrality and weighted

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1Users react to their friends’ posts after they browse these posts in the case of browsing stimulating creating; and for creating’s stimulation effects on browsing, users check their social media homepage more often after they post.
cross-activity externality factor when the Nash equilibrium is unique. For comparative static studies, the monopoly is always better off for adding new links between users whichever case it is, which explain why social media keeps recommending friends to users. Also, the monopoly gets more profit from users’ higher exogenous willingness to create boosted by daily life events in no cross-activity externality and complementarity; for the substitutability case, a certain condition is needed to guarantee the increasing of the monopoly’s profit. We also analyze the welfare of the monopoly and users in three different cases. The results show that for each user and the monopoly, their welfare is decreasing as cross activity factor decreases.

This paper also investigates the case when the monopoly sends advertisements to users whenever they are creating or browsing. The results show that the advertisement densities sent to the users are linear in their weighted Katz-Bonacich centrality. For the network with mixed externalities or heterogeneous externality factors, the uniqueness of Nash equilibrium is conditioning on that the accumulative complementary effect of the network and the accumulative substitutable effect which is amplified by the network complementarity are not too large.

2 Brief literature review

Some empirical studies show users’ behaviors on social media. Benevemuto et al. (2009) and Schneider et al. (2009) show that users spend most of their time on browsing, not creating on social media. Lewis et al. (2008) showed that network effect on users’ behavior: a user with a higher degree and centrality would be more active on social media. Our model provides a constructive way to think about users’ behavior on social media.

Seminal work by Katz and Shapiro (1985) and Farrell and Saloner (1985) studied network externality of global nature. Here our study concerns about the local interaction between agents.

A strand of literature concerns individuals’ local network interaction when their best replies are linear. The largest eigenvalue of the network matrix determines agents’ equilibrium strategy in the game of complements (e.g. Ballester et al. (2006) and Corbo et al. (2007)); the smallest eigenvalue does in the game of substitutes (e.g. Bramoullé et al. (2014)). These two eigenvalues are still central in our analysis.

The most relevant works are Candogan et al. (2012) and Bloch and Quéréou (2013). With linear best reply setting, their studies showed that for a monopoly’s pricing in an undirected network, the price is uniform\(^2\). Candogan et al. (2012) also studied the case of allowing for discriminate consumers with a full pricing and a discount price and found it was an NP-hard problem. Our contribution is showing that network information is useful for monopoly’s pricing in undirected networks when players have two activities.

\(^2\)In Candogan et al. (2012)’s model, the price is uniform when valuation is the same for all consumers
Literature of Multiple activities on networks is rare. Two paper we find are as follows: Chen et al. (2015) studied the case where connected players with multiple interdependent activities experience local network externality of the same activities of their neighbours. Gagnon and Goyal (2016) is interested in explaining the pattern of interaction between market and social ties. In their case, players choose the market action and the network action.

There is other research considering pricing with local network externality in different specific settings (Leduc et al. (2015), Sääskilähti (2007), Ghiglino and Goyal (2010)). Also, Our paper enriches platform competition literature (Caillaud and Jullien (2003), Rochet and Tirole (2003) and Armstrong (2006)) by introducing that one side (user side) of the platform is networked and experiencing local network externality.

3 The Model

3.1 Basic Setting

Consider $n$ users connected in a network and $N$ is the set of all users. $G$ is an undirected and unweighted $n \times n$ adjacency matrix which states the connectedness of the network. $g_{ij}$ is the element of matrix $G$ in row $i$ and column $j$. For $i \neq j$, $g_{ij} = g_{ji} = 1$ if user $i$ and user $j$ is connected as friends; otherwise, $g_{ij} = 0$. $g_{ii}$ is set to be 0. We abuse notation by using $N(i) = \{\forall j \in N \mid g_{ij} = 1\}$ to denote $i$’s friends on the network.

Users has a pair of strategy $\mathbf{x}_i = (t_i, v_i)$: $t_i$ is the time user $i$ chooses to create his/her posts and $v_i$ is the time to browse content on social media. Monopoly chooses a advertisement density plan $\mathbf{p} = (p_1, p_2, ..., p_n)$, such that $p_i$ is the advertisement density for user $i$. We define $\mathbf{t}_{-i}$ as a vector of time spending on creating of all users except for user $i$. The utility function of user $i$ is defined as following:

$$u_i(t_i, t_{-i}, v_i, p_i) = a_i t_i + v_i \sum_{N(i)} t_j - \frac{1}{2} c_i (t_i + v_i)^2 + (c_i + d_i) t_i v_i - p_i v_i \quad (1)$$

in which $a_i t_i$ captures the utility user $i$ get from creating his/her own content($a_i > 0$); the production $v_i \sum_{N(i)} t_j$ is the utility user $i$ gains from browsing. $\frac{1}{2} c_i (t_i + v_i)^2$ indicates the cost from spending time on this social media network($c_i > 0$); it could also be written as $c_i t_i v_i + \frac{1}{2} c_i (t_i^2 + v_i^2)$, and $c_i t_i v_i$ describes the cross-activity substitutability from limitation of time spending. $(c_i + d_i) t_i v_i$($c_i + d_i > 0$) describes the cross-activity complementarity on social media. Therefore, the aggregate cross-activity externality is represented as $d_i t_i v_i$. $d_i$ is named as externality factor of user $i$.

To simplify the analysis for now, we set the cross-activity externality to be the same for all users, such that $c_i = c$ and $d_i = d$, $\forall i \in n$. Also, $|d| < c$, which implies the externalities between $t_i$ and $v_i$ do not exceed marginal cost.
of spending time. When \( d = 0 \), no externalities exist; When \( d < 0 \), \( t_i \) and \( v_i \) are substitutes; They are complements when \( d > 0 \). In extension of mixed externality, general cases of \( \exists i, j \) such that \( d_i \neq d_j \) and \( c_i \neq c_j \) will be discussed.

\( p_i v_i \) are the total times of viewing the advertisements which cause \( i \)'s disutility. The monopoly gets profit from advertisers for users’ times of watching the advertisements. Assume that advertisers’ demand to reach each is high enough, such that all the advertisement spaces can be sold and each viewing can be sold at \$1, the opportunity cost of outside advertisement sponsoring option. Let \( v = (v_1, v_2, ..., v_n) \). Monopoly’s profit function is represented as: \( \pi = p^T v \).

The game is of complete information, and the timing of events is: At the first stage, the monopoly chooses the pricing plan \( p_i \); \( \forall i \in n \), user \( i \) chooses the time to create \( t_i \) at the second stage and the time of browsing \( v_i \) at the final stage.

### 3.2 Nash Equilibria and optimal pricing

As it is a study of canonical economics settings, the utility functions yield best reply functions by which Nash equilibria are characterized. Then, we solve the monopoly’s profit maximization problem to see whether \( p \) will be based on network information or not.

We start with the simplest case— no cross activity externality and then consider cross-activity complementarity case. For the last substitutability case, a potential function of the game is introduced to characterize Nash equilibria.

#### 3.2.1 No Cross-Activity Externalities

When \( d = 0 \), only interpersonal local browsing externality exists. Utility function is \( u_i (t_i, t_{-i}, v_i, p_i) = a_i t_i + v_i \sum_{N(i)} t_j - \frac{1}{2} c_i (t_i^2 + v_i^2) - p_i v_i \). Without cross-activity externality, two activities are intrapersonally independent from each other in the first order condition of the utility function. More specifically, user \( i \)'s strategic choice of \( t_i \) does not depend on any other variables but \( a_i \); \( v_i \) depends on the difference between sum of his/her neighbours’ creating time \( \sum_{N(i)} t_j \) and the monopoly’s choice of \( p_i \). If we think of no upper bound condition for monopoly’s pricing strategy \( p \), user \( i \)'s browsing time \( v_i \) will be its lower bound 0 when \( p_i \) is big enough. Naturally, the solution is as follows:

Taking first order condition on utility function w.r.t \( t_i \) and \( v_i \) leads to the best replies as follows:

\[
\begin{bmatrix}
  t_i \\
  v_i
\end{bmatrix}
= BR_i (x_{-i}) := \begin{bmatrix}
  \frac{a_i}{c} \\
  \frac{\sum_{N(i)} t_j - p_i}{c}
\end{bmatrix}
\text{ if } \sum_{N(i)} t_j - p_i > 0, \quad (5.8)
\]

and

\[
\begin{bmatrix}
  t_i \\
  v_i
\end{bmatrix}
= BR_i (x_{-i}) := \begin{bmatrix}
  \frac{a_i}{c} \\
  0
\end{bmatrix}
\text{ if } \sum_{N(i)} t_j - p_i \leq 0. \quad (5.9)
\]
For $v_i$, it is equal to 0 if the advertisement density to $i$ is high enough ($\sum_{N(i)} t_j - p_i \leq 0$); otherwise, $v_i$ is positive and is proportional to difference between neighbours’ sum of creating time and density of advertisement sent to user $i$. Let $Q = \left\{ i \in N \mid \sum_{j \in N(i)} a_{ij} - p_i \leq 0 \right\}$ denote the set of users who spend no time on browsing due to high advertisement densities sent to them, and $N \setminus Q$ be the set which contains all users except for users in set $Q$. Moreover, $v^*_Q$ is the a vector contain browsing times of users in set $Q$, and $v^*_{N \setminus Q}$ contains all users’ browsing time except users in set $Q$. Also, Let $a = (a_1, a_2, ..., a_n)^T$ be the vector of users’ exogenous willingness to create, and $t = (t_1, t_2, ..., t_n)^T$ be the vector of all users’ time spending on creating. Replace $\sum_{N(i)} t_j$ with $\frac{1}{2} \sum_{N(i)} a_{ij}$, and we have the unique Nash equilibrium. Writing the best replies in matrix leads to the first proposition.

