Continuity of Equilibria for Two-Person Zero-Sum Games with Noncompact Action Sets and Unbounded Payoffs

February 16, 2017

Eugene A. Feinberg¹, Pavlo O. Kasyanov², and Michael Z. Zgurovsky³

Abstract

The paper extends Berge’s maximum theorem for possibly noncompact action sets and unbounded cost functions to minimax problems and studies applications of these extensions to two-player zero-sum games with possibly noncompact action sets and unbounded payoffs. It provides the results on the existence of values and solutions for such games and on the continuity properties of the values and solution multifunctions.

1 Introduction

Berge’s maximum theorem provides sufficient conditions for the continuity of a value function and upper semi-continuity of a solution multifunction. This theorem plays an important role in control theory, optimization, game theory, and mathematical economics. The major limitation of the classic Berge’s maximum theorem is the assumption that the sets of available controls at each state are compact. Feinberg et al. [5, 6, 8] generalized Berge’s maximum theorem and related results to possibly noncompact sets of actions and introduced the notions of $\mathbb{K}$-inf-compact functions for metric spaces and $\mathcal{K}_N$-inf-compact functions for Hausdorff topological spaces. These generalizations led to the developments of general optimality conditions for Markov decision processes in Feinberg et al. [7], partially observable Markov decision processes in Feinberg et al. [9], and inventory control in Feinberg [4] and Feinberg and Lewis [10].

This paper presents the results for metric spaces on continuity of the value function and solution multifunctions, when the minimax problem is considered instead of the optimization problem. The results are applied to one-step zero-sum games of two players with possibly noncompact action sets and unbounded payoffs. The rest of this introduction contains definitions and propositions useful for the understanding of

¹Department of Applied Mathematics and Statistics, Stony Brook University, Stony Brook, NY 11794-3600, USA, eugene.feinberg@sunysb.edu
²Institute for Applied System Analysis, National Technical University of Ukraine “Kyiv Polytechnic Institute”, Peremogy ave., 37, build. 35, 03056, Kyiv, Ukraine, kasyanov@i.ua.
³National Technical University of Ukraine “Kyiv Polytechnic Institute”, Peremogy ave., 37, build, 1, 03056, Kyiv, Ukraine, zgurovsm@hotmail.com
the future material. Section 2 presents results relevant to Berge’s maximum theorem for noncompact action sets. Section 3 describes continuity properties of minimax. In particular, Theorem 3.11 is the extension of Berge’s maximum theorem for metric spaces with possibly noncompact action sets and unbounded costs to the minimax. Section 4 presents results on preserving $K$-inf-compactness of a function, when action or state sets are extended to the sets of probability measures on these sets. Section 5 deals with two-person zero-sum games with possibly noncompact action sets and unbounded payoffs. The definitions and preliminary facts for games are introduced in Subsection 5.1. In particular the classes of safe and unsafe strategies are defined, and the value is defined. Of course, in the case of bounded payoffs, all the strategies are safe. Theorem 5.11 of Subsection 5.2 states the existence of the value. Subsection 5.3 introduces sufficient conditions for the existence of solutions for the game. The sufficient conditions, that imply that two players have solutions lead to the situation that one of the players uses compact actions. This is consistent with the approach undertaken in Jaśkewicz and Nowak [12], where the most general available results were obtained for stochastic games with compact action sets and unbounded payoffs, and the optimality conditions for one of the players were provided. Subsection 5.4 describes continuity properties of the value and solution multifunctions for the game. Section 6 clarifies that pure strategies are sufficient for games with perfect information, that is, the second player knows the move of the first player. Therefore, the results of Section 3 describe the properties of solutions for such games.

Let $\mathbb{R} := \mathbb{R} \cup \{ \pm \infty \}$ and $\mathbb{S}$ be a metric space. For a nonempty set $S \subseteq \mathbb{S}$, the notation $f : S \subseteq \mathbb{S} \to \mathbb{R}$ means that for each $s \in S$ the value $f(s) \in \mathbb{R}$ is defined. In general, the function $f$ may be also defined outside of $S$. The notation $f : \mathbb{S} \to \mathbb{R}$ means that the function $f$ is defined on the entire space $\mathbb{S}$. This notation is equivalent to the notation $f : \mathbb{S} \subset \mathbb{S} \to \mathbb{R}$, which we do not write explicitly. For a function $f : S \subseteq \mathbb{S} \to \mathbb{R}$ we sometimes consider its restriction $f|_S : S \subseteq \mathbb{S} \to \mathbb{R}$ to the set $S \subseteq S$. Throughout the paper we denote by $\mathbb{K}(\mathbb{S})$ the family of all nonempty compact subsets of $\mathbb{S}$ and by $S(\mathbb{S})$ the family of all nonempty subsets of $\mathbb{S}$.

We recall that, for a nonempty set $S \subseteq \mathbb{S}$, a function $f : S \subseteq \mathbb{S} \to \mathbb{R}$ is called lower semi-continuous at $s \in S$, if for each sequence $\{s^{(n)}\}_{n=1,2,...} \subset S$, that converges to $s$ in $S$, the inequality $\lim \inf_{n \to \infty} f(s^{(n)}) \geq f(s)$ holds. A function $f : S \subseteq \mathbb{S} \to \mathbb{R}$ is called upper semi-continuous at $s \in S$, if $-f$ is lower semi-continuous at $s \in S$. Consider the level sets

$$D_f(\lambda; S) := \{ s \in S : f(s) \leq \lambda \}, \quad \lambda \in \mathbb{R}.$$ 

The level sets $D_f(\lambda; S)$ satisfy the following properties that are used in this paper:

(a) if $\lambda_1 > \lambda$, then $D_f(\lambda_1; S) \subseteq D_f(\lambda_1; S)$;

(b) if $g, f$ are functions on $S$ satisfying $g(s) \geq f(s)$ for all $s \in S$ then $D_g(\lambda; S) \subseteq D_f(\lambda; S)$.

A function $f : S \subseteq \mathbb{S} \to \mathbb{R}$ is called lower / upper semi-continuous, if $f$ is lower / upper semi-continuous at each $s \in S$. A function $f : S \subseteq \mathbb{S} \to \mathbb{R}$ is called inf-compact on $S$, if all the level sets $\{ D_f(\lambda; S) \}_{\lambda \in \mathbb{R}}$ are compact in $\mathbb{S}$. A function $f : S \subseteq \mathbb{S} \to \mathbb{R}$ is called sup-compact on $S$, if $-f$ is inf-compact on $S$.

Each nonempty subset $S$ of a metric space $\mathbb{S}$ can be considered as a metric space with the same metric. Let $\mathbb{K}(S)$ be the family of all compact subsets of the metric space $S$.

**Remark 1.1.** For each nonempty subset $S \subseteq \mathbb{S}$ the following equality holds:

$$\mathbb{K}(S) = \{ C \subseteq S : C \in \mathbb{K}(\mathbb{S}) \}.$$
Remark 1.2. It is well-known that a function \( f : S \to \mathbb{R} \) is lower semi-continuous if and only if the set \( D_f(\lambda; S) \) is closed for every \( \lambda \in \mathbb{R} \); see e.g., Aubin [1, p. 12, Proposition 1.4]. For a function \( f : S \subseteq \mathbb{R} \), let \( \tilde{f} \) be the function \( f : S \to \mathbb{R} \), defined as \( \tilde{f}(s) := f(s) \), when \( s \in S \), and \( \tilde{f}(s) := +\infty \) otherwise. Then the function \( \tilde{f} : S \to \mathbb{R} \) is lower semi-continuous if and only if for each \( \lambda \in \mathbb{R} \) the set \( D_\lambda(f; S) \) is closed in \( S \).

Let \( X \) and \( Y \) be metric spaces. For a set-valued mapping \( \Phi : X \to 2^Y \), let

\[ \text{Dom } \Phi := \{ x \in X : \Phi(x) \neq \emptyset \}. \]

A set-valued mapping \( \Phi : X \to 2^Y \) is called strict if \( \text{Dom } \Phi = X \), that is, \( \Phi : X \to S(Y) \) or, equivalently, \( \Phi(x) \neq \emptyset \) for each \( x \in X \). For \( Z \subseteq X \) define the graph of a set-valued mapping \( \Phi : X \to 2^Y \), restricted to \( Z \):

\[ \text{Gr}_Z(\Phi) = \{ (x, y) \in Z \times Y : x \in \text{Dom } \Phi, y \in \Phi(x) \}. \]

When \( Z = X \), we use the standard notation \( \text{Gr}(\Phi) \) for the graph of \( \Phi : X \to 2^Y \) instead of \( \text{Gr}_X(\Phi) \).

Throughout this section assume that \( \text{Dom } \Phi \neq \emptyset \). The following definition introduces a notion of a \( K \)-inf-compact function defined on \( \text{Gr}_X(\Phi) \) for \( \Phi : X \to 2^Y \), while in [8] such functions are defined for \( \Phi : X \to S(Y) \).

Definition 1.3. (cf. Feinberg et al. [8, Definition 1.1]) A function \( f : \text{Gr}(\Phi) \subseteq X \times Y \to \mathbb{R} \) is called \( K \)-inf-compact on \( \text{Gr}(\Phi) \), if for every \( C \in K(\text{Dom } \Phi) \) this function is inf-compact on \( \text{Gr}_C(\Phi) \).

Definition 1.4. A function \( f : \text{Gr}(\Phi) \subseteq X \times Y \to \mathbb{R} \) is called \( K \)-sup-compact on \( \text{Gr}(\Phi) \) if the function \( -f \) is \( K \)-inf-compact on \( \text{Gr}(\Phi) \).

Remark 1.5. According to Remark 1.1, a function \( f : \text{Gr}(\Phi) \subseteq X \times Y \to \mathbb{R} \) is \( K \)-inf-compact / \( K \)-sup-compact on \( \text{Gr}(\Phi) \) if and only if \( f : \text{Gr}(\Phi) \subseteq \text{Dom } \Phi \times Y \to \mathbb{R} \) is \( K \)-inf-compact / \( K \)-sup-compact on \( \text{Gr}(\Phi) \), where \( \text{Dom } \Phi \) is considered as a metric space with the same metric as on \( X \).

The topological meaning of \( K \)-inf-compactness of a function on a graph of a strict set-valued mapping \( \Phi : X \to S(Y) \) is explained in Feinberg et al. [8, Lemma 2.5]; see also Feinberg et al. [5, Lemma 2] and [6, p. 1041].

Lemma 1.6. (Feinberg et al. [8, Lemma 2.5] and Feinberg and Kasyanov [5, Lemma 2]) Let \( \Phi : X \to S(Y) \) be a strict set-valued mapping. Then the function \( f : \text{Gr}(\Phi) \subseteq X \times Y \to \mathbb{R} \) is \( K \)-inf-compact on \( \text{Gr}(\Phi) \) if and only if the following two assumptions hold:

(i) for each \( \lambda \in \mathbb{R} \) the set \( D_\lambda(f; \text{Gr}(\Phi)) \) is closed in \( X \times Y \);

(ii) if a sequence \( \{ x^{(n)} \}_{n=1,2,...} \) with values in \( X \) converges and its limit \( x \) belongs to \( X \), then each sequence \( \{ y^{(n)} \}_{n=1,2,...} \) with \( y^{(n)} \in \Phi(x^{(n)}) \), \( n = 1, 2, \ldots \), satisfying the condition that the sequence \( \{ f(x^{(n)}, y^{(n)}) \}_{n=1,2,...} \) is bounded above, has a limit point \( y \in \Phi(x) \).

The following lemma provides necessary and sufficient conditions for \( K \)-inf-compactness of a function \( f : \text{Gr}(\Phi) \subseteq X \times Y \to \mathbb{R} \) for a possibly non-strict set-valued mapping \( \Phi : X \to 2^Y \).
Lemma 1.7. The function $f : \text{Gr}(\Phi) \subseteq X \times Y \to \overline{R}$ is K-inf-compact on $\text{Gr}(\Phi)$ if and only if the following two assumptions hold:

(i) $f : \text{Gr}(\Phi) \subseteq X \times Y \to \overline{R}$ is lower semi-continuous;

(ii) if a sequence $\{x^{(n)}\}_{n=1,2,\ldots}$ with values in $\text{Dom} \Phi$ converges in $X$ and its limit $x$ belongs to $\text{Dom} \Phi,$ then each sequence $\{y^{(n)}\}_{n=1,2,\ldots}$ with $y^{(n)} \in \Phi(x^{(n)}),$ $n = 1, 2, \ldots,$ satisfying the condition that the sequence $\{f(x^{(n)}, y^{(n)})\}_{n=1,2,\ldots}$ is bounded above, has a limit point $y \in \Phi(x).$

Proof. According to Remark 1.5, the function $f : \text{Gr}(\Phi) \subseteq X \times Y \to \overline{R}$ is K-inf-compact on $\text{Gr}(\Phi)$ if and only if the function $f : \text{Gr}(\Phi) \subseteq \text{Dom} \Phi \times Y \to \overline{R}$ is K-inf-compact on $\text{Gr}(\Phi),$ where $\text{Dom} \Phi$ is considered as a metric space with the same metric as on $X.$ Therefore, Lemma 1.6, being applied to $X = \text{Dom} \Phi,$ $Y = Y,$ $f = f,$ and $\Phi = \Phi|_{\text{Dom} \Phi},$ yields that the function $f : \text{Gr}(\Phi) \subseteq X \times Y \to \overline{R}$ is K-inf-compact on $\text{Gr}(\Phi)$ if and only if the following two assumptions hold:

(a) for each $\lambda \in \overline{R}$ the set $D_f(\lambda; \text{Gr}(\Phi))$ is closed in $\text{Dom} \Phi \times Y;$

(b) assumption (ii) of Lemma 1.7 holds.

The rest of the proof establishes that, under assumption (b), assumption (a) holds if and only if assumption (i) of Lemma 1.7 holds.

Let us prove that assumptions (a) and (b) imply assumption (i) of Lemma 1.7. Consider a sequence $\{(x^{(n)}, y^{(n)})\}_{n=1,2,\ldots} \subseteq \text{Gr}(\Phi)$ that converges to $(x, y) \in \text{Gr}(\Phi).$ Then either $\liminf_{n \to \infty} f(x^{(n)}, y^{(n)}) = +\infty$ or there exists a subsequence $\{(x^{(n_k)}, y^{(n_k)})\}_{k=1,2,\ldots} \subseteq \{(x^{(n)}, y^{(n)})\}_{n=1,2,\ldots}$ such that $\{(x^{(n_k)}, y^{(n_k)})\}_{k=1,2,\ldots}$ is eventually in $D_f(\lambda; \text{Gr}(\Phi))$ for each real $\lambda > \liminf_{n \to \infty} f(x^{(n)}, y^{(n)}).$ Since the set $D_f(\lambda; \text{Gr}(\Phi))$ is closed in $\text{Dom} \Phi \times Y,$ then $(x, y) \in D_f(\lambda; \text{Gr}(\Phi))$ for each real $\lambda > \liminf_{n \to \infty} f(x^{(n)}, y^{(n)})$ and, therefore, $f(x, y) \leq \liminf_{n \to \infty} f(x^{(n)}, y^{(n)}),$ that is, assumption (i) of Lemma 1.7 holds.

