A Matching Theory of Organization and Network

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Abstract

To explore the formation of organization and network with their various structures, a general theory of matching evolves from the classic paradigm of Gale and Shapley in the following directions: constructing its connectivity in which two or more matched teams could have some common elements through matching with their functions rather than partitions of agents; Incorporating the directionality of hierarchy through matching with the orders of functions; Introducing the paradigm of organizational rationality consisting of its self-fulfilling and preference system into the theory of matching. The static notion of matching is not universal since there does not always exist a stable one. The property of universal stability is constructed within a matching evolution organizing with a series of matching between organizations matched in its sub-games and free agents no matter if the game is stable in its static matching.

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1 Introduction

The evolution of matching theory in this paper is motivated by the following two purposes: to model the formation of organization and network with their various structures, and to construct the universality of matching such that every game could have a stable one.

Organization and network are a duality in modern economic and social life. An organization is a compact network consisting of relations between its

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elements with various structures such as the bureaucratic and flat ones. Organizational structure is made of hierarchy and specialization of functions. An organization interacts with other organizations in an external network. Network, as an open organization of its elements, is a real or virtual format to capture relations among various organizations and agents to carry out their interactions. Any relation in a network also is an organization. Organizational behavior and interaction within network are two of elementary economic and social activities. With informative innovations such as internet and web development, social organization and network emerge into a new format such as sharing economy. There has been a long-standing interest to explore formation of organization and network with their various structures in social and economic life, which is a complicated process in which organizations and their networks interact together. An organization is structured not only by arranging its internal elements but also by interacting within its environment. A network is not only made of the relations among organizations, but also by being affected by the internal structure of its organizations. This is a causality dilemma of the chicken or the egg.

The matching theory based on the classic paradigm of Gale and Shapley is one of potential choices for us to employ into the exploration of the formation of organization and network since the seminal work of Gale and Shapley (1962), as its theoretic origin of stable matching, was just to explore the formation of marriages, one of most fundamental social organizations and relations. Matching theory is one of natural paradigms to deal with the problem of resource allocation with its most limited assumption, a preference order only. It does not only incorporate all of theoretic models such as auctions, and also have wide application in economics from college admission to macroeconomic issues. The deferred acceptance algorithm proposed by Gale and Shapley is not only construct the scheme of computing economics, but also provide an organizing vision of social and economic life.

However, to be a fundamental language for the theory of organization and network, the classic models of matching faces several challenges coming from its deficiency of connectivity, directionality, organizational behavior and universality, which are inherent properties of organization and network. First, connectivity is the property of organization and network that associate with their internal relations among their elements and external relations with others. Organization and network exhibit the property since variation of internal and external connections construct one of foundations of organization and network. The theory of matching deficits its connectivity, that matched teams such couples of men and women are disjoint, since there is a exogenous partition of agents in classic models. Second, Hierarchy, as an element of organizational structure and network, could be measured by order of its functions or called as a direction. The property of directionality is also failure for the classic theory of matching in which there is no room to place the order of functions. Third, organization, as a social innovation, behaviors to deal with goal conflict with collective decision. Its behavior is not modeled well in the classic matching theory. One of fundamental challenges to the classic matching theory is its deficiency of universality. In social and economic life, there are a various of organizations and networks
with their functions and hierarchies. However, in classic theory, there does not always exist a stable matching as the roommates problem proposed by Gale and Shapley (1962). Those theoretic challenges motivate an evolution in matching theory.

Kenneth Arrow (1974) states "Organizations are a means of achieving the benefits of collective action in situations where the price system fails". Organizations are a social innovation for people to deal with goal conflict in their collective behaviors. It is interesting to examine how organizations could behave in matching games and what impacts organizational behaviors could have with the challenges in classic models of matching.

The first contribution of this paper is to develop the matching theory through integrating the property of connectivity. The connectivity of matching, in which two or more teams have some common elements or agents, could be incorporated through removing the partition requisition of agents and enlarging agents’s capacities. In most classic models, there are a partition requisition, e.g., the partition of men and women in marriage matching. In the general theory, each agent could play in different functions with a quota.

The second contribution of this paper is to incorporate the property of directionality or hierarchy into the matching theory. Hierarchy, as structure of organization and network, could be measured with the orders of their functions or a direction. The directionality of a game could be constructed through matching with its functional orders. Formation of some organizations and networks is equivalent to the mappings of stable matching into a directed hypergraph. The number of functions or dimensions of a matching game could be an arbitrary positive integer. The directionality of matching implies that there exists a measure, direction, to capture orders of functions carried by agents in matched teams. A game could be matched following both its functions and functional orders, also its matching could be due to its functions only.

The third contribution of this paper is employ directed hypergraph to present matching theory. A matching is a mapping of a game into a directed hypergraph. The directions of a hypergraph could be various rather than one in most literatures.

The fourth contribution of this paper is to explore the formation of organization and network. Organization and network are two of elements of social and economic life. An organization, as a sub-group of agents teamed in a match, and a network, a structure of relations among various organizations, could be described in terms of the general theory naturally. Formation of some organizations and networks are modeled as the mappings of stable matchings into a hypergraph.

The fifth contribution of this paper is to explore organizational rationality and behaviors in problems of complicated matching. Herbert A. Simon (1972) states: "One plausible distinction between them is that a theory of organizational rationality must treat the phenomena of goal conflict, while a theory of individual rationality need not. This is only partly correct, for goal conflict may be important in individual as in group behavior...". Organizations are a social innovation for people to deal with goal conflict in their
collective behaviors. The paradigm of organizational rationality consists of two elements: its self fulfillment and preference system. An organization is self fulfilling if its welfare is not less than welfare sum of all their subsidiaries. A rational organization behaves in matching game as an agent with its own preference system. Organizations are a social innovation for people to deal with goal conflict in their collective behaviors. The paradigm of organizational rationality consists of two elements: its self fulfillment and preference system. An organization is self fulfilling if its welfare is not less than welfare sum of all their subsidiaries. A rational organization behaves as an agent following its own preference system. A matching theory could incorporate organizational behavior through introducing the paradigm of organizational rationality.

The final contribution is construct the universality of matching theory. In social and economic life, there are a lot of organizations and networks with their various functions and hierarchies. However, the classic notion of matching is not universal since there does not always exist a stable matching showed by Gale and Shapley (1962). With the paradigm of organizational rationality, we overcome the shortage and construct universal existence of stable matching through transformation of a game without stable matching into a series of organizing matching of organizations formed in sub-games with stable matching with free agents through adopting the recursive and adaptive schemes.

The structure of rest part of this paper is as follows. In the second section, Organization and network are two formats of representations of social or economic relations. Connectivity, existence of relations between agents and organizations, is one of common properties of organization and network. In this section, we construct the connectivity of matching through both removing the partition of agents in classic models and enlarging the capacities of agents. First we introduce a solo mentorship game in which an agent could match with others as both a mentor and montee respectively once following her strictly ranking preference. Then, we introduce the concept of directed graph as a preliminary for a general mentorship. We generalize the solo mentorship into a general matching through enlarging their capability in matching. The set of stable matching is not empty for a mentorship. We studies the relation between stable matching and network formation. Employing the theory of mentorship, the romantic circle structure among high school students could be modeled as a stable matching. Mentorship is a generalized framework to study two-dimensional matching problems such as college admissions, stable network and flows.

In third section, to explore the complicated structure of organizations and networks, we develop another two properties of directionality and universality of matching. That is to construct a game with arbitrary size of functions, or universality, in which organizations are matched follow their
orders of functions, or directionality. First, we revisit the problem of mentorship and introduce a directed mentorship matching. Second, we introduce the concept of directed hypergraph as a preliminary language. Third, we propose a multi-dimensional directed matching in which organizations are matched due to its directional set. A matching, a special case of directed matching, is a game in which organizations are matched due to its functional set. Final, we discuss the relation between directed matching and organizational structures.

In the fourth section, we will introduce the paradigm of organizational rationality to present the will and determination of organizations. Organizational rationality consists two parts: its self fulfillment and preference system. Following the paradigm of organizational rationality, we construct a concept of organizational matching through studying a bipartite matching games between organizations and free agents.

Until now, all of existence of stable matching we discussed could be be reduced into a bipartite problem. In the fifth section, first we discuss one of challenges of classic theory that there does not always exist a stable matching. Second, with the paradigm of organizational rationality and matching, we will construct the existence concept of universal stability that the set of stable matching is not empty for every game no matter if it is in two dimensions though developing two schemes: recursive and adaptive matchings.

2 Mentorship Matching and Network

2.1 On Connectivity

Organization and network are a duality of social or economic relations. Organization is a network with a boundary, which is made of its internal relations among its elements and its external relations with others. Network is an organization of its elements without an boundary, which is a real or virtual formate to capture relations among various organizations and agents to carry out their interactions.

A direct relation between two elements such agents is their connection which is their common organization made of them. An indirect relation between two agents is a path of several organizations between them. Connectivity is the profile of connections between agents and organizations, which is one of common properties of organization and network.

With employing the classic models of matching, external relations of an organization and network can not be modeled well due to the partition requisition of agents such as the partition of men and women in a marriage game. The connectivity is failure to classic models.

The connectivity of matching could be build through both removing the partition of agents in classic models and enlarging the capacities of agents.

In social and economic life, an agent could play various roles or functions within its interaction with others. A person could be both a speaker and listener with same and/or different person in a conversation or several conversations. Also she could chat herself. In an economy, an agent could be both a seller and buyers.
In this section, first we introduce a solo mentorship game in which an agent could match with others as both a mentor and mentee respectively once following her strictly ranking preference. Since an agent could be a mentor in a mentorship and a mentee in other one, the two matched mentorships are connected by the agent. Since a couple of agents in a mentorship could play various functions, the order of their functions could indict characteristic of the mentorship, called a direction.

Second, we introduce the concept of directed graph as a preliminary for a general mentorship.

Third, we generalize the solo mentorship into a general matching through enlarging their capability in matching. With introducing quota in function, each agent could play same role in different team. If a person has three of his mentor quota and five of his mentee quota, he could be mentors in three different pairs and be mentees in five teams. The set of stable matching is not empty for a mentorship.

Fourth, We studies the relation between stable matching and network formation. Employing the theory of mentorship, the romantic circle structure among high school students could be modeled as a stable matching. Mentorship is a generalized framework to study two-dimensional matching problems such as college admissions, stable network and flows.

