Abstract

In this paper we examine two voting paradoxes. The first one arises if alternative $x$ has been elected by a given electorate then, ceteris paribus, another alternative, $y$, may be elected if additional voters join the electorate whose favorite alternative is $x$. The second occurs if alternative $y$ has not been elected by a given electorate then, ceteris paribus, $y$ may be elected if additional voters join the electorate whose least preferred alternative is $y$. Following Felsenthal and Tideman [5] and Felsenthal and Nurmi [4], we refer to the first as P-TOP and to the latter as P-BOT paradox. To the best of our knowledge, out of all Condorcet consistent voting correspondences proposed in the literature so far, only the Minimax rule is immune to both of the above paradoxes. We introduce a new voting correspondence that is Condorcet consistent and not affected by the P-TOP and P-BOT paradoxes. We then provide a necessary condition for a Condorcet consistent voting correspondence to be not affected by these paradoxes and show that there is a Condorcet consistent voting function that is immune to P-TOP and P-BOT paradoxes.

Keywords: Elections, Condorcet consistency, voting paradoxes, no show paradox
1 Introduction

In the eighteenth century, the Marquis of Condorcet proposed one of today’s most important principles in social choice theory. Condorcet’s principle requires that if there is a candidate that, in pairwise comparison, is favoured by a majority of voters over all other candidates, then this candidate should be declared to be the winner. Further, whenever there is a Condorcet winner then s/he must be elected [2]. Clearly, the appeal of Condorcet’s principle stems from its democratic character.

We say a voting correspondence is Condorcet consistent, if it is consistent with Condorcet’s principle, i.e. whenever there is a Condorcet winner, s/he must be elected. Moulin [9] shows that for more than three alternatives any Condorcet consistent voting function suffers from the no show paradox. The no show paradox is also described by Brams and Fishburn [6] and it occurs in a situation in which a voter benefits from abstaining. Jimeno et al. [7] extend Moulin’s result to voting correspondences. A voting correspondence that is immune to this paradox makes it a dominant strategy for each voter to vote sincerely, rather than to abstain. Due to the above results the democratic character of Condorcet consistent voting correspondences seems to be threatened.

In this paper we consider two stronger versions of the no show paradox, that were extremely curious. The first one arises, if candidate \( x \) has been elected by a given electorate then, ceteris paribus, another candidate, \( y \), may be elected if additional voters join the electorate who rank \( x \) at the top of their preference ordering. The second one occurs if candidate \( y \) has not been elected by a given electorate then, ceteris paribus, \( y \) may be elected if additional voters join the electorate who rank \( y \) at the bottom of their preference ordering. Following Felsenthal and Nurmi [4], as well as Felsenthal and Tideman [5] we refer to the first as P-TOP and to the latter as P-BOT. These two paradoxes are also known as the positive strong no show paradox and the negative strong no show paradox, respectively [10,11]. Pérez [11], as well as Felsenthal and Nurmi [4] show that there exist Condorcet consistent voting correspondences that are free from both or at least from one of these paradoxes, namely the Minimax rule and Young’s rule, respectively. Duddy [3] allows voters’ preferences to be represented by a weak ordering over alternatives and shows that if there are more than three alternatives, then all Condorcet consistent voting functions suffer from both of the stronger instances of the no show paradox.
In this paper we assume that voters’ preferences are represented by a linear ordering over all alternatives. We introduce two properties, namely Top (TP) and Bot property (BP). The Top property says that if an alternative \(x\) is chosen under some preference profile and a new voter whose favorite alternative is \(x\) joins the electorate then under the new profile, \(x\) is still chosen. If a voting correspondence satisfies this property then it is immune to the P-TOP paradox. The Bot property however, requires that if an alternative, let’s say \(x\) is not chosen under some preference profile and a new voter whose least favorite alternative is \(x\) joins the electorate then under the new profile, \(x\) is still not chosen.

The paper is structured as follows. In section two we provide the basic terminology. In section three we contribute to the literature by providing new proofs, showing that the Minimax rule indeed satisfies Condorcet consistency, TP and BP. Then we introduce a new voting correspondence that satisfies all required properties. We then provide a necessary condition for a Condorcet consistent voting correspondence to satisfy TP and BP. Further, we give an example for a new voting function, other then the Minimax rule, that satisfies Condorcet consistency and is immune to P-TOP and P-BOT. Section three concludes.

2 Notation

Given is a (finite) set \(A\) of alternatives and a set of potential voters \(\mathbb{N}\). Each voter is endowed with a linear ordering \(R_i\) on \(A\). The intended interpretation of \(x \succ_i y\) is \(x\) is preferred over \(y\) by voter \(i\). We denote by \(L\) the set of all linear orderings. Given a finite set of voters \(N \subseteq \mathbb{N}\), a profile \(R^N \in L^N\) assigns a linear ordering to each voter \(i \in N\). A voting correspondence \(C\) assigns to each set of voters \(N \subseteq \mathbb{N}\) and each profile \(R^N \in L^N\) a nonempty subset of alternatives. A voting function \(F\) is a map that assigns to each \(N \subseteq \mathbb{N}\) and each \(R^N \in L^N\) an alternative \(F(R^N) \in A\). A selection \(f\) from \(C\) is a voting function such that \(f(R^N) \in C(R^N)\) for all \(N \subseteq \mathbb{N}\) and \(R^N \in L^N\).