Let assumption (b) and assumption (i) of Lemma 1.7 hold. Then (a) holds. Indeed, we fix an arbitrary $\lambda \in \overline{R}$ and prove that the level set $D_f(\lambda; \text{Gr}(\Phi))$ is closed in $\text{Dom} \Phi \times Y.$ Indeed, let $\{(x^{(n)}, y^{(n)})\}_{n=1,2,\ldots} \subseteq D_f(\lambda; \text{Gr}(\Phi))$ be a sequence that converges and its limit $(x, y)$ belongs to $\text{Dom} \Phi \times Y.$ Assumption (b) implies that $(x, y) \in \text{Gr}(\Phi).$ Moreover, since $f : \text{Gr}(\Phi) \subseteq X \times Y \to \overline{R}$ is lower semi-continuous, then this function is lower semi-continuous at $(x, y) \in \text{Gr}(\Phi).$ Therefore, the following inequalities hold:

$$f(x, y) \leq \liminf_{n \to \infty} f(x^{(n)}, y^{(n)}) \leq \lambda,$$

that is, $(x, y) \in D_f(\lambda; \text{Gr}(\Phi)).$ Thus we established that the set $D_f(\lambda; \text{Gr}(\Phi))$ is closed in $\text{Dom} \Phi \times Y$ for arbitrary $\lambda \in \overline{R}.$ Assumption (a) holds.

The following corollary establishes that assumption (i) in Lemma 1.6 can be substituted by lower semi-continuity of $f : \text{Gr}(\Phi) \subseteq X \times Y \to \overline{R}.$

Corollary 1.8. Let $\Phi : X \to S(Y)$ be a strict set-valued mapping and $f : \text{Gr}(\Phi) \subseteq X \times Y \to \overline{R}$ be a function satisfying assumption (ii) of Lemma 1.6. Then for each $\lambda \in \overline{R}$ the set $D_f(\lambda; \text{Gr}(\Phi))$ is closed in $X \times Y$ if and only if the function $f : \text{Gr}(\Phi) \subseteq X \times Y \to \overline{R}$ is lower semi-continuous.
Proof. The statement of this corollary follows directly from Lemmas 1.6 and 1.7.

A set-valued mapping $F : X \rightarrow 2^Y$ is upper semi-continuous at $x \in \text{Dom } F$ if, for each neighborhood $G$ of the set $F(x)$, there is a neighborhood of $x$, say $U(x)$, such that $F(x^*) \subseteq G$ for all $x^* \in U(x) \cap \text{Dom } F$; a set-valued mapping $F : X \rightarrow 2^Y$ is lower semi-continuous at $x \in \text{Dom } F$ if, for each open set $G$ with $F(x) \cap G \neq \emptyset$, there is a neighborhood of $x$, say $U(x)$, such that if $x^* \in U(x) \cap \text{Dom } F$, then $F(x^*) \cap G \neq \emptyset$ (see e.g., Berge [3, p. 109] or Zgurovsky et al. [17, Chapter 1, p. 7]). A set-valued mapping is called upper / lower semi-continuous, if it is upper / lower semi-continuous at all $x \in \text{Dom } F$.

A set-valued mapping $F : X \rightarrow 2^Y$ is K-upper semi-compact if for each $C \in K(\text{Dom } \Phi)$ the set $\text{Gr}_C(F)$ is compact; see e.g. Feinberg et al. [6, Definition 2.3].

Let us provide the sufficient conditions for $K$-inf-compactness.

**Lemma 1.9.** Let $\Phi : X \rightarrow 2^Y$ be a set-valued mapping and $f : \text{Gr}(\Phi) \subseteq X \times Y \rightarrow \mathbb{R}$ be a function. Then the following statements hold:

(a) if $f : \text{Gr}(\Phi) \subseteq X \times Y \rightarrow \mathbb{R}$ is inf-compact on $\text{Gr}(\Phi)$, then the function $f$ is $K$-inf-compact on $\text{Gr}(\Phi)$;

(b) if $f : \text{Gr}(\Phi) \subseteq X \times Y \rightarrow \mathbb{R}$ is lower semi-continuous and $\Phi : X \rightarrow 2^Y$ is upper semi-continuous and compact-valued at each $x \in \text{Dom } \Phi$, then the function $f$ is $K$-inf-compact on $\text{Gr}(\Phi)$.

**Proof.** According to Remark 1.1, Feinberg et al. [8, Lemma 2.1], being applied to $X := \text{Dom } \Phi$, $Y := A$, $u := f$, and $\Phi := \Phi|_X$, yields all the statements of the lemma.

The following lemma provides the necessary and sufficient conditions for $K$-upper semi-compactness of a possibly non-strict set-valued mapping $\Phi : X \rightarrow 2^Y$.

**Lemma 1.10.** A set-valued mapping $\Phi : X \rightarrow 2^Y$ is $K$-upper semi-compact if and only if it is upper semi-continuous and compact-valued at each $x \in \text{Dom } \Phi$.

**Proof.** According to Remark 1.1, Feinberg et al. [6, Theorem 2.5], being applied to $X := \text{Dom } \Phi$, $Y := A$, $u := f$, and $\Psi := \Phi|_X$, yields all the statements of the lemma.

## 2 Continuity Properties of Minima

Let $X, Y$ be metric spaces, $\Phi : X \rightarrow 2^Y$ be a set-valued mapping with $\text{Dom } \Phi \neq \emptyset$ and $f : \text{Gr}(\Phi) \subseteq X \times Y \rightarrow \mathbb{R}$ be a function. Define the value function

$$f^*(x) := \inf_{y \in \Phi(x)} f(x, y), \quad x \in \text{Dom } \Phi, \quad (2.1)$$

and the solution multifunction

$$\Phi^*(x) := \{y \in \Phi(x) : f^*(x) = f(x, y)\}, \quad x \in \text{Dom } \Phi. \quad (2.2)$$

According to Berge’s theorem [3, Theorem 2, p. 116], under assumptions of Lemma 1.9(b), the function $f^*$ is lower semi-continuous if the set-valued mapping $\Phi : X \rightarrow 2^Y$ is strict. For metric spaces $X$ and $Y$, the following theorem generalizes Berge’s theorems from Feinberg et al. [6, Theorems 2.1(ii) and 3.4] and [8, Theorem 3.1] to a possibly non-strict set-valued mapping $\Phi : X \rightarrow 2^Y$. 

5
Theorem 2.1. If a function \( f : \text{Gr}(\Phi) \subseteq X \times Y \to \mathbb{R} \) is \( K \)-inf-compact on \( \text{Gr}(\Phi) \), then the value function \( f^* : \text{Dom} \Phi \subseteq X \to \mathbb{R} \) defined in (2.1) is lower semi-continuous. Moreover, the infimum in (2.1) can be replaced with the minimum and the nonempty sets \( \{ \Phi^*(x) \}_{x \in \text{Dom} \Phi} \) defined in (2.2) satisfy the following properties:

(a) the graph \( \text{Gr}(\Phi^*) \) is a Borel subset of \( X \times Y \);

(b) if \( f^*(x) = +\infty \), then \( \Phi^*(x) = \Phi(x) \), and, if \( f^*(x) < +\infty \), then \( \Phi^*(x) \) is compact; \( x \in \text{Dom} \Phi \).

Proof. According to Remark 1.1, Feinberg et al. [6, Theorems 2.1(ii) and 3.4], being applied to \( X := \text{Dom} \Phi, Y := A, u := f \), and \( \Phi := \Phi|_X \), yields that the value function \( f^* : \text{Dom} \Phi \subseteq X \to \mathbb{R} \) is lower semi-continuous. Moreover, Feinberg et al. [8, Theorem 3.1], being applied to \( X := \text{Dom} \Phi, Y := A, u := f \), and \( \Phi := \Phi|_X \) implies that the infimum in (2.1) can be replaced with the minimum and the nonempty sets \( \{ \Phi^*(x) \}_{x \in \text{Dom} \Phi} \) defined in (2.2) satisfy properties (a) and (b). \( \square \)

The following theorem describes sufficient conditions for upper semi-continuity of the value function \( f^* \) defined in (2.1). A more general result is presented in Feinberg and Kasyanov [5, Theorem 4], which can be generalized to a possibly nonstrict set-valued mapping \( \Phi \). However, for the purposes of this paper we need only the following theorem for metric spaces.

Theorem 2.2. (Hu and Papageorgiou [11, Proposition 3.1, p. 82]) If a set-valued mapping \( \Phi : X \to S(Y) \) is lower semi-continuous and a function \( f : \text{Gr}(\Phi) \subseteq X \times Y \to \mathbb{R} \) is upper semi-continuous, then the value function \( f^* : X \to \mathbb{R} \) defined in (2.1) is upper semi-continuous.

The following theorem describes sufficient conditions for \( K \)-upper semi-compactness of the solution multifunction \( \Phi^* \) defined in (2.2); see also Lemma 1.10.

Theorem 2.3. (Feinberg and Kasyanov [5, Theorem 5] and Feinberg et al. [6, p. 1045]) Let \( \Phi : X \to S(Y) \), a function \( f : \text{Gr}(\Phi) \subseteq X \times Y \to \mathbb{R} \) be \( K \)-inf-compact on \( \text{Gr}(\Phi) \), and the value function \( f^* : X \to \mathbb{R} \cup \{-\infty\} \) defined in (2.1) be continuous. Then the infimum in (2.1) can be replaced with the minimum and the solution multifunction \( \Phi^* : X \to S(Y) \) defined in (2.2) is \( K \)-upper semi-compact.

3 Continuity Properties of Minimax

This section describes continuity properties of minimax and solution multifunctions. For metric spaces the presented results can be viewed as extensions of Berge’s maximum theorem for noncompact image sets and relevant statements for optimization problems from Feinberg et al. [6, 8] to minimax settings.

Let \( X, A \) and \( B \) be metric spaces, \( \Phi_A : X \to S(A) \) and \( \Phi_B : \text{Gr}(\Phi_A) \subseteq X \times A \to S(B) \) be set-valued mappings and \( f : \text{Gr}(\Phi_B) \subseteq X \times A \times B \to \mathbb{R} \) be a function. Define the worst-loss function:

\[
f^\sharp(x, a) := \sup_{b \in \Phi_B(x, a)} f(x, a, b), \quad (x, a) \in \text{Gr}(\Phi_A),
\]

the minimax or upper value function:

\[
\varphi^\sharp(x) := \inf_{a \in \Phi_A(x)} \sup_{b \in \Phi_B(x, a)} f(x, a, b), \quad x \in X,
\]

6
and the solution multifunctions:

\[
\Phi^*_A(x) := \{ a \in \Phi_A(x) : v^f(x) = \sup_{b \in \Phi_B(x,a)} f(x,a,b) \}, \quad x \in \mathcal{X};
\]

\[
\Phi^*_B(x,a) := \{ b \in \Phi_B(x,a) : \sup_{b^* \in \Phi_B(x,a)} f(x,a,b^*) = f(x,a,b) \}, \quad (x,a) \in \text{Gr}(\Phi_A).
\]

We note that the following equalities hold:

\[
v^f(x) = \inf_{a \in \Phi_A(x)} f^\sharp(x,a), \quad \Phi^*_A(x) = \{ a \in \Phi_A(x) : v^f(x) = f^\sharp(x,a) \}, \quad x \in \mathcal{X};
\]

\[
\Phi^*_B(x,a) = \{ b \in \Phi_B(x,a) : f^\sharp(x,a) = f(x,a,b) \}, \quad (x,a) \in \text{Gr}(\Phi_A).
\]

The rest of the section is devoted to:

(i) continuity properties of the worst-loss function \(f^\sharp\) (Theorems 3.2, 3.3, 3.4 and 3.5 and Corollary 3.3);

(ii) continuity properties of the minimax function \(v^f\) (Theorems 3.6, 3.7 and 3.8);

(iii) continuity properties of the solution multifunctions \(\Phi^*_A\) and \(\Phi^*_B\) (Theorems 3.9 and 3.10);

(iv) continuity of the worst-loss function \(f^\sharp\) and the minimax function \(v^f\) and upper semi-continuity of the solution multifunctions \(\Phi^*_A\) and \(\Phi^*_B\) (Theorem 3.11),

when the image sets \(\{ \Phi_A(x) \}_{x \in \mathcal{X}}\) and \(\{ \Phi_B(x,a) \}_{(x,a) \in \text{Gr}(\Phi_A)}\) are possibly noncompact.

To state the main results of this section we introduce the set-valued mapping \(\Phi^{A+B}_B : \mathcal{X} \times \mathcal{B} \to 2^A\) uniquely defined by its graph,

\[
\text{Gr}(\Phi^{A+B}_B) := \{(x, b, a) \in \mathcal{X} \times \mathcal{B} \times A : (x, a, b) \in \text{Gr}(\Phi_B)\},
\]

and the function \(f^{A+B} : \text{Gr}(\Phi^{A+B}_B) \subseteq (\mathcal{X} \times \mathcal{B}) \times A \to \mathbb{R},

\[
f^{A+B}(x, b, a) := f(x, a, b), \quad (x, a, b) \in \text{Gr}(\Phi_B).
\]

According to (3.6), the following equalities hold:

\[
\text{Dom} \Phi^{A+B}_B = \text{proj}_{\mathcal{X} \times \mathcal{B}} \text{Gr}(\Phi_B) = \{(x, b) \in \mathcal{X} \times \mathcal{B} : (x, a, b) \in \text{Gr}(\Phi_B) \text{ for some } a \in A\},
\]

where \(\text{proj}_{\mathcal{X} \times \mathcal{B}} \text{Gr}(\Phi_B)\) is a projection of \(\text{Gr}(\Phi_B)\) on \(\mathcal{X} \times \mathcal{B}\).