**PROPOSITION 1.** When $d = 0$, for any $a$ and $p$, there is a unique Nash Equilibrium such that: $t^* = \frac{1}{c} a$ and $v^*_Q = 0$ and $v^*_{N \setminus Q} = \frac{1}{2c} (Ga)_{N \setminus Q} - \frac{1}{c} p_{N \setminus Q}$.

Any user’s browsing behavior is only influenced by his/her friends. Combined with that creating time is independent of the network, any effect of network structure or users’ behavior is naturally restricted between linked users.

Now we consider the monopoly’s profit problem:

$$\max_p p^T v$$

There is no incentive for the monopoly to choose any $p_i$ such that $v_i$ is zero, since the profit is $p_i v_i$ and $p_i$ influences nothing but $v_i$ in the case of externality factor $d = 0$. By replace $v$ with $\frac{1}{2c} Ga - \frac{1}{c} p$ since $Q = \emptyset$, solving the problem yields to Proposition 2:

**PROPOSITION 2.** When $d = 0$, for any $a$, the monopoly will choose a price plan $p^* = \frac{1}{2c} Ga$ to maximize his/her profit such that $t^* = \frac{1}{c} a$ and $v^* = \frac{1}{2c} Ga$.

Please see the appendix for proofs.

The monopoly’s discriminating pricing plan is based on users’ network position in our case: Pricing plan for each user is now proportional to the sum of his/her neighbours’ exogenous willingness to create, which is different from the optimal uniform pricing plan from Bloch and Quèrou (2013) and Candogan et al. (2012).

The intuition behind the difference between our case and their cases is: When the one activity is divided into two activities, and each activity plays a different role in the utility function, the interaction of two activities provide network information which monopoly can separate from users’ equilibrium strategy and is useful in their pricing plan.
Additionally, a remark is made as follows: the optimal pricing plan \( p^* \) will always be the same with a given network \( G \) and an \( a \) when \( d = 0 \), even if advertisers pay at different prices to reach different users. Consider a new profit function \( \pi = \frac{1}{c} \sum_{i \in n} r_i p_i \left( \frac{1}{c} \sum_{N(i)} a_i - p_i \right) \) (\( r_i \) is cost per impression for user \( i \)). \( r_i \) is cancelled out in first order conditions of the profit function with respect to \( p_i \). This statement will be helpful to simplify the analyses if advertisers have heterogenous demand to reach users.

### 3.2.2 Cross-Activity Complementarity

When \( |d| > 0 \), creating and browsing behaviors become interdependent intrapersonal. In the case of cross-activity complementarity \( d > 0 \), the utility function is \( u_i(t_i, t_{-i}, v_i, p_i) = a_i t_i + v_i \sum_{N(i)} t_j - a_i \left( \frac{1}{2} t_i^2 + v_i^2 \right) + dt_i v_i - p_i v_i \). \( t_i \) and \( v_i \) are boosted by each other due to cross-activity complementarity. We have to solve two simultaneously determined equations for each user and then characterize the Nash equilibria and the monopoly’s discriminating pricing strategy.

To solve this case, again we must consider the upper bound of \( p \). We start from considering the case in which \( p < \frac{1}{c} G a + \frac{d}{c} a \). \( \forall i, p_i < \frac{1}{c} \sum_{N(i)} a_j + \frac{d}{c} a_i \).

To derive the best relies, we take first order condition of utility function on \( t_i \) and \( v_i \), and we have:

\[
t_i = \frac{a_i + dv_i}{c} \quad (2)
\]

\[
v_i = \frac{\sum_{N(i)} t_j + dt_i - p_i}{c} \quad (3)
\]

When \( p < \frac{1}{c} G a + \frac{d}{c} a, \ v > 0 \) since the lower bound for \( \sum_{N(i)} t_j + dt_i \) is \( \frac{1}{c} \sum_{N(i)} a_j + \frac{d}{c} a_i \) from that \( t_i \)’s lower bound is \( \frac{a_i}{c} \).  

Replacing \( v_i \) in (2) with (3) results in best reply of \( t_i \):

\[
t_i = \frac{c}{c^2 - d^2} a_i + \frac{d}{c^2 - d^2} \sum_{N(i)} t_j - \frac{d}{c^2 - d^2} p_i.
\]

Weighted Katz-Bonancich centrality is needed to construct the Nash equilibrium. It is defined as \( m_a(G, \theta) = (I - \theta G)^{-1} a = I + \theta a + \theta^2 G^2 + ... \) in which \( \theta \) is a scalar such that \( I - \theta G \) is invertible and \( a \) has at least two different elements.  

Let \( \lambda_{\text{max}}(G) \) be network matrix \( G \)’s largest eigenvalue. The condition for existence and uniqueness of Nash equilibrium is \( \frac{d}{c^2 - d^2} \lambda_{\text{max}}(G) < 1 \) such that \( I - \frac{d}{c^2 - d^2} G \) is invertible (See (Ballester et al., 2006) (Ballester and Calvó-Armengol, 2010)). Using \( \delta = \frac{1}{c^2 - d^2} \) to simplify the notation, we have:

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3We spend no effort in this paper to discuss the case in which there is no constraint on \( p \), since the monopoly has no incentive to choose \( p_i \) \( \forall i \). please see the appendix for Proposition 3 for details.

4Please see the detail of Katz-Bonancich Centrality in Appendix.
PROPOSITION 3. When $d > 0$, if $d\delta \lambda_{\text{max}}(G) < 1$ and $p < \frac{1}{2}Ga + \frac{d}{\delta}a$, there is a unique Nash Equilibrium such that users’ strategies are $t^* = \delta (I - d\delta G)^{-1} (ca - dp)$ and $v^* = \delta (I - d\delta G)^{-1} (Ga + da - cp)$.

Creating time spending for users are proportional to Katz-Bonanchich centrality weighted by weighted difference between exogenous willingness to create and advertisement density sent to them; Browsing time are proportional to Katz-Bonanchich centrality weighted by $Ga + da - cp$.

The monopoly’s pricing strategy for each user has global impacts on the network now: increasing any user’s advertisement density will make all other users’ creating and browsing time decrease. Further, we solve the monopoly’s profit maximization problem to see the changes of users’ strategies and the monopoly’s pricing strategy, which results in Proposition 4:

PROPOSITION 4. When $d > 0$, if $d\delta \lambda_{\text{max}}(G) < 1$, monopoly’s optimal pricing strategy is $p^* = \frac{1}{2c}(G + dI)a$, and there is a unique Nash equilibrium such that users’ strategies are $t^* = \frac{1}{2c} \left( I + c^2 \delta (I - d\delta G)^{-1} \right) a$ and $v^* = \frac{d}{2} (G + dI)(I - d\delta G)^{-1} a$.

For user $i$, $t_i$ is divided into two parts: one is his/her willingness to create and the other is proportional to Katz-Bonanchich centrality of creating willingness which is boosted by cross-activity complementarity. As for $v_i$, the part time spending boosted by his/her own and his/her neighbour’s own willingness has been reduced because of advertisement send to them; the remaining part is browsing time boosted by cross-activity complementarity.\(^5\) Compared to the no-externalities case, cross-activity complementarity boosts both behaviors into a higher level of time spending. The optimal advertisement density level for each user also increases in this case.

3.2.3 Cross-Activity Substitutability

We move to the last cross-activity substitutability case in which $d < 0$. In Bramoullé et al. (2014), the characterization and multiplicity of Nash equilibria of one activity with network substitutable effects between agents and their neighbours are addressed. The multiplicity of Nash equilibria presents in this case, too. But we only consider the cases with a unique and interior Nash equilibrium, which are simple and are analytic solutions such that we can see clearly how pricing strategy will be different in different cross-activity externalities cases.

In this section, we firstly solve a proxy maximization problem which yields the same set of solutions as the set of Nash equilibria of cross-activity substitutability. Then we introduce the condition for uniqueness of Nash equilibrium

\(^5\)By (3), $cv^* = (G + dI) t - p$: the part of creating boosted by their own willingness to create $\frac{d}{2c} (G + dI)a$ is equal to $p$, and the remaining part is $\frac{d}{2} (G + dI)(I - d\delta G)^{-1} a$. 

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in our two interdependent activity case. At last, we solve the monopoly’s profit maximization problem.