For any set of voters \(N \subseteq \mathbb{N}\) let

\[
n_{yx}(R^N) = |\{i \in N : y \succ_i x\}| - |\{i \in N : x \succ_i y\}|
\]

for all \(x, y \in A, R^N \in L^N\) denote the difference between the number of voters who prefer \(y\) to \(x\) and the number of voters who prefer \(x\) to \(y\). This is called
the resistance of \( y \) to \( x \). Note that \( n_{yx}(R^N) = -n_{xy}(R^N) \). Further, for all \( x, y \in A, N \subseteq \mathbb{N}, R^N \in L^N \)

\[
m_x(R^N) = \max_{y \neq x} n_{yx}(R^N)
\]

denotes the size of the largest resistance against \( x \). We call this the maximal resistance against \( x \).

Remark 1 If \(|N|\) is even, then \( n_{yx}(R^N) \) is even for all \( x \in A \) and \( R^N \in L^N \). Hence, \( m_x(R^N) \) is even as well. If however, \(|N|\) is odd, then \( n_{yx}(R^N) \) is odd for all \( x \in A \) and \( R^N \in L^N \). Hence, \( m_x(R^N) \) is also odd.

Moreover, let \( t(R^i) \) denote the favorite alternative of agent \( i \), i.e. for all \( i \in N \subseteq \mathbb{N} \):

\[
t(R^i) = x \text{ whenever } xR^iy \text{ for all } y \in A \setminus \{x\}.
\]

Let \( b(R^i) \) denote the least favorite alternative of agent \( i \), i.e. for all \( i \in N \subseteq \mathbb{N} \):

\[
b(R^i) = x \text{ whenever } yR^ix \text{ for all } y \in A \setminus \{x\}.
\]

Let \( N \subseteq \mathbb{N} \) and \( R^N \in L^N \). We write

\[
WCW(R^N) = \{x \in A : m_x(R^N) \leq 0\}
\]

to denote the weak Condorcet winner set. Note that \( m_x(R^N) < 0 \) implies \( WCW(R^N) = \{x\} \). We say that \( x \) is the strong Condorcet winner.

The Minimax set \( M(R^N) \) is defined by

\[
M(R^N) = \{x \in A : m_x(R^N) \leq m_y(R^N)\}
\]

for all \( y \in A, N \subseteq \mathbb{N}, R^N \in L^N \). In what follows we will refer to the correspondence that always selects the Minimax set as the Minimax rule.\footnote{In the literature this correspondence is also known as the Simpson-Kramer rule.}

Definition 1 (Condorcet consistency / CC) A correspondence \( C \) is Condorcet consistent if for all \( N \subseteq \mathbb{N}, R^N \in L^N, WCW(R^N) \neq \emptyset \) implies \( C(R^N) \subseteq WCW(R^N) \).

In words, a voting correspondence is Condorcet consistent if for any electorate and any profile, it chooses a subset of all Condorcet winners, if there are any.
Definition 2 (Strong Condorcet consistency/ SCC) A correspondence $C$ is strongly Condorcet consistent if for all $N \subseteq \mathbb{N}$, $R^N \in L^N$, $WCW(R^N) \neq \emptyset$ implies $C(R^N) = WCW(R^N)$.

In words, a voting correspondence is strongly Condorcet consistent if for any electorate and any profile, it chooses all Condorcet winners, if there are any.

Definition 3 (Top property/ TP) A correspondence $C$ satisfies the top property if: if $x \in C(R^N)$ then $x \in C(R^N, R')$ for all $R' \in L$ with $t(R') = x$ and for all $x \in A$, $N \subseteq \mathbb{N}$ and $R^N \in L^N$.

That is, a correspondence satisfies the top property if an alternative, let’s say $x$ is chosen under some preference profile and a new voter whose favorite alternative is $x$, joins the electorate then under the new profile, $x$ is still chosen.

Definition 4 (Bot property/ BP) A correspondence $C$ satisfies the bot property if: if $x \notin C(R^N)$ then $x \notin C(R^N, R')$ for all $R' \in L$ with $b(R') = x$ and for all $x \in A$, $N \subseteq \mathbb{N}$ and $R^N \in L^N$.

That is, a correspondence satisfies the bot property if an alternative, let’s say $x$ is not chosen under some preference profile and a new voter joins the electorate, that has $x$ as least favorite alternative, then under the new profile, $x$ is still not chosen.

Definition 5 (Independence of irrelevant alternatives/ IIA) A selection $f$ from a correspondence $C$ satisfies IIA with respect to $C$ if: For $N, M \subseteq \mathbb{N}$ and any two profiles $R^N \in L^N$ and $\bar{R}^M \in L^M$, if $C(\bar{R}^M) \subseteq C(R^N)$ and $f(R^N) \in C(\bar{R}^M)$ then $f(\bar{R}^M) = f(R^N)$

In words, a selection satisfies IIA if, if it selects an alternative from a set, then it selects this alternative from any subset that contains it.