**Remark 3.1.** According to Lemma 1.7, the function \(f^{A+B} : \text{Gr}(\Phi^{A+B}_B) \subseteq (\mathcal{X} \times \mathcal{B}) \times A \to \mathbb{R}\) defined in (3.7), where \(\Phi^{A+B}_B\) is defined in (3.6), is \(\mathcal{K}\)-inf-compact on \(\text{Gr}(\Phi^{A+B}_B)\) if and only if the following two conditions hold:

(i) the function \(f : \text{Gr}(\Phi_B) \subseteq \mathcal{X} \times A \times \mathcal{B} \to \mathbb{R}\) is lower semi-continuous;

(ii) if a sequence \(\{ x^{(n)}, b^{(n)} \}_{n=1,2,...} \) with values in \(\text{Dom} \Phi^{A+B}_B\) converges and its limit \((x, b)\) belongs to \(\text{Dom} \Phi^{A+B}_B\), then each sequence \(\{ a^{(n)} \}_{n=1,2,...} \) with \((x^{(n)}, a^{(n)}, b^{(n)}) \in \text{Gr}(\Phi_B), n = 1, 2, \ldots,\) satisfying the condition that the sequence \(\{ f(x^{(n)}, a^{(n)}, b^{(n)}) \}_{n=1,2,...} \) is bounded above, has a limit point \(a \in \Phi_A(x)\).
The following theorem and its corollary establish the sufficient conditions for $\mathbb{K}$-inf-compactness and lower semi-continuity of the worst-loss function $f^\sharp : \text{Gr}(\Phi) \subseteq X \times A \rightarrow \mathbb{R}$ defined in (3.1), when the image sets $\{\Phi(x)\}_{x \in X}$ and $\{(\Phi_b(x, a))_{(x, a) \in \text{Gr}(\Phi)}\}$ are possibly noncompact.

**Theorem 3.2.** ($\mathbb{K}$-inf-compactness of the worst-loss function) Let $\Phi_B : \text{Gr}(\Phi) \subseteq X \times A \rightarrow S(B)$ be a lower semi-continuous set-valued mapping and the function $f^{A \leftrightarrow B} : \text{Gr}(\Phi_B^{A \leftrightarrow B}) \subseteq (X \times B) \rightarrow \mathbb{R}$ defined in (3.7), where $\Phi_B^{A \leftrightarrow B}$ is defined in (3.6), be $\mathbb{K}$-inf-compact on $\text{Gr}(\Phi_B^{A \leftrightarrow B})$. Then the worst-loss function $f^\sharp : \text{Gr}(\Phi) \subseteq X \times A \rightarrow \mathbb{R}$ defined in (3.1) is $\mathbb{K}$-inf-compact on $\text{Gr}(\Phi)$.

**Proof.** Since the function $f^{A \leftrightarrow B} : \text{Gr}(\Phi_B^{A \leftrightarrow B}) \subseteq (X \times B) \rightarrow \mathbb{R}$ defined in (3.7), where $\Phi_B^{A \leftrightarrow B}$ is defined in (3.6), is $\mathbb{K}$-inf-compact on $\text{Gr}(\Phi_B^{A \leftrightarrow B})$, then properties (i) and (ii) from Remark 3.1 hold.

To prove that the function $f^\sharp$ is $\mathbb{K}$-inf-compact on $\text{Gr}(\Phi)$, we fix arbitrary $C \in \mathbb{K}(X)$, $\lambda \in \mathbb{R}$, and $\{(x^{(n)}, a^{(n)})\}_{n=1,2,...} \subseteq \text{Gr}_C(\Phi)$ such that

$$f^\sharp(x^{(n)}, a^{(n)}) \leq \lambda, \quad (3.9)$$

for each $n = 1, 2, \ldots$, and establish that the sequence $\{(x^{(n)}, a^{(n)})\}_{n=1,2,...}$ has a limit point $(x, a) \in \text{Gr}_C(\Phi)$ satisfying

$$f^\sharp(x, a) \leq \lambda. \quad (3.10)$$

The proof consists of the following two steps: Step 1 proves that each sequence $\{(x^{(n)}, a^{(n)})\}_{n=1,2,...} \subseteq \text{Gr}_C(\Phi)$ satisfying inequality (3.9) has a limit point $(x, a) \in \text{Gr}_C(\Phi)$. Step 2 establishes inequality (3.10) for a limit point $(x, a) \in \text{Gr}_C(\Phi)$ of a sequence $\{(x^{(n)}, a^{(n)})\}_{n=1,2,...} \subseteq \text{Gr}_C(\Phi)$ that converges and satisfies (3.9).

**Step 1.** Let $\{(x^{(n)}, a^{(n)})\}_{n=1,2,...} \subseteq \text{Gr}_C(\Phi)$ be a sequence satisfying inequality (3.9). Let us prove that this sequence has a limit point from $\text{Gr}_C(\Phi)$. Since $C \in \mathbb{K}(X)$, then, without loss of generality, we may assume that the sequence $\{x^{(n)}\}_{n=1,2,...}$ converges in $X$ and its limit $x$ belongs to $C$. To prove that the sequence $\{a^{(n)}\}_{n=1,2,...}$ has a limit point $a \in \Phi(x)$, we fix an arbitrary $b \in \Phi_B(x, a)$ and note that there exists a sequence $\{b^{(n)}\}_{n=1,2,...}$ with $b^{(n)} \in \Phi_B(x^{(n)}, a^{(n)})$, $n = 1, 2, \ldots$, that converges and its limit equals to $b$ because the set-valued mapping $\Phi_B : \text{Gr}(\Phi) \subseteq X \times A \rightarrow S(B)$ is lower semi-continuous. Then, according to (3.1) and (3.9), the sequence $\{f(x^{(n)}, a^{(n)}, b^{(n)})\}_{n=1,2,...}$ is bounded above by $\lambda$. Therefore, property (ii) from Remark 3.1 yields that the sequence $\{a^{(n)}\}_{n=1,2,...}$ has a limit point $a \in \Phi(x)$. Therefore, the sequence $\{(x^{(n)}, a^{(n)})\}_{n=1,2,...}$ has a limit point $(x, a) \in \text{Gr}_C(\Phi)$.

**Step 2.** Let $\{(x^{(n)}, a^{(n)})\}_{n=1,2,...} \subseteq \text{Gr}_C(\Phi)$ be a convergent sequence satisfying (3.9). Let us prove that its limit point $(x, a) \in \text{Gr}_C(\Phi)$ satisfies inequality (3.10). For this purpose fix an arbitrary $b \in \Phi_B(x, a)$ and note that the lower semi-continuity of the set-valued mapping $\Phi_B : \text{Gr}(\Phi) \subseteq X \times A \rightarrow S(B)$ implies that there exists a sequence $\{b^{(n)}\}_{n=1,2,...}$, with $b^{(n)} \in \Phi_B(x^{(n)}, a^{(n)})$, $n = 1, 2, \ldots$, that converges to $b$. Moreover, the following inequalities hold:

$$f(x^{(n)}, a^{(n)}, b^{(n)}) \leq f^\sharp(x^{(n)}, a^{(n)}) \leq \lambda \quad (3.11)$$

for each $n = 1, 2, \ldots$, where the first inequality follows from (3.1) and the second inequality follows from (3.9). Property (i) from Remark 3.1 and (3.11) imply that

$$f(x, a, b) \leq \liminf_{n \rightarrow \infty} f(x^{(n)}, a^{(n)}, b^{(n)}) \leq \lambda \quad (3.12)$$
because the sequence \( \{(x^{(n)}, a^{(n)}, b^{(n)})\}_{n=1,2,...} \subseteq \text{Gr}(\Phi_B) \) converges to \((x, a, b) \in \text{Gr}(\Phi_B)\). To finish the proof we note that (3.12) yields inequality (3.10) because \(b \in \Phi_B(x, a)\) is arbitrary (see, also, (3.1)). 

**Corollary 3.3.** (Lower semi-continuity of the worst-loss function) Let assumptions of Theorem 3.2 hold. Then the worst-loss function \( f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow \mathbb{R} \) defined in (3.1) is lower semi-continuous.

**Proof.** Theorem 3.2 yields that the worst-loss function \( f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow \mathbb{R} \) is \( K \)-inf-compact on \( \text{Gr}(\Phi_A) \). Therefore, Lemma 1.7(i), being applied to \( X := X, Y := A, \Phi := \Phi_A, \) and \( f := f^\sharp \), implies that \( f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow \mathbb{R} \) is lower semi-continuous. 

The following theorem establishes sufficient conditions for upper semi-continuity of the worst-loss function \( f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow \mathbb{R} \) defined in (3.1) and basic properties for the solution multifunction \( \Phi_B^\ast \) defined in (3.4), when the image sets \( \{\Phi_A(x)\}_{x \in X} \) and \( \{\Phi_B(x, a)\}_{(x,a) \in \text{Gr}(\Phi_A)} \) are possibly noncompact.

**Theorem 3.4.** (Upper semi-continuity of the worst-loss function) If a function \( f : \text{Gr}(\Phi_B) \subseteq (X \times A) \times B \rightarrow \mathbb{R} \) is \( K \)-sup-compact on \( \text{Gr}(\Phi_B) \), then the worst-loss function \( f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow \mathbb{R} \) defined in (3.1) is upper semi-continuous. Moreover, the supremum in (3.1) can be replaced with the maximum and the nonempty sets \( \{\Phi_B^\ast(x, a)\}_{(x,a) \in \text{Gr}(\Phi_A)} \) defined in (3.4) (see also the last equality in (3.5)) satisfy the following properties:

(a) the graph \( \text{Gr}(\Phi_B^\ast) \) is a Borel subset of \( X \times A \times B \);

(b) if \( f^\sharp(x, a) = -\infty \), then \( \Phi_B^\ast(x, a) = \Phi_B(x, a) \), and, if \( f^\sharp(x, a) > -\infty \), then \( \Phi_B^\ast(x, a) \) is compact.

**Proof.** Since the function \( f : \text{Gr}(\Phi_B) \subseteq (X \times A) \times B \rightarrow \mathbb{R} \) is \( K \)-sup-compact on \( \text{Gr}(\Phi_B) \), then Theorem 2.1, being applied to \( X = X \times A, Y = B, \Phi = \Phi_B \) and \( f = -f \), yield all the statements of Theorem 3.4. 

The following theorem describes sufficient conditions for continuity of the worst-loss function \( f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow \mathbb{R} \) defined in (3.1), when the image sets \( \{\Phi_A(x)\}_{x \in X} \) and \( \{\Phi_B(x, a)\}_{(x,a) \in \text{Gr}(\Phi_A)} \) are possibly noncompact.

**Theorem 3.5.** (Continuity of the worst-loss function) Let \( \Phi_B : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow S(B) \) be a lower semi-continuous set-valued mapping, \( f : \text{Gr}(\Phi_B) \subseteq (X \times A) \times B \rightarrow \mathbb{R} \) be a \( K \)-sup-compact function on \( \text{Gr}(\Phi_B) \), and the function \( f^{A \ast B} : \text{Gr}(\Phi_B^{A \ast B}) \subseteq (X \times B) \times A \rightarrow \mathbb{R} \) defined in (3.7), where \( \Phi_B^{A \ast B} \) is defined in (3.6), be \( K \)-inf-compact on \( \text{Gr}(\Phi_B^{A \ast B}) \). Then the worst-loss function \( f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow \mathbb{R} \) defined in (3.1) is continuous.

**Proof.** Corollary 3.3 yields that the worst-loss function \( f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow \mathbb{R} \) is lower semi-continuous. Theorem 3.4 implies that \( f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow \mathbb{R} \) is upper semi-continuous. Therefore, \( f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \rightarrow \mathbb{R} \) is continuous.

The following theorem describes sufficient conditions for lower semi-continuity of the minimax function \( v^\sharp \) defined in (3.2) and basic properties for the solution multifunction \( \Phi_A^\ast \) defined in (3.3), when the image sets \( \{\Phi_A(x)\}_{x \in X} \) and \( \{\Phi_B(x, a)\}_{(x,a) \in \text{Gr}(\Phi_A)} \) are possibly noncompact.
Theorem 3.6. (Lower semi-continuity of minimax) Let $\Phi_B : \text{Gr}(\Phi_A) \subseteq X \times A \to S(B)$ be a lower semi-continuous set-valued mapping and the function $\mathbf{1}^{A+B} : \text{Gr}(\Phi_A^{A+B}) \subseteq (X \times B) \times A \to \mathbb{R}$ defined in (3.7), where $\Phi_A^{A+B}$ is defined in (3.6), be $\mathcal{K}$-inf-compact on $\text{Gr}(\Phi_A^{A+B})$. Then the minimax function $\mathbf{v}^d : X \to \overline{\mathbb{R}}$ defined in (3.2) is lower semi-continuous. Moreover, the infimum in (3.2) can be replaced with the minimum and the nonempty sets $\{\Phi_A^*(x)\}_{x \in X}$ defined in (3.3) satisfy the following properties:

(a) the graph $\text{Gr}(\Phi_A^*)$ is a Borel subset of $X \times A$;

(b) if $v^d(x) = +\infty$, then $\Phi_A^*(x) = \Phi_A(x)$, and, if $v^d(x) < +\infty$, then $\Phi_A^*(x)$ is compact.

Proof. Theorem 3.2 yields that the worst-loss function $\mathbf{1}^d : \text{Gr}(\Phi_A) \subseteq X \times A \to \mathbb{R}$ defined in (3.1) is $\mathcal{K}$-inf-compact on $\text{Gr}(\Phi_A)$). Therefore, Theorem 2.1, being applied to $X := X$, $Y := A$, $F := \Phi_A$ and $f := \mathbf{1}^d$, implies all the statements of Theorem 3.6.

The following theorem describes sufficient conditions for upper semi-continuity of the minimax function $\mathbf{v}^d$ defined in (3.2) and basic properties for the solution multifunction $\Phi_A^*$ defined in (3.4), when the image sets $\{\Phi_A^*(x)\}_{x \in X}$ and $\{\Phi_B(x,a)\}_{(x,a) \in \text{Gr}(\Phi_A)}$ are possibly noncompact.

Theorem 3.7. (Upper semi-continuity of minimax) Let $\Phi_A : X \to S(A)$ be a lower semi-continuous set-valued mapping and $\mathbf{1} : \text{Gr}(\Phi_B) \subseteq (X \times A) \times B \to \mathbb{R}$ be a $\mathcal{K}$-sup-compact function on $\text{Gr}(\Phi_B)$. Then the minimax function $\mathbf{v}^d : X \to \overline{\mathbb{R}}$ defined in (3.2) is upper semi-continuous. Moreover, the supremums in (3.1) and (3.2) can be replaced with the maximums and the nonempty sets $\{\Phi_B^*(x,a)\}_{(x,a) \in \text{Gr}(\Phi_A)}$ defined in (3.4) (see also the last equality in (3.5)) satisfy properties (a) and (b) of Theorem 3.4.

Proof. Theorem 3.4 yields that the worst-loss function $\mathbf{1}^d : \text{Gr}(\Phi_A) \subseteq X \times A \to \mathbb{R}$ defined in (3.1) is upper semi-continuous on $\text{Gr}(\Phi_A)$, the supremums in (3.1) and (3.2) can be replaced with the maximums, and the nonempty sets $\{\Phi_B^*(x,a)\}_{(x,a) \in \text{Gr}(\Phi_A)}$ defined in (3.4) (see also the last equality in (3.5)) satisfy properties (a) and (b) of Theorem 3.4. The upper semi-continuity of the minimax function $\mathbf{v}^d : X \to \overline{\mathbb{R}}$ follows from Theorem 2.2, being applied to $X := X$, $Y := A$, $F := \Phi_A$ and $f := \mathbf{1}^d$, because a set-valued mapping $\Phi_A : X \to S(A)$ is lower semi-continuous and the function $\mathbf{1}^d : \text{Gr}(\Phi_A) \subseteq X \times A \to \mathbb{R}$ is upper semi-continuous.

