Polyequilibrium

Igal Milchtaich∗
March 2017

Polyequilibrium is a generalization of Nash equilibrium that is applicable to any strategic game, whether finite or otherwise, and to dynamic games, with perfect or imperfect information. It differs from equilibrium in specifying strategies that players do not choose and by requiring an after-the-fact justification for the exclusion of these strategies rather than the retainment of the non-excluded ones. Specifically, for each excluded strategy of each player there must be a non-excluded one that responds to every profile of non-excluded strategies of the other players at least as well as the first strategy does. A polyequilibrium’s description of the outcome of the game may be more or less specific, depending on the number and the identities of the retained, non-excluded strategy profiles. A particular result (e.g., Pareto efficiency of the payoffs) is said to hold in a polyequilibrium if it holds for all non-excluded profiles. Such a result does not necessarily hold in any Nash equilibrium in the game. In this sense, the generalization proposed in this work extends the set of justifiable predictions concerning a game’s results.

Keywords: Polyequilibrium, Polystrategy, Coarsening of Nash equilibrium, Subgame perfection, Bayesian perfection.

1 Introduction

A Nash equilibrium is a self-enforcing strategy profile. Each player $i$ is assigned a strategy $x_i$ that is an optimal choice for $i$ if all the other players choose the strategies assigned to them. Viewed from a different perspective, a Nash equilibrium excludes all but a single strategy for each player $i$. The exclusion is justified in that, if none of the other players chooses an excluded strategy, player $i$ also has no incentive to do so; choosing any excluded strategy would not make the player better off in comparison with choosing the unique non-excluded one.

The first, conventional view of Nash equilibrium generalizes to rationalizability (Bernheim 1984, Pearce 1984). A rationalizable strategy is a best response to some belief about the other players’ play that assigns positive probability only to strategies that are themselves rationalizable. Thus, unlike Nash equilibrium, the self-referring rationalizability condition potentially involves a set of strategies for each player rather than a single strategy. The same is true for the related solution concept of curb set (for Closed Under Rational Behavior; Basu and Weibull 1991). However, whereas rationalizability provides justification for the inclusion of the strategies in a player’s set,
a curb set can be described as requiring justification for the exclusion of the strategies outside it, similarly to the above alternative view of Nash equilibrium. Specifically, in a curb set, each excluded strategy must not be a best response to any belief about the other players’ strategies that assigns positive probability only to non-excluded strategies.¹

Polyequilibrium is similar to curb set in being an “excluding” set-valued solution concept but differs from it and from rationalizability in not involving beliefs, i.e., probability distributions over the other players’ strategy sets. Furthermore, like pure-strategy Nash equilibrium, polyequilibrium is a purely ordinal concept, invariant to arbitrary player-specific increasing transformations of the payoff functions. It requires that, for each excluded strategy of each player, there is a non-excluded one that yields the same or higher payoff against every profile of non-excluded strategies. Note that this requirement is weaker than requiring the excluded strategies to be (weakly) dominated, because (a) it only considers strategies of the other players that are not themselves excluded and (b) it allows for selection, that is, choosing among equally good strategies.

In short, polyequilibrium may be described as a self-enforcing subgame. A subgame by definition restricts each player $i$ to a designated set of allowable strategies, and polyequilibrium requires the restriction to be self-enforcing in the sense that every strategy $x_i$ outside the player’s designated set has an adequate substitute within it: an allowable strategy $x_i'$ that responds at least as well as $x_i$ does to every profile of allowable strategies for the other players. Note that this requirement is a substantially stronger kind of self-enforcement than that employed by another set-valued generalization of Nash equilibrium, the Nash retract (Kalai and Samet 1984). The latter’s definition changes the order of logical quantifiers and only requires that, against any given profile of allowable strategies for the other players, every strategy $x_i$ has an adequate substitute $x_i'$ among the player’s allowable strategies.

Polyequilibrium and the corresponding notion of self-enforcement are essentially a straightforward generalization of Shapley’s (1964) notion of generalized saddle point in the context of finite two-player zero-sum games. More precisely, generalized saddle point is a special case of strict polyequilibrium (see Section 2) and its weak version is a special case of polyequilibrium. See Section 4.1.

The condition defining polyequilibrium ostensibly allows any strategy of any player to be included – justification is only required for the excluded strategies. This lenience in the definition is counterbalanced by a unanimity requirement when it comes to stating that a particular property of the game’s outcome holds for a particular polyequilibrium. That is, a property is said to hold only if all strategy profiles in the polyequilibrium possess it. In particular, a player’s strategy is said to be a polyequilibrium strategy if there is some polyequilibrium where the player’s other strategies are all excluded. This means that, for example, a strategy in a finite

¹ Another set-valued solution concept is strategic stability (Kohlberg and Mertens 1986). However, the solution in this case includes only strategy profiles that are themselves Nash equilibria.
game that is played with positive probability in some mixed-strategy equilibrium – and is therefore automatically rationalizable – is not necessarily a polyequilibrium strategy (because the support of the player’s mixed strategy also includes other pure strategies). However, a polyequilibrium may be able to specify certain aspects of the game’s outcome without singling out a unique strategy profile. It may specify, for example, that a particular player does not take a particular action, that the player’s payoff is positive, that the outcome is Pareto optimal, and so on. The polyequilibrium concept thus represents a somewhat different philosophy than Nash equilibrium and certain other solution concepts that are designed to be completely specific, or at least as specific as possible, about the players’ play. A polyequilibrium does not have to satisfy any set requirements in terms of its predictive power. Indeed, the collection of all strategy profiles in a game is a polyequilibrium (which immediately settles the question of existence). A polyequilibrium is, however, as good as the predictions it makes. Crucially, which predictions are “good”, or interesting, is not determined by any objective measure of interest but is entirely context-dependent and, ultimately, subjective.

As an illustration of the difference between (Nash) equilibrium and polyequilibrium, consider the following two statements about a particular $2 \times 2$ game: “$T$ is an equilibrium strategy for player 1” and “$T$ is a polyequilibrium strategy for player 1”. Both statements do not mention player 2, which suggests that player 1 is the main subject of interest. The first statement means that it is possible to assign player 2 a strategy that is a best response to 1’s $T$ and vice versa. It just does not spell out whether that strategy is $L$ or $R$ (or whether there are two equilibria, one with $L$ and the other with $R$). The second statement, by contrast, does not necessarily refer to a specific strategy of player 2 even implicitly. The polyequilibrium it refers to may specify that 2 plays $L$ or $R$, but it may also leave 2’s strategy unspecified. There is of course nothing unfamiliar about justifying a player’s choice of strategy without specifying the other players’ play. Choosing $T$ is justifiable if it is a dominant strategy. Thus, the notion of dominant strategy essentially also comes under the umbrella of polyequilibrium. A strictly dominated strategy, by contrast, is never a polyequilibrium strategy.

2 Definitions and Basic Facts

A (strategic) game $\Gamma$ is specified by a set of players and, for each player $i$, a nonempty set of strategies $S_i$ and a payoff function $u_i$ that determines $i$’s payoff for each strategy profile $x \in S \equiv \prod_i S_i$. The game is finite if so are its set of players and each player’s strategy set.

A strategy $x'_i$ of player $i$ responds to a strategy profile $x''$ at least as well as strategy $x_i$ does if

$$u_i(x'' | x'_i) \geq u_i(x'' | x_i),$$

where the argument on each side is the strategy profile obtained from $x''$ by replacing $i$’s strategy $x''_i$ with the indicated one. Strategy $x'_i$ responds to a set of strategy profiles $X$ at least as well as $x_i$ does if inequality (1) holds for all $x'' \in X$. If, in addition, at least some of the inequalities are strict or all of them are so, then $x'_i$ (weakly) dominates or strictly dominates $x_i$. 


respectively, relative to \( X \). Strategy \( x_i' \) is a **best response** to a strategy profile or set of strategy profiles if it responds to it at least as well as every other strategy of player \( i \) does. A **never-best-response** strategy relative to a set of strategy profiles \( X \) is a strategy that is not a best response to any strategy profile in \( X \). Such a strategy in not necessarily dominated relative to \( X \) or vice versa. However, a strategy that is **strictly** dominated (by some other strategy) relative to \( X \) is clearly also a never-best-response strategy relative to it. In each of the above expressions involving the phrase “relative to (a set of strategy profiles) \( X \)”, the latter may be dropped, and in this case, it is understood that the expression refers to the entire collection of strategy profiles, that is, \( X = S \).

A strategy profile \( x' \) responds to a strategy profile \( x'' \) at least as well as strategy profile \( x \) does if (1) holds for every player \( i \), and responds to a set of strategy profiles \( X \) at least as well as \( x \) does if the previous condition holds for all \( x'' \in X \). A strategy profile is a best response to a strategy profile or set of strategy profiles if it responds to it at least as well as every other strategy profile does. A game has the **best-response existence property** if there is a best response to every strategy profile. Clearly, all finite games and all mixed extensions of finite games have this property.

For a player \( i \) in a game \( \Gamma \), a **polystrategy** is any nonempty set of strategies, \( \emptyset \neq X_i \subseteq S_i \). Player \( i \)'s entire strategy set \( S_i \) is referred to as his **trivial polystrategy**. A polystrategy that is a singleton, \( \{x_i\} \), may be identified with the strategy \( x_i \). A **polystrategy profile** \( X \) is a Cartesian product of polystrategies, one polystrategy \( X_i \) for each player \( i \). Equivalently, it is a nonempty rectangular subset of \( S \). If the subset is a singleton, \( \{x\} \), it may be identified with its single strategy profile \( x \). Every polystrategy profile \( X \) defines a **subgame** of \( \Gamma \), denoted \( \Gamma^X \), in which the players are as in \( \Gamma \) but each player \( i \) can only choose among the strategies in \( X_i \) and his payoff function is the restriction of \( u_i \) to \( X \). For polystrategy profiles \( X' \) and \( X'' \) with \( X' \subseteq X'' \), the **interval** \([X', X''] \) is the collection of all polystrategy profiles \( X \) with \( X' \subseteq X \subseteq X'' \).

A polystrategy profile \( X \) is obtained by **successive elimination** of (weakly) dominated strategies if there is a nonincreasing finite sequence of polystrategy profiles \( S = X^0 \supseteq X^1 \supseteq \cdots \supseteq X^L = X \), with \( L \geq 1 \), such that for every \( 1 \leq l \leq L \), player \( i \) and (eliminated) strategy \( x_i \in X^l_i \setminus X^{l-1}_i \) there is some \( x'_i \in X^l_i \) that dominates \( x_i \) relative to \( X^{l-1} \).

**Definition 1.** A polystrategy profile \( X \) is a **polyequilibrium** if for every strategy profile \( x \notin X \) there is some \( x' \in X \) that responds to \( X \) at least as well as \( x \) does. \( X \) is a **strict polyequilibrium** if for every player \( i \) and strategy \( x_i \notin X_i \) there is some \( x'_i \in X_i \) that strictly dominates \( x_i \) relative to \( X \).

A polystrategy profile \( X \) is a **simple polyequilibrium** if it includes some strategy profile \( x' \) that is a best response to \( X \) (and, necessarily, an equilibrium). A polyequilibrium or strict polyequilibrium is **minimal** if it does not contain any other polyequilibrium or strict polyequilibrium, respectively. A polyequilibrium \( X \) is **small** or **large** if there is no polyequilibrium that can be obtained from \( X \) by the deletion or addition, respectively, of any single strategy of a single player.

The seven facts below easily follow from the definitions.
Fact 1. Every game has at least one (strict) polyequilibrium, namely, the trivial polyequilibrium $S$, which consists of all strategy profiles.

Fact 2. A polystrategy profile that is a singleton, $\{x\}$, is a polyequilibrium or strict polyequilibrium if and only if its single element $x$ is a (Nash) equilibrium or strict equilibrium, respectively.

Fact 3. A polystrategy profile $X$ that is obtained by (any number of rounds of) successive elimination of (any number of) dominated strategies is a polyequilibrium.

Fact 4. A sufficient condition for a polystrategy profile $X$ in a game $\Gamma$ to be a polyequilibrium is that all strategy profiles in $X$ are equilibria. However, this condition is not necessary. A polyequilibrium $X$ satisfies it if and only if each player’s payoff in the subgame $\Gamma^X$ is independent of his own strategy, that is, $u_i(x) = u_i(x \mid x')$ for all $i, x \in X$ and $x' \in X_i$.

Fact 5. A sufficient condition for a polystrategy profile $X$ in a subgame $\Gamma'$ of a game $\Gamma$ to be a polyequilibrium in $\Gamma'$ is that $X$ is a polyequilibrium in $\Gamma$. If the subgame is of the form $\Gamma' = \Gamma^X$, where $X'$ is a polyequilibrium in $\Gamma$, then this condition is also necessary.