3 Compatibility of (S)CC, TP and BP

3.1 The Minimax Rule

First, let us examine a voting rule that satisfies SCC, TP and BP, namely the Minimax rule. Although this result is already known, we contribute to the literature by providing a new proof.²

²There are already some papers containing this result. We refer the reader for instance to [4], [5] and [11]
Theorem 1  \( M \) satisfies SCC, TP and BP.

Proof. That the Minimax rule satisfies SCC is obvious. First we show that the Minimax rule satisfies TP. Suppose \( x \in M(R^N) \). Then \( m_x(R^N) \leq m_y(R^N) \) for all \( y \in A \setminus \{x\} \). Add a preference \( R' \in L \) such that \( t(R') = x \). Then \( m_x(R^N, R') = m_x(R^N) - 1 \) and \( m_y(R^N, R') \geq m_y(R^N) - 1 \), for all \( y \in A \setminus \{x\} \). It holds that \( m_x(R^N, R') \leq m_y(R^N, R') \) for all \( y \in A \setminus \{x\} \).

It remains to show that the Minimax rule satisfies BP. Suppose \( x \notin M(R^N) \). Choose \( y \) with \( m_x(R^N) > m_y(R^N) \). Add a preference \( R' \in L \) such that \( b(R') = x \). Then \( m_x(R^N, R') = m_x(R^N) + 1 \) and \( m_y(R^N, R') \leq m_y(R^N) + 1 \). Hence, \( m_x(R^N, R') > m_y(R^N, R') \). □

Absence of the Top property is called the P-Top Paradox [4]. The P-TOP Paradox says that it is possible that if candidate \( x \) has been elected by a given electorate then, ceteris paribus, another candidate, \( y \), may be elected if one additional voter joins the electorate whose favorite alternative is \( x \). One possible interpretation is then that for this voter, abstaining was beneficial. The P-BOT Paradox on the other hand states that it is possible that if candidate \( y \) has not been elected by a given electorate then, ceteris paribus, \( y \) may be elected if an additional voter joins the electorate who ranks \( y \) at the bottom of her preference ordering. Again for this voter it was beneficial to abstain. Note that the P-TOP and the P-BOT Paradoxes are then stronger instances of the no show paradox. The no show paradox arises in situations where voters can benefit from abstaining. Again, the existence of the paradox is interpreted as a lack of participation. According to our opinion, when considering voting correspondences, in order to judge whether abstaining is beneficial or not, one must derive preferences over sets from the given preferences over alternatives. We refer the reader to [7] that gives an extension of Moulin’s result for voting correspondences.

As the Minimax rule is not a function by definition, in what follows, we focus on selections from the Minimax set. A very common way to break ties is to first fix a tiebreaking ordering, either by some external authority or by copying the preferences of a distinguished voter. As selecting according to a fixed order is a natural and easy way, we will give further insights into it.

First we will establish Lemma 1 in order to be able to prove Proposition 1, just to then provide the main result in this section, namely that any selection \( f \) from \( M \) that satisfies IIA also satisfies TP and BP. First we provide a further definition.
Definition 6 A correspondence \( C \) is surjective if for every \( \emptyset \neq B \subseteq A \) there is a finite \( N \subseteq \mathbb{N} \) and \( R^N \in L^N \) such that \( C(R^N) = B \).

Lemma 1 Suppose \( C \) satisfies SCC. Then \( C \) is surjective.

Proof. Let \( \emptyset \neq B = \{x_1, ..., x_m\} \subseteq A \) and let \( N = \{1, 2\} \). Let \( R_1 \) be a preference such that \( x_1R_1x_2R_1...R_1x_mR_1y \) for all \( y \in A \setminus B \) and let \( R_2 \) be a preference such that \( x_mR_2x_{m-1}R_2...R_2x_1R_2y \) for all \( y \in A \setminus B \). Then \( m_x(R^N) = 0 \) for every \( x \in B \) and \( m_y(R^N) = 2 \) for every \( y \in A \setminus B \). Hence, \( WCW(R^N) = B \), so that \( C(R^N) = B \). \(\square\)

Remark 2 Notice, that we can construct the proof in an analogous way for any \( N \subseteq \mathbb{N} \) with \( |N| \) being an even number. If however, the number of voters is odd, this is not possible. Take \( N \subseteq \mathbb{N} \) with \( |N| \) being an odd number. We show that if \( WCW(R^N) \neq \emptyset \) then \( |WCW(R^N)| = 1 \) for all \( R^N \in L^N \). Suppose \( WCW(R^N) \neq \emptyset \). By \( |N| \) being odd and Remark 1, \( m_x(R^N) \neq 0 \) for all \( x \in A \). Since \( WCW(R^N) \neq \emptyset \), there is \( x \in A \) such that \( m_x(R^N) < 0 \). Then \( x \) is strong Condorcet winner. As mentioned before, this is unique.