The following theorem describes sufficient conditions for continuity of the minimax function $\mathbf{v}^d$ defined in (3.2), when the image sets $\{\Phi_A^*(x)\}_{x \in X}$ and $\{\Phi_B(x,a)\}_{(x,a) \in \text{Gr}(\Phi_A)}$ are possibly noncompact.

Theorem 3.8. (Continuity of minimax) Let $\Phi_A : X \to S(A)$ and $\Phi_B : \text{Gr}(\Phi_A) \subseteq X \times A \to S(B)$ be lower semi-continuous set-valued mappings, $\mathbf{1} : \text{Gr}(\Phi_B) \subseteq (X \times A) \times B \to \mathbb{R}$ be a $\mathcal{K}$-sup-compact function on $\text{Gr}(\Phi_B)$, and the function $\mathbf{1}^{A+B} : \text{Gr}(\Phi_A^{A+B}) \subseteq (X \times B) \times A \to \mathbb{R}$ defined in (3.7), where $\Phi_A^{A+B}$ is defined in (3.6), be $\mathcal{K}$-inf-compact on $\text{Gr}(\Phi_A^{A+B})$. Then the minimax function $\mathbf{v}^d : X \to \overline{\mathbb{R}}$ defined in (3.2) is continuous.

Proof. Theorem 3.6 yields that the function $\mathbf{v}^d : X \to \overline{\mathbb{R}}$ is lower semi-continuous. Theorem 3.7 implies that the function $\mathbf{v}^d : X \to \overline{\mathbb{R}}$ is upper semi-continuous. Thus, the function $\mathbf{v}^d : X \to \overline{\mathbb{R}}$ is continuous.

The next theorem describes $\mathcal{K}$-upper semi-compactness of the solution multifunction $\Phi_A^*$ defined in (3.3), when the image sets $\{\Phi_A^*(x)\}_{x \in X}$ and $\{\Phi_B(x,a)\}_{(x,a) \in \text{Gr}(\Phi_A)}$ are possibly noncompact.
Theorem 3.8. (Continuity properties for solution multifunction $\Phi^*_A$) Let $v^\sharp : X \to \mathbb{R} \cup \{ -\infty \}$ defined in (3.2) be a continuous function, $\Phi_B : \text{Gr}(\Phi_A) \subseteq X \times A \to S(B)$ be a lower semi-continuous set-valued mapping, and the function $f^{A \times B} : \text{Gr}(\Phi_B^{A \times B}) \subseteq (X \times B) \times A \to \mathbb{R}$ defined in (3.7), where $\Phi_B^{A \times B}$ is defined in (3.6), be $K$-inf-compact on $\text{Gr}(\Phi_B^{A \times B})$. Then the infimum in (3.2) can be replaced with the solution and the solution multifunction $\Phi^*_A : X \to S(A)$ defined in (3.3) is upper semi-continuous and compact-valued.

Proof. Theorem 3.2 yields that the worst-loss function $f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \to \mathbb{R}$ defined in (3.1) is $K$-inf-compact on $\text{Gr}(\Phi_A)$. Since $v^\sharp : X \to \mathbb{R} \cup \{ -\infty \}$ defined in (3.2) is a continuous function, then Theorem 2.3, being applied to $X := X$, $Y := A$, $\Phi := \Phi_A$ and $f := f^\sharp$, implies that the infimum in (3.2) can be replaced with the minimum and the solution multifunction $\Phi^*_A : X \to S(A)$ defined in (3.3) is upper semi-continuous and compact-valued.

The following theorem provides sufficient conditions for $K$-upper semi-compactness of the solution multifunction $\Phi^*_B$ defined in (3.4), when the image sets $\{ \Phi_A(x) \}_{x \in X}$ and $\{ \Phi_B(x,a) \}_{(x,a) \in \text{Gr}(\Phi_A)}$ are possibly noncompact.

Theorem 3.10. (Continuity properties of the solution multifunction $\Phi^*_B$) Let $f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \to \mathbb{R} \cup \{ +\infty \}$ defined in (3.1) be a continuous function on $\text{Gr}(\Phi_A)$ and $f : \text{Gr}(\Phi_B) \subseteq (X \times A) \times B \to \mathbb{R}$ be a $K$-sup-compact function on $\text{Gr}(\Phi_B)$. Then the supremums in (3.1) and (3.2) can be replaced with the maximums and the solution multifunction $\Phi^*_B : \text{Gr}(\Phi_A) \subseteq X \times A \to S(B)$ defined in (3.4) is upper semi-continuous and compact-valued.

Proof. According to Remark 1.1, the statements of the theorem follow from Theorem 2.3, being applied to $X := \text{Gr}(\Phi_A)$, $Y := B$, $\Phi := \Phi_B$ and $f := -f$.

For metric spaces the following theorem can be viewed as an extension of Berge’s maximum theorem for noncompact image sets from Feinberg et al. [6, Theorem 1.4] to the minimax formulation.

Theorem 3.11. (Continuity of the worst-loss function $f^\sharp$ and the minimax function $v^\sharp$ and upper semi-continuity of the solution multifunctions $\Phi^*_A$ and $\Phi^*_B$) Let $\Phi_A : X \to S(A)$ and $\Phi_B : \text{Gr}(\Phi_A) \subseteq X \times A \to S(B)$ be lower semi-continuous set-valued mappings, $f : \text{Gr}(\Phi_B) \subseteq (X \times A) \times B \to \mathbb{R}$ be a $K$-sup-compact function on $\text{Gr}(\Phi_B)$, and the function $f^{A \times B} : \text{Gr}(\Phi_B^{A \times B}) \subseteq (X \times B) \times A \to \mathbb{R}$ defined in (3.7), where $\Phi_B^{A \times B}$ is defined in (3.6), be $K$-inf-compact on $\text{Gr}(\Phi_B^{A \times B})$. Then the worst-loss function $f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \to \mathbb{R}$ defined in (3.1) is continuous and the minimax function $v^\sharp : X \to \mathbb{R}$ defined in (3.2) is continuous. Moreover, the following two properties hold:

(a) the infimum in (3.2) can be replaced with the minimum, and the solution multifunction $\Phi^*_A : X \to S(A)$ defined in (3.3) is upper semi-continuous and compact-valued;

(b) the supremums in (3.1) and (3.2) can be replaced with the maximums, and the solution multifunction $\Phi^*_B : \text{Gr}(\Phi_A) \subseteq X \times A \to S(B)$ defined in (3.4) is upper semi-continuous and compact-valued.

Proof. Theorem 3.5 yields that the worst-loss function $f^\sharp : \text{Gr}(\Phi_A) \subseteq X \times A \to \mathbb{R}$ defined in (3.1) is continuous on $\text{Gr}(\Phi_A)$. Continuity of the minimax function $v^\sharp : X \to \mathbb{R}$ defined in (3.2) follows from Theorem 3.8. Theorems 3.9 and 3.10 yield statements (a) and (b) respectively.
4 Preserving Properties of $K$-inf-compact functions

In Section 3 we have considered the problems, in which players select actions deterministically. In other words, players play pure strategies. The previous section describes the continuity properties for objective functions and solution multifunctions for such problems with possibly unbounded payoffs, and noncompact action sets. In general, it is known that, if the second player knows the decision of the first players, pure strategies are sufficient. In Section 6 we show that pure strategies are indeed sufficient for the problem studied in the previous section. However, if players make decisions simultaneously, pure strategies usually are not sufficient, and the players should choose randomized strategies, which are probability distributions on the sets of actions. The remarkable fact is that the property of $K$-inf-compactness is preserved when randomized strategies are used instead of pure ones. This section describes such results. Most of them were derived in Feinberg et al. [9] for studying partially observable Markov decision processes.

For a metric space $S$, let $B(S)$ be a Borel $\sigma$-field on $S$, that is, the $\sigma$-field generated by all open sets of the metric space $S$. For a nonempty Borel subset $S \subset S$, denote by $B(S)$ the $\sigma$-field whose elements are intersections of $S$ with elements of $B(S)$. Observe that $S$ is a metric space with the same metric as on $S$, and $B(S)$ is its Borel $\sigma$-field. For a metric space $S$, let $P(S)$ be the set of probability measures on $(S, B(S))$ and $P_f(S)$ denotes the set of all probability measures whose supports are finite subsets of $S$. A sequence of probability measures $\{\mu^n\}_{n=1,2,...}$ from $P(S)$ converges weakly to $\mu \in P(S)$ if for each bounded continuous function $f$ on $S$

$$\int_S f(s)\mu^n(ds) \to \int_S f(s)\mu(ds) \quad \text{as} \quad n \to \infty.$$  

Note that the set $P_f(S)$ is dense in a separable metric space $P(S)$ with respect to the weak convergence topology for probability measures, when $S$ is a separable metric space; Parthasarathy [15, Chapter II, Theorem 6.3].

Let $X, Y$ be nonempty Borel subsets of respective Polish spaces (complete separable metric spaces). The following lemma, three theorems, and a corollary describe preserving properties for lower semi-continuous, inf-compact, and $K$-inf-compact functions.

**Lemma 4.1.** (Feinberg et al. [9, Lemma 6.1]) If the function $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is bounded from below and lower semi-continuous, then the function $\hat{f} : X \times P(Y) \to \mathbb{R} \cup \{+\infty\}$

$$\hat{f}(x, z) := \int_Y f(x, y)z(dy), \quad x \in X, z \in P(Y), \quad (4.1)$$

is bounded from below with the same constant as $f$ and lower semi-continuous.

**Theorem 4.2.** (Feinberg et al. [9, Theorem 6.1]) If $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is an inf-compact function on $X \times Y$, then the function $\hat{f} : X \times P(Y) \to \mathbb{R} \cup \{+\infty\}$ defined in (4.1) is inf-compact on $X \times P(Y)$.

**Corollary 4.3.** If $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is a $K$-inf-compact function on $X \times Y$, then the function $\hat{f} : X \times P(Y) \to \mathbb{R} \cup \{+\infty\}$ defined in (4.1) is $K$-inf-compact on $X \times P(Y)$. 

12
Proof. According to Definition 1.3, the function \( \hat{f} : \mathbb{X} \times \mathbb{P}(\mathbb{Y}) \to \mathbb{R} \cup \{+\infty\} \) defined in (4.1) is \( \mathbb{K} \)-inf-compact on \( \mathbb{X} \times \mathbb{P}(\mathbb{Y}) \) if and only if for every \( C \in \mathbb{K}(\mathbb{X}) \) this function is inf-compact on \( C \times \mathbb{P}(\mathbb{Y}) \).

Let us prove that the function \( \hat{f} : \mathbb{X} \times \mathbb{P}(\mathbb{Y}) \to \mathbb{R} \cup \{+\infty\} \) defined in (4.1) is inf-compact on \( C \times \mathbb{P}(\mathbb{Y}) \) for each \( C \in \mathbb{K}(\mathbb{X}) \). For this purpose we fix an arbitrary \( C \in \mathbb{K}(\mathbb{X}) \) and note that the function \( f|_C : C \times \mathbb{Y} \to \mathbb{R} \cup \{+\infty\} \) is inf-compact on \( C \times \mathbb{Y} \) because this function is \( \mathbb{K} \)-inf-compact on \( \mathbb{X} \times \mathbb{Y} \). Theorem 4.2 yields that the function \( \hat{f} \) defined in (4.1) is inf-compact on \( C \times \mathbb{P}(\mathbb{Y}) \). Therefore, this function is \( \mathbb{K} \)-inf-compact on \( \mathbb{X} \times \mathbb{P}(\mathbb{Y}) \) because \( C \in \mathbb{K}(\mathbb{X}) \) is arbitrary. \( \square \)

**Theorem 4.4.** (Feinberg et al. [9, Theorem 3.3]) If the function \( f : \mathbb{X} \times \mathbb{Y} \to \mathbb{R} \cup \{+\infty\} \) is bounded from below and \( \mathbb{K} \)-inf-compact on \( \mathbb{X} \times \mathbb{Y} \), then the function \( \tilde{f} : \mathbb{P}(\mathbb{X}) \times \mathbb{P}(\mathbb{Y}) \to \mathbb{R} \cup \{+\infty\} \),

\[
\tilde{f}(z, y) := \int_{\mathbb{X}} f(x, y)z(dx), \quad z \in \mathbb{P}(\mathbb{X}), \ y \in \mathbb{P}(\mathbb{Y}),
\]

is bounded from below with the same constant as \( f \) and \( \mathbb{K} \)-inf-compact on \( \mathbb{P}(\mathbb{X}) \times \mathbb{P}(\mathbb{Y}) \).

**Theorem 4.5.** If the function \( f : \mathbb{X} \times \mathbb{Y} \to \mathbb{R} \cup \{+\infty\} \) is bounded from below and \( \mathbb{K} \)-inf-compact on \( \mathbb{X} \times \mathbb{Y} \), then the function \( \tilde{f} : \mathbb{P}(\mathbb{X}) \times \mathbb{P}(\mathbb{Y}) \to \mathbb{R} \cup \{+\infty\} \),

\[
\tilde{f}(z^X, z^Y) := \int_{\mathbb{X}} \int_{\mathbb{Y}} f(x, y)z^Y(dy)z^X(dx), \quad z^X \in \mathbb{P}(\mathbb{X}), \ z^Y \in \mathbb{P}(\mathbb{Y}),
\]

(4.2)
is bounded from below with the same constant as \( f \) and \( \mathbb{K} \)-inf-compact on \( \mathbb{P}(\mathbb{X}) \times \mathbb{P}(\mathbb{Y}) \).

**Proof.** Lemma 4.1, being applied to \( f : \mathbb{X} \times \mathbb{Y} \to \mathbb{R} \cup \{+\infty\} \), yields that the function \( \tilde{f} : \mathbb{X} \times \mathbb{P}(\mathbb{Y}) \to \mathbb{R} \cup \{+\infty\} \) defined in (4.1) is bounded from below with the same constant as \( f \). Then, Lemma 4.1, being applied to \( \hat{f} : \mathbb{X} \times \mathbb{P}(\mathbb{Y}) \to \mathbb{R} \cup \{+\infty\} \), implies that the function \( \tilde{f} : \mathbb{P}(\mathbb{X}) \times \mathbb{P}(\mathbb{Y}) \to \mathbb{R} \cup \{+\infty\} \) is bounded from below with the same constant as \( f \).