Fact 6. For a polyequilibrium $X$ in a game with the best-response existence property, and for any strategy profile $x^1 \in X$, some $x^2 \in X$ is a best response to $x^1$. Successive use of this fact yields a best-response sequence $x^1, x^2, x^3, \ldots$ where each entry, except the first one, is an element of $X$ that is a best response to its immediate predecessor.

Fact 7. For polystrategy profiles and $X' \subseteq X''$, $[X', X'']$ is an interval of polyequilibria (in the sense that all its elements are so) if and only if for every strategy profile $x$ there is some $x' \in X'$ that responds to $X''$ at least as well as $x$ does. The interval of polyequilibria is maximal (in the sense that it is not a subset of another one) if and only if the polyequilibria $X'$ and $X''$ are small and large, respectively. A polyequilibrium $X$ is small if and only if, for every $x \in X$, there is no other strategy profile $x' \in X$ that responds to $X$ at least as well as $x$ does.

Note that the definitions and facts above are all ordinal in the sense that they are invariant to arbitrary increasing transformations of the players’ payoff functions. None of them requires cardinal utilities or, fundamentally, any utilities, for everything could be alternatively formulated in terms of preferences over strategy profiles rather than payoffs. Correspondingly, the polyequilibrium concept does not involve randomization or beliefs and is thus a generalization of pure-strategy Nash equilibrium.

It could be argued that, with cardinal utilities, the definition of polyequilibrium should be extended to include also polystrategy profiles $X$ where for every excluded strategy $x_i$ of a player

---

$^2$ An example of such a polyequilibrium is the dominance solution of a dominance solvable finite game (Moulin 1979), which is obtained by the successive elimination of all dominated strategies.
there is a *mixed* strategy that responds to $X$ at least as well as $x_i$ does and whose support is included in $X_i$. At least, it may seem justifiable to exclude strategies that are strictly dominated with respect to $X$ by such a mixed strategy. However, an implicit assumption in this assertion is that the player is actually able to play the mixed strategy. But in this case, it (or an equivalent strategy) should have been included in the strategy set $X_i$, for otherwise the latter is mislabeled as it is not a complete specification of the player’s possible choices. Put differently, allowing mixed strategies in a game $\Gamma$ effectively turns it into another game, namely, the mixed extension $\Gamma^*$, and in this case, the relevant polyequilibria are those in $\Gamma^*$.

The connections between the set of polyequilibria in a finite game $\Gamma$ and in its mixed extension $\Gamma^*$ are studied in Section 4.

### 2.1 Strategy Substitution

A polyequilibrium may be alternatively described in terms of strategy substitution. Whereas an equilibrium prescribes one, specific strategy for each player, a polyequilibrium may be viewed as a prescription of a suitable substitute for each of the player’s strategies.

A prescription of substitute strategies for a player $i$ is expressed by a function $\phi_i: S_i \to S_i$. Any family of such functions, one for each player, defines a substitution function $\phi: S \to S$ by $(\phi(x))_i = \phi_i(x_i)$ for all $i$. A substitution function $\phi$ is rational if, for all $x$ and $i$, 

$$u_i(\phi(x)) \geq u_i(\phi(x) \mid x_i).$$

The inequality means that it is acceptable for player $i$ to use the recommended substitute $\phi_i(x_i)$ to strategy $x_i$ if all the other players also follow their recommendations. This formulation differs from Definition 1 in that it combines the specification of the players’ polystrategies with the justification for them. Specifically, the logical relation between the two concepts is as follows.

**Fact 8.** A polystrategy profile $X$ is a polyequilibrium if and only if it is the image of some rational substitution function $\phi$ (that is, $\phi(S) = X$).

A number of continuity results immediately follow from the definition. For example, in a game where the players’ strategy sets are topological spaces and their payoff functions are continuous (with respect to the product topology), any substitution function that is the pointwise limit of rational substitution functions is also rational.

### 3 Polyequilibrium Results

Polyequilibrium is a predictive solution concept, not a prescriptive or normative one. As a polyequilibrium generally includes multiple strategy profiles, there is no general sense in which it may be “played”. Instead, a polyequilibrium predicts certain outcomes, or results, of the

---

3 It may be natural to require the function $\phi_i$ to be idempotent, which means that any strategy that is some other strategy’s substitute is also its own substitute. Adding this requirement would not affect any of the assertions below.
players’ choice of strategies.

Formally, a result $R$ in a game $\Gamma$ is any set of strategy profiles. Its negation $\sim R$ is the complementary set $S \setminus R$. A result $R$ holds in a polystate profile $X$ if $X \subseteq R$, and it is a polyequilibrium result if it holds in some polyequilibrium in $\Gamma$. A result may also be specified implicitly, as a particular property or consequence of strategy profiles (for example, “player 1’s payoff is higher than 2’s payoff”). In this case, $R$ is the collection of all strategy profiles with the specified property, so that the result holds in a polyequilibrium $X$ if and only if all strategy profiles $x \in X$ have the property. In particular, a real number $v_i$ is a polyequilibrium payoff for a player $i$ if there is some polyequilibrium $X$ with $u_i(x) = v_i$ for all $x \in X$, and a strategy $x_i$ is a polyequilibrium strategy if there is some polyequilibrium $X$ with $X_i = \{x_i\}$. A generalization of the first concept is polyequilibrium payoff interval for player $i$, which is any convex set of real numbers $E$ such that “$i$’s payoff lies in $E$” is a polyequilibrium result, that is, $u_i(X) \subseteq E$ for some polyequilibrium $X$. Another generalization is limit polyequilibrium payoff, which is any extended real number $v_i$ (that is, a number, $\infty$ or $-\infty$) such that every convex neighborhood of $v_i$ is a polyequilibrium payoff interval for player $i$. Similar definitions may be applied to payoff vectors.

The concept of result may be used also for special kinds of polyequilibria, and in particular for equilibria. Every equilibrium result is also a polyequilibrium result but not conversely. There are also some logical differences between the two concepts. In particular, for any result $R \neq S$, the proposition “$R$ holds in every polyequilibrium” is false (because the result does not hold in the trivial polyequilibrium) but the proposition “there does not exist a polyequilibrium where $\sim R$ holds” may or may not be false. Thus, the two propositions are not logically equivalent, even though they would be so if ‘polyequilibrium’ were replaced by ‘equilibrium’. The reason, of course, is the possibility that, in a polyequilibrium $X$, both $R$ and its negation do not hold. For $R$ that is the collection of all strategy profiles with a particular property, this is so if and only if some, but not all, strategy profiles in $X$ have the property.

**Example 1.** In the finite (that is, pure-strategy) version of matching pennies, the only polyequilibrium is the trivial one. Therefore, there is no polyequilibrium where player 1 plays Heads and no polyequilibrium where he does not play Heads (equivalently, plays Tails).

A game is PE-plain if every polyequilibrium in it includes at least one equilibrium, equivalently, if every polyequilibrium result is also an equilibrium result. In a PE-plain game, a polyequilibrium where all strategy profiles have a particular property exists only if some equilibrium has the property. Conversely, in a non-PE-plain game, some result that does not hold in any equilibrium holds in a polyequilibrium. The next three examples present such games.

**Example 2.** For both players in the finite game

$$
\begin{array}{ccc}
T & 1,1 & 0,0 & 0,0 \\
M & 0,0 & 2,3 & 3,2 \\
B & 0,0 & 3,2 & 2,3 \\
\end{array}
$$
(Basu and Weibull 1991) the only equilibrium payoff is 1, but [2,3] is also a polyequilibrium payoff interval. The former corresponds to the game’s unique (pure-strategy) equilibrium \( \{T, L\} \) and the latter to the polyequilibrium \( \{M, B\} \times \{C, R\} \). If a third player were added to the game, who does not take any meaningful action and whose payoff is the average payoff of the original two players, then only 1 would be an equilibrium payoff for that player but 2.5 would be a polyequilibrium payoff. The latter is obviously also a mixed-equilibrium payoff for all players (see Proposition 4 below).

**Example 3. Bilateral trade.** A buyer has to offer a price \( p \) to the owner of an item whose worth of 1 to the buyer and 0 to the owner, and the latter has to decide whether to sell at that price. The strategy “accept any price greater than zero” is a dominant strategy for the owner, yet it is not an equilibrium strategy because the buyer does not have a best response to it: offering any \( p > 0 \) is less profitable than offering, say, half that price. Thus, the intuitive idea that the buyer’s response should be to offer “as little as possible”, or “an \( \epsilon \)”, is incompatible with the definition of equilibrium. However, the idea is compatible with polyequilibrium. Indeed, for any \( 0 < \epsilon \leq 1 \), the buyer’s polystrategy \( 0 < p \leq \epsilon \) (“offer a positive price not higher than \( \epsilon \)”) and the owner’s strategy of accepting any positive price together constitute a polyequilibrium.

**Example 4.** In a symmetric two-player game, each player must choose a positive integer \( y \) and his payoff is

\[
z - \left| 1 - \frac{2z}{y} \right|
\]

where \( z \) is the number chosen by the other player. To receive his highest payoff of \( z \), a player must choose \( y = 2z \). However, such a choice prevents the other player from receiving his highest payoff of \( y \) (which would require \( z = 2y \)), and therefore an equilibrium does not exist. In fact, as the total payoff is easily seen to be at most \( y + z - 2 \), even an \( \epsilon \)-equilibrium does not exist, for all \( 0 < \epsilon < 1 \). However, the nonexistence of equilibrium arguably does not reflect a significant misalignment of interests. In particular, if the players alternately escalate their “bids” by doubling that of their rival, both payoffs spiral upwards. This observation is reflected in the fact that for every positive integer \( L \) the (symmetric) polystrategy profile where both players’ polystrategy is \( y > L \) (“choose a number greater than \( L \)”) is a strict polyequilibrium, in which both of them receive at least \( L \). Thus, infinity is a limit polyequilibrium payoff.

### 4 Finite Games and Their Mixed Extensions

The game in Example 2 has three polyequilibria, which coincide with the supports of its three mixed-strategy equilibria (one of which is pure) as well as with the game’s three curb sets. However, such coincidences are the exception rather than the rule. In particular, that game is special in that all its polyequilibria are strict, and as can easily be seen, a strict polyequilibrium in a finite game is always a curb set. This observation aligns with the fact that the concept of curb set is meant as a generalization of strict, rather than Nash, equilibrium (Basu and Weibull 1991).
Example 5. The finite game

$$
\begin{pmatrix}
T & L & C & R \\
-1,2 & 0,0 & 2,-1 \\
M & 0,0 & 0,0 & 0,0 \\
B & 2,-1 & 0,0 & -1,2
\end{pmatrix}
$$

has two supports of mixed-strategy equilibria, two polyequilibria and two curb sets. None of the pairs coincides with another. The first, pure-strategy, equilibrium \((M, C)\) is a polyequilibrium but it is not a curb set. The support \(\{T, B\} \times \{L, R\}\) of the second mixed-strategy equilibrium is not a polyequilibrium but it is a curb set. The trivial polyequilibrium is (trivially) also a curb set.

Example 5 shows that, in general, a curb set in a finite game may not even contain a polyequilibrium and vice versa, and the support of a mixed-strategy equilibrium may not contain a polyequilibrium. However, the converse of the last assertion is false: a polyequilibrium always contains (although it does not necessarily coincide with) the support of some mixed-strategy equilibrium.

Proposition 1. In a finite game, every polyequilibrium contains the support of some mixed-strategy equilibrium.

This result is an immediate corollary of Proposition 3 below, which identifies a specific connection between polyequilibria in a finite \(n\)-player game \(\Gamma\) and in its mixed extension \(\Gamma^*\).

By definition, an unqualified ’strategy’ in \(\Gamma\) or \(\Gamma^*\) is a pure or mixed strategy, respectively, in \(\Gamma\), and similarly for ‘equilibrium’. As the collection \(S_i\) of all (pure) strategies for a player \(i\) in \(\Gamma\) may be viewed as a subset of the player’s strategy set \(S_i^*\) in \(\Gamma^*\), and similarly for the collections \(S\) and \(S^*\) of all strategy profiles in \(\Gamma\) and \(\Gamma^*\), respectively, \(\Gamma\) may be viewed as a subgame of \(\Gamma^*\), with the same symbol \(u_i\) denoting the payoff function of a player \(i\) in both games. The set of equilibria in \(\Gamma\) is thus identified with a subset of that in \(\Gamma^*\), namely, the pure-strategy equilibria.

A polystrategy \(X_i\) for a player \(i\) in \(\Gamma^*\) is a nonempty subset of \(S_i^*\), that is, a collection of mixed strategies in \(\Gamma\). It is a pure polystrategy if \(X_i \subseteq S_i\). By the first part of Fact 5, every pure-polystrategy polyequilibrium in \(\Gamma^*\) is a polyequilibrium also in \(\Gamma\). However, as the following example and proposition show, the converse is false. In particular, \(S\), the trivial polyequilibrium in \(\Gamma\), is usually not a polyequilibrium in \(\Gamma^*\).