Proposition 1 Let \( C \) be surjective and let \( f \) be a selection from \( C \). Then \( f \) satisfies IIA with respect to \( C \) if and only if \( f \) chooses the maximal alternative from \( C(R^N) \) according to some ordering \( Q \in L \), for all \( N \subseteq \mathbb{N} \) and \( R^N \in L^N \).

Proof. \( \Rightarrow \) : To show: If a selection \( f \) from \( C \) satisfies IIA then there exists an ordering \( Q \in L \) and \( f \) chooses the maximal alternative from \( C(R^N) \) according to it. Take \( N \subseteq \mathbb{N} \) and \( R^N \in L^N \) such that \( C(R^N) = A \), then \( f(R^N)Qx \) for all \( x \in A \setminus \{f(R^N)\} \). Then take \( N \subseteq \mathbb{N} \) and \( R^N \in L^N \) such that \( C(R^N) = A \setminus \{f(R^N)\} \) and let \( f(R^N)Qf(R^N)Qx \) for all \( x \in \{f(R^N) \cup f(R^N)\} \). And so on. Now, we will show that for any arbitrary set \( B \subseteq A \), \( f \) selects from \( C \) according to \( Q \). Take any \( N \subseteq \mathbb{N} \) and \( R^N \in L^N \) such that \( C(R^N) = B \). We have to show that \( f(R^N) \) is the \( Q \)-maximal element of \( B \). Take \( b \in B \) such that \( bQx \) for all \( x \in B \). Now define \( D = \{x : bQx \forall x \in A\} \cup \{b\} \) and take \( R^N \) such that \( C(R^N) = D \) then, by construction of \( Q \), \( f(R^N) = b \). By IIA, \( f(R^N) = b \).

\( \Leftarrow \) : To show: If there is an ordering \( Q \in L \) and \( f \) chooses the maximal alternative from \( C(R^N) \) according to it then \( f \) satisfies IIA. Suppose \( N, M \subseteq \mathbb{N}, R^N \in L^N, R^M \in L^M, C(R^M) \subseteq C(R^N) \), and \( f(R^N) \in C(R^M) \). Then, since \( f(R^N)Qx \) for all \( x \in C(R^N) \), we also have \( f(R^N)Qx \) for all
$x \in C(\bar{R}^M)$. Hence, $f(R^N) = f(\bar{R}^M)$. □

**Corollary 1** If $C$ satisfies SCC then a selection $f$ from $C$ satisfies IIA with respect to $C$ if and only if $f$ chooses the maximal alternative from $C(R^N)$ according to some ordering $Q \in L$, for all $N \subseteq \mathbb{N}$ and $R^N \in L^N$.

**Proof.** This follows from Lemma 1 and Proposition 1. □

**Theorem 2** Every selection $f$ from $M$ that satisfies IIA with respect to $M$ satisfies TP and BP.

**Proof.** First, we show that every selection $f$ from $M$ that satisfies IIA also satisfies TP. Let $R^N \in L^N$, $x = f(R^N)$, and $R' \in L$ with $t(R') = x$. By IIA of $f$ with respect to $M$, it is sufficient to show that $M(R^N, R') \subseteq M(R^N) \cup M(R^N)$ and $x \in M(R^N, R')$. Let $y \in A$ with $y \notin M(R^N)$. Then $m_y(R^N) > m_x(R^N)$. Then $m_y(R^N, R') \geq m_y(R^N) - 1 > m_x(R^N) - 1 = m_x(R^N, R')$, so that $y \notin M(R^N, R')$. Hence, $M(R^N, R') \subseteq M(R^N) \cup M(R^N)$. Further, for every $z \in A$, $m_x(R^N, R') = m_x(R^N) - 1 \leq m_z(R^N) - 1 \leq m_z(R^N, R')$, so that $x \in M(R^N, R')$.

Now we show that every selection $f$ from $M$ that satisfies IIA satisfies BP. Suppose $x \notin M(R^N)$. Then $f(R^N) \neq x$. Add a preference $R'$ such that $b(R') = x$. By Theorem 1, $x \notin M(R^N, R')$. Hence, $f(R^N, R') \neq x$.

Suppose $x \in M(R^N)$ but $f(R^N) \neq x$. Then there is $y \in M(R^N)$ such that $m_x(R^N) = m_y(R^N) \leq m_z(R^N)$ for all $z \in A \setminus \{x, y\}$. Add a preference $R'$ such that $b(R') = x$. Then $m_x(R^N, R') = m_x(R^N) + 1$ and $m_y(R^N) - 1 \leq m_y(R^N, R') \leq m_y(R^N) + 1$ for all $y \in A \setminus \{x\}$. We then have to consider two cases. Suppose $x \notin M(R^N, R')$. Then $f(R^N, R') \neq x$. Hence, BP is satisfied. Now assume that $x \in M(R^N, R')$. Then $m_x(R^N, R') = m_x(R^N) + 1 = m_y(R^N) + 1 \geq f(R^N, R')$ for all $y \in M(R^N)$. As $f(R^N) \neq x$ and $f$ chooses according to a fixed order $Q$, there is $y \in M(R^N, R')$ such that $yQx$. Hence, $f(R^N, R') \neq x$. □

Subsequently, we show that a selection from $M$ that satisfies CC and BP need not satisfy TP and that a selection from $M$ that satisfies CC and TP need not satisfy BP. As Theorem 2 already indicates, these selections must violate IIA.
For $R^N \in L^N$ and $x \in A$ let $b_m(R^N)(x)$ denote the number of voters who rank $x$ last.