Theorem 4.4, being applied to \( f : \mathbb{X} \times \mathbb{Y} \to \mathbb{R} \cup \{+\infty\} \), yields that the function \( \tilde{f} : \mathbb{P}(\mathbb{X}) \times \mathbb{Y} \to \mathbb{R} \cup \{+\infty\} \) defined in (4.2) is \( \mathbb{K} \)-inf-compact on \( \mathbb{P}(\mathbb{X}) \times \mathbb{Y} \). Therefore, Corollary 4.3, being applied to \( \tilde{f} : \mathbb{P}(\mathbb{X}) \times \mathbb{Y} \to \mathbb{R} \cup \{+\infty\} \), implies that the function \( \tilde{f} : \mathbb{P}(\mathbb{X}) \times \mathbb{P}(\mathbb{Y}) \to \mathbb{R} \cup \{+\infty\} \) is \( \mathbb{K} \)-inf-compact on \( \mathbb{P}(\mathbb{X}) \times \mathbb{P}(\mathbb{Y}) \). \( \square \)

## 5 Two-Person Zero-Sum Games with Simultaneous Moves

In this section we provide sufficient conditions for continuity of the value functions, upper semi-continuity of solution multifunctions, and compactness of solution sets for zero-sum stochastic games with possibly uncountable and noncompact action sets and unbounded payoff functions.

### 5.1 Preliminaries

Let \( S \) be a metric space. An integral \( \int_S f(s)\mu(ds) \) of a measurable \( \mathbb{R} \)-valued function \( f \) on \( S \) over the measure \( \mu \in \mathbb{P}(S) \) is well-defined if either \( \int_S f^-(s)\mu(ds) > -\infty \) or \( \int_S f^+(s)\mu(ds) < +\infty \), where \( f^-(s) = \min\{f(s), 0\}, f^+(s) = \max\{f(s), 0\}, s \in S \). If the integral is well-defined, then \( \int_S f(s)\mu(ds) := \int_S f^+(s)\mu(ds) + \int_S f^-(s)\mu(ds) \).
Remark 5.2. If a triplet 

\[ \{ A, B, c \} \]

is the set of mixed strategies

\[ A \text{ is the space of actions for Player I, which is a nonempty Borel subset of a Polish space; } \]

\[ B \text{ is the space of actions for Player II, which is a nonempty Borel subset of a Polish space; } \]

\[ \text{the payoff to Player II, } -\infty < c(a, b) < +\infty, \text{ for choosing actions } a \in A \text{ and } b \in B, \text{ is a measurable function on } K := A \times B; \]

\[ \text{for each } b \in B \text{ the function } a \to c(a, b) \text{ is bounded from below on } A; \]

\[ \text{for each } a \in A \text{ the function } b \to c(a, b) \text{ is bounded from above on } B. \]

Remark 5.3. Assumptions (iv) and (v) for the game payoffs are possibly undefined even if the one of the players chooses a pure strategy.

The game is played as follows:

\( \text{• a decision-makers (Players I and II) choose simultaneously respective actions } a \in A \text{ and } b \in B; \)

\( \text{• the result } (a, b) \text{ is announced to both of them; } \)

\( \text{• Player I pays Player II the amount } c(a, b). \)

Mixed strategies for Players I and II are probability measures \( \pi^A \in P(A) \) and \( \pi^B \in P(B). \) Moreover, \( \pi^A \) (\( \pi^B \)) is called pure, if the probability measure \( \pi^A(\cdot) \) (\( \pi^B(\cdot) \)) is concentrated at one point. Note that \( P(A) \) is the set of mixed strategies for Player I, and \( P(B) \) is the set of mixed strategies for Player II.

Remark 5.2. If a triplet \( \{ A, B, c \} \) is two-person zero-sum game defined above, then the triplet \( \{ B, A, -c^{A\leftrightarrow B} \}, \)

where \( c^{A\leftrightarrow B}(b, a) = c(a, b) \) for each \( a \in A \) and \( b \in B, \) is also a game satisfying conditions in Definition 5.1.

\[ \text{Let us set } \]

\[ c^+(a, b) := \max\{c(a, b), 0\}, \quad c^-(a, b) := \min\{c(a, b), 0\}, \]

for each \( (a, b) \in K, \)

\[ \hat{c}^+(\pi^A, \pi^B) := \int_A \int_B c^+(a, b) \pi^B(db) \pi^A(da), \quad \hat{c}^-(\pi^A, \pi^B) := \int_A \int_B c^-(a, b) \pi^B(db) \pi^A(da), \]

for each \( (\pi^A, \pi^B) \in K := P(A) \times P(B). \) Then the expected payoff to Player II

\[ \hat{c}(\pi^A, \pi^B) := \hat{c}^+(\pi^A, \pi^B) + \hat{c}^-(\pi^A, \pi^B), \]

is well-defined if either \( \hat{c}^+(\pi^A, \pi^B) < +\infty \) or \( \hat{c}^-(\pi^A, \pi^B) > -\infty; \) \( (\pi^A, \pi^B) \in P(A) \times P(B). \) Of course, when the function \( c \) is unbounded both below as well as above, the quantity \( \hat{c}(\pi^A, \pi^B) \) is possibly undefined for some \( (\pi^A, \pi^B) \in P(A) \times P(B). \) We denote:

\[ P^S_{\pi^A}(B) := \{ \pi^B \in P(B) : \hat{c}(\pi^A, \pi^B) \text{ is well-defined} \}, \quad \pi^A \in P(A); \]

\[ P^S_{\pi^B}(A) := \{ \pi^A \in P(A) : \hat{c}(\pi^A, \pi^B) \text{ is well-defined} \}, \quad \pi^B \in P(B). \]

Further, if a measure \( \pi^A \in P(A) \) is concentrated at a point \( a \in A, \) then we will write \( \hat{c}(a, \pi^B) \) instead of \( \hat{c}(\pi^A, \pi^B) \) for each \( \pi^B \in P(B). \) Similarly, if a measure \( \pi^B \in P(B) \) is concentrated at a point \( b \in B, \) then we will write \( \hat{c}(\pi^A, b) \) instead of \( \hat{c}(\pi^A, \pi^B) \) for each \( \pi^A \in P(A). \)
Remark 5.4. Assumption (iv) for the game \(\{A, B, c\}\) yields that \(c^-(\pi_A, b) > -\infty\) for each \(\pi_A \in P(A)\) and \(b \in B\). Therefore, \(P^{fs}(B) \subset P^S(\pi_A(B)) \text{ for each } \pi_A \in P(A)\) and, since \(P^{fs}(B)\) is dense in \(P(B)\), then \(\cap_{\pi_A \in P(A)} P^S(\pi_A(B))\) is dense in \(P(B)\).

Remark 5.5. Assumption (v) for the game \(\{A, B, c\}\) implies that \(\hat{c}^+(a, \pi^B) < +\infty\) for each \(a \in A\) and \(\pi^B \in P(B)\). Thus, \(P^{fs}(A) \subset P^S(\pi_B(A)) \text{ for each } \pi_B \in P(B)\) and, since \(P^{fs}(A)\) is dense in \(P(A)\), then \(\cap_{\pi_B \in P(B)} P^S(\pi_B(A))\) is dense in \(P(A)\).

The set of mixed strategies for each player is partitioned into the sets of safe strategies \(P^S(A)\) and \(P^S(B)\) (strategies, for which the expected payoff is well-defined for all strategies played by another player) and unsafe strategies \(P^U(A)\) and \(P^U(B)\):

\[
P^S(A) := \{\pi_A \in P(A) : P^S(\pi_A(B)) = P(B)\}, \quad P^U(A) := \{\pi_A \in P(A) : P^S(\pi_A(B)) \neq P(B)\};
\]

\[
P^S(B) := \{\pi_B \in P(B) : P^S(\pi_B(A)) = P(A)\}, \quad P^U(B) := \{\pi_B \in P(B) : P^S(\pi_B(A)) \neq P(A)\}.
\]

Remark 5.6. We note that \(P(A) = P^S(A) \cup P^U(A)\), \(P(B) = P^S(B) \cup P^U(B)\), \(P^S(A) \cap P^U(A) = \emptyset\), and \(P^S(B) \cap P^U(B) = \emptyset\). Moreover, \(P^{fs}(A) \subset P^S(A)\) (see Assumption (iv) for the game \(\{A, B, c\}\) and Remark 5.4) and \(P^{fs}(B) \subset P^S(B)\) (see Assumption (v) for the game \(\{A, B, c\}\) and Remark 5.5). Therefore, \(P^S(A)\) is dense in \(P(A)\) and \(P^S(B)\) is dense in \(P(B)\).

Let us introduce the following notations:

\[
\hat{c}^\circ(\pi_A) := \sup_{b \in B} \hat{c}(\pi_A, b), \quad P^S(\pi_A) := \{\pi_A \in P(A) : \hat{c}^\circ(\pi_A) \leq \alpha\}, \quad \alpha \in \mathbb{R},
\]

\[
\hat{c}^\circ(\pi_B) := \inf_{a \in A} \hat{c}(a, \pi_B), \quad P^S(\pi_B) := \{\pi_B \in P(B) : \hat{c}^\circ(\pi_B) \geq \beta\}, \quad \beta \in \mathbb{R}.
\]

(5.1)

Remarks 5.4 and 5.5 imply respectively that \(\hat{c}^\circ(\pi_A) > -\infty\) for all \(\pi_A \in P(A)\) and \(\hat{c}^\circ(\pi_B) < +\infty\) for all \(\pi_B \in P(B)\).

Theorem 5.7. Let \(\{A, B, c\}\) be a two-person zero-sum game introduced in Definition 5.1 and \((\pi_A, \pi_B) \in P(A) \times P(B)\). Then the following two equalities hold:

\[
\hat{c}^\circ(\pi_A) = \sup_{\pi_B \in P^S(B)} \hat{c}(\pi_A, \pi_B); \quad \hat{c}^\circ(\pi_B) = \inf_{\pi_A \in P^S(A)} \hat{c}(\pi_A, \pi_B),
\]

(5.2)

where \(\hat{c}^\circ\) and \(\hat{c}^\circ\) are defined in (5.1).

Proof. It is sufficient to establish equality (5.2) for each \(\pi_A \in P(A)\). Indeed, if we apply this statement to the game \(\{B, A, -c^{A+B}\}\), where \(c^{A+B}(b, a) = c(a, b)\) for each \(a \in A\) and \(b \in B\), and each \(\pi_B \in P(B)\), then we obtain equality (5.3).

Let us prove that equality (5.2) holds for each \(\pi_A \in P(A)\). Fix an arbitrary \(\pi_A \in P(A)\).

According to Remark 5.4, the expected payoff \(\hat{c}(\pi_A, b)\) to Player II is well-defined for each \(b \in B\). Then the inequality

\[
\sup_{\pi_B \in P^S(B)} \hat{c}(\pi_A, \pi_B) \geq \sup_{b \in B} \hat{c}(\pi_A, b) = \hat{c}^\circ(\pi_A)
\]

15
holds because each pure strategy for Player II can be interpreted as the mixed strategy concentrated in a point. Now let us prove that
\[ \hat{c}^d(\pi^A) \leq \sup_{\pi^B \in P^S_\pi^A(B)} \hat{c}(\pi^A, \pi^B). \]

If \( \sup_{b \in B} \hat{c}(\pi^A, b) = +\infty \), then the inequality
\[ \sup_{\pi^B \in P^S_\pi^A(B)} \hat{c}(\pi^A, \pi^B) \leq \sup_{b \in B} \hat{c}(\pi^A, b) \] (5.4)
obviously holds. Let \( \sup_{b \in B} \hat{c}(\pi^A, b) < +\infty \). Inequality (5.4) holds if and only if
\[ \hat{c}(\pi^A, \pi^B) \leq \sup_{b \in B} \hat{c}(\pi^A, b) \] (5.5)
for each \( \pi^B \in P^S_\pi^A(B) \). The rest of the proof establishes inequality (5.5).

Let us fix an arbitrary \( \pi^B \in P^S_\pi^A(B) \). Since either \( \hat{c}^-(\pi^A, \pi^B) > -\infty \) or \( \hat{c}^+(\pi^A, \pi^B) < +\infty \), then the Fubini-Tonelli theorem yields that
\[ \hat{c}(\pi^A, \pi^B) = \int_B \hat{c}(\pi^A, b) \pi^B(db), \]
which implies (5.5). Inequality (5.4) is proved.

Remark 5.8. According to (5.1) and Assumptions (iv) and (v) for the game \( \{A, B, c\} \) (see also Remarks 5.4 and 5.5 and Theorem 5.7), the inequality
\[ \hat{c}^d(\pi^B) \leq \hat{c}^d(\pi^A) \] (5.6)
holds for all \( \pi^A \in P(A) \) and for all \( \pi^B \in P^S(B) \). Indeed, for \( \pi^B \in P^S(B) \) and for \( \pi^A \in P(A) \),
\[ \hat{c}^d(\pi^B) = \inf_{\pi^A \in P(A)} \hat{c}(\pi^A, \pi^B) \leq \hat{c}(\pi^A, \pi^B) \leq \sup_{\pi^B \in P^S_\pi^A(B)} \hat{c}(\pi^A, \pi^B) = \hat{c}^d(\pi^A). \]

Since it is not clear whether inequality (5.6) holds for \( \pi^B \in P^U(B) \), the following definition introduces the value in the asymmetric form.

Definition 5.9. If the equality
\[ \sup_{\pi^B \in P^S(B)} \hat{c}^d(\pi^B) = \inf_{\pi^A \in P(A)} \hat{c}^d(\pi^A) \] (5.7)
holds, then we say that this common is the value of the game \( \{A, B, c\} \). We denote this value by \( v \).

Let us set
\[ P^S_{<\alpha}(A) := \{\pi^A \in P(A) : \hat{c}^d(\pi^A) < \alpha\}, \quad \alpha \in \mathbb{R}, \]
\[ P^S_{>\beta}(B) := \{\pi^B \in P(B) : \hat{c}^d(\pi^B) > \beta\}, \quad \beta \in \mathbb{R}, \]
where \( \hat{c}^d \) and \( \hat{c}^d \) are defined in (5.1).

Lemma 5.10. Let \( \{A, B, c\} \) be a two-person zero-sum game introduced in Definition 5.1. Then the following statements hold:
(a) the function $\hat{c}^\#$ is convex on $P(\mathcal{A})$;

(b) the function $\hat{c}^\prime$ is concave on $P(\mathcal{B})$;

(c) the sets $P^\#_\alpha(\mathcal{A})$, $P^\#_{=\alpha}(\mathcal{A})$, $P^\#_\beta(\mathcal{B})$, and $P^\#_{\geq\beta}(\mathcal{B})$ are convex for all $\alpha, \beta \in \mathbb{R}$.