2.2 The Solo Mentorship

Organization and network are two formats of representations of social or economic relations. Connectivity, existence of relations between agents and organizations, is one of common properties of organization and network. First, we construct the connectivity of matching through removing the partition of agents in classic models by introducing a solo mentorship as follows.

For a group of agents, one could be matched with others as both mentor and mentee with their preferences. Here we assume that mentorship is solo, i.e., anyone could be only one mentor and only one mentee in a mentorship matching.

A solo mentorship game is a three-tuple \( \{N, F, \succ\} \), in which there is a set of agents or players \( N = \{1, \ldots, i, \ldots, n\} \) without any pervious partition: There a two-element functional set \( F = \{+, -\} \) such as mentor and mentee, denoted as + and -. Each agent \( i \in N \) could be both mentor and mentee and plays each role once. Let \( \lambda \) to be a strictly order of \( N \) and \( \Lambda \) to be the collection of all strictly orders of \( N \). The preference system of the game is a mapping \( \succ : N \times F \rightarrow \Lambda \). For \( i \in N \), if he plays as a mentor, he has a mentor or positive preference system \( \succ_i^+ \) on \( N \), which is a strict order of \( N \) as its potential mentees. In mentee or negative preference system \( \succ_i^- \), every \( i \in N \), as a mentee, has a preference \( \succ_i^- \) on \( N \), which is a strict order of \( N \) as its potential mentors.

Without preference system, the notion of matching of this game or solo mentorship is introduced as follows. The map \( \mu : N \times F \rightarrow N \) is called a solo mentorship (matching) if \( \mu(\mu(x, +), -) = \mu(\mu(x, -), +) = x \) for all \( x \in N \). \( x \in N \) is self matched if \( \mu(x, +) = \mu(x, -) = x \). For a mentorship
(matching) \( \mu, \mu(N) \) is the set of its all mentor teams or matched partners, 
\( \mu(N) = \{(x, y) \in N^2 \mid \mu(x, +) = y, \mu(y, -) = x\} \). Let \( \mathcal{M} \) to be the set of all mentorships.

Within preference system of a game, the notion of its stable matching or mentorship is constructed. An agent \( x \in N \) prefers matching \( \mu \) to matching \( \nu \) if and only if \( \mu(x, f) \succeq_x f \nu(x, f) \forall f \in \{+,-\} \), where \( \mu(x, f) \sim_x f \nu(x, f) \) if and only if \( \mu(x, f) = \nu(x, f) \in N \), and there exists at least one \( f \) such that \( \mu(x, f) \neq \nu(x, f) \).

A matching \( \mu \) is individual rational if there does not exist any \( x \in N \) and \( f \in \{+,-\} \) such that \( x \succ_x f \mu(x, f) \). A matching \( \mu \) is pairwise stable if there does not exist any pair \( x, y \in N \) such that \( y \succ_y f \mu(x, f) \) and \( x \succ_x g \mu(y, g) \), for \( f, g \in \{+,-\} \).

A solo mentorship is stable if it is both individual rational and pairwise stable. A matching is stable if it cannot be improved upon by either any individual or any pair of agents.

**Example 2.1.0: The marriage game** The marriage game is a special case of solo mentorship game in which men are the part of agents with only + function and women are other part of agents with only – function in a solo mentorship.

**Example 2.1.1**
Let \( N = \{a, b, c\} \), their preference system (mentor, mentee). The agent a, as mentor, ranks its potential mentee: b first, c second, itself last; Then the agent a, as mentee, ranks its potential mentor: b first, c second, itself last. \( \Lambda(3, 3) = \begin{pmatrix} 3,3 & 1,2 & 2,1 \\ 2,1 & 3,3 & 1,2 \\ 1,2 & 2,1 & 3,3 \end{pmatrix} \)

In this game, \( i \in N, \succ^+ (i, j) \Rightarrow \succ^- (j, i) \). The stable matching mentors initiate proposing is \( \{(a, b), (b, c), (c, a)\} \), which is a circle in which here first and second pairs connected at b, second and third pairs connected at c, The third and first pairs connected at a and direction abc.

The stable matching mentees initiate proposing is \( \{(b, a), (c, b), (a, c)\} \), which is a circle in which here first and third pairs connected at a, third and second pairs connected at c, The second and first pairs connected at b and direction cba which is the reverse one of mentor proposing.

**Example 2.1.2** \( \Lambda(3, 3) = \begin{pmatrix} 3,1 & 1,1 & 2,1 \\ 2,3 & 3,2 & 1,2 \\ 1,2 & 2,3 & 3,3 \end{pmatrix} \)

**Proposition 1. [The Existence Theorem]** Every solo mentorship game has a stable matching. The stable matching is unique if their preference system is systemic on mentors and mentees, i.e., \( \lambda^+(i, j) = \lambda^-(i, j) \).

The proof of the proposition. Solo mentorship is a game consisting of two functions: mentor and mentee. It is equivalent to a bipartite game such as a marriage a game. It could be proved following the deferred acceptance algorithm proposed by Gale and Shapley (1962).

**Example 2.1.3** \( \Lambda(3, 3) = \begin{pmatrix} 3,3 & 1,1 & 2,2 \\ 2,2 & 3,3 & 1,1 \\ 1,1 & 2,2 & 3,3 \end{pmatrix} \)
2.3 A fair Algorithm

Employing the deferred acceptance algorithm proposed by Gale and Shapley (1962), a stable matching always be found for every marriage game. However, a matching is optimal to the group who propose in the algorithm. There are a great interest to explore alternative algorithms to be fair in procedure or outcome. Here, we outline a natural algorithm evolved from the algorithm of Gale and Shapley.

**Definition 1. [Offer-Deferred-Acceptance (ODA) Algorithm]** For a mentorship \(N, (\succ^+, \succ^-)\), there is a offer-deferred-acceptance algorithm as follows:

**Step 1.** Every agent \(i \in N\) offers a pair of proposals for its mentee and its mentee to its top candidates according to its strict preference system \((\succ^+, \succ^-)\) respectively. At same time, it may receive proposals to be mentee or mentor from others.

**Step 2.**
1. a pair of mentor and mentee matched if one of its offers coincides with another. The functions matched for this pair will be out in next round.
2. an agent rejects all unmatched proposals offered by others but itself.

**Step 3.** Every agent, but fully matched ones, \(i \in N\) offers a pair of proposals for its mentee and its mentee, but its matched function, to its next top candidates according to its strict preference system \((\succ^+, \succ^-)\) respectively. At same time, it may receive proposals to be mentee or mentor from others.

**Step 4.**
1. a pair of mentor and mentee matched if one of its offers coincides with another. The functions matched for this pair will be out in next round.
2. An agent holds an offer proposed by another and pull back its own offer if the other’s offer is better.
3. Otherwise, an agent rejects all offers proposed by others but itself.

**Step 5.**
1. The process indicate by the ODA algorithm will exist for a settlement if for a group of agents, they pull back their offers as counterparties at same time.
2. Otherwise, the game will reach a matching or a settlement through repeating Step 3, 4 and 5.

**Proposition 2.** A matching generated by the offer-deferred-acceptance algorithm is fair.

**Example 2.1.4 (Gale and Shapley (1962))** \(A(3, 3) = \begin{pmatrix} 1,3 & 2,2 & 3,1 \\ 3,1 & 1,3 & 2,2 \\ 2,2 & 3,1 & 1,3 \end{pmatrix}\)

Its fair matching is \{\((\alpha, B), (\beta, C), (\gamma, A)\)\} with same rankings \((2, 2)\), which the solution of the game with the offer-deferred-acceptance algorithm.

Example 2.1.1. is an example that there does not exist a fair matching and has to be settled with the offer-deferred-acceptance algorithm.
2.4 Mentorship Matching

Here we generalize the solo mentorship matching into a general matching through enlarging the capacity of a function for agents through introducing a size measure, quota. A matching could exhibit property of connectivity if its quota is more than one since an agent could be in more than one matched sub-groups.

A mentorship matching game is a 4-tuple \( \{ N, F, q, \succ \} \), consisting of (a) an agent set \( N = \{ 1, \ldots, i, \ldots, n \} \); (b) There a two-element functional set \( F = \{ +, - \} \), which two functions denote the notation + such as a mentor, speaker, or seller and the notation - such as a mentee, audience, or buyer. (c) Let’s introduce quota system \( q \) to measure the capability of roles each agent can play in each function, \( q : N \times F \rightarrow \mathbb{Z}_q^2 \). For \( i \in N \), there a pair of quotas in two functions \( q_i = (q_i^+, q_i^-) \). A game has a \( 2 \times n \) quota matrix \( q = (q^+, q^-) = \{ q_i^+, q_i^- \} \) \( i \in N \). (d) Let \( \lambda \) to be a strictly order of \( N \) and \( \Lambda \) to be the collection of all strictly orders of \( N \). The preference system of the game is a mapping \( \succ \) : \( N \times F \rightarrow \Lambda \). For \( i \in N \), if he plays as a mentor, he has a mentor or positive preference system \( \succ^+_i \) on \( N \), which is a strict order of \( N \) as its potential mentees. In mentee or negative preference system \( \succ^-_i \), every \( i \in N \), as a mentee, has a preference \( \succ^-_i \) on \( N \), which is a strict order of \( N \) as its potential mentors.