By using potential theory, we have:

**LEMMA 1.** If \( d < 0 \), the set of Nash equilibria for a given \( c, d, a, p \) and \( G \) corresponds to the set of maxima and saddle points of the potential function \( \varphi(t; c, d, a, p, G) = \delta t^T (ca + dp) - \frac{1}{2} t^T (I - dG) t \) on \([0, \frac{a_i}{c}]^n\).

The multiplicity of Nash equilibria complicates the analysis of users’ behavior. It causes meaningless selection problem of Nash equilibrium in which one monopoly gets the highest profit. Therefore, we only care about the games of cross-activity substitutability with a unique Nash equilibrium. We have:

**PROPOSITION 5.** When \( d < 0 \), if \( 1 - d\delta \lambda_{\text{min}}(G) > 0 \), there exists a unique Nash equilibrium. If \( \forall i \frac{d}{c^2} \sum_{j \in N(i)} a_j < a_i \) and the Nash equilibrium is an interior solution, the monopoly’s optimal pricing strategy is presented as \( p^* = \frac{1}{c} (G + dI) a \), and users’ strategies are \( t^* = \frac{1}{2c} \left( I + c^2 \delta (I - dG)^{-1} \right) a \) and \( v^* = \frac{\delta}{2} (G + dI) (I - dG)^{-1} a \).

Conditions \(-\frac{d}{c^2}Ga < a\) are used to guarantee that the creating time spending for each user is positive. For details of the proof, please see the appendix.

Compared to the case of no cross-activity externality, advertisement density level decreases. The individual change of creating time and browsing time are unclear. Aggregately, we have precise results that the total creating time decreases and browsing time increases, compared with those in no cross-activity externalities.

For the comparison with cross-activity complementarity, if absolute values of \( d \) in cross-activity complementarity and substitutability are the same, both time spending on two activities in cross-activity complementarity case is greater than that in cross-activity substitutability case separately. In the following section comparative static studies and welfare analysis, the strategies of users will also be analyzed for the cross-activity substitutability case.

From three cases above, it is not hard to see that the analytical expressions of the three cases are the same. In fact, whatever \( d \) is, the game can be solved by the potential function \( \varphi \) (See also Bramouillé et al. (2014)). The reason why we still discussed them separately is that putting them together does not make the discussion simpler: When \( d > 0 \), the condition for no-ever-increasing play \( d\delta \lambda_{\text{max}}(G) < 1 \) is still needed; When \( d < 0 \), without lower bound, Existence of

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6In the two cases discussed before, extraordinarily high advertisement density is allowed when we characterize Nash equilibrium, but it is not a choice of monopoly’s pricing plan to her goal of maximizing profit. However, in the case of substitutability, some users spend all their time on creating, and no time on browsing due to their high advertisement densities in the profit-maximizing Nash equilibrium. It is unrealistic for users to use a social media where they only create and never browse. The equilibrium with the highest profit but no browsing users is a loss of interest. Please see the appendix for examples.

7Please see the appendix for proof.
Nash equilibria is guaranteed and conditions for uniqueness of Nash equilibrium  
\( d \lambda_{\min} (G) < 1 \) is still needed to mention. In order to the following studies easy  
to read and to understand, we still consider three different cases and compare  
them.

4 Comparative Static Studies

In this section, we investigate the effect of changes in the network structure or  
the willingness of each user to create on users’ equilibrium strategy in three cases.

4.1 Changes in the network structure

The motivation of comparative static studies on network structure is that friend-  
based social media recommends users friends they may know in their real life  
constantly. Does this service which helps to build a denser network benefit  
the monopoly in all three cases? Moreover, how denser network structures will  
affect users’ behavior in each case? We will discuss each case one by one.

Some assumption should be made before we start our analysis: the monopoly’s  
pricing plan depends on the network structure in all cases. Therefore, it causes  
confusion about whether the monopoly’s pricing plan will change or not with the  
network structures. Thus, we assume that the monopoly’s pricing plan changes  
with the network structure automatically to maximize profit.

Again, start from the simplest no cross-activity externality case.

For the case of no cross-activity externality, independence of creating behav-
ior from network structure and the constrained influence of browsing has been  
discussed before. Straightly, network structure changes affect only browsing  
strategies of the users whose degrees are changed. Let \( G \setminus ij \) denote the set  
of all users except for user \( i \) and user \( j \). We have:

**PROPOSITION 6.** When \( d = 0 \), consider a \( G \) and users’ strategy \( t^* \) and  
\( v^* \) in Equilibrium. Consider a \( G' \) such that \( G' = G + l_{ij} \) for which \( g_{ij} = 0 \) and  
users strategy \( t^* \) and \( v^* \). Then, \( t^* = t^* \) and \( v^*_{G \setminus ij} = v^*_{G' \setminus ij} \), \( v^*_{i,j} < v^*_{i,j} \). For  
monopoly, \( \pi^* < \pi^* \).

The profit increases from the browsing time’s and advertisement densities’  
increments of user \( i \) and user \( j \). The profit gained from other users’ keeps the  
same in \( G \) and \( G' \).

In the case of cross-activity complementarity, \( t^* \) and \( v^* \) are propositional to  
weighted Katz-Bonancock centrality.\(^8\) For every user in the network, his/her  
weighted Katz-Bonanckich Centrality increases with new link between any two  
users. Therefore, each element of \( t^* \) and \( v^* \) increases and each element of \( p^* \)  
does not decrease with more links in the network, it is not hard to deduce:

\[^8\]v^* = \left( G + dI \right) \left( I - dbG \right)^{-1} a \text{ in Proposition 5 could be written as } v^* = \frac{d^2 d^2}{2 d} \left( I - dbG \right)^{-1} a - \frac{1}{2} d a.
PROPOSITION 7. When $d > 0$. Consider a $G$ and $\delta$ such that $\delta \lambda_{\text{max}}(G) < 1$ and users’ strategy $t^*$ and $v^*$ in Equilibrium. Consider a $G'$ such that $G$ is a subgraph of $G'$, and $\delta \lambda_{\text{max}}(G') < 1$. Users strategy presents as $t^*$ and $v^*$. Then, $\forall i t_i^* < t_i^*$ and $v_i^* < v_i^*$. For monopoly, $\pi^* < \pi^*$.

The case of cross-activity substitutability is more complicated than the two cases discussed above. Consider adding a link between user $i$ and user $j$. It may increase $i$ and $j$’s browsing time and decrease their creating time. Then, $i$ and $j$’s friends behave oppositely. And then their friends’ friends... Example 1 below shows the case of the adding links to a five-node circle until it turns into a complete network. The Figure1. states the processes of the six-node-star turning into the complete graph. The Figure 2. shows how the users’ time spending and the monopoly’s profit change with adding links. Parameters related to the analysis are set as: $c = 10$, $d = -0.9$, $a^T = (100, 100, 100, 100, 100, 100)$.

A user creating time drops greatly when there is a new link added between him/her and another users; and a user’s creating time increases when the new link is not linking him/her to another user. The browsing time changes in the opposite way. The aggregate results are much easier to follow: users’ total time for creating decreases and that of browsing increases when the network become denser, and the monopoly’s profit increases. Despite the mess results of users’ individual behavior changing with network structures, can we have the precise results of aggregate behavior of all users and the monopoly’s profit on the network never than less? See Proposition 8:

PROPOSITION 8. When $d < 0$. Consider a $G$, users’ strategy $t^*$ and $v^*$ and monopoly pricing $p^*$ in Equilibrium. Consider a $G'$ such that $G$ is a subgraph of $G'$ . Users strategy presents as $t^*$ and $v^*$. Then, $\sum t_i^* > \sum t_i^*$, $\sum v_i^* < \sum v_i^*$. For monopoly, $\pi^* < \pi^*$.

The aggregate creating time decreases and the aggregate browsing time increases with the network becoming denser.

In different cross-activity externalities, the effects of network vary with influencing scopes and the directions of change for both users’ creating and browsing behavior. However, the monopoly always benefits from denser network structure no matter what kind of cross-activity externality present in the networks.

4.2 Changes in the exogenous willingness to create

The social media users post more when they go out for travelling or their birthday comes. Therefore, It is of particular interest how behaviors of users and monopoly’s profit will be affected by some users’ exogenous willingness to create. We investigate the effect of exogenous willingness to create on users’ behavior and the monopoly’s profit in this section.
Figure 1: An example of network structure changes (From a 6-node-star to a complete graph)
Figure 2: An example of comparative studies of Users' behavior and the monopoly’s profit

With the assumption that in each case monopoly adjust their pricing plan automatically, we firstly consider the case of no-cross activity externality. Again, since the all the change effects are restricted locally, we have:

**PROPOSITION 9.** When \( d = 0 \), consider a \( \mathbf{a} \) and users’ strategy \( \mathbf{t}^* \) and \( \mathbf{v}^* \) in Equilibrium. Consider a \( \mathbf{a}' \) such that \( \forall i, a_i \leq a_i' \) and \( \exists i, a_i < a_i' \), users strategy presents \( \mathbf{t}^* \) and \( \mathbf{v}^* \). Then, \( \forall i, t_i^* \leq t_i'^* \) and \( v_i^* \leq v_i'^* \). For monopoly, \( \pi^* < \pi'^* \).