**Proof.** Let us prove statement (a). Indeed, let $\pi^\alpha_1, \pi^\alpha_2 \in P(\mathcal{A})$ and $\alpha \in (0, 1)$. If either $\hat{c}^\#(\pi^\alpha_1) = +\infty$ or $\hat{c}^\#(\pi^\alpha_2) = +\infty$, then $\hat{c}^\#(\alpha \pi^\alpha_1 + (1 - \alpha) \pi^\alpha_2) \leq \alpha \hat{c}^\#(\pi^\alpha_1) + (1 - \alpha) \hat{c}^\#(\pi^\alpha_2)$. Otherwise, $\pi^\alpha_1, \pi^\alpha_2 \in P^\#_{<\infty}(\mathcal{A})$ and

$$
\alpha \hat{c}^\#(\pi^\alpha_1) + (1 - \alpha) \hat{c}^\#(\pi^\alpha_2) = \alpha \sup_{b \in \mathcal{B}} \hat{c}(\pi^\alpha_1, b) + (1 - \alpha) \sup_{b \in \mathcal{B}} \hat{c}(\pi^\alpha_2, b)
\geq \sup_{b \in \mathcal{B}} \hat{c}(\alpha \pi^\alpha_1 + (1 - \alpha) \pi^\alpha_2, b) = \hat{c}^\#(\alpha \pi^\alpha_1 + (1 - \alpha) \pi^\alpha_2).
$$

(5.8)

Since $\pi^\alpha_1, \pi^\alpha_2 \in P(\mathcal{A})$ and $\alpha \in (0, 1)$ are arbitrary, then (5.8) yields that the worst-loss function $\hat{c}^\#$ is convex on $P(\mathcal{A})$. Statement (a) is proved.

Statement (b) follows from statement (a) applied to $\{\mathcal{B}, \mathcal{A}, -c^{\mathcal{A}+\mathcal{B}}\}$, where $c^{\mathcal{A}+\mathcal{B}}(a, b) := c(a, b)$ for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Statement (c) follows from statements (a) and (b).

\[ \square \]

### 5.2 The Existence of a Value

The following Theorem 5.11 provides sufficient conditions for the existence of a value for a two-person zero-sum game with possibly noncompact action sets and unbounded payoffs and describes the property of the solution set for one of the player under these conditions. For well-defined payoff functions, the proof of the existence of the value is usually based on Sion’s theorem (Mertens et al. [14, Theorem I.1.1]) that requires that at least one of the decision sets is compact. In our situation, both decision sets may not be compact. In addition, the payoff function $c$ may be unbounded above and below, and therefore the payoff function $\hat{c}$ may be undefined for some pairs of mixed strategies. Because of these reasons, our proof of the existence of the value does not use Sion’s theorem. In general, a game on the unit square with bounded measurable payoffs may not have a value; see Yanovskaya [16, p. 527], and the references to counterexamples by Ville, by Wald, and by Sion and Wolfe cited there. Therefore, some conditions on continuity of payoff functions are needed, and Theorem 5.11 requires mild assumptions (i)–(ii).

**Theorem 5.11.** Let a two-person zero-sum game $\{\mathcal{A}, \mathcal{B}, c\}$ introduced in Definition 5.1 satisfy the following assumptions:

(i) for each $b \in \mathcal{B}$ the function $a \to c(a, b)$ is lower semi-continuous;

(ii) there exists $b_0 \in \mathcal{B}$ such that the function $a \to c(a, b_0)$ is inf-compact on $\mathcal{A}$.

Then the game $\{\mathcal{A}, \mathcal{B}, c\}$ has the value $v$, that is, equality (5.7) holds. Moreover, the set $P^\#(\mathcal{A})$ is a nonempty convex compact subset of $P(\mathcal{A})$.

Let $F(S)$ denotes be the family of all finite subsets of a set $S$. The proof of Theorem 5.11 uses the following theorem.
Theorem 5.12. (Aubin and Ekeland [2, Theorem 6.2.2]) Let $A$ and $B$ be nonempty convex subsets of vector spaces and $f : A \times B \to \mathbb{R}$ be a function such that $a \to f(a, b)$ is convex and $b \to f(a, b)$ is concave. Then the following equality holds:

$$
\sup_{b \in B} \inf_{a \in A} f(a, b) = \sup_{F \in \mathcal{F}(B)} \inf_{a \in A} \max_{b \in F} f(a, b). 
$$

(5.9)

Proof of Theorem 5.11. Observe that the following statements hold:

(i) the sets $\mathbb{P}^d_{\leq +\infty}(A)$ and $\mathbb{P}^{fs}(B)$ are nonempty and convex;

(ii) the function $\hat{c} : \mathbb{P}^d_{\leq +\infty}(A) \times \mathbb{P}^{fs}(B) \to \mathbb{R}$ is well-defined and affine in each variable;

(iii) the function $\hat{c}(\cdot, \pi^B) : \mathbb{P}(A) \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous for each $\pi^B \in \mathbb{P}^{fs}(B)$;

(iv) the function $\hat{c}(\cdot, b_0) : \mathbb{P}(A) \to \mathbb{R} \cup \{+\infty\}$ is inf-compact on $\mathbb{P}(A)$.

Let us prove statements (i)–(iv).

(i) According to Remark 5.5, $\hat{c}^d(\pi^A) < +\infty$ for some $\pi^A \in \mathbb{P}(A)$. Thus the set $\mathbb{P}^d_{\leq +\infty}(A)$ is not empty. Lemma 5.10(c) yields that the set $\mathbb{P}^d_{\leq +\infty}(A)$ is convex. The set $\mathbb{P}^{fs}(B)$ is not empty since the set of pure strategies for Player II is not empty and each pure strategy for Player II belongs to $\mathbb{P}^{fs}(B)$. The set $\mathbb{P}^{fs}(B)$ is convex because a convex combination of two probability measures on $B$ with finite supports is a probability measure on $B$ with a finite support. Statement (i) is proved.

(ii) Let $\pi^A \in \mathbb{P}^d_{\leq +\infty}(A)$ and $\pi^B \in \mathbb{P}^{fs}(B)$. The definition of $\mathbb{P}^{fs}(B)$ yields the existence of $M = 1, 2, \ldots, \{\beta(m)\}_{m=1,2,\ldots,M} \subseteq [0, 1]$, and $\{b(m)\}_{m=1,2,\ldots,M} \subseteq B$ such that $\beta(1) + \beta(2) + \ldots + \beta(M) = 1$ and $\pi^B(B) = \beta(1)\mathbf{I}\{b(1) \in B\} + \beta(2)\mathbf{I}\{b(2) \in B\} + \ldots + \beta(M)\mathbf{I}\{b(M) \in B\}$ for each $B \in \mathcal{B}(B)$, where $\mathbf{I}\{b \in B\} = 1$ whenever $b \in B$ and $\mathbf{I}\{b \in B\} = 0$ otherwise. Since the function $a \to c(a, b)$ is bounded from below on $A$ for each $b \in B$, then

$$
\hat{c}(\pi^A, \pi^B) = \int_A \left( \sum_{m=1}^M \beta(m) c^- (a, b(m)) \right) \pi^A(da) \geq \sum_{m=1}^M \beta(m) \inf_{a \in A} c^-(a, b(m)) > -\infty,
$$

(5.10)

which implies that $\hat{c}(\pi^A, \pi^B)$ is well-defined for all $\pi^A \in \mathbb{P}^d_{\leq +\infty}(A)$ and for all $\pi^B \in \mathbb{P}^{fs}(B)$. Furthermore,

$$
\hat{c}(\pi^A, \pi^B) \leq \hat{c}^d(\pi^A) < +\infty,
$$

(5.11)

for all $\pi^A \in \mathbb{P}^d_{\leq +\infty}(A)$ and for all $\pi^B \in \mathbb{P}^{fs}(B)$, where the first inequality follows from the definition of the function $\hat{c}^d$, equality (5.2) and Remark 5.4. The second inequality from (5.11) follows from the definition of the set $\mathbb{P}^d_{\leq +\infty}(A)$.

Inequalities (5.10) and (5.11) yield that the function $\hat{c}(\cdot, \cdot)$ takes finite values on $\mathbb{P}^d_{\leq +\infty}(A) \times \mathbb{P}^{fs}(B)$. This function is affine in each variable on $\mathbb{P}^d_{\leq +\infty}(A) \times \mathbb{P}^{fs}(B)$ because of the basic properties of the Lebesgue integral. Statement (ii) is proved.

(iii) Let us fix an arbitrary $\pi^B \in \mathbb{P}^{fs}(B)$. As shown in the proof of (i), there exist $M = 1, 2, \ldots$, $\{\beta(m)\}_{m=1,2,\ldots,M} \subseteq [0, 1]$, and $\{b(m)\}_{m=1,2,\ldots,M} \subseteq B$ such that $\beta(1) + \beta(2) + \ldots + \beta(M) = 1$, and $\pi^B(B) = \beta(1)\mathbf{I}\{b(1) \in B\} + \beta(2)\mathbf{I}\{b(2) \in B\} + \ldots + \beta(M)\mathbf{I}\{b(M) \in B\}$ for each $B \in \mathcal{B}(B)$. Since
\( \hat{c}(\pi^A, \pi^B) = \beta(1)\hat{c}(\pi^A, b(1)) + \beta(2)\hat{c}(\pi^A, b(2)) + \ldots + \beta(M)\hat{c}(\pi^A, b(M)) \) for each \( \pi^A \in \mathbb{P}(A) \), then it is sufficient to prove that the function \( \hat{c}(\cdot, b) : \mathbb{P}(A) \to \mathbb{R} \cup \{+\infty\} \) is lower semi-continuous for each \( b \in B \) because a convex combination of a finite number of bounded below lower semi-continuous functions is lower semi-continuous. Lemma 4.1, being applied to \( S_1 = \{ b \} \), \( S_2 = \mathbb{A} \), \( f(s_1, s_2) = c(s_2, s_1) \), and \((s_1, s_2) \in S_1 \times S_2 \), implies that the function \( \hat{c}(\cdot, b) : \mathbb{P}(A) \to \mathbb{R} \cup \{+\infty\} \) is inf-compact on \( \mathbb{P}(A) \). Statement (i4) is proved.

Assumption (i) and Theorem 4.2, being applied to \( S_1 = \{ b \} \), \( S_2 = \mathbb{A} \), \( f(s_1, s_2) = c(s_2, s_1) \), \((s_1, s_2) \in S_1 \times S_2 \), yield that the function \( \hat{c}(\cdot, b_0) : \mathbb{P}(A) \to \mathbb{R} \cup \{+\infty\} \) is inf-compact on \( \mathbb{P}(A) \). Statement (i2) is proved.

Let us prove that the equality (5.7) holds. In view of inequality (5.6), it is sufficient to prove that
\[
\inf_{\pi^A \in \mathbb{P}(A)} \hat{c}^2(\pi^A) \leq \sup_{\pi^B \in \mathbb{P}^S(B)} \hat{c}^2(\pi^B). \tag{5.12}
\]
We denote the left-hand side of inequality (5.12) by \( v^\flat \) and the right-hand side of inequality (5.12) by \( v^\sharp \). In virtue of \( \mathbb{P}^{fs}(B) \subseteq \mathbb{P}^S(B) \),
\[
\sup_{\pi^B \in \mathbb{P}^{fs}(B)} \hat{c}^2(\pi^B) \leq v^\sharp. \tag{5.13}
\]
Since \( \mathbb{P}^{fs}(B) \subseteq \mathbb{P}^S(B) \) (see Remark 5.6), then (5.1) and (5.3) yield that for each \( \pi^B \in \mathbb{P}^{fs}(B) \)
\[
\hat{c}^2(\pi^B) = \inf_{a \in A} \hat{c}(a, \pi^B) = \inf_{\pi^A \in \mathbb{P}(A)} \hat{c}(\pi^A, \pi^B). \tag{5.14}
\]
Remark 5.5 and the definition of the set \( \mathbb{P}_{<+\infty}^A \) imply that each pure strategy of Player I belongs to \( \mathbb{P}_{<+\infty}^A \subseteq \mathbb{P}(A) \). Therefore, (5.14) implies
\[
\hat{c}^2(\pi^B) = \inf_{\pi^A \in \mathbb{P}_{<+\infty}^A} \hat{c}(\pi^A, \pi^B), \tag{5.15}
\]
for each \( \pi^B \in \mathbb{P}^{fs}(B) \). Inequality (5.13) and equality (5.15) yield
\[
\sup_{\pi^B \in \mathbb{P}^{fs}(B)} \inf_{\pi^A \in \mathbb{P}_{<+\infty}^A} \hat{c}(\pi^A, \pi^B) \leq v^\sharp. \tag{5.16}
\]
In view of properties (i1) and (i2), Theorem 5.12, with \( A = \mathbb{P}_{<+\infty}^A \), \( B = \mathbb{P}^{fs}(B) \), and \( f = \hat{c} \), yields
\[
\sup_{\pi^B \in \mathbb{P}^{fs}(B)} \inf_{\pi^A \in \mathbb{P}_{<+\infty}^A} \hat{c}(\pi^A, \pi^B) = \sup_{F \in \mathbb{P}(\mathbb{P}^{fs}(B))} \inf_{\pi^A \in \mathbb{P}_{<+\infty}^A} \max \hat{c}(\pi^A, \pi^B). \tag{5.17}
\]
Let \( F_0(\mathbb{P}^{fs}(B)) \) denotes the family of all finite subsets of \( \mathbb{P}^{fs}(B) \) containing the pure strategy of Player II concentrated at the point \( b_0 \in B \), whose existence is stated in Assumption (ii). Since \( \mathbb{P}_{<+\infty}^A \subseteq \mathbb{P}(A) \) and \( F_0(\mathbb{P}^{fs}(B)) \subset \mathbb{P}(\mathbb{P}^{fs}(B)) \), then
\[
v^* := \sup_{F \in F_0(\mathbb{P}^{fs}(B))} \inf_{\pi^A \in \mathbb{P}(A)} \max \hat{c}(\pi^A, \pi^B) \leq \sup_{F \in \mathbb{P}(\mathbb{P}^{fs}(B))} \inf_{\pi^A \in \mathbb{P}_{<+\infty}^A} \max \hat{c}(\pi^A, \pi^B). \tag{5.18}
\]
If
\[
v^\flat \leq v^*, \tag{5.19}
\]
then formulae (5.16)–(5.18) imply (5.12). Therefore, if inequality (5.19) holds, then equality (5.7) holds.