There is a preference system \( \succ = \{ \succ^+, \succ^- \} = \{ \succ^+_i, \succ^-_i \} \) \( i \in N \). \( \lambda^+_i,j \) preference of leader or mentor \( i \) on follower or mentee \( j \). \( \lambda^-_{i,j} \) preference of follower or mentee \( i \) on leader or mentor \( j \). In a mentorship game, for any agent \( i \in N \), its preference is separable on the two functions such mentor and mentee due to the assumption that the two functions of mentor and mentee are not alternatives to it.

\[
\Lambda(n, n) = \begin{pmatrix}
(\lambda^+_1, 1, \lambda^-_1, 1) & (\lambda^+_1, 2, \lambda^-_1, 2) & \cdots & (\lambda^+_1, n, \lambda^-_1, n) \\
(\lambda^+_2, 1, \lambda^-_2, 1) & (\lambda^+_2, 2, \lambda^-_2, 2) & \cdots & (\lambda^+_2, n, \lambda^-_2, n) \\
\vdots & \vdots & \ddots & \vdots \\
(\lambda^+_n, 1, \lambda^-_n, 1) & (\lambda^+_n, 2, \lambda^-_n, 2) & \cdots & (\lambda^+_n, n, \lambda^-_n, n)
\end{pmatrix}
\] (2.1)

Let’s introduce the notion of matching into mentorship game without its preference system, in which their quota could be more than one. The map \( \mu(\cdot, \cdot, q(\cdot, \cdot)) : N \times F \rightarrow \mathcal{M} \) is called a mentorship (matching) if for all \( x \in N \), \( \mu(x, +, q(x, +)) \in \mathcal{M} \), \( \mu(x, -) \leq q(x, -) \) \( \in \mathcal{M} \). For a mentorship (matching) \( \mu \), \( \mu(N) \) is the set of its all mentor teams or matched partners, \( \mu(N) = \{(x, y) \in N^2 \mid y \in \mu(x, +, q(x, +)), x \in \mu(y, -, q(y, -))\} \). Let \( \mathcal{M} \) be the set of all mentorships of the game. A mentorship is solo if \( q : N \times F \rightarrow \{+\} \). Within preference system of a game, the notion of its stable matching or mentorship is constructed. A matching \( \mu \) is individual rational if there does not exist any \( x \in N \), for any \( f \in \{+,-\} \) such that there exists
A matching $\mu$ is pairwise stable if there does not exist any pair $x, y \in N$ and $y \notin \mu(x, f(q(x, f)))$ and $x \notin \mu(y, g(q(y, g)))$, for $f, g \in \{+, -\}$ such that there exists $y' \in \mu(x, f(q(x, f)))$ and $x' \in \mu(y, g(q(y, g)))$, such that $y \succ_{x} f x y'$ and $x \succ_{y} g y x'$.

A matching or mentorship is stable if it is both individual rational and pairwise stable.

**Theorem 1.** [The Existence Theorem] The set of stable mentorship is not empty.

**The proof of theorem 1.** This is a corollary of Proposition 1. It could be showed by the offer-deferred-acceptance algorithm when $q \geq 2$.

**Proposition 3.** The set of stable matching indicated by the offer-deferred-acceptance algorithm is not empty when $q \geq 0$.

In the practice with the offer-deferred-acceptance algorithm, $i \in N$ gives $q^i - 1$ offers to others and leave 1 for receiving offers from others.

**Example 3.1.0. College Admission**

The game of college admission proposed by Gale and Shapley (1962) is a special case of mentorship game in which let colleges to be a part of agents with only + function and their quotas $q(., +) > 1$, and applicants to be other part of agents with only $-$ function and their quotas $q(., -) = 1$.

2.5 A Preliminary: Directed Graph

A graph is a couple tuple $G = (N, E)$, where $N$ is the set of nodes or vertices, and $E$ is a set of edges, which are pairs of elements of $N$. In a graph $G$, two vertices $u, v \in N$ constructs an edge if and only if $uv \in E$.

A directed graph, or digraph is a pair $G = (N, A) = (N, H \times T)$ where $N$ is a finite set of nodes $N = \{1, ..., n\}$ and $A$ is the set of directed arcs. For a directed arc $a \in A$ is an ordered pair $(\alpha, \beta)$, $\alpha = head(a) \in H \subset N$, $\beta = tail(a) \in T \subset N$. The direction of $a \in A$ is its functional order from head to tail.

For $N, A$ is a real-valued $n \times n$ matrix

$$A(n, n) = \begin{pmatrix}
    a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
    a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{pmatrix} \quad (2.2)$$

A digraph $G$ could be presented by a $n \times n$ real-valued matrix $A(n, n)$, that is there exists an one to one mapping $f$ such that $f : G \rightarrow A(n, n)$ Where $a_{ij} = 1$, 0 represents the directed relation from two nodes $i, j \in N$. $\forall i, j \in N, a_{ij} = 1$ if there is $(i, j) \in A$, an arc between the pair of nodes with the direction from $i$ to $j$. $\forall i, j \in N, a_{ij} = 0$ if $(i, j)$ is not an arc. A pair arcs $a, b \in A$ connected at a node $i \in N$ if the node $i$ is an element in both $a$ and $b$. A pair of arcs connected at heads (or at tails) if $a_{ij} = a_{k,j} = 1$ (or $a_{j,i} = a_{k,i} = 1$), a pair of arcs connected in path if $a_{i,j} = a_{k,i} = 1$. 


2.6 Mentorship and its Network

A solo mentorship matching is a map \( m \), such that

\[
m : \{N, \succ\} \rightarrow \{N, M^+ \times M^-\},
\]

where \( m \in M \) is an ordered pair \((i, j)\), where its mentor \( i \in M^+ \subseteq N \), its mentee \( j \in M^- \subseteq N \).

A mentorship matching is is a mapping \( m, m : \{N, q, \succ\} \rightarrow \{N, D\} = \{N, D^+ \times D^-\} \), an element \( d \in D \) is an ordered pair \((i, j)\), where its head \( i \in D^+ \subseteq N \), its tail \( j \in D^- \subseteq N \).

In a matching, a mentorship is an ordered pair \( m = (i, j) \), i.e., \( \text{mentor}(m) = i \) and \( \text{mentee}(m) = j \). The direction of a mentorship is its order of roles, from mentor to mentee. The direction of a mentorship implies that the relation or interaction between mentor \( i \) and mentee \( j \) is orientational, for example, in a mentorship, its mentor \( i \) is issuer +, its mentee \( j \) is receiver -. A direction of a mentorship from its mentee to mentor is negative since its order of roles is reversed.

The directional property of mentorship matching is order of function of each matched pair: positive direction of from mentor to mentee, negative direction from mentee to mentor.

The connectivity of mentorship matching is that one agent could be a mentor in a matched pair and a mentee in other pair.

**Proposition 4.** A mentorship is equivalent to a network which could be presented in a digraph.

\[
\begin{align*}
M & \xrightarrow{N,F,q} D(N,A) \\
\text{Network} & \quad N,A
\end{align*}
\]

A network is a set of relations of its elements. Matching is to construct relations between elements. For a network, there is a matching to formate it. And a matching formats a network. \( \mu : N, F, q(\ldots) \leftrightarrow \text{network} \). A graph is a structural presentation. A matching is drawing. For a graph, there exists a matching to work out it. A matching corresponds a graph. \( \mu : N, F, q(\ldots) \leftrightarrow D(N,A) \). A network is equivalent to a graph. \( D(N,A) \leftrightarrow \text{network} \).

**Proposition 5.** The network of a mentorship is circle if and only if the matching is stable.

Let \( M = \{N, q, \succ\} \) to be a dual matching game, \( D = (N, A) \) is its dual digraph where the node set is coincides with the agent. A network \( A^* \) in digraph \( D \) is formatted by a stable matching of dual game, there is stable matching function \( f \) such that \( f(M) = A^* \).

Stable matching mapping is a mapping \( f, f : M \rightarrow D \), or \( f : N \rightarrow A \) such that satisfies \( q \) conditions and stable conditions.

If a digraph \( A^* \) is formatted by a stable matching, i.e., \( A^* = f(M) \), the \( a_{i,j} = 1 \), the directed edge from \( i \) to \( j \) implies the pair of \( i \) in + function and \( j \) in - function in the matching.

For a stable matching, \( q_i^+ \) is the non-zero numbers of column for \( i \), \( q_i^- \) is row number of non-zero for \( i \).
Example 2.4.1: The Romantic Cycle Bearman et al (2004) studied the romantic relationship in students of high school based on the Add Health data set and found a cycle structure of the romantic relationship. Employing the functional matching game in which $m^i : q^+_i = 2, q^-_i = 0; w^i : q^+_i = 0, q^-_i = 2$, its stable matching could generate the network structure of the romantic cycle.

2.7 Its Applications and Related Literatures

Let’s study two special cases of matching of mentorship: auction and marriage games. In acquisition or art markets, auction is the mechanism to match a target seller with one of a group of potential buyers.

An auction problem: $N = \{s\} \cup B \cup \emptyset$, for $i \in B$, its preference $\succ_i$ represented by its bid $x^i \in [v^i, 0]$. Let $X = \{x^i\}_{i \in B}$, The preference of seller $s$ on $N$ is presented as $\max\{\max(X), v_s\}$.

An auction could be designed like the blind auction or Vickrey auction.

Marriage matching game is a matching of mentorship in which its group of agents divided into bipartite: mentors and mentees.

The functional matching is a generalization from the cases of college admission, as well stable Network and flow problems.

3 Directed Matching and Organizational Structure

3.1 On Organizational Structure and Directionality

In addition to connectivity, directionality is other inherent properties of organizational structure and network. Directionality is the orientational profile of organization and network in which there are orders or directions of their functions. Since network is an organization of organizations, we will explore the implication of directionality of organization and network through focusing organizational structures.

An organization, as a network with boundary, exhibits its internality and externality. Its externality includes its external functions such as business for corporation, its interface within its environments such as boundary for club, its organizational behavior such as movements between governments, and so on. Organizational internality includes its structure, operation, and culture, and so on. In this paper, we focus organizational structure.

There are two classic schools on organizational structure: bureaucracy and organic organization.

3.1.1 On Bureaucracy

The school of bureaucracy or mechanistic organization leaded by Weber. Weber (1948) states that the fully developed bureaucratic mechanism compares with other organizations exactly as does the machine compare with the non-mechanical modes of production. Precision, speed, unambiguity, strict subordination, reduction of friction and of material and personal costs- these
are raised to the optimum point in the strictly bureaucratic administration. Bureaucratic structures and mechanistic organization carries tow properties: hierarchy and specialization of functions.

A hierarchy is an arrangement of functions of units of an organization into related levels of different weights ranks, which is captured by their functional orders or directions. To be a language to deal with a organization with hierarchy and specialization of functions, matching theory should incorporate the directionality through introducing measures of functional orders or directions into its models.

3.1.2 On Organic Organization

The theory of organic organization was proposed by Burns and Stalker (1961). An organic organization is a flat structure, as opposed the hierarchy in mechanistic organization, in which the agents should have equal levels, without special classification. Some of modern marriages are organic.

A theory of matching should be introduced to place the organic organization with its flat structure and no pervious partition.