We provide a selection from $M$ that satisfies BP but violates TP. Define

$$M(R^N) = \{x \in M(R^N) : b_m(R^N)(x) \leq b_m(R^N)(y) \text{ for all } y \in M(R^N)\}$$

for all $N \subseteq \mathbb{N}$, $R^N \in L^N$. Let $Q \in L$. Define

$$\bar{f}(R^N) = \begin{cases} x \in M(R^N) : xQy \text{ for all } y \in M(R^N) \setminus \{x\} & \text{if } |N| \text{ is even} \\ x \in M(R^N) : yQx \text{ for all } y \in M(R^N) \setminus \{x\} & \text{if } |N| \text{ is odd} \end{cases}$$

for all $N \subseteq \mathbb{N}$, $R^N \in L^N$.

We show that $\bar{f}$ satisfies BP but not TP. First we provide a proof that $\bar{f}$ violates TP. In order to see this, let $\{a, b, c\}$ and take $Q \in L$ as follows: $aQbQc$. Consider the following preference profile $R^N \in L^N$:

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First note that $|N|$ is even. Further, $n_{ab}(R^N) = n_{bc}(R^N) = 2$ and $n_{ac}(R^N) = 2$. Then $m_a(R^N) = m_b(R^N) = m_c(R^N) = 2$. Also $M(R^N) = \bar{M}(R^N) = \{a, b, c\}$. Hence, $\bar{f}(R^N) = a$. Now add a preference $R' \in L$ such that $aR'cR'b$. Notice that $b_m(R^N)(a) = m_b(R^N, R')(a) = b_m(R^N)(c) = m_b(R^N, R')(c)$. Further $m_a(R^N, R') = m_c(R^N, R') = 1$ and $m_c(R^N, R') = 3$. Then $M(R^N, R') = \bar{M}(R^N, R') = \{a, c\}$. Now, as $|N|$ is odd, $\bar{f}(R^N, R') = c$. Thus we have a contradiction against TP.

We have to show that $\bar{f}$ satisfies BP. By Theorem 1, if $x \not\in M(R^N)$ and $b(R') = x$, then $x \not\in M(R^N, R')$ and we are done. Suppose $x \in M(R^N)$ but $\bar{f}(R^N) \neq x$. Then either $x \not\in \bar{M}(R^N)$, i.e there is $y \in M(R^N)$ such that $b_m(R^N)(x) > b_m(R^N)(y)$, or $x \in \bar{M}(R^N)$ and there is $y \in M(R^N)$ such that $yQz$ for all $z \in M(R^N) \setminus \{y\}$. First suppose $x \not\in \bar{M}(R^N)$ and add $R' \in L$ such that $b(R') = x$. Then $b_m(R^N, R')(x) = b_m(R^N) + 1$ and $b_m(R^N, R')(y) = b_m(R^N)(y)$ for all $y \in M(R^N, R') \setminus \{x\}$. First, we show that there is $y \in M(R^N, R')$ such that $y \in M(R^N)$. Then $m_y(R^N) \leq m_y(R^N) + 2$ for all $z \not\in M(R^N)$. By adding a preference $R'$ we have $m_y(R^N, R') \leq m_y(R^N) + 1$ and $m_z(R^N, R') \geq m_z(R^N) - 1$. Then $m_y(R^N, R') \leq m_z(R^N, R')$.
for all \( z \in A \). Thus, \( y \in M(R^N, R') \). Also, \( b_m(R^N, R')(x) = b_m(R^N) + 1 > b_m(R^N)(y) = b_m(R^N, R')(y) \) for all \( y \in M(R^N, R') \setminus \{x\} \). Hence, \( \tilde{f}(R^N, R') \neq x \). Now suppose \( x \in \bar{M}(R^N) \), but \( \tilde{f}(R^N) \neq x \). Then, there is \( y \in \bar{M}(R^N) \) such that \( yQz \) for all \( z \in \bar{M}(R^N) \setminus \{y\} \) if |\( N \)| is even or \( y\bar{Q}z \) for all \( z \in \bar{M}(R^N) \setminus \{y\} \). Also, \( t_m(R^N)(x) = t_m(R^N)(y) \). Add a preference \( R' \) such that \( b(R') = x \). Then \( b_m(R^N, R')(x) = b_m(R^N)(x) + 1 \) while \( b_m(R^N, R')(y) = b_m(R^N) \). Then \( b_m(R^N, R')(x) = b_m(R^N)(x) + 1 > b_m(R^N)(y) = b_m(R^N, R')(y) \). Then \( x \notin \bar{M}(R^N, R') \). By the same argument as above, there is \( y \in \bar{M}(R^N, R') \), that is in \( M(R^N) \). Hence, \( \tilde{f}(R^N) \neq x \). We conclude that BP is satisfied.