Let us prove (5.19). Statements (i3) and (i4) yield that the function \( \max_{\pi \in \mathcal{P}} \hat{c}(\cdot, \pi^B) \) is inf-compact on \( \mathbb{P}(A) \) for each \( F \in \mathbb{F}_0(\mathbb{P}^f(B)) \). Therefore, for each \( F \in \mathbb{F}_0(\mathbb{P}^f(B)) \) there exists \( \pi^A_F \in \mathbb{P}(A) \) such that \( \pi^A_F = \arg \min_{\pi^A \in \mathbb{P}(A)} \max_{\pi \in \mathcal{P}} \hat{c}(\pi^A, \pi^B) \). The definition of \( v^* \) given in (5.18) implies that \( \pi^A_F \in \cap_{\pi \in \mathcal{P}} \mathcal{D}_{\hat{c}(\cdot, \pi)}(v^*) \) for each \( F \in \mathbb{F}_0(\mathbb{P}^f(B)) \). Thus, for each \( F \in \mathbb{F}_0(\mathbb{P}^f(B)) \),

\[
\cap_{\pi \in \mathcal{P}} \mathcal{D}_{\hat{c}(\cdot, \pi)}(v^*) \neq \emptyset.
\]  \( (5.20) \)

Statement (i3) and Remark 1.2 yield that the set \( \mathcal{D}_{\hat{c}(\cdot, \pi)}(v^*) \) is closed for each \( \pi^B \in \mathbb{P}^f(B) \). Statement (i4) implies that the set \( \mathcal{D}_{\hat{c}(\cdot, \pi^B)}(v^*) \) is compact. Since, as follows from (5.20), the collection \( \{ \mathcal{D}_{\hat{c}(\cdot, \pi)}(v^*) \cap \mathcal{D}_{\hat{c}(\cdot, \pi^B)}(v^*) \}_{\pi \in \mathbb{P}^f(B)} \) of closed subsets of the compact set \( \mathcal{D}_{\hat{c}(\cdot, \pi^B)}(v^*) \) satisfies the finite intersection property, then this collection has a nonempty intersection, that is, there exists \( \pi^A \in \mathbb{P}(A) \) such that \( \pi^A \in \cap_{\pi \in \mathbb{P}^f(B)} \mathcal{D}_{\hat{c}(\cdot, \pi)}(v^*) \). Thus \( \hat{c}(\pi^A, \pi^B) \leq v^* \) for all \( \pi^B \in \mathbb{P}^f(B) \), and therefore

\[
\sup_{\pi \in \mathbb{P}^f(B)} \hat{c}(\pi^A, \pi^B) \leq v^*.
\]  \( (5.21) \)

We note that the inequality

\[
\hat{c}^\sharp(\pi^A) \leq \sup_{\pi \in \mathbb{P}^f(B)} \hat{c}(\pi^A, \pi^B)
\]  \( (5.22) \)

holds because each pure strategy of Player II is a probability measure on \( B \), which support is a singleton, and therefore each strategy of Player II belongs to \( \mathbb{P}(B) \).

Inequalities (5.21) and (5.22) and the definition of \( \hat{c}^\sharp \) imply inequality (5.19), which yields inequality (5.12). Thus, equality (5.7) holds.

Let us prove that the set \( \mathbb{P}^f(A) \) is a nonempty convex compact subset of \( \mathbb{P}(A) \). The nonemptyness of the set \( \mathbb{P}^f(A) \) follows from (5.21) and (5.22), because \( v^* = \hat{v}^* = v \), where \( v \) is defined in Definition 5.9. Moreover, according to properties (i2)–(i4), the set \( \mathcal{D}_{\hat{c}(\cdot, \pi^B)}(v) \) is a convex compact subset of \( \mathbb{P}(A) \) and the set \( \mathcal{D}_{\hat{c}(\cdot, \pi)}(v) \) is a convex closed subset of \( \mathbb{P}(A) \) for each \( \pi^B \in \mathbb{P}^f(B) \). Therefore,

\[
\mathbb{P}^f(A) = \cap_{\pi \in \mathbb{P}^f(B)} \mathcal{D}_{\hat{c}(\cdot, \pi)}(v) = \cap_{\pi \in \mathbb{P}(B)} \mathcal{D}_{\hat{c}(\cdot, \pi)}(v)
\]

is a nonempty convex compact subset of \( \mathbb{P}(A) \).

The following example describes two-person zero-sum game with noncompact action sets and unbounded payoffs satisfying the assumptions of Theorem 5.11.

**Example 5.13.** Let \( A = B = \mathbb{R} \), \( c(a, b) = a^2 - b^2 \), \( (a, b) \in \mathbb{R}^2 \). Then the game \( \{A, B, c\} \) satisfies the conditions of Theorem 5.11 and \( v = 0 \).

Example 5.13 admits the following interpretation in the form of a simple game of timing (see Yanovskaya [16, Section 6]) with noncompact decision sets. Two teams work on a project consisting of two independent tasks, each performed by one of the teams. The project should be completed on a target date. The project is completed when both tasks are completed, and they should be completed simultaneously. The penalty, in the amount of \( t^2 \) paid to another team for completing its task by \( t \) units of time later or earlier than the
target date, creates incentives to the teams to complete their tasks exactly on time. Of course, there are other payoff functions including \(|t|\) that provide incentives to achieve the same goal.

If

\[
\sup_{\pi^B \in \mathbb{P}^U(B)} \hat{c}^-(\pi^B) \leq \sup_{\pi^A \in \mathbb{P}(A)} \hat{c}^+ (\pi^A),
\]

then the existence of the value \(v\) defined in (5.7) implies that the equality

\[
\sup_{\pi^B \in \mathbb{P}(B)} \hat{c}^+ (\pi^B) = \inf_{\pi^A \in \mathbb{P}(A)} \hat{c}^-(\pi^A)
\]

holds. In particular, (5.23) and (5.24) hold if \(\hat{c}^-(\pi^B) = -\infty\) for all \(\pi^B \in \mathbb{P}^U(B)\). The following example demonstrates that it is possible that under the condition, that the function \((b, a) \rightarrow c(a, b)\) is \(\mathbb{K}\)-inf-compact on \(\mathbb{B} \times \mathbb{A}\), which is a stronger condition than the assumptions of Theorem 5.11, it is possible that \(\hat{c}^+(\pi^B) > -\infty\) for some \(\pi^B \in \mathbb{P}^U(B)\).

**Example 5.14.** The function \((b, a) \rightarrow c(a, b)\) is \(\mathbb{K}\)-inf-compact on \(\mathbb{B} \times \mathbb{A}\), the function \((a, b) \rightarrow c(a, b)\) is \(\mathbb{K}\)-sup-compact on \(\mathbb{A} \times \mathbb{B}\), and there exists \(\pi^B \in \mathbb{P}^U(B)\) such that \(\hat{c}^+(\pi^B) > -\infty\).

Let us set \(A := B := \{1, 2, \ldots\}, c(a, b) := 6^a 4^b I\{b < a\} - 6^b 4^a I\{a < b\}, \pi^B(\{b\}) := \frac{11}{12^b}, b = 1, 2, \ldots\). We consider the discrete metrics on \(\mathbb{A}\) and \(\mathbb{B}\).

The function \((b, a) \rightarrow c(a, b)\) is \(\mathbb{K}\)-inf-compact on \(\mathbb{B} \times \mathbb{A}\) because \(c(a, b) \rightarrow +\infty\), as \(a \rightarrow \infty\), for each \(b = 1, 2, \ldots\). Here we note that a set \(K \subset \mathbb{B}\) is compact if and only if \(K\) is finite. The function \((a, b) \rightarrow c(a, b)\) is \(\mathbb{K}\)-sup-compact on \(\mathbb{A} \times \mathbb{B}\) because \(c(a, b) \rightarrow -\infty\), as \(b \rightarrow \infty\), for each \(a = 1, 2, \ldots\).

We notice that for each \(b = 1, 2, \ldots\)

\[\hat{c}^-(a, \pi^B) = - \sum_{b=a+1}^{\infty} 6^b 4^a \frac{11}{12^b} = -11 \cdot 4^a \sum_{b=a+1}^{\infty} \frac{1}{2^b} = -11 \cdot 2^a,\]

\[\hat{c}^+(a, \pi^B) = \sum_{b=1}^{a-1} 6^a 4^b \frac{11}{12^b} = 11 \cdot 6^a \sum_{b=1}^{a-1} \frac{1}{3^b} = \frac{33}{2} 2^a.\]

Therefore, \(\hat{c}(a, \pi^B) = \frac{11}{2} 2^a - \frac{33}{2} 2^a\) for each \(a = 1, 2, \ldots\). Since \(\hat{c}(a, \pi^B) \rightarrow +\infty\), as \(a \rightarrow \infty\), then \(\hat{c}^+(\pi^B) = \inf_{a \in \mathbb{A}} \hat{c}(a, \pi^B) > -\infty\).

Let us set \(\pi^A(\{a\}) := \frac{1}{2^a}, a = 1, 2, \ldots\). Since \(\pi^A \in \mathbb{P}(\mathbb{A})\) and

\[\hat{c}^-(\pi^A, \pi^B) = - \sum_{a=1,2,\ldots} 11 \cdot 2^a \frac{1}{2^a} = -\infty,\]

\[\hat{c}^+(\pi^A, \pi^B) = \sum_{a=1,2,\ldots} \left( \frac{11}{2} 2^a - \frac{33}{2} 2^a \right) \frac{1}{2^a} = +\infty,\]

then \(\pi^B \in \mathbb{P}^U(B)\).

### 5.3 The Existence of a Solution

This subsection provides the definition of a solution of a two-person zero-sum game with possibly non-compact actions and unbounded payoff. Theorem 5.17 establishes sufficient conditions for the existence of a solution for such game.
**Definition 5.15.** The pair of mixed strategies \((\pi^A, \pi^B) \in P(S(A)) \times P(S(B))\) for Players I and II is called a solution (saddle point, equilibria) of the game \(\{A, B, c\}\), if

\[
\hat{c}(\pi^A, \pi^B) \leq \hat{c}(\pi^A, \pi^B) \leq \hat{c}(\pi^A, \pi^B)
\]  

(5.25)

for each \(\pi^*_A \in P(A)\) and \(\pi^*_B \in P(B)\).

**Remark 5.16.** If the solution \((\pi^A, \pi^B) \in P(S(A)) \times P(S(B))\) of the game \(\{A, B, c\}\) exists, then the number \(v := \hat{c}(\pi^B) = \hat{c}(\pi^A)\) uniquely defines the value of this game.

The following theorem provides sufficient conditions for the existence of a solution.

**Theorem 5.17.** Let two-person zero-sum game \(\{A, B, c\}\) introduced in Definition 5.1 satisfy the following assumptions:

(a) the function \((b, a) \rightarrow c(a, b)\) is H-inf-compact on \(B \times A\);

(b) the function \((a, b) \rightarrow c(a, b)\) is H-sup-compact on \(A \times B\);

(c) the function \((a, b) \rightarrow c(a, b)\) is bounded from below.

Then the following statements hold:

(i) the game \(\{A, B, c\}\) has a solution \((\pi^A, \pi^B) \in P^1(A) \times P^1(B)\);

(ii) the sets \(P^1(A)\) and \(P^1(B)\) are nonempty convex compact subsets of \(P(A)\) and \(P(B)\) respectively;

(iii) a pair of strategies \((\pi^A, \pi^B) \in P(A) \times P(B)\) is a solution of the game \(\{A, B, c\}\) if and only if \(\pi^A \in P^1(A)\) and \(\pi^B \in P^1(B)\).

**Proof.** Assumptions (b) and (c) yield that the space of actions \(B\) for Player II is compact. Therefore, \(P^S(A) = P(A)\) and \(P^S(B) = P(B)\). According to Theorem 5.7 and Lemma 5.10, statement (iii) follows from Aubin and Ekeland [2, Proposition 1, Chapter 6]. Therefore, Theorem 5.11, being applied twice to the games \(\{A, B, c\}\) and \(\{B, A, -c^A \otimes B\}\) respectively, where \(c^A \otimes B\) \((b, a) := c(a, b)\) for each \(a \in A\) and \(b \in B\), implies statements (i) and (ii). \(\square\)

**Remark 5.18.** Assumptions (b) and (c) of Theorem 5.17 yield that the space of actions \(B\) for Player II is compact.

### 5.4 Continuity Properties of Equilibria

Define families of games with action sets and payoff functions depending on a parameter. Let \(X, A, B\) be Borel subsets of Polish spaces, \(K_A \in B(X \times A)\), where \(B(X \times A) = B(X) \otimes B(A)\), \(K_B \in B(X \times B)\), where \(B(X \times B) = B(X) \otimes B(B)\). It is assumed that for each \(x \in X\) the sets \(K_A\) and \(K_B\) satisfy the following two conditions:

\[
A(x) := \{a \in A : (x, a) \in K_A\} \neq \emptyset \quad \text{and} \quad B(x) := \{b \in B : (x, b) \in K_B\} \neq \emptyset.
\]

Let

\[
K := \{(x, a, b) \in X \times A \times B : x \in X, a \in A(x), b \in B(x)\}.
\]
Remark 5.19. We note that $Gr(A) = K_A$, $Gr(B) = K_B$, and $K = Gr(A \times B)$, where $(A \times B)(x) := \{(a, b) : a \in A(x), b \in B(x)\}$, $x \in X$. We note also that $K = Gr(\tilde{B})$, where $\tilde{B}(x, a) := B(x)$, $(x, a) \in K_A$. If we set $\tilde{A}(x, b) := Ax$, $(x, b) \in K_B$, then $Gr(\tilde{A}) = \{(x, b, a) : (x, a, b) \in K\}$ and $K = \{(x, a, b) : (x, b, a) \in Gr(\tilde{A})\}$.

Consider the family of two-person zero-sum games

$$\{\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in X\}.$$  

satisfying for each $x \in X$ all the assumptions from Definition 5.1. Define the function $c^{A\times B} : Gr(\tilde{A}) \subseteq (X \times B) \times A \rightarrow \mathbb{R}$,

$$c^{A\times B}(x, b, a) := c(x, a, b), \quad (x, a, b) \in K.$$  \hspace{1cm} (5.26)

In this subsection we consider the following assumptions:

(A1) the function $c^{A\times B} : Gr(\tilde{A}) \subseteq (X \times B) \times A \rightarrow \mathbb{R} \cup \{+\infty\}$ defined in (5.26) is $K$-inf-compact on $Gr(\tilde{A})$;

(A2) the function $c : K \subseteq (X \times A) \times B \rightarrow \mathbb{R} \cup \{-\infty\}$ is $K$-sup-compact on $K$;

(A3) $A : X \rightarrow S(\mathbb{A})$ is a lower semi-continuous set-valued mapping;

(A4) $B : X \rightarrow S(\mathbb{B})$ is a lower semi-continuous set-valued mapping.

Remark 5.20. According to Lemma 1.7 and Remarks 3.1 and 5.19, Assumption (A1) holds if and only if the following two conditions hold:

(i) the mapping $c : K \subseteq X \times A \times B \rightarrow \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous;

(ii) if a sequence $\{x^{(n)}, b^{(n)}\}_{n=1,2,...}$ with values in $K_B$ converges and its limit $(x, b)$ belongs to $K_B$, then each sequence $\{a^{(n)}\}_{n=1,2,...}$ with $(x^{(n)}, a^{(n)}, b^{(n)}) \in K$, $n = 1, 2, \ldots$, satisfying the condition that the sequence $\{c(x^{(n)}, a^{(n)}, b^{(n)})\}_{n=1,2,...}$ is bounded above, has a limit point $a \in A(x)$.