In the pervious section, we propose the concept of connectivity of matching through introducing a mentorship. To explore the directionality of organizations and networks, a natural evolution of matching theory is to extend its models of two functions to the models with arbitrary size of functions and directions. First, we revisit the problem of mentorship and introduce a directed mentorship matching. Second, we introduce the concept of directed hypergraph as a preliminary language. Third, we propose a multi-dimensional directed matching in which organizations are matched due to its directional set. A matching, a special case of directed matching, is a game in which organizations are matched due to its functional set.

3.2 On Directed Mentorship

In pervious two sections, the direction or order of functions of mentorship is determined, from mentor to mentee. However, in real life, structure of organization has various hierarchies. In addition to the hierarchy from mentor, seller, to mentee, buyer, there other two structures: reversed hierarchy from mentee, buyer to mentor, seller, equal (flat) structure, no order among mentor (seller) and mentee (buyer).

A directed mentorship game is a 5-tuple \( \{N, F, R, q, \succ\} \), consisting of

(a) an agent set \( N = \{1, \ldots, i, \ldots, n\} \); (b) a functional set \( F = \{f_1, f_2\} \), where every agent could play two functions, \( f_1 \) and \( f_2 \); (c) a directional collection \( R = \{r_0, r_1, r_2\} \), where there are three directions: \( r_1 = (f_1, f_2) \): the head function \( h(r_1) = f_1 \), the tail function \( t(r_1) = f_2 \); \( r_2 = (f_2, f_1) \): the head function \( h(r_2) = f_2 \), the tail function \( t(r_2) = f_1 \) and \( r_0 = (f_1, f_2) \), where \( (f_1, f_2) \) implies an indifferent order. (d) Let’s introduce quota system \( q \) to measure the capability of roles each agent can play in each function, \( q : N \times F \rightarrow \mathbb{Z}_2^+ \). For \( i \in N \), there a pair of quotas in two functions \( q_i = (q_{f_1}, q_{f_2}) \). A game has a \( 2 \times n \) quota matrix \( q = (q_{f_1}, q_{f_2}) = \{q_i^{f_1}, q_i^{f_2}\}_{i \in N} \).
Here there is no previous quota allocation on directions, but for \( i \in N \), and \( f \in F \), \( \sum_{r \in R} q^i_f(r) \leq q^i_f \). (e) For \( N \) and \( R \),

\[
N(R) = (N, R) = \{(r, x)\}_{x \in N}.
\] (3.1)

Let \( \lambda(R) \) or \( \lambda \) to be a strict order on \( N(R) \) and \( \Lambda(R) \) or \( \Lambda \) to be the collection of all strict orders on \( N(R) \). In a directed mentorship game, for any agent \( i \in N \), its preference is not separable on the three directions or orders of functions due to the assumption that the directions are alternatives to it here. The separability of preference on directions will be studied later.

The preference system of the game is a mapping \( \succ : N \times F \rightarrow \Lambda(R) \). For \( i \in N \), if he plays as a mentor, he has a mentor or positive preference system \( \succ^+_i \) on \( N(R) \), which is a strict order of \( N(R) \) as its potential mentees \( y \in N \) with directions \( r \in \{r_0, r_1, r_2\} \). In mentee or negative preference system \( \succ^-_i \), every \( i \in N \), as a mentee, has a preference \( \succ^-_i \) on \( N(R) \), which is a strict order of \( N(R) \) as its potential mentors \( y \in N \) with directions \( r \in \{r_0, r_1, r_2\} \).

The notion of matching is developed for a directed mentorship game without its preference system. The map \( \mu(\ldots, q(\ldots)) : N \times F \times R \rightarrow N \) is called a directed mentorship (matching) if for all \( x \in N \), \( f \in F = \{f_1, f_2\} \), \( r \in R = \{r_0, r_1, r_2\} \), \( \mu(x, f, r, q(x, f)) \subseteq N \) and \( \mu(x, f, r, q(x, f)) = q(x, f, r) \) and \( \sum_{r \in R} q(x, f, r) \leq q(x, f) \); \( \forall y \in \mu(x, f, r, q(x, f)) \), \( \forall z \in \mu(x, g, t, q(x, g)) \), \( f, g \in F \), \( f \neq g \), and \( r, t \in R \) (\( r = t \), or \( r \neq t \)), \( x \in \mu(y, g, r, q(y, g)) \), \( x \in \mu(z, f, r, q(z, f)) \), \( N \) is the collection of all subsets of \( N \), \( x \in N \) is self matched if both \( x \in \mu(x, f, r, q(x, f)) \), and \( x \in \mu(x, g, r, q(x, g)) \).

For a directed mentorship (matching) \( \mu \), \( \mu(N) \) is the set of its all mentor teams or matched partners with their directions, \( \mu(N) = \{(x, y)_r \in N^2 \mid y \in \mu(x, f, r, q(x, f)) \} \). \( \forall \alpha = (x, y)_r \in \mu(N) \), \( \mu^2(\alpha) = \alpha \).

Let \( \mathcal{M}(R) \) to be the set of all directed mentorships of the game.

Within preference system of a game, the notion of its stable directed matching or mentorship is constructed. A matching \( \mu \) is individual rational if there does not exist any \( x \in N \), for any \( f \in F = \{f_1, f_2\} \), \( r \in \{r_0, r_1, r_2\} \) and \( x \notin \mu(x, f, r, q(x, f)) \), there exists \( y \in \mu(x, f, r, q(x, f)) \) such that \( x \succ^+_x y \). A matching \( \mu \) is pairwise stable if there does not exist any pair \( x, y \in N \) and \( x \notin \mu(x, f, r, q(x, f)) \) and \( x \notin \mu(y, g, r, q(y, g)) \), \( f, g \in \{f_1, f_2\} \) such that there exists \( y' \in \mu(x, f, r, q(x, f)) \) and \( x' \in \mu(y, g, r, q(y, g)) \), such that \( y \succ^+_x y' \) and \( x \succ^+_y x' \), if there does not exist \( r, t \in R \) such that \( \alpha(t) \in \mu(N) \) and \( \alpha(r) \notin \mu(N) \), but \( y(r) \succ^+_y y(t) \) and \( x(r) \succ^+_y x(t) \).

A directed mentorship is stable if it is both individual rational and pairwise stable.

A directed mentorship with a direction \( R \) implies the hierarchy of mentorship. \( R^1 = \{f_1, f_2\} \) indicates a mentor is the leader; \( R^2 = \{f_2, f_1\} \) indicates a mentee is the leader; and in \( R^0 = \{f_1, f_2\}_0 \), both mentor and mentee are in equal positions.

**Example 3.1:** A game of 3-agent, 2-function, 3-direction As a comparison, games of 3-agent, 2-function and 1-direction are presented as follows. For example, when its direction is given in \( R^1 = \{f_1, f_2\} \), its preference system is showed in example 2.1.1; when its direction is given in \( R^2 = \{f_2, f_1\} \), its preference system is showed in example 2.1.2; when its
direction is given in flat $R^0 = (f_1, f_2)_0$, its preference system is shown in example 2.1.3.

A game of 3-agent, 2-function, 3-direction is $\{N, F, R, \succ\}$, where an agent set $N = \{\alpha, \beta, \gamma\}$, a functional set $F = \{f_1, f_2\}$, and its directional collection $R = \{r_1, r_2, r_0\}$, and $r_1 = (f_1, f_2)$ indicates a mentor is the leader; $r_2 = (f_2, f_1)$ indicates a mentee is the leader; and in $r_0 = (f_1, f_2)_0$, both mentor and mentee are in equal positions. The preference system $\succ = \{\succ_i\}_{i \in \mathbb{N}} = \{(\succ_i^f)_{f \in F}\}_{i \in \mathbb{N}}$, i.e., for $i \in \mathbb{N}$ and $f \in F$, there exists a strictly preference order $\succ_i^f$ on

$$
\begin{pmatrix}
\text{Agent} & \text{Preference order} \\
\alpha(f_1) & \beta(r_1) & \gamma(r_1) & \gamma(r_2) & \cdots \\
\beta(f_1) & \gamma(r_2) & \alpha(r_2) & \alpha(r_0) & \cdots \\
\gamma(f_1) & \alpha(r_0) & \beta(r_0) & \alpha(r_2) & \cdots \\
\alpha(f_2) & \beta(r_1) & \gamma(r_0) & \gamma(r_2) & \cdots \\
\beta(f_2) & \gamma(r_2) & \alpha(r_1) & \alpha(r_0) & \cdots \\
\gamma(f_2) & \beta(r_2) & \beta(r_0) & \alpha(r_2) & \cdots
\end{pmatrix}
$$

(3.2)

A stable directed mentorship is $\{(\alpha, \beta)_1, (\beta, \gamma)_2, (\gamma, \alpha)_0\}$, in which, first matched pair implies: $\alpha$ is its mentor, $\beta$ is its mentee, its direction is $r_1$, i.e., mentor first and mentee second.

**Theorem 2.** [The Existence Theorem] Every directed mentorship game has a stable matching.

**Proof of the Theorem 2.** A directed mentorship is transformed into mentorship through rearranging its preference system of iii, iv., and v into a strict preference $\succ^i, \succ^j$. This is a corollary of theorem 1.

### 3.3 Directed and Functional Matching

The theory of mentorship will be generalized in two directions. First, two functions of a mentorship could extend to any size of functions, a functional matching. Second, to correspond the directionality carried by hierarchy of organization and network, we construct the framework of directed matching through extending the directed mentorship. Functional matching is a special case of directed matching without directionality.

**Definition 2.** [Directed Matching] A directed matching is a 5-tuple $G = \{N, F, R, q, \succ\}$. Where (a) $N = \{x_1, ..., x_i, ..., x_n\}$ is a set of agents or players with $n = |N|$; (b) $F = \{f_1, ..., f_l\}$ is a set of functions with its size $l = |F|$. Let’s define a functional operation $F \cdot N = \times_{f \in F} N_f \equiv N^F$, where $N_f$ is the agent set in function $f$. $F^{-1} : \times_{f \in F} N_f \rightarrow N$ such that for a team $\alpha \in \times_{f \in F} N_f$, $f^{-1}(\alpha) = x$ implies $x \in N$ is in $f$ function of the team, $\forall f \in F$. (c) $r(F)$ is a functional order or direction of $F$, $R(F)$ to be a directional set of all directions of $F$, $l = |F|$, $R(F) = \{r_j \mid j = 0, ... l!, \}$, $|R(F)| = l! + 1$. 