For the case of cross-activity complementarity, an increasing in \( a_i \), \( \forall i \in n \), causes weighted Weighted Katz-Bonancich Centralities increase for all users in the network. Therefore, both users’ creating and browsing time increase. So does monopoly’s profit. We have:

**PROPOSITION 10.** When \( d > 0 \), consider a \( \mathbf{a} \) and users’ strategy \( \mathbf{t}^* \) and \( \mathbf{v}^* \) in Equilibrium. Consider a \( \mathbf{a}' \) such that \( \forall i, a_i \leq a_i' \) and \( \exists i, a_i < a_i' \). Users strategy presents as \( \mathbf{t}^* \) and \( \mathbf{v}^* \). Then, \( \forall i, t_i^* < t_i'^* \) and \( v_i^* < v_i'^* \). For monopoly, \( \pi^* < \pi'^* \).

The case of cross-activity substitutability is more complicate as before. We investigate the aggregate level of creating and browsing time, and the result is showed in Proposition 11:
Proposition 11. When $d<0$. Consider a $a$ and users’ strategy $t^*$ and $v^*$ in Equilibrium. Consider an $a'$ such that $\forall i$, $a_i \leq a_i'$ and $\exists i$, $a_i < a_i'$. Users strategy presents as $t^*_i$ and $v^*_i$. Then, $\sum t^*_i < \sum t^*_i$; If $\forall i$, $\sum_{N(i)} a_j + da_i \leq \sum_{N(i)} a'_j + da'_i$, $v^*_i \leq v^*_i$ and $\pi^* < \pi^*$.

Higher exogenous willingness to create results in no decreasing creating time spending individually in first two cases and aggregately in the last case. The browsing is no decreasing as well, and the monopoly is better off in first two cases. For the last case, a certain condition is needed to guarantee the non-decreasing aggregate browsing time and increasing of monopoly’s profit.

5 Welfare Analysis

In this section, we discuss how the welfare of agents will be different as the externality factor $d$ increases or decreases.

As we know from section 3, the analytic solutions of advertisement density $p$ and users’ behavior $t$ and $v$ are the same if certain condition are satisfied. So do the profit of the monopoly and users’ utilities. The monopoly’s profit is $\pi^* = \frac{1}{2\epsilon} a^T (G + dI)^2 (I - d\delta G)^{-1} a$ and users’ utilities are $U^* = \frac{3}{8\epsilon} a^2 + \frac{3\delta}{8} \left( (I - d\delta G)^{-1} a \right)^2 + \frac{c\delta^2}{8} \left( G (I - d\delta G)^{-1} \right)^2$ if conditions are met in Proposition 4 when $d > 0$ and conditions are met in Proposition 5 when $d < 0$.

For all three cases we discuss above, the monopoly’s profit can be present as $\pi^* = \frac{3}{4\epsilon} a^T (G + dI)^2 (I - d\delta G)^{-1} a$ and users’ utilities are $U^* = \frac{3}{8\epsilon} a^2 + \frac{3\delta}{8} \left( (I - d\delta G)^{-1} a \right)^2 + \frac{c\delta^2}{8} \left( G (I - d\delta G)^{-1} \right)^2$.

We take the first derivative with respect to $d$ on both $\pi^*$ and $U^*$:

$$\frac{\partial \pi^*}{\partial d} = \frac{\delta}{2\epsilon} a^T (G + dI)^2 \left( dI + \frac{c^2 + d^2}{2} \delta^2 G (I - d\delta G)^{-1} \right) (I - d\delta G)^{-1} a$$

and

$$\frac{\partial U}{\partial d} = \frac{dc\delta}{2} \left( G (I - d\delta G)^{-1} a \right)^2 + \frac{c\delta^2}{4} \left( G^2 (I - d\delta G)^{-2} a \right)^2$$

$$+ \frac{dc\delta^2}{4} \left( (I - d\delta G)^{-1} a \right)^2 + \frac{c\delta}{8} \left( G (I - d\delta G)^{-2} a \right)^2$$

When $d > 0$, $\frac{\partial U}{\partial d} > 0$ and $\frac{\partial \pi^*}{\partial d} > 0$. When $d < 0$ and $(I - d\delta G)^{-2} a > 0$, $\frac{\partial U}{\partial d} > 0$. Therefore:
PROPOSITION 12. When $d > 0$ and $d\delta\lambda_{\text{max}}(G) < 1$. As $d$ increases, the monopoly’s profit increases, and each user’s utility increases as well. When $d < 0$, $(I - d\delta G)^{-1} a > 0$, and Nash equilibrium is unique and interior, as $d$ increases, each users’ utility decreases, too.

When $d > 0$, it is not very difficult to see that the monopoly’s profit is increasing with $d$, since both the browsing time $v$ and advertisement density $p$ both monotonically increase with $d$. So does the each user’s utility, even though the monopoly prices each user with a higher advertisement density compared to that in no cross-activity case. Both monopoly and users benefit from cross-activity complementarity.

When $d < 0$, when the Nash equilibrium is unique

The Figure3. shows an example the users’ welfare and the monopoly’s profit changes with $d$ in which $c = 10$, $a^T = (100, 100, 100, 100, 100)$:

6 Extension

In the earlier chapters, we only consider the case in which the monopoly can only monetize users’ browsing time. However, monetizing creating time is a technically available option. The practical reason why social media give up the option is that too many advertisements lead to permanently losing users. In section 6.1, we consider the monopoly pricing users’ total time spent on the network. One of the aims is to compare the difference between pricing browsing time and pricing total time. How does pricing total time will influence users’ activities? How will the monopoly’s profit be influenced? In addition, we consider two more general cases to understand the effect of dividing one activity to two serving different parts activities on the pricing strategy. Understand the change from a uniform pricing plan to a network information relevant pricing plan.

In section 6.2, we consider a more general case in which users’ cross-activity externalities are heterogeneous. As we discussed before, users’ cross-activity externalities could be different. For two connected users $i$ and $j$, $i$’s creating could boost $j$’s creating due to $d_j > 0$ but $j$’s creating could redacting $i$’s creating since $d_i$ is negative. We concern how asymmetrical influences between users will affect the equilibrium.

6.1 Pricing with the same advertisement density for both activities of each user

Now consider the monopoly pricing total time each user spent on the network such that $\pi = p^T (t + v)$. User $i$’s utility function is:

$$u_i(t_i, t_{-i}, v_i, p_i) = a_i t_i + v_i \sum_{N(i)} t_j - \frac{1}{2} c (t_i + v_i)^2 + (c + d) t_i v_i - p_i (t_i + v_i)$$

(4)
(a) The network structure

(b) Users’ welfare and the monopoly’s profit

Figure 3: An Example of welfare analysis
<table>
<thead>
<tr>
<th>$t^*$</th>
<th>$v^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>In Prop. 3.</td>
<td>$\delta (I - d\delta G)^{-1} (ca - dp)$</td>
</tr>
<tr>
<td>In Prop. 15.</td>
<td>$\delta (I - d\delta G)^{-1} (ca - (c + d)p)$</td>
</tr>
</tbody>
</table>

Table 1: Comparison between pricing users’ total time and only browsing time

Here $d$ is nonnegative. We focus on complementarity case since the analytic expressions of cross-activity complementarities and substitutability are the same when Nash equilibrium is unique and interior, and discussion on corner solutions of substitutability leads to losing focus.

$p_i$ now enters into the both first order conditions to maximize $u_i$. For the same $p$, the levels of both creating time and browsing time of pricing total time should be lower than those of pricing only browsing time. It is hard to deduce that how users’ behavior in Nash equilibrium and the monopoly’s pricing strategy will be different in this case. Using the same step we have used for several time, we have Proposition 15:

**PROPOSITION 13.** In the case of users’ utility function is (4) and the monopoly’s profit is $p^T (t + v)$, when $d > 0$ and $d\delta \lambda_{\text{max}} (G) < 1$, the monopoly’s optimal pricing strategy is $p^* = \frac{1}{2} a - \frac{1}{d} \left( I + \frac{1}{2(c+d)} G \right)^{-1} a > 0$, and there is a unique interior Nash equilibrium such that users’ strategies are $t^* = \delta (I - d\delta G)^{-1} (ca - (c + d)p) > 0$ and $v^* = \frac{c^2}{d^2} (I - d\delta G)^{-1} (ca - (c + d)p) - \frac{1}{d^2} (a - p) > 0$.

$p^*$ is based on network information. For user $i$ and $j$ such that $a_i = a_j$, the user having higher weighted Katz-Bonacich Centrality with scalar $-\frac{i}{2(c+d)}$ is charged by less advertisement densities. $p^*_i = \frac{d}{2c+d} a + \frac{1}{2c} Ga$ in Proposition 4 ($b$ is used to indicate that pricing browsing time). $p^* = \frac{1}{2} a - \frac{1}{d} \left( I + \frac{1}{2(c+d)} G \right)^{-1} a$ could be written as $p^* = \frac{1}{2} a + \frac{d}{2(c+d)} Ga + O \left( \frac{d^2}{(c+d)^2} \right)$. When pricing for the total time, the weights are more on each users’ exogenous willingness to create. When pricing for the browsing time, the weights are more on neighbours’ sum of exogenous willingness ro create.