Example 6. The finite game

$$
\begin{pmatrix}
T & L & R \\
3,-3 & 0,0 \\
M & 1,-1 & 1,-1 \\
B & 0,0 & 3,-3
\end{pmatrix}
$$

has a single polyequilibrium, the trivial one \(S\). The mixed extension of the game has two small polyequilibria, which are the equilibrium \(x' = \left(\frac{1}{2}T + \frac{1}{2}B, \frac{1}{2}L + \frac{1}{2}R \right)\) and

\[
X' = \{ (pT + (1 - p)B, qL + (1 - q)R) \mid 0 \leq p, q \leq 1 \},
\]
and two large equilibria, which are
\[ X'' = \{ (pT + (1 - 2p)M + pB, 1/2 L + 1/2 R) \mid 0 \leq p \leq 1/2 \} \]
and the trivial polyequilibrium \( S^* \). Its entire set of polyequilibria is the union of two disjoint intervals: the interval of simple polyequilibria \( \{[x'], X''\} \) and that of strict polyequilibria \( [X', S^*] \).

**Proposition 2.** A necessary and sufficient condition for a polyequilibrium \( X \) in a finite game \( \Gamma \) to be a polyequilibrium also in the mixed extension \( \Gamma^* \) is that it is simple. In particular, the trivial polyequilibrium in \( \Gamma \) is a polyequilibrium also in \( \Gamma^* \) if and only if every player has a strategy that is a best response to all strategy profiles.

The proof of the proposition uses the following result.

**Lemma 1.** In the mixed extension of a finite game, every finite polyequilibrium \( X \) is simple.

**Proof.** For each player \( i \), let \( x'_i \) be some strategy that is not (weakly) dominated relative to \( X \) by any other strategy in \( X_i \). Necessarily, each \( \bar{x}_i \in X_i \) is of one of two kinds: either \( u_i(x'' | \bar{x}_i) = u_i(x'' | x'_i) \) for all \( x'' \in X \), or \( u_i(x'' | \bar{x}_i) < u_i(x'' | x'_i) \) for some \( x'' \in X \). Note that there is some \( 0 < \epsilon^* < 1 \) such that, for every strategy \( \bar{x}_i \in X_i \) that is of the second kind, some \( x'' \in X \) satisfies \( u_i(x'' | \bar{x}_i) < u_i(x'' | (1 - \epsilon^*)x'_i + \epsilon^* x_i) \) for all \( x_i \in S_i^* \). Consider any strategy \( x_i \in S_i^* \). Since \( X \) is a polyequilibrium, there is some \( \bar{x}_i \in X_i \) that responds to \( X \) at least as well as \( (1 - \epsilon^*)x'_i + \epsilon^* x_i \), and therefore also at least as well as \( x_i \). Thus, each player \( i \) has a strategy in \( X_i \) (namely, \( x'_i \)) that is a best response to \( X \), which proves that the latter is a simple polyequilibrium.

**Proof of Proposition 2.** If \( X \) is simple, so that every player \( i \) has a strategy \( x'_i \in X_i \) that responds to \( X \) at least as well every (pure) strategy \( x_i \in S_i \) does, then, by the linearity of \( u_i \), player \( i \)'s own strategy, the same is true also for every \( x_i \in S_i^* \), which proves that \( X \) is a simple polyequilibrium also in \( \Gamma^* \). The opposite direction follows from Lemma 1.

Proposition 2 shows that the identity between the equilibria in \( \Gamma \) and those equilibria in \( \Gamma^* \) that only involve pure strategies does not extend to polyequilibria. Specifically, the set of polyequilibria in \( \Gamma \) does not generally coincide with but is rather a (usually, proper) superset of the (possibly, empty) set of pure-polystrategy polyequilibria in \( \Gamma^* \). There is, however, a simple one-to-one correspondence between the former and another set of polyequilibria in \( \Gamma^* \). This correspondence, indicated by the next proposition, matches each polyequilibrium in \( \Gamma \) with its convex hull.

---

4 This difference between equilibrium and polyequilibrium means that, for the latter, even if the particular strategies examined are all pure, it still matters whether or not mixed strategies are considered admissible alternatives. Unlike for equilibrium, the choice between the two possibilities is consequential.
The convex hull of a polystrategy $X_i$ for a player $i$ in $\Gamma^*$, denoted $\text{conv} X_i$, is itself a polystrategy in $\Gamma^*$. If $X_i$ is pure (equivalently, a polystrategy in $\Gamma$), $\text{conv} X_i$ consists of all mixed strategies whose supports are subsets of $X_i$. For a polystrategy profile $X = X_1 \times X_2 \times \cdots \times X_n$ in $\Gamma^*$ or (as a special case) in $\Gamma$, the convex hull of $X$ is the polystrategy profile in $\Gamma^*$ given by

$$\text{conv} X = \text{conv} X_1 \times \text{conv} X_2 \times \cdots \times \text{conv} X_n.$$  

**Proposition 3.** For every polyequilibrium $X$ in a finite game $\Gamma$ or in its mixed extension $\Gamma^*$, $\text{conv} X$ is a polyequilibrium in $\Gamma^*$. This polyequilibrium (hence, every polyequilibrium in $\Gamma^*$ that consists of convex polystrategies) includes at least one equilibrium (which is a mixed-strategy equilibrium in $\Gamma$).

**Proof.** Consider any polyequilibrium $X$ in $\Gamma$ or $\Gamma^*$. For each player $i$, select for each (pure) strategy $x_i \in S_i$ some $x_i' \in X_i$ such that (1) holds for all $x'' \in X$, and let $X_i' \subseteq \text{conv} X_i$ be some polytope (that is, the convex hull of a finite number of strategies) that includes each of these (finitely many) strategies $x_i'$. Every (mixed) strategy in $S_i$ is a convex combination of elements in $S_i$. It follows, by the linearity of $u_i$ in player $i$’s own strategy, that for every $x_i \in S_i$ there is some $x_i' \in X_i$ such that (1) holds for all $x'' \in X$, and therefore, by the multilinearity of $u_i$, also for all $x'' \in \text{conv} X$. Since the polystrategy profile $X' = \prod_i X_i'$ is a subset of $\text{conv} X$, the last conclusion proves that both polystrategy profiles are polyequilibria in $\Gamma^*$ (see Fact 7). The subgame $\Gamma^* = \Gamma''$ of $\Gamma^*$, where each player $i$ is restricted to strategies in the polytope $X_i'$, is (identifiable with) the mixed extension of a finite game, and therefore has at least one equilibrium. By the second part of Fact 5, each such equilibrium is an equilibrium also in $\Gamma^*$.  

As indicated, one corollary of Proposition 3 is Proposition 1. Another corollary is the following connections between the equilibrium and polyequilibrium payoffs in a finite game and in its mixed extension.

**Proposition 4.** For a player $i$ in a finite game $\Gamma$, with mixed extension $\Gamma^*$, the following inclusions and equalities hold, and the inclusions may be strict:

$$\text{equilibrium payoffs in } \Gamma \subseteq \text{polyequilibrium payoffs in } \Gamma \subseteq \text{mixed-equilibrium payoffs in } \Gamma = \text{equilibrium payoffs in } \Gamma^* = \text{polyequilibrium payoffs in } \Gamma^*.$$  

Moreover, in both $\Gamma$ and $\Gamma^*$, every polyequilibrium payoff interval for player $i$ includes at least one of the player’s mixed-equilibrium payoffs in $\Gamma$.

**Proof.** Example 2 shows that both inclusions above may be strict. The first inclusion and the first equality are trivial, and the second ones are special cases of the second part of the proposition. To prove the latter, consider a polyequilibrium payoff interval $E$ for player $i$ in either $\Gamma$ or $\Gamma^*$ and a corresponding polyequilibrium $X$, such that $u_i(X) \subseteq E$. By Proposition 3, there is some equilibrium payoff $v_i$ for player $i$ in $\Gamma^*$ such that $v_i \in u_i(\text{conv} X) \subseteq \text{conv} u_i(X) \subseteq E$.  

Proposition 4 shows that, in the mixed extension $\Gamma^*$ of a finite game $\Gamma$, the only polyequilibrium payoffs are the equilibrium ones. It follows from the second part of Theorem 1 below that for
A generically stronger proposition holds. Namely, all the polyequilibrium results in \( \Gamma^* \) are also PE-equilibrium results. In other words, mixed extensions of finite games are generically PE-plain. However, these games are not always PE-plain. Specifically, as the next example shows, the inclusion indicated by the second part of Proposition 3 does not necessarily hold for polyequilibria with non-convex polystategies.

**Example 7.** A polyequilibrium that does not include an equilibrium. In the mixed extension of the finite game

\[
\begin{array}{c|cc}
T & L & R \\
\hline
M & -1, -3 & 1, -4 \\
B & 1, 3 & -1, 4 \\
\end{array}
\]

a strategy profile is an equilibrium if and only if players 1 and 2 play \( T \) and \( L \), respectively, with probability \( 1/2 \). However, regardless of player 2’s strategy, any strategy \( pT + p'M + (1 - p - p')B \) of player 1 yields him the same payoff as the non-equilibrium strategy

\[
f(t)T + (f(t) - t)M + (1 - 2f(t) + t)B,
\]

where \( t = p - p' \) and

\[
f(t) = \begin{cases} (t + 1)/2, & 1/3 \leq t \leq 1 \\ t, & 0 < t < 1/3 \\ 0, & -1 \leq t \leq 0 \end{cases}
\]

Therefore, the following polystategy profile \( X = X_1 \times X_2 \) is a polyequilibrium: \( X_1 \) consists of all strategies of player 1 except the equilibrium ones and \( X_2 \) consists of all strategies of player 2. A smaller, minimal polyequilibrium is obtained by including in \( X_1 \) only the strategies of the form (2), with any \(-1 \leq t \leq 1 \) and \( f \) given by (3).\(^5\) It is easy to check that, in this polyequilibrium, player 2’s payoff satisfies \( |u_2| \geq 1 \). His unique equilibrium payoff, by contrast, is \( 0 \).\(^6\)

**Theorem 1.** If a polyequilibrium \( X \) in the mixed extension \( \Gamma^* \) of a finite game \( \Gamma \) does not include any equilibrium, then its convex hull \( \text{conv} \, X \) must include infinitely many equilibria. Therefore, if \( \Gamma^* \) has only finitely many equilibria, then it is PE-plain: every polyequilibrium includes at least one equilibrium.

**Proof.** It has to be shown that, for any polyequilibrium \( X \) in \( \Gamma^* \) and any finite set \( A \subseteq \text{conv} \, X \setminus X \), the set \( \text{conv} \, X \setminus A \) includes an equilibrium. If \( A = \emptyset \), the inclusion follows from Proposition 3. Suppose then that \( A = \{x^1, \ldots, x^L\} \), with \( L \geq 1 \). For each \( 1 \leq l \leq L \), let \( \bar{x}^l \in X \) (necessarily,

---

\(^5\) A rational substitution function (see Section 2.1) whose image is this polyequilibrium is essentially defined by \( f \).

\(^6\) Note that, by the second part of Proposition 4, it is not be possible to find a similar example where player 2’s payoff (rather than its absolute value) in some polyequilibrium \( X \) is greater than 0 while his unique equilibrium payoff is 0.
\( \bar{x}^i \neq x^i \) be a strategy profile that responds to \( X \), hence also to \( \text{conv} \, X \), at least as well as \( x^i \) does. For each player \( i \), let \( X'_i \subseteq \text{conv} \, X_i \) be a polytope as in the proof of Proposition 3, with the additional requirement that \( \{ x^i_1, ..., x^i_L, \bar{x}^i_1, ..., \bar{x}^i_L \} \subseteq X'_i \). As shown there, every equilibrium \( x^* \) in the subgame \( \Gamma' \) of \( \Gamma^* \) obtained by restricting each player \( i \) to strategies in \( X'_i \) is an equilibrium also in \( \Gamma^* \). Therefore, it suffices to show that some such equilibrium satisfies \( x^* \notin A \). Note that, by construction, for every player \( i \) and \( 1 \leq l \leq L \)

\[
u_i(x | \bar{x}^i_l) \geq \nu_i(x | x^i_l), \quad x \in X', \tag{4}\]

where \( \nu_i \) is player \( i \)'s payoff function in \( \Gamma^* \) (and in \( \Gamma' \)) and \( X' = \prod_i X'_i \).