15
Let’s introduce quota system $q$ to measure the capability each agent can play in each function, $q : N \times F \rightarrow \mathbb{Z}_+^{n\times l}$. For $i \in N$, there are quotas in $l$ functions $q_i = \{q^{i,f}_l\}_{f \in F}$. A game has an $l \times n$ quota matrix $q = \{q^{i,f}_l\}_{i \in N, f \in F}$. Here there is no previous quota allocation on directions, but for $i \in N$, and $f \in F$, $\sum_{r \in R(F)} q^{i,f}_l(r) \leq q^i$. For $N^F$ and $R(F)$, $N(R) = (N^F, R(F)) = \{xR\}_{x \in N^F}$. Let $\lambda(R)$ or $\lambda$ to be a strict order on $N(R)$ and $\Lambda(R)$ or $\Lambda$ to be the collection of all strict orders on $N(R)$. The preference system of the game is a mapping $\triangleright: N \times F \rightarrow \Lambda(R)$.

Discussion on directed matching game. Functions, for example, $m = 4$, $F = \{investor, CEO, engineer, worker\}$, the president and its cabinet $\{p, s_1, ..., s_{25}\}$, where $s_1$ is the secretary of state. Direction. For a pair of ordered sets: $\{..., f_i, f_j, ...,\} \neq \{..., f_j, f_i, ...,\}$, for instance, a functional set of investor, entrepreneur, and worker. The ordered set $\{I, E, W\}$ is different from the other ordered one $\{E, I, W\}$: where $S_0$ is without functional order or indifference, e.g., $S_0 = \{I, E, W\}_0$. Organization $\alpha$ is a matched team introduced in the notion of directed matching, which is represented by three-tuple $\{N(\alpha), F(\alpha), r(\alpha)\}$, $F(\alpha)$ is its set of functions, $N(\alpha)$ is its set of agents, an organization is normal if $F^{-1}(\alpha) \neq \emptyset$, and for any pair $f, g \in F$, $f^{-1}(\alpha) \neq g^{-1}(\alpha)$. An organization is one of the smallest normal organizations if it is normal and its cardinalities of functional set and agent set coincide, i.e., $|N(\alpha)| = |F(\alpha)|$, $r(\alpha)$ is its direction or functional order, which measure its hierarchy. An organization $\alpha$ (if it is not agent) has a collection of its subsidiaries or sub-organizations $S(\alpha)$, for an subsidiary $\beta \in S(\alpha)$ has its three-tuple $\{N(\beta), F(\beta), r(\beta)\}$. For an agent $\beta$, as an elementary organization, its function is equal to its direction. For a subsidiary or an agent in an organization, they have some free directions or functions which are not used. When $q : N \times F \rightarrow 1$, a normal organization $\alpha$, has $|N(\alpha)| = |F(\alpha)|$ of free functions, i.e., each agent has one. A self organization has one given direction, indifference and $l!$ of free directions.

The notion of directed matching is introduced without its preference system. A correspondence or set map $\mu(\ldots, q(\ldots)) : N \times F \times R(F) \rightarrow N^{F^\{f\}} - 1$ is a directed matching if $\forall x \in N, f \in F, r \in R(F), \mu(x, f, r, q(x, f)) \in N^{F^\{f\}}(r) \cup \emptyset$. Let $q(x, f, r) = |\mu(x, f, r, q(x, f))|$, $\mathbf{1}^{F^\{f\}} = \sum_{g \in F^\{f\}} g(\mu(x, f, r, q(\alpha, f)))|$, and $\sum_{r \in R(F)} q(x, f, r) \leq q(x, f)$; $\forall y_g \in g(\mu(x, f, r, q(x, f)))$, for all $g \in F^\{f\}$, then $x \in f(\mu(y_g, g, r, q(y_g, g)))$, $\forall f \in F$, $r(\mu(S)) = x \in N_f$. $N$ is the collection of all subsets of $N, N^l = N \times \ldots \times N; x \in N$ is self matched in $r$ if $x^{F^\{f\}} \in \mu(x, f, r, q(x, f))$, $\forall f \in F$. $x \in N$ is self matched if it is self matched in $r \forall r \in R(F)$.

For a directed matching $\mu$, $\mu(N, R)$ is the set of its all matched teams or organizations with their directions, $\mu(N, R) = \{\alpha(r) \in N^l, r \in R \mid \alpha^{F^\{f\}}(r - 1) \in \mu(f^{-1}(\alpha), f, r, q(f^{-1}(\alpha), f)), \forall f \in F\}$, where $\alpha^{F^\{f\}}(r - 1)$ is a sub-organization of $\alpha(r)$. For a directed correspondence $\mu$, we can construct a vector correspondence $\nu \equiv \{\mu_f\}_{f \in F}$ such that $\nu^2(\alpha) = \alpha$ if and only if $\alpha \in \mu(N)$. Let $\mathcal{M}(R)$ to be the set of all directed matchings.

Within preference system of a game, the stability of directed matching is constructed. A matching $\mu$ is individual rational if there does not exist
any \( x \in \mathcal{N} \) such that \( x^i \notin \mu(\mathcal{N}) \), but for \( f \in F \) and \( r \in R(F) \) and there exists \( \beta \in \mu(x, f, r, q(x, f)) \in N^r \setminus \{f\} \) such that \( x \succ \beta g(\beta), \forall g \in F \setminus \{f\} \). A matching \( \mu \) is **coalitional stable** if there does not exist a subset \( S \subset \mathcal{N} \) that \( \beta(S, r) \notin \mu(\mathcal{N}, \mathcal{R}) \), where \( \beta(S, r) \) is an organization making of \( S \) with in direction \( r \in R. \forall f \in F \), and \( \forall g \in F \setminus \{f\} \) such that there exists \( y(g, t) \in g^{-1}(\mu(f^{-1}(\beta), f, t, q(f^{-1}(\beta), f))) \), such that \( g^{-1}(\beta(S, r)) \succ \beta y(g, t), \forall f \in F, t \in R \). The coalitional stability implies that no one can be better offer through their coalition of restructuring, redirection or both.

A directed matching is **stable** if it is both individual rational and coalitional stable. A directed matching \( m : \mathcal{G} \to \mathbb{H} \) is **stable** if it cannot be improved upon either any individual or any sub-group of agents.

**Example** A 4-agent, 3-function, 7-direction, \( q : \mathcal{N} \times F \to 1 \). \( \mathcal{N} = \{\alpha, \beta, \gamma, \delta\}, F = \{f_1, f_2, f_3\}, R = \{r_1, r_2, r_3\} \). For an agent \( i \in \mathcal{N} \) with a function \( f \in F \), its space of partners \( (N^2, R) \) consisting of \( 4 \times 4 \times 7 = 112 \) of elements. Given \( q_i^j = 1, \forall i \in \mathcal{N}, \forall f \in F \), a matching consists of at most 4 full matched teams or organizations.

\[
\begin{pmatrix}
\text{Agent} & \text{Preference} & \text{order} \\
\alpha(f_1) & \beta(\gamma(r_1)) & \gamma(\delta(r_4)) & \delta(\gamma(r_5)) & \cdots \\
\beta(f_1) & \gamma(\delta(r_0)) & \delta(\alpha(r_6)) & \alpha(\gamma(r_0)) & \cdots \\
\gamma(f_1) & \delta(\alpha(r_2)) & \delta(\beta(r_0)) & \delta(\alpha(r_3)) & \cdots \\
\delta(f_1) & \alpha(\beta(r_3)) & \gamma(\beta(r_4)) & \alpha(\gamma(r_5)) & \cdots \\
\alpha(f_2) & \delta(\beta(r_3)) & \gamma(\delta(r_4)) & \delta(\gamma(r_5)) & \cdots \\
\beta(f_2) & \alpha(\gamma(r_1)) & \delta(\alpha(r_6)) & \alpha(\gamma(r_0)) & \cdots \\
\gamma(f_2) & \beta(\delta(r_0)) & \delta(\beta(r_0)) & \delta(\alpha(r_3)) & \cdots \\
\delta(f_2) & \gamma(\alpha(r_2)) & \gamma(\beta(r_4)) & \alpha(\beta(r_5)) & \cdots \\
\alpha(f_3) & \gamma(\delta(r_2)) & \gamma(\delta(r_4)) & \delta(\gamma(r_5)) & \cdots \\
\beta(f_3) & \delta(\alpha(r_3)) & \delta(\alpha(r_6)) & \alpha(\gamma(r_0)) & \cdots \\
\gamma(f_3) & \alpha(\beta(r_1)) & \delta(\beta(r_0)) & \delta(\alpha(r_3)) & \cdots \\
\delta(f_3) & \beta(\gamma(r_0)) & \gamma(\beta(r_4)) & \alpha(\gamma(r_5)) & \cdots \\
\end{pmatrix}
\] (3.3)

A stable directed matching is \( \{ (\alpha, \beta, \gamma), (\delta, \alpha, \beta), (\gamma, \delta, \alpha), (\beta, \gamma, \delta) \} \), in which, first matched team with first direction \( (f_1, f_2, f_3) \) has: \( \alpha \) in \( f_1 \), \( \beta \) in \( f_2 \), \( \gamma \) in \( f_3 \). The connections between two matched organization are a subset of two common agents with different functions and directions.

**Proposition 6.** For a game with a common orientational preference, in which \( \forall T \in F, \forall i \in S \subset \mathcal{N} \), their preferences \( >_T \) coincide on \( S \), its set of stable matching is non-empty.

Let’s introduce the notion of functional matching as a special case of directed matching. A **functional matching** is a \( 4-\)tuple \( \mathcal{G} = \{N, F, q, >\} \). Where (a) \( N = \{x_1, \ldots, x_n\} \) is a set of agents or players with \( n = |N| \); (b) \( F = \{f_1, \ldots, f_m\} \) is set of functions with \( m = |F| \), for example, \( m = \)}
4, \( F = \{ \text{investor, CEO, engineer, worker} \} \), the president and its cabinet \( \{ p, s_1, \ldots, s_{25} \} \), where \( s_1 \) is the secretary of state.

(c) Let’s introduce quota system \( q \) to measure the capability each agent can play in each function, \( q : N \times F \to \mathbb{Z}_+^m \). For \( i \in N \), there are quotas in \( m \) functions \( q_i = (q_i^f)_{f \in F} \). A game has a \( m \times n \) quota matrix \( q = (q_i^f)_{i \in N} \).