Comparing creating and browsing behavior in Proposition 15 to Proposition 4. See Table 1. . For creating time in Nash equilibrium, pricing total time makes the negative impact of $p$ on $t^*$ greater than pricing on only browsing time by $c\delta (I - d\delta G)^{-1} p$ (viewed as the influence from pricing the creating time). For browsing time, overall effect from pricing creating time is $-\frac{c^2}{d^2} (I - d\delta G)^{-1} p + \frac{1}{d} p < 0$.

The Figure4 shows an example of how users’ behavior and the monopoly’s pricing strategy and profit will be different in a 6-node circle case. The Horizontal axis is $d$. In the example, $c = 10, a_i^T = (100, 100, 100, 100, 100, 100, 100)$. What we should notice is advertisement densities for pricing total is larger than those for pricing only browsing time.

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Figure 4: An example of comparison between pricing users’ total time and only browsing time
Now we compare pricing total time with the case in which users have only one network activity. If users have one activity and their utility presents as

\[ u_i(x_i, x_{-i}, p_i) = a_i x_i + x_i \sum_{N(i)} x_j - \frac{1}{2} c x_i^2 - p_i x_i \]

or \( u_i(t_i, v_i, t_{-i}, v_{-i}, p_i) = a_i(t_i + v_i) + (t_i + v_i) \sum_{N(i)} (t_j + v_j) - \frac{1}{2} c (t_i + v_i)^2 - p_i (t_i + v_i) \), \( p^* = \frac{1}{2} a \) (For more details, please see Candogan et al. (2012)). In our case, \( p^* = \frac{1}{2} a - \frac{1}{\kappa} \left( I + \frac{1}{2(c+d)} G \right)^{-1} a \). Despite the same part \( \frac{1}{2} a \), \( \frac{1}{\kappa} \left( I + \frac{1}{2(c+d)} G \right)^{-1} a \) captures the overall effect from dividing one behavior \( a_i x_i + x_i \sum_{N(i)} x_j \) as present in the utility function to two \( a_i t_i + v_i \sum_{N(i)} t_j \) as present in the utility function) and cross-activity externality from \(-c t_i v_i \) to \( d t_i v_i \).

More generally, we consider a auxiliary utility function: \( u_i(t_i, t_{-i}, v_i, p_i) = a_i(t_i + \kappa v_i) + (v_i + \kappa t_i) \sum_{N(i)} (t_j + \kappa v_j) - \frac{1}{2} c (t_i + v_i)^2 + (1 - \kappa) (c + d) t_i v_i - p_i (t_i + v_i) \), in which \( 0 \leq \kappa \leq 1 \). The optimal pricing plan for the monopoly is \( p = \frac{1}{2} (1 + \kappa) a - \frac{1 + \kappa}{\kappa} \left( I + \frac{1}{2(c+d)} G \right)^{-1} a \). \( \frac{1}{\kappa} \kappa a - \frac{1 + \kappa}{\kappa} \left( I + \frac{1}{2(c+d)} G \right)^{-1} a \leq 0 \) captures the overall effect that the behavior are divided to play different roles but one behavior plays \( \kappa \) part in the other behavior’s role \( a_i(t_i + \kappa v_i) + (v_i + \kappa t_i) \sum_{N(i)} (t_j + \kappa v_j) \) as present in the utility function. When \( \kappa = 0 \), it is our case of pricing total time of users on network such that \( p^* = \frac{1}{2} a - \frac{1}{\kappa} \left( I + \frac{1}{2(c+d)} G \right)^{-1} a \). When \( \kappa = 1 \), it is Candogan et al. (2012)’s case such that \( p = \frac{1}{2} a \).

Consider the following general auxiliary utility function to make pricing only browsing time and one network activity case comparable. The monopoly prices only \( 0 \leq \kappa \leq 1 \) portion of creating time but the whole browsing time of users such that \( \pi = p^T (\kappa t + v) \). The utility function for user i is now:

\[
\begin{align*}
  u_i(t_i, t_{-i}, v_i, p_i) &= a_i(t_i + \kappa v_i) + (v_i + \kappa t_i) \sum_{N(i)} (t_j + \kappa v_j) - \frac{1}{2} c (t_i + v_i)^2 \\
  &
  + (1 - \kappa) (c + d) t_i v_i - p_i (\kappa t_i + v_i)
\end{align*}
\]

\( p^* = \frac{1}{2} \left( \frac{1 + \kappa^2 (1 - \kappa)}{1 + \kappa^2 (1 - \kappa) c + 2 \kappa d} G + \frac{\kappa (1 + \kappa^2 c + (1 + \kappa^2) d)}{1 + \kappa^2 (1 - \kappa) c + 2 \kappa d} I \right) a \) is the monopoly’s optimal pricing strategy if users’ utility functions are (5). Adjusting \( \kappa \) leads to reallocating weights between \( Ga \) and \( a \). When \( \kappa = 0 \), it is the case \( p = \frac{1}{2c} (G + dI) a \); When \( \kappa = 1 \), it is the case in which \( p = \frac{1}{2} a \).

6.2 Mixed Externalities

In the cases we studied in the earlier sections, \( \forall i, d_i = d \) and \( c_i = c \). As users’ cross-activity externality factors and coefficients of quadratic cost could be heterogenous, we consider a more general case. The utility function is now:

\[
\begin{align*}
  u_i(t_i, t_{-i}, v_i, p_i) &= a_i t_i + v_i \sum_{N(i)} t_j - \frac{1}{2} c_i (t_i + v_i)^2 + (c_i + d_i) t_i v_i - p_i v_i
\end{align*}
\]

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Firstly, we consider it when $\forall i \ d_i \geq 0$. Even though asymmetrical network effect of differences of users’ cross-activity externalities presents, the uniqueness of Nash equilibrium is supported with supermodularity of this game. To present the Nash equilibrium of mixed externalities, let $H$ denote a diagonal matrix in which $h_{ii} = \frac{d_i}{c_i - d_i^2}$, $K$ denote a diagonal matrix in which $k_{ii} = \frac{c_i}{c_i - d_i^2}$, $L$ denote a diagonal matrix in which $l_{ii} = \frac{1}{c_i - d_i^2}$, $M$ denote a diagonal matrix in which $m_{ii} = \frac{1}{c_i}$, and $O$ denote a diagonal matrix in which $o_{ii} = \frac{d_i}{c_i}$. Then:

**Lemma 3.** When users’ cross-activity externality factors are not equal but all positive such that $\exists i, j \ d_i \neq d_j$ and $\forall i \ d_i > 0$, there exists a unique interior Nash equilibrium if $\lambda_{\text{max}}(HG) < 1$, such that the monopoly’s pricing plan is $p = (I - (HG)^T) \left(2I - (HG + (HG)^T)\right)^{-1} (OI + MG)a$, and the users’ strategies in equilibrium are $t^* = (I - HG)^{-1}(Ka - Hp)$ and $\upsilon^* = (I - HG)^{-1}(LGa + Ha - Kp)$.

Please see the appendix for proof. Now $\lambda_{\text{max}}(HG)$ please a key role in determining the existence and uniqueness of the Nash equilibrium. When $\lambda_{\text{max}}(HG) < 1$, $(I - HG)^{-1}$ is a positive matrix with entries upper bounded. As $(I - HG)^{-1}$ is not a symmetric matrix, we have $(HG)^T$ present when the monopoly’s profit function is taken first order condition for maximization.

Now we move to the case when $\forall i \ d_i < 0$. The users’ strategies and the monopoly’s pricing strategy is the same as in Lemma 3 if the Nash equilibrium is unique and interior:

**Lemma 4.** When users’ externality factors are not equal but all negative such that $\exists i, j \ d_i \neq d_j$ and $\forall i \ d_i < 0$, if $\lambda_{\text{max}}(HG) < 1$, there exists a unique Nash equilibrium; if the Nash Equilibrium is interior, the monopoly’s pricing plan is $p = (I - (HG)^T) \left(2I - (HG + (HG)^T)\right)^{-1} (OI + MG)a$, and the users’ strategies in equilibrium are $t^* = (I - HG)^{-1}(Ka - Hp)$ and $\upsilon^* = (I - HG)^{-1}(LGa + Ha - Kp)$.

When both positive and negative externality factors exist, the Nash equilibria exist conditioning that the accumulative network effect from complementarity is small enough. For accumulative network effect from substitutability, the strategies of all users with negative externality factors are naturally bounded.

Let $C = \{i \in N \mid d_i > 0\}$ denote the set of users whose cross-activity externality factors are positive, we have the conditions for the existence of Nash equilibria.

**Lemma 5.** When users’ cross-activity externality are not equal such that $\exists i, j$ and $d_i \neq d_j$, if $\lambda_{\text{max}}(HC\cdot G_C) < 1$, there exists at least one Nash equilibrium.
This is very intuitive. \( \lambda_{\text{max}}(H_CG_C) < 1 \) guarantee that the accumulative complementary effect does not drive the strategies of users who cross-activity factors are positive to infinite. Therefore, the strategy sets for users are compact and convex. By Fixed point theorem, the Nash equilibria exist.