Now we present a selection from \( M \) that satisfies TP but not BP. For \( N \subseteq \mathbb{N} \), \( R^N \in L^N \) let

\[
\bar{M}(R^N) = \{x \in M(R^N) : b_m(R^N)(x) \geq b_m(R^N)(y) \text{ for all } y \in M(R^N)\}.
\]

Let \( \tilde{f} \) choose from \( \bar{M}(R^N) \) according to a fixed order \( Q \in L \).

We show that \( \tilde{f} \) satisfies TP but not BP.

Let \( A = \{a, b, c, d, e\} \) and take \( Q \in L \) such that \( aQbQcQdQe \). Consider the following profile \( R^N \in L^N \):

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Then \( n_{ab}(R^N) = 0 \), \( n_{ca}(R^N) = n_{da}(R^N) = n_{ea}(R^N) = 2 \), \( n_{cb}(R^N) = n_{db}(R^N) = n_{eb}(R^N) = 2 \), \( n_{cd}(R^N) = n_{de}(R^N) = n_{ec}(R^N) = 4 \). Then, \( M(R^N) = \{a, b\}, M(R^N) = \{a, b\} \). Hence, \( \tilde{f}(R^N) = a \). Add a voter whose preference is such that \( xR'aR'b \) for all \( x \in A \setminus \{a, b\} \). Then \( \bar{M}(R^N, R') = \{b\} \), hence \( \tilde{f}(R^N, R') = b \). Thus, BP is violated.

Now we show that \( \tilde{f} \) satisfies TP. Take \( N \subseteq \mathbb{N} \), \( R^N \in L^N \) and \( Q \in L \) such that \( \tilde{f}(R^N) = x \). Then \( m_x(R^N) \leq m_y(R^N) \) for all \( y \in A \). Then add a preference \( R' \) such that \( xR'y \) for all \( y \in A \). Then, by Theorem 1 \( x \in M(R^N, R') \) and \( M(R^N, R') \subseteq M(R^N) \) (Proof Theorem 2). It is sufficient to show that \( \bar{M}(R^N, R') \subseteq \bar{M}(R^N) \) and \( \tilde{f}(R^N) \in \bar{M}(R^N, R') \). By \( x \in \bar{M}(R^N) \), \( m_x(R^N) \leq m_y(R^N) \) for all \( y \in A \) and \( b_m(R^N)(x) \geq t_m(R^N)(y) \)
for all \( y \in M(R^N) \). Also, \( b_m(R^N)(x) = b_m(R^N, R')(x) \) as \( b(R') \neq x \). Suppose there is \( y \in \hat{M}(R^N, R') \setminus \hat{M}(R^N) \), i.e. \( b_m(R^N, R')(y) = b_m(R^N)(y) + 1 = b_m(R^N)(x) \). This implies that \( b(R') = y \). Then \( m_y(R^N, R') = m_y(R^N) + 1 \) while \( m_x(R^N, R') = m_x(R^N) - 1 \). Hence \( y \notin M(R^N, R') \) and therefore \( y \notin \hat{M}(R^N, R') \). Also \( b_m(R^N, R')(x) \geq b_m(R^N, R')(y) \) for all \( y \in M(R^N) \). Hence, \( \hat{M}(R^N, R') \subseteq \hat{M}(R^N, R') \) and \( \hat{f}(R^N, R') \in \hat{M}(R^N, R') \). By Proposition 1 \( \hat{f}(R^N, R') = x \).

3.2 A new voting correspondence

Our main contribution in this section is, that we provide a necessary condition for a voting correspondence to satisfy TP and SCC. In order to do so, we present a new correspondence \( H \) that satisfies TP and SCC and then show that it is the maximal one satisfying both properties. So any correspondence that satisfies SCC and TP selects from \( H \). Moreover, we show that this correspondence also satisfies BP. Further, we show that there exists a selection from it, other than a selection from \( M \), that satisfies CC, TP and BP.

\[
H(R^N) = \{ x \in A : \text{for all } y \in A \text{ with } m_y(R^N) < m_x(R^N) \text{ we have } n_{yz}(R^N) < m_y(R^N) \}
\]

for all \( N \subseteq \mathbb{N} \) and \( R^N \in L^N \).

Remarks. For any electorate \( N \subseteq \mathbb{N} \) and any profile \( R^N \in L^N \), \( H(R^N) \) contains all alternatives with minimal maximal resistance. So, \( H(R^N) \supseteq \hat{M}(R^N) \). Further, of those alternatives \( x \in A \) that do not have minimal maximal resistance, \( H(R^N) \) contains the ones where for all alternatives \( y \in A \setminus \{ x \} \) with smaller maximal resistance, the resistance of \( y \) against \( x \) is smaller than the maximal resistance against \( y \).