Remark 5.21. According to Lemma 1.7 and Remark 5.19, Assumption (A2) holds if and only if the following two conditions hold:

(i) the mapping $c : K \subseteq X \times A \times B \rightarrow \mathbb{R} \cup \{-\infty\}$ is upper semi-continuous;

(ii) if a sequence $\{x^{(n)}, a^{(n)}\}_{n=1,2,...}$ with values in $K_A$ converges and its limit $(x, a)$ belongs to $K_A$, then each sequence $\{b^{(n)}\}_{n=1,2,...}$ with $(x^{(n)}, a^{(n)}, b^{(n)}) \in K$, $n = 1, 2, \ldots$, satisfying the condition that the sequence $\{c(x^{(n)}, a^{(n)}, b^{(n)})\}_{n=1,2,...}$ is bounded from below, has a limit point $b \in B(x)$.

Remark 5.22. Assumptions (A1) and (A2) imply that the payoff to Player II, $-\infty < c(x, a, b) < +\infty$, for choosing actions $a \in A(x)$ and $b \in B(x)$ in a state $x \in X$, is a continuous function.

Remark 5.23. Let $\{\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in X\}$ be the family of two-person zero-sum games. Further let $\hat{c}^x(x)$ and $\check{c}^x(x)$ be defined in (5.1) and $v(x)$ denote the value of the game $\{A(x), B(x), c(x, \cdot, \cdot)\}$ if it exists, $x \in X$. 

23
The following theorem provides the sufficient conditions for the lower semi-continuity of the value for a family of two-person zero-sum games with possibly noncompact action sets and unbounded payoffs.

**Theorem 5.24.** Let the family of two-person zero-sum games \( \{\{A(x), B(x), c(x, \cdot, \cdot)\} : x \in X\} \) satisfy Assumptions (A1) and (A4). Then the following statements hold:

(i) for each \( x \in X \) the following equality holds:

\[
\sup_{\pi^B \in P(B(x))} \partial^y(x, \pi^B) = \inf_{\pi^A \in P(A(x))} \partial^x(x, \pi^A) \quad (=: v(x)).
\] (5.27)

Moreover, \( v : X \rightarrow \mathbb{R} \) is a lower semi-continuous function.

(ii) the sets \( \{P^v(x)(A(x)) : x \in X\} \) satisfy the following properties:

(a) for each \( x \in X \) the set \( P^v(x)(A(x)) \) is a nonempty convex compact subset of \( P(A) \);

(b) the graph \( Gr(P^v(\cdot)(A(\cdot))) = \{(x, \pi^A) : x \in X, \pi^A \in P^v(x)(A(x))\} \) is a Borel subset of \( X \times P(A) \);

(c) there exists a measurable mapping \( \phi^A : X \rightarrow P(A) \) such that \( \phi^A(x) \in P^v(x)(A(x)) \) for each \( x \in X \).

**Proof.** Assumption (A1) and Corollary 4.3, being applied to \( X := X \times B, Y := A, f := c^A+B \) on \( Gr(\tilde{A}) \), and \( f := +\infty \) on the complement of \( Gr(\tilde{A}) \), yield that the mapping \( c^A+B : Gr(P(\tilde{A}(\cdot, \cdot))) \subseteq (X \times B) \times P(A) \rightarrow \mathbb{R} \cup \{+\infty\} \), where

\[
c^A+B(x, b, \pi^A) = \int_{A(x)} c(x, a, b)\pi^A(da), \quad (x, b) \in K_B, \pi^A \in P(A(x), b),
\]
is \( K \)-inf-compact on \( Gr(P(\tilde{A}(\cdot, \cdot))) \). Equality (5.27) follows from Theorem 5.11. The least statements follow from Theorem 3.6, being applied to \( X := X, A := P(A), B := B, \Phi_A(\cdot) := P(A(\cdot)), \Phi_B(x, \pi^A) := B(x), x \in X, \) and \( f(x, \pi^A, b) := c(x, \pi^A, b), (x, \pi^A, b) \in \{(x, \pi^A, b) : x \in P(A) \times B : (x, b) \in K_B, \pi^A \in P(A(x))\} \), and from Feinberg et al. [8, Theorem 3.3].

The following example describes a family of two-person zero-sum games satisfying Assumptions (A1) and (A4). Payoff functions are unbounded and decision sets are noncompact for the games in this family.

**Example 5.25.** Let \( X = A = B = \mathbb{R}, K_A = K_B = \mathbb{R}^2, K = \mathbb{R}^3, c(x, a, b) = \varphi_X(x) + \varphi_A(a) + \varphi_B(b), (x, a, b) \in K, \) where \( \varphi_X, \varphi_A, \varphi_B : \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions such that \( \varphi_A(a) \rightarrow +\infty \) as \( |a| \rightarrow \infty \). Then \( c \) is a continuous function on \( \mathbb{R}^3 \) and it satisfies Assumption (A1). Indeed, let a sequence \( \{x(n), b(n)\}_{n=1,2,\ldots} \) with values in \( \mathbb{R}^2 \) converges and its limit \( (x, b) \) belongs to \( \mathbb{R}^2 \), a sequence \( \{a(n)\}_{n=1,2,\ldots} \) with \( (x(n), a(n), b(n)) \in \mathbb{R}^3, n = 1, 2, \ldots, \) satisfy the condition that the sequence \( \{c(x(n), a(n), b(n))\}_{n=1,2,\ldots} \) is bounded above. Then the sequence \( \{\varphi_A(a(n))\}_{n=1,2,\ldots} \) is bounded above and, since \( \varphi_A(a) \rightarrow +\infty \) as \( |a| \rightarrow \infty \), then the sequence \( \{a(n)\}_{n=1,2,\ldots} \) has a limit point \( a \in A(x) = \mathbb{R} \). Therefore, Assumption (A1) holds. Assumption (A4) holds, because the multi-valued mapping \( \Phi : \mathbb{R} \rightarrow S(\mathbb{R}), \Phi(s) = \mathbb{R}, s \in \mathbb{R}, \) is lower semi-continuous on \( \mathbb{R} \).
The following theorem and its corollary describes sufficient conditions for the continuity of the value function and upper semi-continuity of the solution multifunctions for a family of two-person zero-sum games with possibly noncompact action sets and unbounded payoffs.

**Theorem 5.26.** (Continuity of equilibria) *Let a family of two-person zero-sum games* \( \{ \{ A(x), B(x), c(x, \cdot, \cdot) \} : x \in X \} \) *satisfy Assumptions (A1)–(A4) and* \( B \) *be a compact. Then the following statements hold:

(i) for each* \( x \in X \) *the game* \( \{ A(x), B(x), c(x, \cdot, \cdot) \} \) *has a solution* \( (\pi^A, \pi^B) \in \mathbb{P}^v_{v(x)}(A(x)) \times \mathbb{P}^b_{v(x)}(B(x)) \).

Moreover, \( v : X \to \mathbb{R} \) *is a continuous function;

(ii) the sets* \( \{ \mathbb{P}^d_{v(x)}(A(x)) : x \in X \} \) *satisfy the following properties:

(a) for each* \( x \in X \) *the set* \( \mathbb{P}^d_{v(x)}(A(x)) \) *is a nonempty convex compact subset of* \( \mathbb{P}(A) \);

(b) the multifunction* \( \mathbb{P}^d_{v(\cdot)}(A(\cdot)) : X \to \mathcal{K}(\mathbb{P}(A)) \) *is upper semi-continuous;

(iii) the sets* \( \{ \mathbb{P}^b_{v(x)}(B(x)) : x \in X \} \) *satisfy the following properties:

(a) for each* \( x \in X \) *the set* \( \mathbb{P}^b_{v(x)}(B(x)) \) *is a nonempty convex compact subset of* \( \mathbb{P}(B) \);

(b) the multifunction* \( \mathbb{P}^b_{v(\cdot)}(B(\cdot)) : X \to \mathcal{K}(\mathbb{P}(B)) \) *is upper semi-continuous.

**Proof.** According to Theorem 5.17 and Remark 5.18, Theorem 5.24, being applied to \( \{ \{ A(x), B(x), c(x, \cdot, \cdot) \} : x \in X \} \) *and* \( \{ \{ B(x), A(x), -c^{A \leftrightarrow B}(x, \cdot, \cdot) \} : x \in X \} \), *where* \( c^{A \leftrightarrow B}(x, b, a) := c(x, a, b) \) *for each* \( x \in X \), \( a \in A(x) \) *and* \( b \in B(x) \), *yields all the statements of the theorem. \( \square \)

**Corollary 5.27.** *Let a family of two-person zero-sum games* \( \{ \{ A(x), B(x), c(x, \cdot, \cdot) \} : x \in X \} \) *satisfy assumptions of Theorem 5.26. Then there exist measurable mappings* \( \phi^A : X \to \mathbb{P}(A) \) *and* \( \phi^B : X \to \mathbb{P}(B) \) *such that* \( \phi^A(x) \in \mathbb{P}^d_{v(x)}(A(x)) \) *and* \( \phi^B(x) \in \mathbb{P}^b_{v(x)}(B(x)) \) *for all* \( x \in X \). *Moreover, for each* \( x \in X \) *a pair of strategies* \( (\pi^A(x), \pi^B(x)) \) *in* \( \mathbb{P}(A(x)) \times \mathbb{P}(B(x)) \) *is a solution of the game* \( \{ A(x), B(x), c(x, \cdot, \cdot) \} \) *if and only if* \( \pi^A(x) \in \mathbb{P}^d_{v(x)}(A(x)) \) *and* \( \pi^B(x) \in \mathbb{P}^b_{v(x)}(B(x)) \).

**Proof.** *All statements directly follow from statements (ii) and (iii) of Theorem 5.26. \( \square \)

### 6 Notes on One-Step Two-Person Zero-Sum Stochastic Games with Perfect Information

This section shows that for the sequential one-step game studied in Section 3, it is sufficient for the both players to use only pure strategies.

Let \( X, A, \) and \( B \) be Borel subsets of Polish spaces, \( \Phi_A : X \to S(A) \) and \( \Phi_B : \text{Gr}(\Phi_A) \subseteq X \times A \to S(B) \) be set-valued mappings and \( f : \text{Gr}(\Phi_B) \subseteq X \times A \times B \to \mathbb{R} \) be a function. A *one-step two-person zero-sum stochastic game with perfect information* is a tuple \( \{ X, A, B, \Phi_A, \Phi_B, f \} \) satisfying the following assumptions:

(i) \( X \) is the state space;
(ii) \( A \) is the action space of the Player I;

(iii) \( B \) is the action space of the Player II;

(iv) \( \text{Gr}(\Phi_A) \in B(\mathbb{X} \times A) \), where \( B(\mathbb{X} \times A) = B(\mathbb{X}) \otimes B(A) \), is the constrained set for the Player I. It is assumed the existence of a measurable mapping \( \phi_A : \mathbb{X} \to A \) such that \( \phi_A(x) \in \Phi_A(x) \) for each \( x \in \mathbb{X} \). A nonempty Borel subset \( \Phi_A(x) \) of \( A \) represents the set of admissible actions of the Player I in the state \( x \in \mathbb{X} \);

(v) \( \text{Gr}(\Phi_B) \in B(\mathbb{X} \times A \times B) \), where \( B(\mathbb{X} \times A \times B) = B(\mathbb{X}) \otimes B(A) \otimes B(B) \), is the constrained set for the Player II. It is assumed the existence of a measurable mapping \( \phi_B : \mathbb{X} \times A \to B \) such that \( \phi_B(x, a) \in \Phi_B(x, a) \) for each \( (x, a) \in \text{Gr}(\Phi_A) \). A nonempty Borel subset \( \Phi_B(x, a) \) of \( B \) represents the set of admissible actions of the Player II in the state \( x \in \mathbb{X} \) when Player I choose an action \( a \in \Phi_A(x) \);

(vi) the stage cost for Player I, \(-\infty \leq f(x, a, b) \leq +\infty\), for choosing actions \( a \in \Phi_A(x) \) and \( b \in \Phi_B(x, a) \) in a state \( x \in \mathbb{X} \), is a Borel function on \( \text{Gr}(\Phi_B) \).

The decision process proceeds as follows:

- the current state \( x \in \mathbb{X} \) is observed by each player;
- Player I choose an action \( a \in \Phi_A(x) \);
- the result \( a \) is announced to Player II;
- Player II choose an action \( b \in \Phi_B(x, a) \);
- the result \( b \) is announced to Player I;
- Player I pays Player II the amount \( f(x, a, b) \).

For a one-step two-person zero-sum stochastic game with perfect information \( \{ \mathbb{X}, A, B, \Phi_A, \Phi_B, f \} \) let \( f^2 \) be the worst-loss function (for Player I) defined in (3.1), \( v^\ast \) be the minimax function defined in (3.2), and \( \Phi_A^\ast \) and \( \Phi_B^\ast \) be the solution multifunctions defined in (3.3) and (3.4) respectively. If for each \( (x, a) \in \text{Gr}(\Phi_A) \) the function \( b \to f(x, a, b) \) is bounded from above, then, according to Theorem 5.7, the following equalities hold:

\[
\sup_{\pi^B \in \Pi(\Phi_B(x, a))} \int_{\Phi_B(x, a)} f(x, a, b) \pi^B(db) = \sup_{b \in \Phi_B(x, a)} f(x, a, b) = f^2(x, a), \quad (6.1)
\]

for each \( (x, a) \in \text{Gr}(\Phi_A) \). Moreover, if for each \( x \in \mathbb{X} \) the function \( a \to f^1(x, a) \) is bounded from below, then, according to Theorem 5.7, the following equalities additionally hold:

\[
\inf_{\pi^A \in \Pi(\Phi_A(x))} \int_{\Phi_A(x)} f^2(x, a) \pi^A(da) = \inf_{a \in \Phi_A(x)} f^1(x, a) = v^\ast(x), \quad (6.2)
\]

for each \( x \in \mathbb{X} \). Therefore, all theorems and corollary from Section 3 hold for stochastic one-step two-person zero-sum stochastic game with perfect information \( \{ \mathbb{X}, A, B, \Phi_A, \Phi_B, f \} \) when each player possibly choose mixed strategies. According to equalities (6.1) and (6.2), the optimas for each player attain on the sets of respective pure strategies.

Acknowledgements. The authors thank William D. Sudderth for his valuable comments on von Neumann’s and Sion’s minimax theorems. Research of the first author was partially supported by NSF grants CMMI-1335296 and CMMI-1636193.
References