(d) Let \( \lambda \) to be a strict order on \( N \) and \( \Lambda \) to be the collection of all strict orders on \( N \).

The preference system of the game is a mapping \( \succ : N \times F \to \Lambda \).

For a directed game \( G \), its matching is a map of the game into a directed hypergraph, that is \( m : G \to H \). The matching is \( M(n, f) = (N, F, R, A) \), where \( F \) is a collection of subsets of \( F \),

\[
A = \{ A(S) = \prod_{i \in S} A_i \subset N^S \mid A_i \subseteq N, S \in F \} \tag{3.4}
\]

### 3.4 Another Preliminary: Directed Hypergraph

A hypergraph is an ordered set \( H = (N, \mathcal{E}) \) such that \( N \) is a set (of vertices, nodes of \( H \)) and \( \mathcal{E} \) is a family of subsets of \( V \) (called edges of \( H \)). The rank of \( H \) is the cardinality of the largest edge of \( H \). The number of vertices in \( H \) is its order.

Let \( F = (1, \ldots, f) \) be a set of colors. \( f = |F| \) is the size of color set. When \( f = 2 \), \( F \) has two elements: head and tail. \( F \) may be a set of fix colors: red, yellow, blue, green, white and black.

A directed hypergraph is a four tuple \( H(n, f) = (N, F, \mathcal{F}, A) \), where \( \mathcal{F} \) is a collection of subsets of \( F \), For example, \( F = (1, 2, 3, 4) \), \( \mathcal{F} = \{(1, 2, 3), (1, 3), (2, 4), (1, 2, 3, 4)\} \).

\[
A = \{ A(T) = \prod_{i \in T} A_i \subset N^T \mid A_i \subseteq N, \forall T \in \mathcal{F} \} \tag{3.5}
\]

A hyperarc \( a \in \prod_{i \in T} A_i \) be an ordered set of vertices positioned following the color order of \( T \). The direction (orientation) of a directed hyperarc is the color order of its vertices. In this paper, we assume that for any directed hyperarc, the size of its vertices coincides with the number of its colors, i.e., for any directed hyperarc \( a \), \( v(a) = f(a) \). In general, if \( f < n \), its directions is \( C_n^f f! \). In a lot of literature, only case of \( f = 2 \) is studied. \( t \) number of vertices and colors could generate \( t! \) of hyperarc with different directions. For example, if \( f = 2 \) or \( F = (h, t) \), a directed hyperarc \( a = (x, y) \) implies that \( x = \text{head}(a) \), and \( y = \text{tail}(a) \); and \( b = (y, x) \) is reverse direction with \( y = \text{head}(b) \), and \( x = \text{tail}(b) \). For \( f = 3 \) or \( F = \{ \text{yellow, green, blue} \} \), the set of vertices \( N = \{1, 2, 3\} \) could generate \( 3! = 6 \) of directed hyperarcs \((1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1)\).

A directed hypergraph is perfect to \( N \) and \( F \) if \( \mathcal{F} = \{ F \} \) and \( \forall i \in F \) \( A^i = N \).

Let \( G(N, F) = \{ G(N, T) \}_{T \subseteq F} \), \( G(N, T) = matrix_{n!} \). For \( i = 2, matrix_{n^2} \) is a \( n \times n \) matrix, \( i = 3, matrix_{n^2} \) is a \( n \times n \times n \) matrix, \( i = j, matrix_{n^3} \) is a \( n^2 \) matrix.
A directed hypergraph \( H(n, f) = (N, F, \mathcal{F}, \mathcal{A}) \) could be written in \( G(N, F) = \{G(N, T)\}_{T \subseteq \mathcal{F}} \).

For a directed game \( G \), its matching is a map of the game into a directed hypergraph, that is \( m : G \to H \). The matching is \( M(n, f) = (N, F, \mathcal{R}, \mathcal{A}) \), where \( \mathcal{F} \) is a collection of subsets of \( F \).

\[
\mathcal{A} = \{A(R) = \prod_{i \in \mathcal{R}} A^i \subseteq N^R \mid A^i \subseteq N, \forall R \in \mathcal{R}\}
\]

(3.6)

A directed matching \( m : G \to H \) is stable if it cannot be improved upon either any individual or any sub-group of agents.

Remarks: Difference between directed matching and hypergraph.

- In concept of directed hypergraph, The directional set \( \mathcal{R} = (1, 2, 3, \ldots, r) \) and its orientational collection \( \mathcal{R} \) are ordered. For instance, \( r = 2 \), (head, tail) two direction, In general for \( \mathcal{R} \), there exists \( r! \) directions.
- In concept of functional matching, \( F = (f_1, \ldots, f_f) \) and its collection \( \mathcal{F} \) are not ordered. For \( N \), there exists \( C^n_f \) of choices.
- In concept of directed matching, there exists a order of function, \( C^n_r r! \).

4 On Organizational Rationality and Matching

4.1 On Organizational Behaviors

Kenneth Arrow (1974):” Organizations are a means of achieving the benefits of collective action in situations where the price system fails”. Herbert A. Simon (1972): ” One plausible distinction between them is that a theory of organizational rationality must treat the phenomena of goal conflict, while a theory of individual rationality need not. This is only partly correct, for goal conflict may be important in individual as in group behavior...”

In this paper, formation of network and restructure of organization are studied together in a framework. Organization, as an internal network with boundary, is not only one of elements of its external network, and also a bridge between other elements in the network. Organizational behavior plays one of very important roles in the process of network formation and organizational restructure.

Organization \( \alpha \) is a matched team introduced in the notion of directed matching, which is represented by three-tuple \( \{N(\alpha), F(\alpha), r(\alpha)\} \). \( F(\alpha) \) is its set of functions, \( N(\alpha) \) is its set of agents, an organization is normal if the cardinalities of functional set and agent set coincide, i.e., \( |N(\alpha)| = |F(\alpha)| \), \( r(\alpha) \) is its direction or functional order, which measure its hierarchy. An organization \( \alpha \) (if it is not agent) has a collection of its subsidiaries or sub-organizations \( S(\alpha) \), for an subsidiary \( \beta \in S(\alpha) \) has its three-tuple \( \{N(\beta), F(\beta), r(\beta)\} \). For an agent \( \beta \), as an elementary organization, its function is equal to its direction. For a subsidiary or an agent in an organization, they have some free directions or functions which are not used.

When \( q : N \times F \to 1 \), a normal organization \( \alpha \), has \( |N(\alpha)| = |F(\alpha)| \) of free functions, i.e., each agent has one.
We will introduce the paradigm of organizational rationality to present
the will and determination of organizations in the theory of matching. Organ-
izational rationality consists of two parts: its self fulfillment and preference
system.

Following the paradigm of organizational rationality, we explore organi-
zational matching between organizations and free agents.

4.2 Paradigm of Organizational Rationality

An organization is self fulfilling if the welfare of the organization is not
less than welfare sum of its any decomposition, even its decomposition of
individuals

\[ S \succ \sum_{i \in S} i \] (4.1)

or its decomposition of subsidiaries or sub-organizations \( T(S) = \{ T^i \subset
S, \dim(T^i \cap T^j) < \min(\dim(T^i), \dim(T^j)) \} \),

\[ S \succ \cup_{T^i \in T(S)} T^i \] (4.2)

The self fulfillment of organizations is one of pretermitted foundations
of stable matching theory. The property implies the phase "One plus one
is more than two". An organization will spin-off or dissolve if it is not self
fulfilling.

A rational organization has its preference system consisting of its or-
ganizational and sub-organizational preferences on counter parties it
interacts. Given its principle of self fulfilling, an organization and its subsi-
diaries behave with their preferences. In this paper, we focus the matching
between organizations and free agents.

Let \( \alpha \) to be an organization with its original direction \( r_0(\alpha) \) with its
functional size \( l(\alpha) \) and \( S(\alpha) \) to be the collection of its all subsidiaries or
sub-organizations. \( R \) is the set of target directions of the matching game.
For a functional subset \( F' \subset F, R(F') \) is its collection of directions generated
by \( F' \); For a direction \( r \in R(F) \), let \( F(r) \) to be its set of function with same
functional size of \( l \). Let \( N \) to be a set of free agents. Let \( N(\alpha, R(F)) =
(N(\alpha), R(F)) = (N^{l-\ell(\alpha)} \times \alpha, R(F)) \), where \( N(\alpha) = N^{l-\ell(\alpha)} \times \alpha = \{ S \in
N^l | N(\alpha) \subset S \} \), \( N(\alpha) \) is a set including all agents of \( \alpha \). \( \lambda \) is a strict order
of \( N(\alpha, R(F)) \), \( \Lambda \) is the set of all strict orders of \( N(\alpha, R(F)) \).

Let \( r_t(\alpha) \) to be a target direction of \( \alpha \) in matching and \( R_t(\alpha) \) to be the
set of all target direction in matching. \( r_0(\alpha) \) may not be in \( R_t(\alpha) \). An
organization \( \alpha \) is directional separable to \( R_t(\alpha) \) if its directional choice is
separable. Otherwise, it is not directional separable. When an organization
is directional separable, its organizational preference is

\[ \succ: A \times R_t(A) \rightarrow \Lambda \] (4.3)

When an organization is not directional separable, its organizational prefer-
ence is

\[ \succ: A \rightarrow \Lambda \] (4.4)
With the development of matching theory, the decision objectives evolve from agents with partition such as men and women to functions of agents such as mentors and mentees, as well directions of organizations. In organizational matching, we assume that an organization is directional separable in general.

For a subsidiary of $\alpha$, $\beta \in S(\alpha)$, let $r_t(\beta)$ to the target direction as a subsidiary of an organization in a matching. $\beta$ is a target set of functions. Let $N(\beta, R(S)) = (N^{r_t(\beta)} \times \beta_t, R(S))$, where $\beta_t$ is a collection of all arrangement of agents of $\beta$ in its target direction $r_t(\beta)$. $\lambda$ is a strict order of $N(\beta, R(S))$, $\Lambda$ is the set of all strict orders of $N(\beta, R(S))$.

The subsidiary preference of $\beta$ with target direction $r_t(\beta)$ is a strict order $\lambda$ on $N(\beta, R(S))$.