First, let us establish a lemma in order to prove our main result. As we require SCC to be satisfied and according to the literature there seems to be a conflict between SCC and TP, it is a natural starting point to examine the effect of a new voter joining the electorate on situations where there is a Condorcet winner.

**Lemma 2** Let \( R^N \in L^N \), \( x \in WCW(R^N) \), and \( R' \in L \) such that \( t(R') = x \). Then \( WCW(R^N, R') = \{ x \} \).
Proof. Note that, \( m_x(R^N, R') = m_x(R^N) - 1 < m_x(R^N) \leq 0 \), hence \( x \) is strong Condorcet winner. \( \square \)

Lemma 2 already indicates in which situations the conflict between SCC and TP arises, namely in situations in which there is no Condorcet winner. This will prove useful in the subsequent course of the paper.

**Definition 7** A correspondence \( C \) that satisfies some properties, is the maximal correspondence satisfying those properties, if for every correspondence \( C' \) that satisfies those properties, it holds that \( C'(R^n) \subseteq C(R^n) \) for all \( N \subseteq \mathbb{N} \) and \( R^N \in L^N \).

**Theorem 3** \( H \) is the maximal correspondence satisfying TP and SCC.

Proof. First, we show that \( H \) satisfies SCC. Let \( R^N \in L^N \) such that \( WCW(R^N) \neq \emptyset \). Then, clearly, \( WCW(R^N) \subseteq H(R^N) \). Suppose there is \( x \in A \) and \( x \notin WCW(R^N) \). Take \( y \in WCW(R^N) \). Then \( m_y(R^N) \leq 0 < m_x(R^N) \). Then \( n_{yx}(R^N) \leq 0 \) that is equivalent to \( n_{yx}(R^N) \geq 0 \). Then, \( n_{yx}(R^N) \geq m_y(R^N) \), so that \( x \notin H(R^N) \).

Now, let us show that \( H \) satisfies TP. In view of Lemma 2 we just have to consider the case \( WCW(R^N) = \emptyset \).

Suppose \( WCW(R^N) = \emptyset \) and \( x \notin H(R^N) \). Let \( R' \in L \) be such that \( t(R') = x \).

We have to show that \( x \in H(R^N, R') \). If \( m_x(R^N, R') \leq m_y(R^N, R') \) for all \( y \in A \) we are done. Now suppose there is \( y \in A \) with \( m_y(R^N, R') < m_x(R^N, R') \).

We have to show that \( n_{yx}(R^N, R') < m_y(R^N, R') \).

**Case 1.** If \( m_y(R^N) < m_x(R^N) \) then, since \( x \in H(R^N) \), \( n_{yx}(R^N) < m_y(R^N) \).

Hence \( n_{yx}(R^N, R') = n_{yx}(R^N) - 1 < m_y(R^N) - 1 \). Since \( m_y(R^N, R') \geq m_y(R^N) - 1 \), we have that \( n_{yx}(R^N, R') < m_y(R^N) - 1 \leq m_y(R^N, R') - 1 \).

**Case 2.** If \( m_y(R^N) \geq m_x(R^N) \) then \( m_x(R^N, R') > m_y(R^N, R') \geq m_y(R^N) - 1 \geq m_x(R^N) - 1 = m_x(R^N, R') \), a contradiction.

It remains to show that \( H \) is the maximal correspondence satisfying SCC and TP. Suppose \( H \) is not maximal. Take \( x \in H(R^N) \setminus H(R^N) \). Then \( m_y(R^N) < m_x(R^N) \) and \( n_{yx}(R^N) \geq m_y(R^N) \) for some \( y \in A \setminus \{x\} \). As \( \tilde{H} \) and \( H \) satisfy SCC and \( x \in \tilde{H} \setminus H \), \( m_x(R^N) > m_y(R^N) > 0 \). Hence, \( n_{yx}(R^N) \geq m_y(R^N) > 0 \). This implies that \( n_{xy} < 0 \), i.e. \( x \) does not constitute largest resistance against \( y \). Add a preference \( R' \in L \) such that \( xR'yR'z \) for all \( z \in A \setminus \{x, y\} \). Then \( m_x(R^N, R') = m_x(R^N) - 1 \) and \( m_y(R^N, R') = m_y(R^N) - 1 \).
1. It holds that $m_y(R^N, R') < m_x(R^N, R')$. Furthermore, $n_{yx}(R^N, R') = n_{yz}(R^N) - 1 \geq m_y(R^N) - 1 = m_y(R^N, R')$. If we keep adding preferences $R'$ $n_{yx}(R^N, R', ..., R')$ decreases but still $n_{yx}(R^N, R', ..., R') > m_y(R^N, R', ..., R')$. Hence, at some point $m_y(R^N, R', ..., R') \leq 0$ while $m_x(R^N, R', ..., R') > m_y(R^N, R', ..., R')$. Therefore, $\tilde{H}$ violates TP. □

Theorem 3 states that for any voting correspondence that satisfies SCC to also satisfy TP it is a necessary condition to choose a (nonempty) subset of $H(R^N)$, for any electorate $N \subseteq \mathbb{N}$ and any profile $R^N \in L^N$.