**Claim 1.** For each player \( i \) there is a continuous function \( g_i: X'_i \rightarrow X'_i \) satisfying the following two conditions:

\[
u_i(x | g_i(x^i_l)) \geq \nu_i(x | x^i_l), \quad x \in X', \tag{5}\]

and, for every \( 1 \leq l \leq L \) with \( x^i_l \neq \bar{x}^i_l \),

\[g_i(x^i_l) \neq x^i_l, \quad x^i_l \in X'_i. \tag{6}\]

The meaning of (5) is that changing player \( i \)'s strategy from any \( x^i_l \) to \( g_i(x^i_l) \) cannot decrease his payoff in \( \Gamma' \). The meaning of (6) is that, if \( x^i_l \neq \bar{x}^i_l \), the image of \( g_i \) does not include \( x^i_l \).

The function in Claim 1 is defined as \( g_i = g^L_i \circ \cdots \circ g^1_i \), the successive composition of \( L \) functions \( g^l_i: X'_i \rightarrow X'_i \) \( (l = 1, ..., L) \), which are constructed as follows. If \( x^i_l = \bar{x}^i_l \), \( g^l_i \) is the identity function, \( g^l_i(x^i_l) = x^i_l \). If \( x^i_l \neq \bar{x}^i_l \), it is defined by

\[
g^l_i(x^i_l) = x^i_l + \alpha^l_i \phi_i(x^i_l) (\bar{x}^i_l - x^i_l), \tag{7}\]

where

\[
\phi_i(x^i_l) = \max\{\alpha \geq 0 \mid x^i_l + \alpha (\bar{x}^i_l - x^i_l) \in X'_i\} \tag{8}\]

and \( 0 < \alpha^l_i < 1 \) is any constant that satisfies the two requirements that are spelled out below. It is not difficult to see that the function \( \phi_i^l: X'_i \rightarrow [0, \infty) \) defined by (8) is continuous and satisfies

\[
\phi_i^l(x^i_l) = \phi_i^l(x^i_l) + \alpha \tag{9}\]

for every \( x^i_l, \bar{x}^i_l \in X'_i \) and \( \alpha \) such that \( x^i_l + \alpha (\bar{x}^i_l - x^i_l) = \bar{x}^i_l \). In particular, if a strategy \( x^i_l \) satisfies \( g_i^l(x^i_l) = x^i_l \) (hence, \( x^i_l + \alpha^l_i \phi_i^l(x^i_l) (\bar{x}^i_l - x^i_l) + (\bar{x}^i_l - x^i_l) = \bar{x}^i_l \)), then \( \phi_i^l(x^i_l) = \phi_i^l(\bar{x}^i_l) + \alpha^l_i \phi_i^l(x^i_l) + 1 \), and therefore

\[
\alpha^l_i \leq 1 - 1/\phi_i^l(x^i_l) \leq 1 - 1/M, \tag{10}\]

where \( M \) is the maximum of the function \( \phi_i^l \). The first requirement that the constant \( \alpha^l_i \) has to satisfy is that it is, in fact, greater than \( 1 - 1/M \). This requirement guarantees that
\[ g_i^l(x_i) \neq x_i^l, \quad x_i \in X_i'. \tag{10} \]

The second requirement is that \( \alpha_i^l \) is sufficiently close to (but smaller than) 1 to make the inequality \((1 - \alpha_i^l)M < \phi_i^l(x_i')\) hold for all \( l' \neq l \) with \( \phi_i^l(x_i') > 0 \). This requirement guarantees that, for every \( l' \neq l \) and strategy \( x_i \),

\[ g_i^l(x_i) = x_i' \Rightarrow x_i = x_i'. \tag{11} \]

This is because, if \( g_i^l(x_i) = x_i' \), then by (9) \( \phi_i^l(x_i) = \phi_i^l(x_i') + \alpha_i \phi_i^l(x_i) \), so that \( \phi_i^l(x_i') = (1 - \alpha_i^l) \phi_i^l(x_i) \leq (1 - \alpha_i^l)M \), which by the above requirement implies that \( \phi_i^l(x_i') = 0 \), and therefore also \( \phi_i(x_i) = 0 \), so that \( x_i = g_i^l(x_i) = x_i' \).

It follows from (4) and (7) that each of the functions \( g_i^l \) satisfies a condition similar to (5) (trivially so if \( g_i^l \) is the identity). Since by definition \( g_i(x_i) = g_i^l \cdots (g_i^1(x_i)) \cdots \), (5) itself clearly also holds. It remains to prove that (6) holds for every \( l \) with \( x_i^l \neq \bar{x}_i^l \). Suppose that this is not so: for some \( l' \) with \( x_i^l \neq \bar{x}_i^l \) and some strategy \( x_i' \), \( g_i^{-1}(\cdots (g_i^1(x_i')) \cdots ) = x_i' \). Necessarily, \( l' \neq l \), since an equality would violate (10) for \( l = L \). Therefore, by (11) (again with \( l = L \)),

\[ g_i^{-1}(\cdots (g_i^1(x_i')) \cdots ) = x_i' \). A repeated use of the same argument now shows that \( l' \neq l \) also for all \( l < L \). This contradiction completes the proof of Claim 1.

Define a function \( g : X' \to X' \) by \( (g(x))_i = g_i(x_i) \) for all \( i \). Construct a game \( \bar{\Gamma} \) that has the same players and strategy sets as \( \Gamma' \) (but is not necessarily the mixed extension of a finite game) by assigning to each player \( i \) the payoff function \( \bar{u}_i \) defined by

\[ \bar{u}_i(x) = u_i(g(x) \mid x_i). \]

The function \( \bar{u}_i \) is linear in player \( i \)'s own strategy \( x_i \), which implies that the set \( B_i(x) \) of best response strategies to any strategy profile \( x \) is a nonempty convex subset of the player's strategy set \( X_i' \). The continuity of \( g \) and of (the multilinear function) \( u_i \) implies that the correspondence \( x \mapsto B_i(x) \) has a closed graph. It therefore follows from Kakutani fixed-point theorem that \( \bar{\Gamma} \) has some equilibrium \( \bar{x} \). To complete the proof of the theorem, it remains to establish the following.

**Claim 2.** The strategy profile \( x^* = g(\bar{x}) \) satisfies \( x^* \notin A \), and it is an equilibrium in \( \Gamma' \).

Consider any \( 1 \leq l \leq L \). Since \( x_i^l \neq \bar{x}_i^l \), there is some \( i \) such that \( x_i^l \neq \bar{x}_i^l \). By (6), \( g_i(\bar{x}_i) \neq x_i^l \), which proves that \( x^* = g(\bar{x}) \neq x^l \). Thus, \( x^* \notin A \). For every player \( i \) and strategy \( x_i \in X_i' \),

\[ u_i(x^*) = u_i(g(\bar{x}) \mid x_i) = u_i(g(\bar{x}) \mid g_i(\bar{x}_i)) \geq u_i(g(\bar{x}) \mid \bar{x}_i) = \bar{u}_i(\bar{x} \mid x_i) = u_i(g(\bar{x}) \mid x_i) = u_i(x^* \mid x_i), \]

where the first inequality follows from (5), the second inequality holds since \( \bar{x} \) is an equilibrium in \( \bar{\Gamma} \), and all the equalities follow from the definitions. This proves that \( x^* \) is an equilibrium in \( \Gamma' \).
4.1 Zero-Sum Games

Shapley (1964) called a strict polyequilibrium in a finite two-player zero-sum game a \textit{generalized saddle point}, and called a polyequilibrium a \textit{weak generalized saddle point}.\(^7\) He showed that, in a game of this kind, the intersection of any number of generalized saddle points is also a generalized saddle point, so that the intersection of all of them, called the \textit{saddle}, is the game’s unique minimal strict polyequilibrium. For weak generalized saddle points, this is not so. Two of them may have a nonempty intersection that is not a weak generalized saddle point and does not even contain one, and every equilibrium is a minimal polyequilibrium. However, it follows as a conclusion from the next proposition that a \textit{unique} equilibrium is necessarily also the game’s unique minimal polyequilibrium. By Theorem 1, the same is true for every game that is the mixed extension of a finite game. However, the conclusion here and the proposition from which it follows concern any two-player zero-sum game: finite, the mixed extension of a finite game, or otherwise.

\textbf{Proposition 5.} A two-player zero-sum game is PE-plain if and only if it has an equilibrium.

\textit{Proof.} It has to be shown that if the game has an equilibrium \(x\), with payoff vector \((v, -v)\), then every polyequilibrium \(X\) includes some equilibrium. In fact, any strategy profile \(x' \in X\) that responds to \(X\) at least as well as \(x\) does is an equilibrium, because if player 2, for example, plays \(x'_2\), player 1 cannot get a higher payoff than \(v\). If there were a strategy \(x''_1\) that yielded a higher payoff, then this would be so also for every strategy in \(X_1\) that responds to \(X\) at least as well as \(x''_1\) does. However, the existence of a strategy in \(X_1\) that yields more than \(v\) contradicts the assumption that strategy \(x'_2\) responds to \(X\) at least as well as the equilibrium strategy \(x_2\) does, so that when player 2 plays \(x'_2\) against any strategy in \(X_1\), he gets at least \(-v\). The contradiction proves that a strategy \(x''_1\) as above does not exist, so that \(x'\) is indeed an equilibrium. \(\blacksquare\)

A finite two-player zero-sum game \(\Gamma'\) may or may not have a (pure-strategy) equilibrium. (The former holds, for example, if \(\Gamma'\) can be presented as an extensive form game with perfect information. The latter holds for the game in Example 6.) However, the mixed extension of such a game, that is, a \textit{matrix game}, always has an equilibrium. It therefore follows from Proposition 5 that matrix games are PE-plain: the sets of equilibrium and polyequilibrium results always coincide (which, as Example 7 demonstrates, is not true for their non-zero-sum counterparts, the \textit{bimatrix games}). The actual collections of equilibria and polyequilibria vary in their relative sizes. In rock-scissors-paper, the game’s unique equilibrium is also its only non-trivial polyequilibrium. Other matrix games with a unique equilibrium, such as the mixed extension of the game in Example 6, have larger, richer sets of polyequilibria.

\(^7\) Duggan and Le Breton (1996) use the last term in a somewhat different sense.
5 Dynamic Games

The defining property of polystrategy is that a player’s course of action may be only partially specified. In a dynamic context, this may mean that the specification is restricted to only some of the player’s information sets.

As for strategic games, a polystrategy $X_i$ of a player $i$ in a dynamic game $G$ with either perfect or imperfect information is any nonempty set of strategies. The meaning of ‘strategy’ here is viewed as part of the game’s specification. The term may refer to either pure strategies, which prescribe a single action at each of the player’s information sets, or behavior strategies, which prescribe a probability distribution over actions at each information set. A polystrategy $X_i$ is said to exclude a particular action or distribution over actions at a particular information set if none of the strategies in $X_i$ prescribes it (in other words, if every strategy that does prescribe it is excluded). A polystrategy is rectangular if it includes all the strategies that do not prescribe excluded actions or distributions over actions at any of the player’s information sets. A profile of rectangular polystrategies may be viewed as a polystrategy profile in the agent normal form of the game.

The simplest kind of dynamic game is an (either perfect- or imperfect-information) extensive form game, which is one that can be described by a finite game tree, possibly with chance nodes. As in the case of general dynamic games, it still needs to be specified whether all behavior strategies or only pure strategies are allowed. Any statement where this is not specified (or can be understood from the context) is to be interpreted as referring to both cases.

**Example 8. The centipede game** (Rosenthal 1981). In this extensive form game with perfect information, there are $m \geq 2$ decision nodes, numbered from 1 to $m$ (see Figure 1). The odd- and even-numbered nodes are controlled by player 1 and 2, respectively. At each node, the controlling player has to choose between Stop and Continue, except that at node $m$ only Stop can be chosen. The payoffs are determined by the first node $k$ at which Stop was chosen: the player controlling that node receives $2^k$ and the other player receives $2^k / 3$.

Consider the version of the centipede game where only pure strategies may be used. Effectively, a strategy is described by the index $1 \leq k \leq m + 1$ of the first node at which the player chooses...
Stop, with \( k = m + 1 \) standing for the strategy of never stopping (which is relevant only for the player not controlling the last node \( m \)). Therefore, a polystrategy profile is any subset of \( \{1, 2, \ldots, m + 1\} \) (specifically, the collection of “first Stop” nodes) that includes at least one odd number and at least one even number. A necessary condition for such a subset to be a polyequilibrium is that it is of the form \( \{1, 2, \ldots, l\} \), for some \( 2 \leq l \leq m + 1 \). This is because, by Fact 6, a polyequilibrium that includes any strategy \( 2 \leq k \leq m + 1 \) must also include the unique best response to it, which is strategy \( k − 1 \). The above condition is also sufficient for polyequilibrium. This is because, for any \( 2 \leq l \leq m + 1 \), the strategy of first stopping at \( l \) or \( l − 1 \) (depending on the player’s identity and the evenness or oddness of \( l \)) responds to \( \{1, 2, \ldots, l\} \) at least as well as any strategy that prescribes a later stopping time does, which means that the latter may be legitimately excluded. Thus, a polystrategy profile is a non-trivial polyequilibrium if and only if the two players’ polystrategies are to stop no later than node \( l \), for some (fixed, common) \( 2 \leq l \leq m \). The game therefore has \( m \) nested polyequilibria: the largest polyequilibrium is the trivial one, and the smallest (which is also the only small polyequilibrium) is the game’s unique equilibrium (corresponding to \( l = 2 \)). Thus, the game is PE-plain.