Last, we consider the condition that the uniqueness of Nash equilibrium in mixed externalities. Firstly, we consider an extension of Lemma 4’s setting: a new user \( w \) whose externality factor is positive is added into the network and connected to two different users \( i \) and \( j \). The negative network effect of \( i \) on \( j \) is enhanced by this new user \( w \). Creating time of user \( i \) boosts user \( w \)’s creating activity and boosted creating activity of user \( w \) reduces user \( j \)’s creating behavior. Adding a new user with positive externality factor into a network with connecting two users with negative externality factor could be seen as building a link between these two users. It changes the accumulative network effect of the original network. A new network matrix should be introduced to describe the new asymmetric network effect.

Now consider the new user as a new set of users with positive externality factor and the links are between different nodes in the order user set and the new user set. The effect of user \( i \) on user \( j \) will be amplified by the postive externality factor network. Let \( S = \{ i \in N \ | \ d_i < 0 \} \) denote the set of users whose cross-activity externality are negative. We define a \( n \times n \) matrix \( G'_S \) such that the entry at row \( j \) and column \( w \) is \( g'_{S,jw} = \frac{d_i}{c_i-d_j-d_i} \left( g_{jw} + \sum_{w'\in N(j)\cap C} \sum_{r\in C} m_{wr} h_{rr} g_{wr} \right) \)

in which \( j, w \in S \). \( G'_S \) describes aggregate effect of users in \( S \) from the substitutability network and the newly added complementarity network. We get

**PROPOSITION 14.** When users’ externality factors are not equal such that \( \exists i, j \) and \( d_i \neq d_j \), if \( \lambda_{\text{max}}(H_CG_C) < 1 \) and \( \lambda_{\text{max}}(G'_S) < 1 \), there exists a unique Nash equilibrium; if the Nash equilibrium is interior , the monopoly’s pricing plan is \( p = \left( I - (HG)^T \right) \left( 2I - (HG + (HG)^T)^T \right)^{-1} (OI + MG) a \), and the users’ strategies in equilibrium are \( t^* = (I - HG)^{-1} (Ka - Hp) \) and \( v^* = (I - HG)^{-1} (LGa + Ha - Kp) \).

For the detail of proof and \( G'_S \), please see the appendix.

## 7 Conclusion and Discussion

This paper contributes to the strand of paper which try to understand the monopoly’s discriminating pricing strategy and targeting users based on their network position. We consider the case of social media owner pricing their users by modelling the multiple interdependent behaviors of users and the way of pricing the browsing behavior. By using game-theoretical approach, we analyze users’ behavior in Nash equilibria of three different type of cross-activity externality: No cross-activity externality, cross-activity complementarity and
cross-activity substitutability. The striking results show that even though users’
best replies are linear, the network information is usable for the monopoly to
discriminate prices across users: advertisement density for each user is propor-
tional to his/her weighted degree.

When the one activity is divided into two activities, and each activity plays a
different role in utility function, the interaction of two activities provide network
information which monopoly can differentiate from users’ equilibrium strategies
, and is useful in their pricing plan.

This paper also tries to explain social media’s benefits from her services, such
as recommending friends and events notification for users, by comparative static
studies and welfare analysis. The results show that the monopoly always gets
more profit from establishing more links in the network whichever cross-activity
it is. Also, higher exogenous willingness to create results in no decreasing cre-
ating time spending individually in first two cases and aggregately in the last
case. The browsing is no decreasing as well, and the monopoly is better off in
first two cases.

In the extension section, we consider the other way monopoly could price.
In section 6.1, we can consider the case where the monopoly chooses the same
advertisement density for each user’s both creating and browsing. The results
show that the advertisement density sent to the users is linear in their weighted
Katz-Bonanchich centrality.

The more general case of mix cross-activity externality among users is dis-
cussed as well. The uniqueness of Nash equilibrium is conditioning on that the
accumulative complementary effect of the network and the accumulative sub-
stitutable effect which is amplified by the network complementarity are not too
large.

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Appendix

A. Katz-Bonancich Centrality:

Katz-Bonancich centrality (see (Katz, 1953), (Bonacich, 1987)) is defined as follows: Consider a scalar θ and a network G such that $I - \theta G$ is invertible, a matrix is defined as :

$$M = (I - \theta G)^{-1} = \sum_{k=0}^{+\infty} \theta^k G^k = I + \theta G + \theta^2 G^2 + \ldots + \theta^n G^n + \ldots$$

Katz-Bonancich centrality is defined as $m(G, \theta) = (I - \theta G)^{-1} 1$.

Weighted Katz-Bonancich centrality is defined as $m_a(G, \theta) = (I - \theta G)^{-1} a$, in which $a > 0$ and there are at least two different elements in $a$.

B. Proof

Proof of Proposition 2.

We firstly check that monopoly has no incentive to choose $p_i = \sum_{j} t_{ij}$ which makes $v_i = 0$ first. Suppose monopoly choose $p^*_i = \sum_{j} t_{ij}$ for user $i$ which maximizes monopoly’s profit $\pi^*$. The profit generated from user $i$ is $\pi^*_i = 0$. There exist an $\epsilon > 0$ such that $p'_i = p^*_i - \epsilon > 0$ and $p' = (p'_1, p'_2)$. The profit from $i$ is $\pi'_i = p'_i^2 > 0 = \pi^*_i$. The profit from other users are the same. Thus, $\pi' > \pi^*$ which contradicts the assumption.

Solve monopoly’s profit maximization problem $\max_p \frac{1}{2} p^T (Ga - \frac{1}{2} p^T p)$. Thus, $p^* = \frac{1}{2} Ga$ and we have $v^* = \frac{1}{2} Ga$.

Proof of Proposition 4.

We check that monopoly has no incentive to choose $v_i = 0$ first. Suppose monopoly choose $v_i = 0$ for user $i$ which maximizes monopoly’s profit $\pi^*$ with $p^*$
and assume that $p_i^* = \sum_{j \in N(i)} t_j + dt_i$. The profit generated from user $i$ is $\pi_i^* = 0$.

There exist an $\epsilon > 0$ such that $p_i' = p_i^* - \epsilon > 0$ and $p_j' = p_j^* + \frac{\partial c}{\partial \pi^*}$ for $j \in N(i)$. The best responses of $j$ will keep the same. We have $\pi_j' = p_j^* + \frac{\partial c}{\partial \pi^*}$ and $\sum_{j \in N(i)} \pi_j' \geq \sum_{j \in N(i)} \pi_j^*$. The profits from other users keep the same. We have $\pi > \pi^*$ which contradicts the assumption.

When $d\delta\lambda_{\text{max}}(G) < 1$, monopoly’s profit maximization problem is $\max_p \pi = \frac{1}{c} (G + dI) (I - d\delta G)^{-1} (c\delta p^T a - d\delta p^T p) - \frac{1}{2} p^T p$. Maximizing $\pi$ with respect to $p$, we have:

$$\left[(d\delta G - I) (I - d\delta G)^{-1} + c^2 \delta (I - d\delta G)^{-1} + I\right] p = \frac{c\delta}{2} (G + dI) (I - d\delta G)^{-1} a.$$  

Thus $p^* = \frac{1}{2c} (G + dI) a$.

Further, users’ strategies are $t^* = \frac{1}{2c} \left( I + c^2 \delta (I - d\delta G)^{-1} \right) a$ and $v = \frac{\delta}{2} (G + dI) (I - d\delta G)^{-1} a$.

**Proof of LEMMA 1.**

For user $i$, the utility function is now:

$$u_i(t_i, t_{-i}, v_i, p_i) = a_i t_i + v_i \sum_{j \in N(i)} t_j + dt_i v_i - p_i v_i - \frac{1}{2} c (t_i^2 + v_i^2)$$

First order condition with constraints($t_i \geq 0$ and $v_i \geq 0$) on the utility function are:

$$t_i = \frac{a_i + dv_i}{c} \quad \text{if} \quad a_i > -dv_i$$

$$\text{and} \quad t_i = 0 \quad \text{if} \quad a_i \leq -dv_i;$$

Maximize $u_i$ with respect to $v_i$:

$$v_i = \frac{\sum_{j \in N(i)} t_j + dt_i - p_i}{c} \quad \text{if} \quad \sum_{j \in N(i)} t_j + dt_i - p_i > 0$$

$$\text{and} \quad v_i = 0 \quad \text{if} \quad \sum_{j \in N(i)} t_j + dt_i - p_i < 0;$$