An organization $S \subset N$ is rationality if it is self fulfilling and has a rational preference system consisting of its organizational and sub-organizational preference systems.

4.3 On Organizational Matching

4.3.1 Classification of Organizational Play

For an organization, there are two kind of matching plays: structure and sub-structure plays. A free agent, as an element organization, has only structure play in matching game if it does not match with itself.

An organization or an agent is in a structure play if it will match with others, agents, organization, sub-organization, into a new organization, but its original organization could be a sub-organization of the new organization or does not exist as a whole organization anymore. An example od structure play is merger and acquisition between corporations. We will focus the structure play of organizations with free agents in this paper. For a game with its set of organizations $A$ and set of free agents $N$, its structure play is the operation as follows, there exists a couple $\alpha \in A$ and $S \subset N$,

$$\alpha + S \mapsto \beta$$

An organization is in a sub-structure play if it and its sub-organization will match with others, agents, organization, sub-organization, into a new organization, however its original organization still exists and it is not a sub-organization of the new organization. For example, joint venture of corporations is sub-structure play.

For a game with its set of organizations $A$ and set of free agents $N$, for $\alpha \in A$, there exists a collection of all sub-arcs of $\alpha$, denoted $B(\alpha)$. Its sub-structure play is the operation as follows, there exists a couple $\alpha \in A$ and $S \subset N$,

$$\alpha + S \mapsto \alpha \cup \gamma$$

$$\alpha \cap \gamma = \beta \in B(\alpha), \beta \neq \emptyset$$

Where $\beta \in B(\alpha)$ is a sub-organization of $\alpha$, or $\alpha$ itself. For instance, a group of agents may be directors in two firms. It is interesting to explore the relation between two organizations $\alpha$ and $\gamma$ through their common sub-organization $\beta$ in a network.
In this paper, we focus the organizational matching with structure play. Other work on organizational matching will be presented in other paper (Liu (2017)).

4.3.2 Structure Play of Organizations with Free Agents

The structure play of organizations with free agents is illustrated by a solo structure game that a group of same structure organizations play with a group of free agents into a matching of solo structure organizations. A solo structure game is six tuple \( \{A, N, F, r, q, \succ\} \):

(a) \( A \) is a set of organizations with same direction or functional order. For every pair of organizations \( \alpha, \beta \in A \), both have the same size \( |N(\alpha)| = |N(\beta)| \), same functional set \( F(\alpha) = F(\beta) = F(A) \), \( |F(A)| = a \) and direction \( r(\alpha) = r(\beta) = r(A), F(r(A)) = F(A) \). Let \( A \) to be the set of agents of \( A \); (b) \( N = \{1, \ldots, n\} \) is a set of free agents; (c) \( F = \{f_1, \ldots, f_m\} \) is set of functions. \( |F| = l \), \( F(A) \subset F \). (d) \( r \) is a solo target direction that all players are going to reach. Its functional set \( F(r) = F \). Its direction, order of functions \( r = \{r_1 \ldots r_l\} \) is given. Let \( r(A) \) to be a target direction of an organization \( \alpha \in A \) indicated by \( r \) and \( R(A) \) to be the set of all \( r(A) \) indicated by \( r \). (e) \( q \) is functional quota matrix on all agents \( A \cup N \) and functions \( F \). To make our story simple, \( q : A \cup N \times F \to 1 \), i.e., for all \( i \in A \cup N \) could play each function \( f \in F \) once. (f) Let \( \Lambda(A, r) \) and \( \Lambda(N, r) \) to be the sets of all strict orders on \( A \times N^{l-a-1} \) and \( N^{l-a} \) respectively. There are a preference system \( \succ \) consisting of organizational preference of \( \alpha \in A \)

\[
\succ : A \times R(A) \to \Lambda(N, r) \tag{4.8}
\]

and directional preference of \( i \in N \)

\[
\succ : N \times F \to \Lambda(A, r) \tag{4.9}
\]

The notion of organizational matching of structure play is introduced as follows. The map \( \mu : A \times R(A) \to N^{l-a} \) and \( \mu : N \times F \to A \times N^{l-a-1} \) is called a organizational matching of structure play with free agents if

\[
r^{-\alpha}(\mu(f^{-1}(\mu(\alpha, r, t)), r, f)) = \alpha \text{ for all } \alpha \in A, \text{ and } f \in F \setminus F(R(A)),
\]

\[
f^{-1}(\mu(r^{-\alpha}(\mu(x, r, f)), r, t)) = x \text{ for all } x \in N, t \in R(A) \text{ and } f \in F \setminus F(R(A)).
\]

For an organizational matching \( \mu, \mu(r) \) is the set of its all new organizations with direction \( r \), \( \mu(r) = \{(\alpha, x) \in A \times N^{l-a} \mid \mu(\alpha, r, t) = x, r^{-\alpha}(\mu(f^{-1}(x), r, f)) = \alpha, \forall f \in F \setminus F(R(A))\} \). Let \( \mathcal{M} \) to be the set of all matching.

The stability of organizational matching is constructed with the preference system of games. In a solo structure game, an organization \( \alpha \in A \) has a set of its self organizations \( \alpha^* \), where \( \alpha^* = \{N(\alpha), F, r\} \). A matching \( \mu \) is individual rational if (1) there does not exist any \( x \in N \) and \( x^f \notin \mu(r) \), but for \( f \in F \) and \( r \) and there exists \( \beta \in \mu(x, f, r, q(x, f)) \in N^{F \setminus \{f\}} \) such that \( x \succ_{f}^{\beta} g(\beta), \forall g \in F \setminus \{f\} \); and (2) there does not exist a \( \alpha \in A \) and its self-organization \( \alpha^* \notin \mu(r) \), but \( f^{-1} \alpha^* \succ_{f}^{\beta} f^{-1} \mu(\alpha, r, t), \forall f \in F \). A matching \( \mu \) is coalitional stable if (1) there does not exist a subset \( S \subset N \) that \( \beta(S, r) \notin \mu(r) \), where \( \beta(S, r) \) an organization making of \( S \) with in direction \( r \in R, \forall f \in F, \text{ and } \forall g \in F \setminus \{f\} \).
such that there exists \( y(g, t) \in g^{-1}(\mu(f^{-1}(\beta), f, t, q(f^{-1}(\beta), f))) \), such that
\( g^{-1}(\beta(S, r)) \succ_\beta y(g, t), \forall f \in F, t \in R \); and (2) there does not exist a pair \( \alpha \in A \) and \( S \subseteq N, \) and \( \beta(\alpha, S) \notin \mu(r), \) but \( S \succ^1_\alpha \mu(\alpha, r, t), \alpha \succ^1_x r^{-a}\mu(x, f, r), \) and \( S \setminus \{x\} \succ^1_x (F \setminus F(\alpha, t) \setminus \{f\})^{-1}\mu(x, f, r). \) The coaltional stability implies that no one can be better offer through their coalition of restructuring, redirection or both.

An organizational matching is **stable** if it is both individual rational and coaltional stable. The **structure matching** is a mapping
\[
m : \{A, N, F, Q, F, \succ\} \rightarrow B(r)
\]
such that \( \alpha + S \mapsto \beta \) where \( \alpha \in A, S \subseteq N, \) and \( \beta \in B(r). \) Let \( a \) be the direction of \( A, r = a + 1 \) implies that the target direction is one of directions of \( \{a_1, \ldots, a_n, b\}. \)

A one-step solo game is five tuple \( \{A, N, F, r, \succ\}: \)

- \( A \) is a set of organizations with same direction or functional order.
- For every pair of organizations \( \alpha, \beta \in A, \) both have the same size \( |N(\alpha)| = |N(\beta)|, \) same functional set \( F(\alpha) = F(\beta) = F(A), \) \( |F(A)| = a \) and direction \( r(\alpha) = r(\beta) = r(A).F(r(A)) = F(A). \) Let \( A \) to be the set of agents of \( A; \) \( N = \{1, \ldots, n\} \) is a set of free agents; \( c \) \( F = F(A) \cup \{g\} \) is set of functions. \( |F| = |F(A)| + 1. \) For all \( i \in A \cup N \) could play each function \( f \in F \) once; \( d \) \( r \) is a **solo target direction** that all players are going to reach. Its functional set \( F(r) = F. \) Its direction, order of functions \( r = r(A) + 1 \) is given. Let \( r(A) \) to be a target direction of an organization \( \alpha \in A \) indicated by \( r \) and \( R(A) \) to be the set of all \( r(A) \) indicated by \( r. \)

- \( f \) Let \( \Lambda(A, r) \) and \( \Lambda(N \cup A, r) \) to be the sets of all strict orders on \( A \) and \( N \cup A \) respectively. There are a preference system \( \succ \) consisting of organizational preference of \( \alpha \in A \)
\[
\succ: A \times R(A) \rightarrow \Lambda(N \cup A, r)
\]
and directional preference of \( i \in N \)
\[
\succ: N \times F \rightarrow \Lambda(A, r)
\]

**Proposition 7.** A one-step solo game of \( \{A, N, F = F(A) \cup \{g\}, r = r(\alpha) + 1, \succ\} \) has a stable structure matching.

**Proof of the Proposition 7.** Since \( \succ \) is separable on \( R(A) \) and \( F, \) the game could be composed by a series of bipartite games of a pair of \( (t, f), \) for all \( t \in R(A) \) and \( f \in F. \) The existence of stable matching of a bipartite game could be induced by the Gale and Shapley theorem.

**Frozen river phenomena** An organizational game is in **frozen river** if it has a stable matching for structure play, but may not have a stable one for its sub-structure play.

### 5 Universality and Organizing Matching

Until now, all of existence of stable matching we discussed could be be reduced into a bipartite problem that Gale and Shapley solved in 1962. It is a natural issues if there exists a stable matching for every game.
In this section, first we discuss the paradox of stale matching in classic theory. Second, with the paradigm of organizational rationality and matching, we will construct the existence concept of universal stability that the set of stable matching is not empty for every game no matter if it is in two dimensions though developing two schemes: recursive and adaptive matchings.

5.1 On Universality of Stable Matching

Gale and Shapley (1962) showed that there does not exist a stable matching for every game. The paradox could be illustrated as following two examples.