As the next example shows, there are also perfect-information extensive form games that are not PE-plain. Moreover, there are such games where a player’s payoffs in some polyequilibrium are higher than his unique equilibrium payoff.

**Example 9. Semi-dictator games.** Players 1 and 2 have $2 to share. They flip a coin, and the winner can either dictate an equal split of the money or ask for the whole sum. If he chooses the latter, however, the other player is allowed to object, and in this case, no one gets anything. Assuming that only pure strategies can be used, each player has four strategies. However, since the only decision that affects a player’s own payoff is the one he makes if he wins the coin toss, any polystrategy that does not exclude any of the two possible decisions there is part of a polyequilibrium. Such a polyequilibrium \( X \) is shown in Figure 2a. Each player’s polystrategy includes two strategies, Black and Gray, which prescribe choosing the actions indicated by black
and gray lines, respectively, in both decision nodes. It is easy to see that none of the four strategy profiles in \(X\) is an equilibrium. The game in Figure 2b is a variant of the first one, and can be described as involving an additional, payoff-irrelevant coin toss. The polystrategy \(X_i\) shown for each player \(i\) includes three strategies, Black, Gray and Thick, which prescribe choosing the action indicated by a line with that property at every decision node. In particular, at the two decision nodes of player \(i\) that immediately follow the chance node \(C\), his polystrategy prescribes three pairs of actions; the only pair missing is choosing the actions indicated by thin lines in both nodes. However, the latter yields player \(i\) the same (expected) payoff as Thick, as long as the other player \(j\) only uses strategies belonging to his polystrategy \(X_j\). This proves that \(X = (X_1, X_2)\) is a polyequilibrium. None of the nine strategy profiles in \(X\) is an equilibrium. Moreover, the outcome of each of them, that is, the distribution it induces over the terminal nodes, is not an equilibrium outcome. Indeed, it is not difficult to check that every strategy profile in \(X\) yields the players a total payoff of either 1 or 1.5, whereas in every (pure-strategy) equilibrium in the game the total payoff is 2. If a third player, whose only role is to get the money if the others do not receive it, were added to the game, that player’s payoff would be greater than 0 in the polyequilibrium \(X\) but 0 in every equilibrium. Parenthetically, it is an immediate corollary of the second part of Proposition 4 that the last statement would not be true if behavior strategies were allowed, as a corresponding equilibrium yielding a positive payoff to the third player does exist. The polyequilibrium \(X\), however, yields such a payoff without requiring the players to randomize.

5.1 Subgame Perfection

A strategy \(x_i\) of a player \(i\) in a dynamic game \(G\) induces a strategy for \(i\) in each subgame of \(G\). That strategy, which may also be denoted by \(x_i\) if the meaning is clear from the context, is obtained by restricting the original strategy to the information sets included in the subgame. \(^8\) These observation and notation convention naturally extend to strategy profiles, polystrategies and polystrategy profiles.

**Definition 2.** A polystrategy profile \(X\) in a dynamic game \(G\) is a weak subgame perfect polyequilibrium if, in every subgame of \(G\), the induced polystrategy profile is a polyequilibrium. The last condition may be expressed more explicitly as follows: for every strategy profile \(x \not\in X\) and every subgame \(G\) there is some \(x' \in X\) that, in the subgame \(G'\), responds to \(X\) at least as well as \(x\) does. A polystrategy profile \(X\) is a subgame perfect polyequilibrium if it satisfies the following stronger condition: for every strategy profile \(x \not\in X\) there is some \(x' \in X\) that in all subgames of \(G\) responds to \(X\) at least as well as \(x\) does.

The difference between subgame perfection and weak subgame perfection is illustrated by Figure 3. (Another example is Figure 2, where the polyequilibrium in a is subgame perfect.

\(^8\) A subgame by definition includes either none or all of the decision nodes in each information set. Note that the meaning of ‘subgame’ in the present, dynamic context (Selten 1975) is different from that in the strategic context (Shapley 1964; see Section 2).
Figure 3. Subgame perfect polyequilibrium (SPP) and weak SPP. a In this one-player game, the singleton \( \{RR'\} \) is a SPP and its complement \( \{RL', LR', LL'\} \) is a weak SPP. b In this two-player game, where only player 1’s payoffs are shown, \( X = \{LR', LL'\} \times \{r, l\} \) is a weak SPP. It is not a SPP because neither of player 1’s strategies responds to \( X \) at least as well as \( RL' \) does both in the whole game (where \( LR' \) does so) and in the subgame starting at the player’s second decision node (where \( LL' \) does so).

whereas that in b is only a weak subgame perfect polyequilibrium.) Both properties are “hereditary” in the sense that a polyequilibrium with either property induces a polyequilibrium with that property in every subgame.

A polystrategy profile that is a singleton is a subgame perfect polyequilibrium if and only if its single element is a subgame perfect equilibrium. In general, however, a subgame perfect polyequilibrium may not include a subgame perfect, or even any, equilibrium (see Figure 2a). Two exceptions to this general rule are presented by the next two propositions.

**Proposition 6.** In a perfect-information extensive form game, every weak subgame perfect polyequilibrium \( X \) where each player’s polystrategy is rectangular includes a subgame perfect equilibrium.

**Proof.** Consider the collection \( \mathcal{X} \) of all subsets of \( X \) that are weak subgame perfect polyequilibria and consist of rectangular polystrategies. This collection is not empty, as \( X \in \mathcal{X} \). For each element of \( \mathcal{X} \), count the number of decision nodes at which at least two actions or distributions over actions are not excluded, and consider some \( X' \in \mathcal{X} \) for which this number is minimal. If the number is zero, then \( X' \) is a singleton, which implies that it is a subgame perfect equilibrium, and the proof is complete. It therefore suffices to assume that the number is greater than zero, and show that this assumption leads to a contradiction. The assumption implies that, for \( X' \), there is some decision node \( v \) such that, (i) at each of the nodes following \( v \), only one action or distribution over actions is not excluded, and (ii) this is not so for \( v \) itself. Expand the set of excluded actions or distributions over actions at \( v \) to include all but a single, optimal one. This modification of \( X' \) gives a new polystrategy profile, \( X'' \subseteq X' \), which is clearly also an element of \( \mathcal{X} \), a contradiction to the minimality assumption concerning \( X' \).

**Proposition 7.** If a perfect-information extensive form game \( G \) has a unique subgame perfect equilibrium \( x \), then every subgame perfect polyequilibrium in \( G \) includes \( x \).
Proof. Suppose that \( x \notin X \) for some subgame perfect polyequilibrium \( X \). To show that this assumption leads to a contradiction, consider a strategy profile \( x' \in X \) that in all subgames responds to \( X \) at least as well as \( x \) does. There is some subgame \( G' \) in which the strategy profiles induced by \( x' \) and by \( x \) differ only at the root, where they prescribe different actions or distribution over actions to the acting player \( i \). Since in \( G' \) strategy \( x'_i \) responds to \( x' \) (hence, to \( x \)) at least as well as \( x_i \) does, and \( x \) is a subgame perfect equilibrium, player \( i \)'s payoff in \( G' \) under \( x' \) must be the same as under \( x \), which implies that both strategy profiles induce subgame perfect equilibria in \( G' \). This conclusion clearly contradicts the assumption that \( G \) has a unique subgame perfect equilibrium. \( \square \)

In the above analysis of the centipede game (Figure 1), ‘strategy’ actually refers to a number of equivalent (pure) strategies, which specify the same first Stop node but differ in their prescription of actions at the player’s later decision nodes. This is so in general: polyequilibrium analysis never requires distinguishing between equivalent strategies, because the payoffs they yield are always identical and therefore any one (or more) of them may be arbitrarily excluded. However, since equivalent strategies may differ in the strategies they induce in subgames, the distinction between them may be important in the context of subgame perfection.

Example 8 (continued). As shown, a polystrategy profile in the centipede game is a polyequilibrium if and only if it has the form \( \{1,2,\ldots,l\} \), for some \( 2 \leq l \leq m + 1 \). For each such polyequilibrium, and for each \( 1 \leq k \leq l \), which represents all equivalent strategies whose first Stop is at node \( k \), consider the representative strategy that prescribes Stop also at each of the player’s later decision nodes. Such a choice of representative strategies makes the polyequilibrium a weak subgame perfect polyequilibrium, since it is not difficult to check that the polystrategy profile it induces in every subgame is again of the general form indicated above. This weak subgame perfect polyequilibrium \( X \) is actually subgame perfectif \( 2 \leq l \leq 5 \). However, for larger \( l \), this is not so. To see this, suppose that \( l \geq 6 \) and consider the strategy \( x_1 \) of player \( 1 \) that instructs him to continue only at his second decision node (no. 3). No strategy \( x'_1 \in X_1 \) responds to \( X \) in all subgames at least as well as \( x_1 \) does. This is because, to do so in the two subgames starting at player \( 1 \)'s first and second decision nodes, \( x'_1 \) must specify the same actions there as \( x_1 \). However, by construction, no strategy in \( X_1 \) does so. This proves that the weak subgame perfect polyequilibrium \( X \) is not subgame perfect if \( l \geq 6 \). Note that \( X \) consists of rectangular polystrategies if and only if \( 2 \leq l \leq 4 \). Nevertheless, for all \( 2 \leq l \leq m + 1 \), \( X \) includes the game’s unique subgame perfect equilibrium, which corresponds to \( l = 2 \) (cf. Propositions 6 and 7).\(^9\)

Dynamic games that are not extensive form ones do not always have subgame perfect equilibria. For such games, the concept of subgame perfect polyequilibrium may be particularly pertinent.

\(^9\) For an analysis of curb sets in the centipede game, see Pruzhansky (2003). If \( m \) is odd, the only curb set is the trivial polyequilibrium.
Example 10. “Almost perfect” information and a continuous action space (Harris et al. 1995, Myerson and Reny 2015). First, players 1 and 2 choose their actions simultaneously. Then, players 3 and 4, who are informed of their predecessors’ choices, do the same. The set of actions for player 1 is the interval $[-1,1]$, and for each of the other players, it is the pair $\{-1,1\}$. Denoting the action of player $i$ by $a_i$, the four players’ payoffs are given by

$$ u_1 = -a_1^2 + 5a_3a_4 - |a_1|a_2a_3 - 5 $$
$$ u_2 = \frac{1}{2}(1 + 3a_2)a_3 $$
$$ u_3 = a_1a_3 $$
$$ u_4 = a_1a_4 $$

It can be shown that, even with all behavior strategies allowed, this game does not have a subgame perfect equilibrium. Roughly, the reason is that, to make the first, second and third term in player 1’s payoff function as large as possible, his action should be, respectively, (i) close to 0, (ii) different from 0, so that the responses of players 3 and 4 will match, and (iii) positive and negative with close-to-equal probabilities, so that player 2 will not be able to match player 3’s action. No mixed strategy of player 1 (equivalently, a behavior strategy; a probability measure on $[-1,1]$) optimally satisfies these three requirements. However, for any $0 < \epsilon \leq 1$, consider the polystrategy $X_1 = \{ \pm a \mid 0 < a \leq \epsilon \}$, where $\pm a$ means playing $a$ with probability 0.5 and $-a$ with probability 0.5. Thus, the smaller is $\epsilon$, the more complete is the specification of player 1’s behavior. For the other players, the behavior is completely specified, as follows: $X_2$ includes only the single strategy $\pm 1$, and $X_3$ and $X_4$ include only the strategy that specifies choosing 1, $-1$ or $\pm 1$ if player 1’s action $a_1$ is positive, negative or zero, respectively. Clearly, the strategy of each of these three players is a best response to the polystrategy profile $X = X_1 \times X_2 \times X_3 \times X_4$, and this is so also in every subgame. To show that $X$ is a subgame perfect polyequilibrium, it remains to show that for every mixed strategy $x_1$ of player 1 there is some $0 < a \leq \epsilon$ such that $\pm a$ responds to $X$ at least as well as $x_1$ does. If player 1 uses $x_1$, his payoff is $-\mathbb{E}a_1^2 - 5\mathbb{P}(a_1 = 0)$, where the expectation and probability are those specified by $x_1$. This payoff is lower than the $-a^2$ player 1 would get from playing $\pm a$, for every $a$ smaller than $\mathbb{E}|a_1|$ (or, if $\mathbb{E}|a_1| = 0$, for any $0 < a \leq \epsilon$).