By substituting, the best reply function of $t_i$ is as follows:

$$t_i = BR_i (t_{-i}) = \begin{cases} 0 & \sum_{j \in N(i)} t_j - p_i > \frac{\delta}{c} a_i \\ \frac{a_i}{c} & \frac{\delta}{c} a_i \leq \sum_{j \in N(i)} t_j - p_i \leq -\frac{\delta}{c} a_i \\ \frac{d}{c} a_i & \sum_{j \in N(i)} t_j - p_i < -\frac{d}{c} a_i \end{cases} \quad (7)$$
Bramoullé et al. (Bramoullé et al., 2014) has shown that the set of Nash equilibria results in the class of local network substitutability game can be obtained by analysing its potential game. Since the game is separated into three stages, we can analyze the game only with users’ creating time strategy as the browsing behavior will correspond with the creating behavior. Monderer and Shapley (Monderer and Shapley, 1996) state that for the payoff functions \( u_i \) which are continuous and twice-differentiable, there exists a potential function if and only if \( \partial^2 u_i / \partial t_i \partial t_j = \partial^2 u_j / \partial t_j \partial t_i \) for all \( i \neq j \in n \). In our case, 
\[
\partial^2 u_i / \partial t_i \partial t_j = \partial^2 u_j / \partial t_j \partial t_i = \frac{d}{2} g_{ij}.
\]
The potential function is:
\[
\varphi (t; c, d, a, p, G) = \delta t^T (ca + dp) - \frac{1}{2} t^T (I - d\delta G) t.
\]
Consider maximizing the potential function:
\[
\max_t \varphi (t; c, d, a, p, G) \text{ s.t.} i \neq i, 0 \leq t_i \leq \frac{a_i}{c}.
\]
\( t^* \) is a Nash equilibrium if and only if \( t^* \) satisfies the Kuhn-Tucker conditions of the problem above. The first order condition of potential function on \( t_i \) is mimicing user \( i \)'s the best response. The Kuhn-Tucker condition of the problem above is for user \( i \): if \( 0 < t_i < \frac{a_i}{c} \), then \( \partial \varphi / \partial t_i = 0 \); if \( t_i = 0 \), then \( \partial \varphi / \partial t_i \leq 0 \); if \( t_i = \frac{a_i}{c} \), then \( \partial \varphi / \partial t_i \geq 0 \).

\textbf{Proof of LEMMA 2.}

The potential function \( \varphi \) has a unique maximal point on \( (0, \frac{a_i}{c})^n \) if and only if \( \varphi \) is a concave function. \( \nabla^2 \varphi = -(I - d\delta G) \), \( \varphi \) is concave if and only if \( I - d\delta G \) is positive definite which means \( 1 - d\delta \lambda_{\text{min}}(G) > 0 \).

\textbf{Proof of PROPOSITION 5.} Since \( 1 - d\delta \lambda_{\text{min}}(G) > 0 \), \( (I - d\delta G) \) is nonsingular.

The upper bound of \( v \) is \( \frac{1}{c} Ga \) When \( Ga < \frac{c^2 \sqrt{\pi}}{\pi} a \). Thus, \( t = \frac{1}{c} (a - dv) \geq \frac{1}{c} (a - \frac{c^2 \sqrt{\pi}}{\pi} Ga) > 0 \).

Such that we have \( v_i = \delta (\sum_{N(i)} a_j + da_i) + d \sum_{N(i)} v_j - cp_i \) \( \forall i \in N \).

When the solution is interior, \( v = \delta (I - d\delta G)^{-1} (Ga + da - cp) \).

For the interior solution, monopoly’s maximization function is now:
\[
\max_p \delta p^T (I - d\delta G)^{-1} ((G + dI) a - cp).
\]
Take the first order condition with respect to \( p \), we get \( p^* = \frac{1}{\delta c} (G + dI) a \), and users’ strategies are \( 0 < t^* = \frac{1}{\delta c} \left( I + c^2 \delta (I - d\delta G)^{-1} \right) a < \frac{1}{\delta} a \) and \( v^* = \frac{\delta}{\pi} (G + dI) (I - d\delta G)^{-1} a > 0 \). Since monopoly’s profit maximization function is continuous and the solution is not a corner solution, \( p^* = \frac{1}{\delta c} (G + dI) a \) is the global profit maximal point.

\textbf{Proof of Footnote 7.}
Consider a \( n \times n \) diagonal matrix \( A \) such that \( A_{ii} = \frac{1}{a_{ii}} \) and a new potential function for \( \psi(x, d) = x^T 1 - \frac{1}{2} x^T A (I - dG) x \). \( \psi(x, G) \) is a concave function and the maximal point of \( \psi(x, d) \) corresponds to the unique equilibrium of substitute case. \( \psi(x^*, d) \) is equivalent to \( \sum t_i^* \), since \( \psi(x^*, G) = \frac{1}{2} a^T (I + dG)^{-1} 1 \), which is the part of \( \sum t_i^* \) varying from different \( G \). \( \psi(x^*, d) > \psi(x^*, d') \) if \( d > d' \). Thus, \( \psi(x^*, d) < \psi(x^*, d') < \psi(x^*, d) \). Therefore, \( \sum x_i^* > \sum x_i^* \) and \( \sum t_i^* > \sum t_i^* \).

For the sum of browsing time on the network:
\[
\psi^T 1 = \frac{1}{2} a^T (G + dI) (I - dG)^{-1} 1 = \frac{1}{2} a^T (I - c^2 \delta (I + dG)^{-1}) 1 = \frac{1}{2} a^T 1 - \frac{1}{2} t^T 1.
\]
Thus, \( \sum v_i^* > \sum t_i^* \).

**Proof of Proposition 8.**

Consider a \( n \times n \) diagonal matrix \( A \) such that \( A_{ii} = \frac{1}{a_{ii}} \) and the potential function for \( \psi(x, G) = x^T 1 - \frac{1}{2} x^T A (I - dG) x \). \( \psi(x, G) \) is a concave function and the maximal point of \( \psi(x, G) \) corresponds to the unique equilibrium of substitute case. \( \psi(x^*, G) \) is equivalent to \( \sum t_i^* \), since \( \psi(x^*, G) = \frac{1}{2} a^T (I + dG)^{-1} 1 \), which is the part of \( \sum t_i^* \) varying from different \( G \). \( \psi(x^*, G) > \psi(x^*, G') \) if \( G \) is a subgraph of \( G' \). Thus, \( \psi(x^*, G') < \psi(x^*, G) < \psi(x^*, G) \). Therefore, \( \sum x_i^* > \sum x_i^* \) and \( \sum t_i^* > \sum t_i^* \).

For the monopoly’s profit, consider a new potential function \( \Phi(x, G) = x^T (G + dI) a - \frac{1}{2} x^T (I - dG) x \). Again \( \Phi(x, G) \) is a concave function and the maximal point of \( \Phi(x, G) \) corresponds to the unique equilibrium of substitute case. \( \Phi(x^*, G) \) is now equivalent to \( \pi^* \)-monopoly’s highest profit at network \( G \), since \( \Phi(x, G) = \frac{1}{2} a^T (G + dI)^2 (I - dG)^{-1} a \). If \( G \) is a subgraph of \( G' \) \(^9\). Thus, \( \Phi(x^*, G') > \Phi(x^*, G') > \Phi(x^*, G) \). Therefore, \( \pi^* > \pi^* \).

**Proof of Proposition 11.**

Use the potential function \( \psi(x, A) = x^T 1 - \frac{1}{2} x^T A (I - dG) x \), since we are now interested in network effect of changes of diagonal matrix \( A \) such that \( A_{ii} = \frac{1}{a_{ii}} \). The concavity of \( \psi(x, A) \) and correspondence of \( \psi(x^*, A) \) to \( t_i^* \) are the same as they are in Proposition 8. \( \psi(x^*, A) < \psi(x^*, A') \), since \( A_{ii} = \frac{1}{a_{ii}} \begin{array}{} \frac{1}{a_{ii}} = A_{ii}. \end{array} \) Thus, \( \psi(x^*, A') \) and \( \psi(x^*, A) \) are not linked with each other in \( G \) but are linked in \( G' \). In matrix form, \( a + d(G + dI) (I - dG)^{-1} a = c^2 \delta (I - dG)^{-1} a > 0 \). Therefore, \( \Phi(x^*, G') > \Phi(x^*, G) > 0 \).

\(^9\) Consider \( \Phi(x^*, G') - \Phi(x^*, G) = \sum_{i,j \in Q} x_i (a_{ij} + \frac{1}{2} dG_{ij}) > \sum_{i,j \in Q} x_i (a_{ij} + dG_{ij}) \). \( Q \) is a set of the pair of users who are not linked with each other in \( G \) but are linked in \( G' \).
For the case of \( \sum v_i^* \), \( v^T 1 = \frac{1}{3} a^T 1 - \frac{\delta}{3} t^T 1 \) can not determine the change of \( \sum v_i^* \), since both items increases. We consider a new potential function \( \phi(x, B) = x^T 1 - \frac{1}{2} x^T B (I - \delta G) x \) in which \( B \) is a diagonal matrix and \( B_{ii} = \frac{1}{\sum_{N(i)} a_j + d_{ai}} \). \( \phi(x, B) \) is concave and \( \phi(x^*, B) \) corresponds to \( \sum v_i^* \).

\( \phi(x^*, B) < \phi(x^*, B') \), since \( B'_{ii} = \frac{1}{\sum_{N(i)} a_j + d_{ai}' - 1} \leq \frac{1}{\sum_{N(i)} a_j + d_{ai}} = B_{ii} \), and \( \phi(x^*, B') = x^T 1 - \frac{1}{2} x^T B' (G + dI) a > x^T 1 - \frac{1}{2} x^T 1 > \phi(x^*, B) \). Thus, \( \phi(x^*, B') < \phi(x^*, B) < \phi(x^*, B) \). Therefore, \( \sum x_i^* \geq \sum x_i^* \), and \( \sum v_i^* > \sum v_i^* \).