**Example 5.1:** There does not always exist a stable matching for one-dimension (or function) game: the roommate problem proposed by Gale and Shapley (1962).

\[
\begin{pmatrix}
\text{Agent} & \cdots & \text{Preference order} \\
1 & \cdots & 234 \\
2 & \cdots & 314 \\
3 & \cdots & 124 \\
4 & \cdots & \text{Arbitrary}
\end{pmatrix}
\quad (5.1)
\]

Tam (1991) provided a necessary and sufficient condition for the existence of complete stable matching for the roommates problem.

**Example 5.2.1** There does not always exist a stable matching for three-dimension (or function) game.

\[
\begin{pmatrix}
\text{Agent} & \cdots & \text{Preference order} \\
\alpha_1 & \cdots & \beta_1\delta_2 & \beta_1\delta_1 & \beta_2\delta_2 & \beta_2\delta_1 \\
\alpha_2 & \cdots & \beta_2\delta_2 & \beta_1\delta_1 & \beta_2\delta_1 & \beta_1\delta_2 \\
\beta_1 & \cdots & \alpha_2\delta_1 & \alpha_1\delta_2 & \alpha_1\delta_1 & \alpha_2\delta_2 \\
\beta_2 & \cdots & \alpha_2\delta_1 & \alpha_1\delta_1 & \alpha_2\delta_2 & \alpha_1\delta_2 \\
\delta_1 & \cdots & \alpha_1\beta_2 & \alpha_1\beta_1 & \alpha_2\beta_1 & \alpha_2\beta_2 \\
\delta_2 & \cdots & \alpha_1\beta_1 & \alpha_2\beta_2 & \alpha_1\beta_2 & \alpha_2\beta_1
\end{pmatrix}
\quad (5.2)
\]

The nature of the universal matching paradox is due to the failure of transitivity of collective decision from cycle-type of preference system of individual agents, which is equivalent to the Arrow’s impossible paradox.

A natural resolution to the paradox of universal stable matching is to enlarge the capacity of matching games, i.e., if the *quota* is big enough, i.e., there are enough resources the paradox is solved. For the example of four persons roommate problem, for every person, increasing its quota of roommate from one to three i.e., a big room could place all persons together.

In next section, we develop a new paradigm of organizational rationality to explore the paradox of universal stable matching.

5.2 One Functional Matching: the Roommates Problem

Morrill (2010) stated that the traditional notion of stability ignores the key physical constraint that roommates require a room. With introducing of
Pareto optimal, he showed that a Pareto optimal assignment always exists in the roommates problem.

Here, we provide another resolution of Paradox of roommates problem. Give a game without stable matching as the example provided by Gale and Shapley, it is common knowledge that there is no stable matching between persons 1, 2 and 3 due to their cycle preference.

A mechanism, for instance a random choice, could be designed by the community of four boys, one of the four will be named by the mechanism to choose his roommate initially. For example, the fourth boy initial the process and anyone of the three, e.g., person 1 and settles the pair (4, 1), then other two will reach a settle (2, 3). Person 1 could not move off with 3 since he knows, by common knowledge, that any default will trigger a not stable cycle between 1, 2, and 3, thus he will band with organizational rationality.

5.3 Static and Organizing Matching

From the classic paradigm proposed by Gale and Shapley, we have developed theory of matching in three directions: functional matching incorporating the property of connectivity, directed matching incorporating the property of directionality of hierarchy, and organizational matching with organizational rationality and behavior. All of these three models are static as the theory of Gale and Shapley.

The notion of static matching is made of two elements: a collection of all matchings and its refinement through the condition of stability. No matter how it could be figured out by any algorithm, a matching is static, independent of any process.

In social and economic life, a lot of organization and networks organized in an evolution. For a game without stability of static matching, we introduce a notion of organizing matching which is an evolution a stable matching organizes through employing a series of organizations matched in its sub-games with free agents. We will explore the organizing matching with two schemes: recursion and adaption. The stability of organizing matching is path-dependent in their evolutions.

5.4 Organizing Matching with Recursive Scheme

For a game of $G = \{N, F, Q, R, A, \succ\}$ and its directed matching following the definition 1. Following the paradigm of organizational rationality and matching, we construct its existence concept of stable matching through the following recursive scheme:

**Step 1.** Given there exists an efficient algorithm to test if there exists a stable directed matching for this game. If the answer is yes, we have a solution and stop. If the answer is no, let’s go to next step.

**Step 2.** We will reduce the elements of its functional set $F$ one by one and adjust its direction $F$ correspondingly until a sub-game with stable matching is found through using the algorithm. There always exists
a sub-game with stable directed matching following the theorem 4, existence of stable matching for a directed mentorship.

**Step 3** Given stable matching of sub-game, with the paradigm of organizational rationality and matching, we to test if there exists a stable directed matching for this new game with the algorithm. If the answer is yes, we have a solution and stop. If the answer is no, let’s go to next step.

**Step 4** Given the functional set $G$ of a sub-game with stable matching, we will reduce the elements of its functional set $F \setminus G$ one by one and adjust its direction $F$ correspondingly until a new sub-game with stable organizational matching is found through using the algorithm according to the proposition 7, the existence of stable matching between organizations and free agents in its structure play.

**Step 5** We could have a stable organizational matching of the game in its structure play through repeating of step 3 and 4.

**Theorem 3.** An organizing game with a recursive scheme has a stable matching.

**Example 5.2.2.** Let’s shift the 3-marriage game Example 5.2.1 to a game of two agents and three functions for a target direction $\{\alpha, \beta, \gamma\}$. It is well known there is no stable matching for the game of example 5.2.1., there is an organizational preference as follows:

\[
\begin{bmatrix}
\text{Agent} & \cdots & \text{Preference order} \\
\alpha_1 & \cdots & \beta_1 & \beta_2 \\
\alpha_2 & \cdots & \beta_2 & \beta_1 \\
\beta_1 & \cdots & \alpha_2 & \alpha_1 \\
\beta_2 & \cdots & \alpha_2 & \alpha_1
\end{bmatrix}
\quad (5.3)
\]

\[
\begin{bmatrix}
\text{Agent} & \cdots & \text{Preference order} \\
\alpha_1\beta_1 & \cdots & \delta_2 & \delta_1 \\
\alpha_1\beta_2 & \cdots & \delta_1 & \delta_2 \\
\alpha_2\beta_1 & \cdots & \delta_2 & \delta_1 \\
\alpha_2\beta_2 & \cdots & \delta_1 & \delta_2 \\
\delta_1 & \cdots & \alpha_1\beta_2 & \alpha_1\beta_1 & \alpha_2\beta_1 & \alpha_2\beta_2 \\
\delta_2 & \cdots & \alpha_1\beta_1 & \alpha_2\beta_2 & \alpha_1\beta_2 & \alpha_2\beta_1
\end{bmatrix}
\quad (5.4)
\]

The matching of $\{\alpha_1\beta_1\delta_2, \alpha_2\beta_2\delta_1\}$ is its organizational stable.

### 5.5 Organizing Matching with Adaptive Scheme

For a game of $G = \{N, F, Q, R, A, \succ\}$ and its directed matching following the definition 1. Following the paradigm of organizational rationality and matching, we construct its existence concept of stable matching through the following adaptive scheme:

**Step 1.** Given there exists an efficient algorithm to test if there exists a stable directed matching for this game. If the answer is yes, we have a solution and stop. If the answer is no, let’s go to next step.
Step 2. If the target direction $F$ is not flat, we have an order of functions. If $F$ is flat, a mechanism to prioritize the functions could be designed by the players. A naive mechanism is a random one.

Step 3 Following the functional order determined in step 2, the set of stable directed matching is not empty for a sub-game with first and second functions of $F$ according to the theorem 4, existence of stable matching for a directed mentorship. We will find a stable matching with an efficient algorithm.

Step 4 Given the stable matching of sub-game of first two function in step 3, with the paradigm of organizational rationality and matching, there exists a stable directed matching between the sub-organizations and free agents for a target direction $\{f_1, f_2, f_3\}$ according to the proposition 4.

Step 5 We could have a stable organizational matching of the game through repeating of step 4 following the functional order $F$ given in step 1.

Theorem 4. An organizing game with an adaptive scheme has a stable matching.

Example 5.2.3. For the game of Example 5.2.1, its organizational stable matching with adaptive scheme coincides its matching with recursive scheme in Example 5.2.2.

6 Literature Review and Concluding Remarks

6.1 Literature Review

This paper to explore the network formation and its structure based on choice theory in the line of Aumann and Myerson (1988) endogenous formation and adopt the matching theory proposed Gale and Shapley (1962).

6.2 Concluding Remarks

The paper is about two themes, the theory of matching and the structure of organization and network within a language of graph theory.

In this paper, both models of matching and structures of organization and network is presented in terms of graph theory. To study the complicated structures of matching and organization, we employ the notion of directed hypergraph with various directions rather than only one direction in most literatures according to our knowledge.

We develop the matching theory from classic partitive matching to mentorship, functional matching, directed matching and organizational matching though maintaining the connectivity, universality and directionality. Through employing the paradigm of organizational behavior, we overcome one of challenges of classic theory and construct the universal existence that every game could generate a stable matching.

The formation and structure of organization and network is one of motivations of this paper. Organization and network are two faces of one thing.
Organization is a network with boundary, and network is an organization without concerning its boundary. The internality of organization, organizational structure such as hierarchy and flat structure is placed in the theory of matching we developed in this paper. Also, the externality of organization, organizational rationality and matching is introduced in our theory to overcome some of challenges in classic theory. The paradigm of organizational matching does not only make the theoretic beauty, but also it is consistence with the externality of organization, as a social innovation, to deal with conflicts of collective behaviors.

The deferred acceptance algorithm is one of shining achievements in the seminal work of Gale and Shapley. It is not only a breakthrough in computing economics, but also creates an organizing path to matching beyond a static version of our world. In evolution of matching theory, we do not only explore the algorithm as introducing the offer-deferring-acceptance algorithm, and also incorporate organizational behavior, as an organizing engine, to capture nature of the world and our real life.

In the concluding remarks of their seminal paper, Gale and Shaley discussed what mathematics is, the old question. Their ”answer, it appears, is that any argument which is carried out with sufficient precision is mathematical”. Mathematics or theory is beautiful due to its logic or precisely inference. However, as Goethe said, theory is gray, but the golden tree of life is green. Theoretic logic is governed by the law of nature and logic of life. Nature or social life always enlighten our thinking. The evolution of matching theory may be a illustration.

References