Another advantage of subgame perfect polyequilibrium over subgame perfect equilibrium manifests itself in dynamic games with many information sets. Whereas a subgame perfect equilibrium must prescribe a carefully selected action at each information set, including those lying far away from the equilibrium path, a polyequilibrium may legitimately ignore all but a relatively small number of relevant information sets.

Example 11. Sequential competition (Milchtaich, Glazer and Hassin 2015). A market for a particular good has a continuum of consumers on one side and two sellers on the other side. The consumers arrive in a steady flow: in any time interval of length $l$, the total mass of arriving consumers is $l$. Each consumer demands a single unit of the good, and leaves the market after buying it or spending a unit of time in the market, whichever comes first. In the first case, the consumer’s payoff is $1 - x - p$, where $x$ is the time he has been in the market and $p$ is the price paid for the good, and in the second case, the payoff is zero. Thus, the longer a consumer waits until he buys the good, the lower is his valuation of it. The two sellers produce the good at zero
cost. Seller 1 arrives at time 1 and seller 2 arrives at time 1.1. An arriving seller announces a price $p$ for the good, sells the demanded quantity $q$, and immediately leaves. His payoff is the revenue $pq$.

For any price $0 < p_1 < 1$ that seller 1 sets, the total mass of the consumers who would receive positive payoff from buying at that price is $1 - p_1$. However, the seller’s actual profit from setting price $p_1$ may be lower than his monopoly profit of $p_1 (1 - p_1)$, because if consumers expect seller 2’s price $p_2$ to be significantly lower than $p_1$, some of them may choose to wait. The wait may in turn affect $p_2$, by changing the demand seller 2 faces. By definition, a strategy for seller 2 prescribes a price $p_2$ for every conceivable demand. Thus, it has to consider all possible partitions of the time interval $[0, 1]$ that ends with seller 1’s arrival into two subsets: the arrival times of the consumers who bought from that seller and of those who did not. One may suspect that these myriad decision nodes are not all equally relevant. This is where the notions of polystrategy and (subgame perfect) polyequilibrium, which legitimize the consideration of only some decision nodes, come in handy.

As shown below, for every price $0.4 \leq p^* \leq 0.5$ there is a subgame perfect polyequilibrium with $p_1 = p^*$ and $p_2 = 0.9$. Since the second price is higher, none of seller 1’s potential customers opts for waiting. Nevertheless, except for in the $p^* = 0.5$ case, seller 1 does not take advantage of this by setting his monopoly price of 0.5. The reason is the credible threat implicit in the consumers’ strategy, which is the following strategy. Consumers always buy at any price $p$ that leaves them with nonnegative payoff, except when it is seller 1’s price and it is higher than $p^*$, in which case they buy only if the resulting payoff is at least $p - 0.2$. Thus, if seller 1 sets a price $p_1 \leq p^*$, all the consumers who arrived in the time interval $[p_1, 1]$ buy from him, but if $p_1 > p^*$, then only those who arrived in $[2(p_1 - 0.1), 1]$ do so (or, if $p_1 > 0.6$, no one does). In the first case, seller 1’s profit is $p_1 (1 - p_1)$, and it thus attains its maximum, which is between 0.24 and 0.25, at $p_1 = p^*$. In the second case, the profit is only $p_1 (1 - 2(p_1 - 0.1))$ (or 0), which is less than 0.16. For player 2, the only relevant decision nodes are those corresponding to the two kinds of time intervals indicated above. In those corresponding to $p_1 \leq p^*$ ($\leq 0.5$), player 2’s profit-maximizing price $p_2$ is 0.9, and in those corresponding to $p_1 > p^*$, $p_2$ is the minimum between $p_1 - 0.1$ and 0.5. In the second case, the difference between $p_2$ and $p_1$ is precisely that required to compensate for the loss of payoff due to the waiting time of 0.1 and guarantee that both the consumers who bought from seller 1 (if there are any such consumers) and those who chose to wait acted optimally. This optimality is the sense in which the threat implicit in the consumers’ strategy is credible, and it is what makes this strategy, together with seller 1’s strategy of selling at price $p^*$ and seller’s 2 rectangular polystrategy just described, a subgame perfect polyequilibrium.

Note that the polyequilibrium in Example 11 has a well-defined path as it prescribes a unique action at every information set that may actually be reached. In this, it differs from every non-singleton polyequilibrium in the centipede game, in which different strategy profiles may specify different paths in the game tree.
5.2 Bayesian Perfection

In the context of dynamic games with imperfect information, which often possess no or only a few subgames other than the game itself, the subgame perfection requirement is arguably too weak. An alternative to subgame perfection is to require each player’s action or distribution over actions at each information set to be optimal with respect to a suitable system of beliefs about the history of play. Such beliefs are described by a belief system $\mu$, which specifies a probability distribution over the nodes in each information set $U$ of each player $i$. The probability assigned to a set of nodes $V \subseteq U$ is denoted $\mu(V)$ (with $\mu(U) = 1$). A minimal consistency requirement is that this probability coincides with that derived from the players’ actual strategy profile $x$ whenever possible, that is,

$$\mu(V) = \frac{P_x(V)}{P_x(U)} \text{ if } P_x(U) \neq 0,$$

where $P_x(V)$ is the probability that, under $x$, one of the nodes in $V$ is reached (and similarly for $P_x(U)$). A belief system satisfying this requirement is said to be weakly consistent with the strategy profile $x$. Under perfect recall, which is assumed below, weak consistency entails that the beliefs concerning an information set $U$ of a player $i$ do not change if only $i$’s strategy changes, provided that the probability that $U$ is reached is positive for both the old and the new strategy. The reason there is no change is that, since player $i$ has a perfect recall of his actions, all nodes in $U$ are preceded by the same sequence of $i$’s actions, and therefore the relative probabilities that the nodes are reached only depend on the probabilities of the other players’ actions. Thus, for $V \subseteq U$ and all strategies $x'_i$ and $x''_i$ of player $i$,

$$\frac{P_{x'|(x'_i)}(V)}{P_{x''|(x'_i)}(U)} = \frac{P_{x''|(x'_i)}(V)}{P_{x''|(x'_i)}(U)} \text{ if } P_{x''|(x'_i)}(U), P_{x''|(x'_i)}(U) \neq 0,$$

where the subscripts refer to the strategy profiles obtained by replacing $i$’s strategy in $x$ with the indicated strategies.

An existing solution concept that may be viewed as a beliefs-based refinement of the polyequilibrium concept is essentially perfect Bayesian equilibrium (Blume and Heidhues 2006). An EPBE may be described as a simple polyequilibrium $X$ that is obtained from a specified strategy profile $x$ by declaring certain information sets irrelevant. The collection of irrelevant information sets must (i) have zero probability of being reached when $x$ is played and (ii) include every information set that follows any irrelevant one. $X$ is then defined as the collection of all strategy profiles that agree with $x$ (in the sense of specifying the same action or distribution over actions) at each of the relevant (that is, not irrelevant) information sets. By requirement (i), any belief system $\mu$ that is weakly consistent with $x$ is also weakly consistent with every other strategy profile in $X$. Requirement (ii) implies that the polystrategy $X_i$ of each player $i$ is a special kind of rectangular polystrategy. The definition of EPBE is completed by the additional requirement that there is some weakly consistent belief system $\mu$ as above (which may be considered part of the EPBE) such that, for every relevant information set $U$ of every player $i$, 

23
strategy $x_i$ is a best response to $X$ in the continuation game starting at $U$ with the distribution over $U$’s nodes specified by $\mu$. This requirement implies that, for each player $i$, strategy $x_i$ is a best response to $X$ also in the whole game, so that $X$ is a simple polyequilibrium.

Essentially perfect Bayesian equilibrium may be regarded a solution to technical problems associated with the perfect Bayesian equilibrium solution concept (Blume and Heidhues 2006). Alternatively, as indicated, it may be viewed as one way of incorporating optimality with respect to a weakly consistent belief system into the polyequilibrium concept. However, EPBS has certain features that, in the general context of polyequilibria in dynamic games, are quite special and restrictive. The first special, albeit arguably desirable, aspect is that an EPBS is required to specify a unique outcome: all strategy profiles in the polyequilibrium give the same distribution over terminal nodes. The second restrictive aspect is that, at each information set, either the acting player’s choice is completely specified or it is left completely unspecified, and in the second case, the same must hold also for each of the following information sets. As Blume and Heidhues (2006) point out, one consequence of this requirement is that, in the (perfect information) centipede game (Example 8), the only EPBE is the subgame perfect equilibrium.

The limitations of the EPBS concept reflect a fundamental tension between the underlying principles of Bayesian perfection and polyequilibrium, which concerns the way the players’ beliefs at the various information sets relate to the actual strategy profile. In a perfect Bayesian equilibrium and a number of related solution concepts, beliefs are reasonable hypotheses, or conjectures, about the events that preceded the arrival at an information set. This theory about the actual history of play is then used to justify the choices of action at off-path information sets, which should not have been reached according to the strategy profile. Different solution concepts have different definitions of “reasonable” hypotheses and different requirements as to how they should reflect the players’ actual strategies. However, they all leave at least some leeway, which means that beliefs are often effectively chosen much like strategies are chosen. It is hard to reconcile this element of choice with the spirit of an excluding solution concept such as polyequilibrium.

There is, however, a simple, natural way to incorporate beliefs in a manner consistent with exclusion rather than choice. In a polyequilibrium $X$, the exclusion of a strategy of a player $i$ requires the existence of an alternative strategy that responds to every strategy profile $x \in X$ at least as well as the first strategy does. This condition may be extended by replacing “every $x \in X$” with “every $x \in X$ and every belief system that is consistent with $x$”. Thus, at each of player $i$’s information sets $U$, the alternative strategy should be an adequate substitute to the excluded one for every strategy profile $x \in X$ and every distribution over $U$’s nodes that is consistent with $x$. If under $x$ the probability that $U$ is reached is positive, the consistency requirement singles out a unique distribution there. Otherwise, there may be multiple consistent distributions, each reflecting a different plausible hypothesis as to why $U$ has been reached. The notion of perfect Bayesian polyequilibrium (PBP), which is described below, is based on the idea that all these hypotheses need to be considered.
Perfect Bayesian polyequilibrium is not a generalization of perfect Bayesian equilibrium but a fundamentally different, stronger concept. The difference is illustrated by Figure 4a. The equilibrium shown is perfect Bayesian, since Player 2’s choice of $r$ rather than $l$ is justified by some belief, namely, that if player 1 switched from his strategy $Out$ to some alternative strategy, it would be $R$ rather than $L$. However, any belief of player 2 at his off-equilibrium information set is arguably consistent with $Out$. Therefore, in a PBP, which requires the exclusion of a strategy to be justified under all consistent beliefs, $l$ cannot be excluded. For player 1, this means that $L$ and $R$ cannot be both excluded, so that there is no PBP where the player plays $Out$. Thus, this perfect Bayesian equilibrium result and player 2’s payoff of 2 are not PBP results.

The last example highlights a basic difference between perfect Bayesian polyequilibrium and other solution concepts such as perfect Bayesian equilibrium, essentially perfect Bayesian equilibrium and sequential equilibrium. The difference is that a PBP is not allowed to specify actions that are justifiable only under particular arbitrary beliefs about the history of play. The choice of such actions (for example, $r$ in Figure 4a) entails the exclusion of one or more alternative actions that under different but equally plausible beliefs are actually better. The disallowance of such exclusions may make it impossible to single out any action. Such an outcome is of course entirely in tune with the central idea of polyequilibrium, which is that the players’ strategies may be only partly specified.