For the monopoly’s profit, consider the potential function:

\[ \Phi(x, a) = x^T (G + dI) a - \frac{1}{2} x^T (I - d\delta G) x. \]

Again \( \Phi(x, a) \) is a concave function and the maximal point of \( \Phi(x^*, a) \) corresponds to the unique equilibrium of substitute case. \( \Phi(x^*, a) \) is now equivalent to \( \pi^* \)-monopoly’s highest profit. \( \Phi(x^*, a) = \Phi(x^*, a^*) \). Thus, \( \Phi(x^*, a^*) > \Phi(x^*, a) \). Therefore, \( \pi^* > \pi^* \).

**Proof of Proposition 13.**

For monopoly’s profit change: we can take the first derivatives of profit function on \( d \) to profit function.

Monopoly’s profit is: \( \pi^* = p^T v = \frac{\delta}{4c} ((G + dI) a)^T (G + dI) (I - \delta G)^{-1} a \).

Thus, \( \pi^* = \frac{\delta}{4c} a^T (G + dI)^2 (I - \delta G)^{-1} a \).

To find out how \( d \) will influence \( \pi^* \), take first derivatives on \( \pi^* \) with respect to \( d \):

\[
\frac{\partial \pi^*}{\partial d} = \frac{\delta}{2c} a^T (G + dI)^2 (I - \delta G)^{-1} a + \frac{\delta}{2c} a^T (G + dI) (I - \delta G)^{-1} \frac{\partial}{\partial d} a
\]

\[
+ \frac{\delta}{4c} a^T (G + dI)^2 ((I - \delta G)^{-1})' a
\]

Solve \( [(I - \delta G)^{-1}]' \):

\[
[(I - \delta G)^{-1}]' = (c^2 + d^2) \delta^2 (G + 2d\delta G^2 + ... + (d\delta)^{n-1} G^n - ...)
\]

\[
= (c^2 + d^2) \delta^2 G \left( (I - \delta G)^{-1} + d\delta G \left( (I - \delta G)^{-1} + d\delta G \ldots \right) \right)
\]

\[
= (c^2 + d^2) \delta^2 G \left( I + \delta G + ... + (d\delta)^{n-1} G^{n-1} \ldots \right)
\]

\[
= (c^2 + d^2) \delta^2 G (I - \delta G)^{-2}.
\]

\[10\text{Consider } \Phi(x^*, a^*) - \Phi(x^*, a) = (a^* - a)^T (G + dI)^2 (I - \delta G)^{-1} a. \text{Since } \sum_{N(i)} a_j + d_{ai} \leq \sum_{N(i)} a_j + d_{ai}', (a^* - a)^T (G + dI)^2 (I - \delta G)^{-1} a > 0 \).
Thus,
\[
\frac{\partial \pi^*}{\partial d} = \frac{\delta}{2c} a^T (G + dI)^2 \left( dI + \frac{\epsilon^2 + d^2}{2} \delta^2 G (I - d\delta G)^{-1} \right) (I - d\delta G)^{-1} a \\
+ \frac{d\delta}{2c} a^T (G + dI) (I - d\delta G)^{-1} a
\]

If \( d > 0, \frac{\partial \pi^*}{\partial d} > 0 \). The monopoly’s profit increasing with \( d \) increasing.

For the users’ utility:

Users’ utilities are:
\[
U = \frac{4}{8c} a^{\circ2} + \frac{c^2}{8} \left( G (I - d\delta G)^{-1} a \right)^{\circ2} + \frac{c^2 \delta}{8} \left( (I - d\delta G)^{-1} a \right)^{\circ2};
\]
take first derivatives on \( U^S \) with respect to \( d \):
\[
\frac{\partial U}{\partial d} = \frac{dc\delta^3}{2} \left( G (I - d\delta G)^{-1} a \right)^{\circ2} + \frac{c^2 \delta^2}{4} \left( G^2 (I - d\delta G)^{-2} a \right)^{\circ2} \\
+ \frac{d\delta^2}{4} \left( (I - d\delta G)^{-1} a \right)^{\circ2} + \frac{c^2 \delta}{8} \left( G (I - d\delta G)^{-2} a \right)^{\circ2};
\]

When \( d > 0, \frac{\partial U}{\partial d} > 0 \). All users’ welfare increase. When \( d < 0 \) and \((I - d\delta G)^{-2} a > 0, \frac{\partial U}{\partial d} > 0 \).

**Proof of LEmma 3.**

With the utility function \( 6 \), first order condition with constraints\( (t_i \geq 0 \text{ and } v_i \geq 0) \) on the utility function are:
\[
t_i = \frac{a_i + d_i v_i}{c_i}
\]
Assuming \( p < K^{-1} (LG + H) a \),
\[
v_i = \frac{\sum_{N(i)} t_j + d_i t_i - p_i}{c_i}.
\]
By substituting, the best reply function of \( t_i \) is as follows:
\[
t_i = \frac{c_i}{c^2_i - d^2_i} a_i + \frac{d_i}{c^2_i - d^2_i} \sum_{N(i)} t_j - \frac{d_i}{c^2_i - d^2_i} p_i.
\]
In matrix form:
\[
t = Ka + Hgt - Hp
\]
When \( I - HG \) is positive definite which means \( \lambda_{\text{max}} (HG) < 1 \), we have
\[
t = (I - HG)^{-1} (Ka - Hp)
\]
Then for the browsing time
\[ v = (I - HG)^{-1} (LGa + Ha - Kp). \]

For the first order condition of monopoly’s pricing strategy:

\[ \frac{d\pi}{dp} = (I - HG)^{-1} (K + LG + H) a - \left( (I - HG)^{-1} + (I - (HG)^T)^{-1} \right) (H + K) p = 0 \]

Solve this condition:

\[ p = \left( I - (HG)^T \right) \left( 2I - (HG + (HG)^T) \right)^{-1} (I + MG) a \]

**Proof of Proposition 14.**

Consider that there are two separate graphs \( G_1 \) and \( G_2 \). The users in set \( C \) belongs to \( G_1 \) such that \( \forall i \in C \ d_i \geq 0 \), and the users in \( S \) belongs to \( G_2 \) such that \( \forall j \in S \ d_j < 0 \). Of course, \( C \) and \( S \) are exclusive.

Now \( G_1 \) and \( G_2 \) merge into a new graph \( G_3 \) such that \( \exists i, j \in G_3 \) in which \( i \in C \) and \( j \in S \), and \( \exists i_1, i_2 \in G_3 \) but \( l_1, i_2 \notin G_1 \) or \( l_1, i_2 \notin G_2 \) in which \( i_1, i_2 \in C \) or \( i_1, i_2 \in S \). There are new links added between \( G_1 \) and \( G_2 \) and no new links added between users in \( C \) or \( S \).

By Lemma 3, the Nash equilibria exist when \( \lambda_{\max}(HCG_C) < 1 \). Suppose a Nash equilibrium \( E \) exists, in which for user \( j \in S \) , \( t^E_j \) is his/her Nash equilibrium strategy in \( E \) and \( \exists i \in C \ g_{3_{ij}} = 1 \). \( g_{3_{ij}} \) is the entry of row \( i \) and column \( j \) in graph \( G_3 \).

For \( i \), the Nash equilibrium strategy can be represented as:

\[ t^E_i = \sum_{r=1}^{\left| C \right|} m_{ir} \left( k_{ir} a_r + h_{rr} \sum_{j \in N(i) \cap S} t^E_j - h_{rr} p_r \right). \]

For \( j \), the best response of creating time is:

\[ t^E_j = \frac{c_j}{c_j - d^2_j} a_j + \frac{d_j}{c_j - d^2_j} \left( \sum_{w \in N(i) \cap S} t^E_w + \sum_{w \in N(i) \cap C} t^E_{i} \right) - \frac{d_j}{c_j - d^2_j} p_j. \]

Reorganizing \( t^E_j \), we have

\[ t^E_j = \frac{c_j}{c_j - d^2_j} t^E \sum_{r \in N(j) \cap C} m_{ir} h_{rr} = \frac{c_j}{c_j - d^2_j} a_j + \sum_{r \in N(j) \cap C} m_{ir} (k_{ir} a_r - h_{rr} p_r) + \frac{d_j}{c_j - d^2_j} \left( \sum_{w \in N(i) \cap S} t^E_w + \sum_{w \in N(i) \cap C} \sum_{r \in C} m_{wr} h_{rr} \sum_{\tau \in N(w) \cap S} t^E_{\tau} \right) - \frac{d_j}{c_j - d^2_j} p_j. \]

By Lemma 4, Nash equilibrium is unique if and only if \( \lambda_{\max}(G^E_S) < 1 \) in which

\[ g^E_{j,w} = \frac{d_j}{c_j - d^2_j} \left( g_{j,w} + \sum_{w \in N(j) \cap C} \sum_{r \in C} m_{wr} h_{rr} g_{wr} \right). \]