Example 12. Only player 1 knows whether it will rain tomorrow. The value of this information for player 2, who would prefer taking an umbrella only if necessary, is $1$. Player 1 gives player 2 a take-it-or-leave-it offer to buy the information, and has to decide on the asking price $p$. Player 2 then has to decide whether to accept or reject the offer, and in the latter case, whether to take an umbrella. For simplicity, only pure strategies are allowed. Every $0 \leq v_1 \leq 1$ is a perfect Bayesian equilibrium payoff for player 1. It is obtained in an equilibrium where $p = v_1$ and player 2 is willing to pay this price but would reject the offer, and take an umbrella, if player 1
asked any other price. This reaction is supported by a belief that a price different from \( v_1 \) indicates rain. By contrast, the only perfect Bayesian polyequilibrium payoff for player 1 is 1. To see this, consider a perfect Bayesian polyequilibrium \( X \) where the payoff is \( v_1 < 1 \). For every \( v_1 < p < 1 \), player 2’s polystrategy \( X_2 \) excludes acceptance of this price, for if it included a strategy prescribing acceptance, it would not be possible to exclude player 1’s strategy of asking \( p \). Consider some strategy \( x_1 \in X_1 \) and some \( v_1 < p < 1 \) that is different from the asking prices specified by \( x_1 \). (In principle, the price may depend on whether or not it will rain tomorrow.) If player 1 uses \( x_1 \), player 2’s information set where he is asked to pay \( p \) is not reached. However, at this information set, there is no action of player 2 that under \( \text{all} \) beliefs does at least as well as the excluded action of acceptance. For example, the action of rejecting the offer and not taking an umbrella is worse than paying \( p \) and acting according to the provided information under the belief that it will surely rain tomorrow. This conclusion contradicts the assumption that \( X \) is a perfect Bayesian polyequilibrium. The contradiction leaves \( v_1 = 1 \) as the only perfect Bayesian polyequilibrium payoff. It is obtained in the polyequilibrium where player 1’s strategy is \( p = 1 \) and player 2’s polystrategy is to accept this price and reject any higher one. The reaction to prices lower than 1 is left unspecified.

Solution concepts that involve a single, specific belief system \( \mu \) usually impose on it certain internal consistency requirements, which express the idea that beliefs at different off-path information sets should not only reflect the players’ strategy profile but also represent a coherent hypothesis about their deviation from it. In particular, beliefs at a player’s information set that follows another information set of the same player should be derived from the beliefs at the latter whenever possible. This requirement is formally expressed by the preconsistency condition (Hendon et al. 1996, Perea 2002), which is based on Fudenberg and Tirole’s (1991) notion of reasonable assessment. Internal consistency between beliefs at information sets belonging to different players is guaranteed by the stronger full consistency condition, which is the central pillar of Kreps and Wilson’s (1982) sequential equilibrium solution concept.\(^{10}\) Perfect Bayesian polyequilibrium does not specify a single belief system, which renders the whole internal consistency requirement moot. This brings about a considerable simplification, since consistency is narrowed down to the “local” requirement that beliefs at a specified information set \( U \) are reconcilable with a specified strategy profile \( x \).

If, under a strategy profile \( x \), the probability that an information set \( U \) of a player \( i \) is reached is positive, the above local requirement is simply the weak consistency condition expressed by (12). However, if the probability is zero, then weak consistency does not specify any beliefs at \( U \). Nevertheless, player \( i \) may actually know a great deal about the history of play there. In particular, he knows that at another information set the acting player \( j \) took a particular action \( a \)

\(^{10}\) In an extensive form game, a belief system \( \mu \) is said to be fully consistent with a strategy profile \( x \) if they are, respectively, the limits of some sequences \( (\mu_1^n)_{n=1}^\infty \) and \( (x^n_1)_{n=1}^\infty \) such that each \( x^n \) is a completely mixed strategy profile, that is, one assigning positive probability for every action in every information set, and \( \mu^n \) is the unique belief system that is weakly consistent with \( x^n \).
if all nodes in \( U \) are preceded by (that information set and) action \( a \). (By the perfect-recall assumption, this is so in particular for each of player \( i \)’s own past actions.) This means that for the set \( \mathcal{A} \) of all actions \( a \) as above the probabilities specified by \( x \) are irrelevant; the effective probabilities are 1.\(^{11}\) Therefore, it only remains for player \( i \) to speculate about the other players’ behavior at information sets that do not involve actions in \( \mathcal{A} \). The simplest hypothesis is that they adhere to \( x \) there. In other words, the hypothesis effectively replaces \( x \) with the strategy profile \( x^\mathcal{A} \) obtained from it by specifying that every action in \( \mathcal{A} \) is taken with probability 1. If, under \( x^\mathcal{A} \), the probability \( \mathbb{P}_{x,\mathcal{A}}(U) \) that \( U \) is reached is positive, this hypothesis yields a unique probability distribution on the nodes in \( U \), which arguably represents the only beliefs at that information set that are consistent with \( x \). This probability distribution \( \mu^\mathcal{A}_i \) is given by

\[
\mu^\mathcal{A}_i(V) = \frac{\mathbb{P}_{x,\mathcal{A}}(V)}{\mathbb{P}_{x,\mathcal{A}}(U)}, \quad V \subseteq U.
\]

However, if even under \( x^\mathcal{A} \) the probability that \( U \) is reached is zero, then reaching it indicates that at least one action \( a \notin \mathcal{A} \) was taken even though the probability assigned to \( a \) by \( x \) is zero. A natural approach in this case is to enlarge \( \mathcal{A} \) in such a way that it becomes minimally sufficient for reaching \( U \) under \( x \), in the sense that the information set is reached with positive probability under \( x^\mathcal{A} \) but not under \( x^\mathcal{B} \), for all \( \mathcal{B} \subset \mathcal{A} \) (which clearly implies that \( x \) assigns probability zero to every action in \( \mathcal{A} \)). A minimally sufficient set of actions \( \mathcal{A} \) induces a probability distribution \( \mu^\mathcal{A}_i \) (defined by (13)) that may be viewed as consistent with the players’ strategy profile \( x \) if \( U \) was reached.\(^{12}\) However, such consistent beliefs are generally not unique, as there may be more than one set of actions that are minimally sufficient for reaching the information set.\(^{13}\)

A deviation from \( x \) that involves a non-minimally sufficient set of actions represents a non-parsimonious hypothesis as to why the information set \( U \) was reached; it assumes more than it has to. Moreover, such a hypothesis may have the troubling aspect that it implies a future deviation from \( x \). This can happen if some players’ information sets include both nodes that precede \( U \) and nodes that follow it, as in Figure 4b. In this game, which is taken from Kreps and Ramey (1987), the players’ order of moves is uncertain – either player 1 or player 3 moves before player 2 moves – and is unknown to them. The only (Nash) equilibrium outcome is that players 1 and 3 both play \( Q \) for sure, so that player 2’s information set is not reached. The

\(^{11}\) Probability 1 here means that the action was actually taken, not that the acting player \( j \) meant it to be played for sure. Moreover, for an action \( a \) that precedes all the nodes in \( U \), there is no effective difference between 1 and any other positive probability. Therefore, if the probability that player \( j \)’s strategy assigns to \( a \) is not 0, it may be left unchanged. Note that the discussion here and the definitions below are relevant both when the players are assumed to use only pure strategies and when all behavior strategies are allowed.

\(^{12}\) It follows immediately from the minimality of \( \mathcal{A} \) that the same distribution on \( U \) would result also if the actions in \( \mathcal{A} \) were assigned arbitrary positive probabilities, that are not necessarily 1.

\(^{13}\) If \( U \) is finite, then the collection of all such sets is also finite. This is because each set of actions that is minimally sufficient for reaching \( U \) necessarily consists of all actions that are assigned probability zero by \( x \) and precede a specific node \( v \in U \).
choice of $Q$ reflects the fact that there is no equilibrium strategy for player 2 that specifies playing either $L$ or $R$ with probability $2/3$ or greater. In particular, neither action is played with probability 1, which means that any equilibrium strategy for player 2 can be justified only by beliefs that assign positive probability to both nodes in his information set. Such beliefs are induced only by strategy profiles in which both player 1 and player 3 deviate from their equilibrium strategies by playing $P$ with positive probability. However, such simultaneous deviations are inconsistent with an assumption that the player acting after player 2 will be using his equilibrium strategy. Thus, this example shows that a non-parsimonious hypothesis about the past may project onto the future. With a parsimonious hypothesis about the deviations from $x$ that led to an information set $U$ being reached, this cannot happen. If a set of actions $𝒜$ is minimally sufficient for reaching $U$ under, then the probability that some information set where $x$ and $x^{\text{adv}}$ disagree is reached after $U$ is reached is zero, both under $x^{\text{adv}}$ and under any strategy profile that differs from it only in the strategy of the player acting at $U$. This is because any history that has positive probability under such a strategy profile and reaches $U$ must, by the minimal-sufficiency assumption, first go through all the actions in $𝒜$, which by the perfect-recall assumption means that it cannot reach any of the corresponding information sets again after $U$. A parsimonious hypothesis is thus structurally consistent with $x$ in the sense of Kreps and Ramey (1987, p. 1338), which is a stronger requirement than Kreps and Wilson’s (1982) notion of structural consistency of beliefs (for which the strategy profile is irrelevant). This is not the case for the beliefs that justify player 2’s equilibrium strategy in the above example, which are only convex structurally consistent. That is, these beliefs can be obtained only as convex combinations of the two structurally consistent ones, which are those that assign probability 1 to one of player 2’s nodes and reflect a hypothesized deviation by only one, particular other player.

The fundamental reason why the problematic structural inconsistency aspect of perfect Bayesian and sequential equilibria is not shared by perfect Bayesian polyequilibrium lies in the logical difference between justifying the choice of a particular strategy and justifying its exclusion. As indicated, justifying the exclusion of a strategy of a player $i$ under a strategy profile $x$ involves examining, for each of the player’s information sets $U$, all probability distributions on $U$’s nodes that are consistent with $x$. However, any such distribution that is the convex combination of two or more of the others may be ignored, because if the examination of each of the latter reveals that the excluded strategy indeed does not do better than the specified alternative strategy, then this is automatically the case also for the former. By contrast, the choice of a strategy may be justified only by the beliefs expressed by certain convex combinations (as in the above example) or even a unique combination. Hence the difference.

The above discussion leads to the following formal definition of PBP, where consistent beliefs are expressed as appropriately modified strategy profiles.\textsuperscript{14} The definition uses the following

\textsuperscript{14} Replacing this notion of consistency, which is the one presented above, with any weaker notion, and in particular with weak consistency, would result in a stronger definition. That is, the set of PBPs would be smaller. The opposite is true for any stronger notion of consistency.
notation. For an information set $U$ of a player $i$, $u_i(x \mid_U x_i')$ denotes the player’s payoff under a strategy profile that differs from $x$ only in that, at $U$ and all the information sets that follow it, player $i$ plays according to strategy $x_i'$.

**Definition 3.** A polystrategy profile $X$ in a dynamic game with perfect recall is a perfect Bayesian polyequilibrium if for every strategy profile $x \notin X$ there is some $x' \in X$ such that, for every $x'' \in X$, player $i$ and information set $U$ of that player, the inequality

$$u_i(x'' \mid_U x'_i) \geq u_i(x'' \mid_U x_i)$$

holds for every set of actions $\mathcal{A}$ that is minimally sufficient for reaching $U$ under $x''$.

The next proposition shows that the PBP concept may be considered a generalization of subgame perfect polyequilibrium. As discussed above, it is not a generalization of perfect Bayesian equilibrium, essentially perfect Bayesian equilibrium or sequential equilibrium. The exact sense in which the last solution concept is weaker than singleton PBP is spelled out by Theorem 2 below.

**Proposition 8.** In an extensive form game with perfect recall, every perfect Bayesian polyequilibrium is also subgame perfect. With complete information, the converse holds too.

**Proof.** For given PBP $X$ and strategy profile $x$, let $x' \in X$ be as in Definition 3, and consider any $x'' \in X$, player $i$ and subgame $G'$. Denote player $i$’s payoff function in $G'$ by $u'_i$. It has to be shown that, in $G'$, strategy $x'_i$ responds to $x''$ at least as well as $x_i$ does. Clearly, a sufficient condition for this is that $u'_i(x'' \mid_U x'_i) \geq u'_i(x'' \mid_U x_i)$ for every information set $U$ of player $i$ in $G'$ that is not preceded by any other such information set and is reached with positive probability in $G'$ when the strategy profile is $x''$. The inequality is equivalent to (14), where $\mathcal{A}$ is the set of all actions preceding the root of $G'$ that are assigned probability zero by $x''$. This completes the proof of the first assertion in the proposition. The second assertion is clear from the fact that an information set in a complete-information game includes only one node, which is the root of a subgame.

**Theorem 2.** In an extensive form game with perfect recall, a strategy profile $x$ is a (singleton) perfect Bayesian polyequilibrium if and only if $(x, \mu)$ is a sequential equilibrium for every belief system $\mu$ that is fully consistent with $x$.

---

15 If $U$ is reached with positive probability under $x''$, then the only minimally sufficient set is $\mathcal{A} = \emptyset$, for which $x'' \mid_{\emptyset} = x''$.

16 Technically, the definition is meaningful even if minimally sufficient sets do not exist, which can happen with a continuum of outcomes to chance moves. However, the last case would be addressed more satisfactorily if sure past outcomes were assigned probability 1, like players’ actions. More generally, the definition of PBP is not completely satisfactory for games with a continuum of actions, where reaching a zero-probability information set may actually be expected and does not necessarily indicate a deviation. See Myerson and Reny (2015) for a discussion of the conceptual difficulties associated with such games.
Note that Theorem 2 only concerns the special case of a singleton PBP. This case is not proposed here as a solution concept in its own right; it is merely an appendage of the full PBP concept.

The proof of the theorem is based on the following lemma, which is of independent interest. The lemma links three collections of beliefs at a particular information set $U$: the collection $\mathcal{B}^A$ of all beliefs that arise from minimally sufficient sets of actions, the collection $\mathcal{B}^F$ of all beliefs arising from fully consistent assessments and the collection $\text{conv} \mathcal{B}^A$ of all convex combinations of beliefs of the first kind. It shows that

$$\mathcal{B}^A \subseteq \mathcal{B}^F \subseteq \text{conv} \mathcal{B}^A. \tag{15}$$

The second inclusion strengthens and extends the Proposition in Kreps and Ramey (1987), which asserts that every fully consistent assessment satisfies convex structural consistency. In general, both inclusions in (15) may be proper. It is easy to see that the second inclusion holds as an equality if and only if the set $\mathcal{B}^F$ is convex.

**Lemma 2.** For a strategy profile $x$ and information set $U$ in an extensive form game, let $\mathcal{A}$ be the (finite) collection of all sets of actions that are minimally sufficient for reaching $U$ under $x$. For each $\mathcal{A} \in \mathcal{A}$, let $\mu_\mathcal{A}^A$ be the probability distribution on $U$ induced by $x^\mathcal{A}$, which is given by (13). Then:

(i) The probability distributions $\{\mu_\mathcal{A}^A\}_{\mathcal{A} \in \mathcal{A}}$ have pairwise disjoint supports.

(ii) Each of them $\mu_\mathcal{A}^A$ coincides with the probability distribution on $U$ specified by some belief system $\mu$ that is fully consistent with $x$, that is,

$$\mu_\mathcal{A}^A(V) = \mu(V), \quad V \subseteq U. \tag{16}$$

(iii) For every belief system $\mu$ that is fully consistent with $x$, the probability distribution on $U$ specified by $\mu$ is a convex combination of the distributions $\{\mu_\mathcal{A}^A\}_{\mathcal{A} \in \mathcal{A}}$, that is,

$$\mu(V) = \sum_{\mathcal{A} \in \mathcal{A}} \lambda^\mathcal{A} \mu_\mathcal{A}^A(V), \quad V \subseteq U \tag{17}$$

for some (unique, by (i)) nonnegative coefficients $\{\lambda^\mathcal{A}\}_{\mathcal{A} \in \mathcal{A}}$ that sum up to 1.

**Proof.** To prove (i), it has to be shown that $\mathbb{P}_{x^\mathcal{A}}(\{v\}) \mathbb{P}_{x^\mathcal{A}'}(\{v\}) = 0$ for all $v \in U$ and $\mathcal{A}, \mathcal{A}' \in \mathcal{A}$ with $\mathcal{A}' \neq \mathcal{A}$. If $\mathbb{P}_{x^\mathcal{A}}(\{v\}) > 0$, then the minimal sufficiency of the set of actions $\mathcal{A}$ implies that all its elements precede node $v$. Since the actions in $\mathcal{A} \setminus \mathcal{A}'$ are assigned probability zero by $x^\mathcal{A}'$, this implies that $\mathbb{P}_{x^\mathcal{A}'}(\{v\}) = 0$.

To prove (ii), consider any $\mathcal{A} \in \mathcal{A}$, with cardinality $|\mathcal{A}| = L (\geq 0)$. Let $\tilde{x}$ be some fixed completely mixed strategy profile and, for $0 < \varepsilon < 1/2$, let $x^\varepsilon$ be the strategy profile that, at each information set, assigns the following probability $x^\varepsilon(a)$ to each action $a$:
\[ x^\varepsilon(a) = (1 - \varepsilon - \varepsilon^{l+1})x(a) + \varepsilon x^d(a) + \varepsilon^{l+1} \bar{x}(a), \]  

where \( x(a), x^d(a) \) and \( \bar{x}(a) \) are the probabilities specified by \( x, x^d \) and \( \bar{x} \). The unique belief system \( \mu^\varepsilon \) that is weakly consistent with \( x^\varepsilon \) satisfies

\[ \mu^\varepsilon(V) = \frac{\mathbb{P}^x \varepsilon(V)}{\mathbb{P}^x \varepsilon(U)}, \quad V \subseteq U. \]

For every \( v \in V \), \( \mathbb{P}^x \varepsilon([v]) = \prod_k x^\varepsilon(a^k) \), where the \( a^k \)'s are all the actions preceding node \( v \).

Using (18), this product can be expressed as a polynomial in \( \varepsilon \). For \( l < L \), the coefficient of \( \varepsilon^l \) is zero, because a positive coefficient would mean that \( \mathbb{P}^x \varepsilon([v]) > 0 \) for some \( B \subseteq \mathcal{A} \), which contradicts the minimal-sufficiency assumption concerning \( \mathcal{A} \). By a similar argument, the coefficient of \( \varepsilon^l \) is \( \mathbb{P}^x \varepsilon([v]) \). It follows that, for \( V \subseteq U \), \( (1/\varepsilon^l) \mathbb{P}^x \varepsilon(V) \to \mathbb{P}^x \varepsilon(V) \) as \( \varepsilon \to 0 \), which implies that the quotient in (19) converges to that in (13). Therefore, if \( (\varepsilon_n)_{n=1}^{\infty} \) is any sequence of positive numbers converging to 0 such that \( (\mu^\varepsilon_n)_{n=1}^{\infty} \) converges to some limit \( \mu \), then that belief system, which is clearly fully consistent with \( x \), satisfies (16). The existence of such a sequence follows from the obvious compactness of the set of all belief system.

To prove (iii), consider any belief system \( \mu \) that is fully consistent with \( x \), and some sequences \((\mu^n)_{n=1}^{\infty}\) and \((x^n)_{n=1}^{\infty}\) as in footnote 10. For every \( \mathcal{A} \in \mathcal{A} \), the minimal-sufficiency condition implies that there is some \( u \in V \) that is preceded by all the actions in \( \mathcal{A} \) and satisfies \( \mathbb{P}^x \varepsilon([u]) = 0 \). For every \( n \geq 1 \), \( \mathbb{P}^x \varepsilon([u]) = \prod_k x^n(a^k) \), where the \( a^k \)'s are all the actions preceding \( u \). Therefore,

\[ \frac{1}{\prod_{a \in \mathcal{A}} x^n(a)} \mathbb{P}^x \varepsilon([u]) = \prod_{a \in \mathcal{A}} x^n(a) \to \mathbb{P}^x \varepsilon([u]). \]

A result similar to (20) holds with \( u \) replaced by any other node \( v \in V \) that is preceded by all the actions in \( \mathcal{A} \), which implies that

\[ \frac{\mu^n_{\mathcal{A}}([v])}{\mu^n_{\mathcal{A}}([u])} = \frac{\mathbb{P}^x \varepsilon([v])}{\mathbb{P}^x \varepsilon([u])} = \lim_{n \to \infty} \frac{\mathbb{P}^x \varepsilon([v])}{\mathbb{P}^x \varepsilon([u])} = \lim_{n \to \infty} \frac{\mu^n([v])}{\mu^n([u])} = \frac{\mu([v])}{\mu([u])} \]

if \( \mu([u]) > 0 \), and if \( \mu([u]) = 0 \), then also \( \mu([v]) = 0 \). Therefore, for \( v \in V \) that is preceded by all the actions in \( \mathcal{A} \), \( \mu([v]) = \lambda^\mathcal{A} \mu^n_{\mathcal{A}}([v]) \), where \( \lambda^\mathcal{A} = \mu([u]) / \mu^n_{\mathcal{A}}([u]) \). As shown in the first part of the proof, such \( v \) also satisfies \( \mu^n_{\mathcal{A}'}([v]) = 0 \) for all \( \mathcal{A}' \in \mathcal{A} \) with \( \mathcal{A}' \neq \mathcal{A} \), so that the equality in (17) holds for \( V = \{v\} \). To prove that the equality holds generally, it remains to note that every \( v \in V \) is preceded by all the actions in \( \mathcal{A} \) since the set of actions preceding \( v \) that are assigned probability zero by \( x \) necessarily has a subset that is minimally sufficient for reaching \( U \) under \( x \). Setting \( V = U \) in (17) proves that the coefficients sum up to 1.

\[ \text{Proof of Theorem 2.} \] By definition, \( x \) is a PBP if and only if, for every player \( i \) and strategy \( y_I \) and information set \( U \) of that player, the inequality

\[ u_i(x^d |_U x_i) \geq u_i(x^{d'} |_U y_i) \]
holds for every set of actions $\mathcal{A}$ that is minimally sufficient for reaching $U$ under $x$. It follows from (iii) in Lemma 2 that a sufficient condition for this is that $(x, \mu)$ is a sequential equilibrium for every belief system $\mu$ that is fully consistent with $x$. Conversely, it follows from (iii) in the lemma that the last condition holds if $x$ is a PBP.

5.3 Beliefs Based on Strategic Reasoning

The notion of consistency implicit in Definition 3 is based on the principle of parsimony: a particular deviation from the players' strategies is assumed to have occurred only if this assumption is needed for explaining why an off-path information set was reached. However, the simplest explanation is not always the most convincing one. In particular, a forward induction argument (Kohlberg and Mertens 1986) may lend credence to inconsistent beliefs. That is, a detected past deviation of another player from his strategy may hint at an additional, unobservable deviation. Unlike consistency, which is a notion based wholly on the game form, forward induction also involves an examination of the strategic interests of the deviating players. For example, in the perfect Bayesian equilibrium and (singleton) polyequilibrium shown in Figure 5a, player 2's choice of $r$ is supported by a belief that player 1 would follow his strategy in the subgame and choose $R$ there. However, if 2's information set is actually reached, which indicates that player 1 deviated from his strategy in the whole game by playing $In$, player 2 may reason that the most likely explanation for the deviation is that player 1 is aiming for the better equilibrium of the subgame, with payoffs 1, which means that he also played $L$ rather than $R$. Such a belief makes $l$ the better choice for player 2.

Past deviations may also be taken as indicators of intended future ones. This possibility is illustrated by the game the differs from that in Figure 5a only in the order of moves in the
equilibrium and perfect Bayesian polyequilibrium. The criterion singles out the same equilibrium for both solution concepts. The exclusion of a restricted action, such as Duel, in the Beer and player’s 2 would choose Duel only as a response to Quiche, may not arise in the same problem does not arise in the same problem does not arise in the example described in the previous subsection does not preclude a belief by the employer that such a choice indicates a high quality worker. Thus, as for equilibria, further restrictions on off-equilibrium beliefs, based on strategic considerations, may be warranted. However, as such restrictions effectively lead to a stronger notion of consistency, they do not eliminate PBPs but can only add new ones (see footnote 14). Thus, in particular, the set of singleton perfect Bayesian polyequilibria, which is typically contained in the sets of perfect Bayesian and sequential equilibria (Theorem 2), expands while the latter contract, which means that the gap between the corresponding sets of results may narrow. The following example illustrates this possibility.

**Example 13.** Consider the Beer–Quiche game shown in Figure 5b, where for simplicity only pure strategies are allowed. There are two (Nash) equilibria, which are both pooling. In one equilibrium, Black (indicated by black lines), types $t_w$ and $t_s$ of player 1 both choose Beer, and in the other, Gray, they choose Quiche. The second equilibrium is eliminated by the intuitive criterion (Cho and Kreps 1987). The criterion is based on a restriction of player 2’s possible beliefs regarding player 1’s type, which in particular precludes beliefs that, following a choice of Beer, attach a positive probability to $t_w$. The reason such beliefs are deemed unreasonable is that this type’s equilibrium payoff of 3 is higher than anything he may get by choosing Beer. The same problem does not arise in the first equilibrium, Black, where both types of player 1 choose Beer and player’s 2 would choose Duel only as a response to Quiche. That response is justified by the unique reasonable belief following a choice of Quiche by player 1, which is that his type is $t_w$ (because $t_s$ would necessarily be harmed by such a choice). Put differently, the unreasonableness of all other beliefs at the off-equilibrium information set justifies the exclusion of the alternative action (Don’t Duel) there. The same argument also shows that a restriction to reasonable beliefs would make Black a PBP. Thus, the logic underlying the intuitive criterion singles out the same equilibrium for both solution concepts, perfect Bayesian equilibrium and perfect Bayesian polyequilibrium. This coincidence contrasts with the situation...
for the original, unmodified definitions, according to which both equilibria are perfect Bayesian equilibria but neither of them is a PBP. It is, however, a rather special outcome, which is due to the fact that the additional reasonableness requirement on off-equilibrium beliefs pins them down uniquely.

References