Hereinafter, we provide further insights with respect to BP.

**Proposition 2** $H$ satisfies BP.

*Proof.* Suppose that $x \in H(R^N, R')$ with $b(R') = x$. We show that $x \in H(R^N)$. Take $x \in H(R^N, R')$. Then for all $y \in A$ with $m_y(R^N, R') < m_x(R^N, R')$ we have $n_{yx}(R^N, R') < m_y(R^N, R')$. Then $n_{yx}(R^N) + 1 < m_y(R^N, R')$. Hence $n_{yx}(R^N) < m_y(R^N, R') - 1 \leq m_y(R^N) + 1 - 1 = m_y(R^N)$. Hence, $x \in H(R^N)$. □

**Corollary 2** $H$ is the maximal correspondence satisfying SCC, TP and BP.

*Proof.* This follows immediately from Theorem 3 and Proposition 2. □

It is already known, that there are several Condorcet consistent correspondences that satisfy BP. Felsenthal and Nurmi [4] as well as Pérez [11] show that besides the Minimax correspondence also Young’s correspondence [12] satisfies BP.

Notice that for instance, Young’s correspondence is not a subcorrespondence of $H$. Let $e_x(R^N)$ denote the number of voters that have to be eliminated in order to yield the largest subprofile of $R^N$ such that $x$ is a weak Condorcet winner in the subprofile. Let the Young’s set $Y(R^N)$ be defined by

$$Y(R^N) = \{x \in A : e_x(R^N) \leq e_y(R^N) \text{ for all } y \in A\}$$

for all $N \subseteq \mathbb{N}$, $R^N \in L^N$. We refer to the correspondence that always selects the Young’s set as the Young’s correspondence $Y$. 

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Consider the following example, which is taken from Nurmi and Felsenthal [4]. Let \( N = \{1, 2, ..., 39\} \) and \( A = \{a, b, c, d, e\} \). Consider the profile \( R^N \):

\[
\begin{array}{ccccccc}
\text{no. of voters} & 11 & 10 & 10 & 2 & 2 & 2 & 1 & 1 \\
\text{preferences} & b & e & a & e & c & d & a \\
 & a & c & c & c & d & b & b \\
 & d & b & d & d & c & a & d \\
 & e & d & b & b & d & a & e \\
 & c & a & e & a & a & e & c \\
\end{array}
\]

Note that \( n_{ab}(R^N) = -17 \), \( n_{ac}(R^N) = 5 \), \( n_{ad}(R^N) = 9 \), \( n_{ae}(R^N) = 11 \), \( n_{be}(R^N) = 15 \), \( n_{bd}(R^N) = 9 \), \( n_{ce}(R^N) = 11 \), \( n_{cd}(R^N) = 9 \), \( n_{ce}(R^N) = -13 \), \( n_{de}(R^N) = 9 \). Then \( m_a(R^N) = 17 \), \( m_b(R^N) = 15 \), \( m_c(R^N) = 13 \), \( m_d(R^N) = 9 \), \( m_e(R^N) = 11 \). Also, \( e_a(R^N) = 17 \), as in order for \( a \) to become a weak Condorcet winner, 17 voters in total (e.g. all the voter that prefer \( b \) the most and seven of the ten voters that prefer \( e \) most) must be removed. In order to make \( b \) a weak Condorcet winner, in total 15 voters (e.g. the ten voters whose most preferred alternative is \( e \) and five of the ten voters whose most alternative is \( a \)) must be removed, i.e. \( e_b(R^N) = 15 \). Also, \( e_c(R^N) = 13 \), e.g. the eleven voters that prefer \( b \) the most, and two of the ten voters that prefer \( e \) most.

In order for \( d \) to become a weak Condorcet winner, a total of 16 voters must be eliminated, e.g. the two voters that prefer \( c \) the most, seven of the eleven voters that prefer \( b \) the most and seven of the ten voters that prefer \( e \) the most. Also, \( e_e(R^N) = 11 \), e.g. one has to eliminate the eleven voters that prefer \( b \) the most. Hence, \( Y(R^N) = e \). Note further, that \( e \not\in H(R^N) \) as \( n_{de}(R^N) = 9 \not< m_d(R^N) = 9 \).

Nurmi and Felsenthal [4] as well as Prez [11] show that \( Y \) does not satisfy TP. In both of the papers, the authors add a group of voters to the electorate in order to get a contradiction. Note that their results hold for our set up as well, as there is no difference in adding the voters one after another, or the whole group at once.

It remains to show that \( Y \) satisfies BP. Suppose \( x \not\in Y(R^N) \). Then there is \( y \in A \) such that \( e_y(R^N) < e_x(R^N) \). Add a preference \( R' \) with \( b(R') = x \). Then \( e_x(R^N, R') = e_x(R^N) + 1 \) and \( e_y(R^N, R') \leq e_y(R^N) + 1 \) for all \( y \in A \setminus \{x\} \).

Then there is \( y \in A \) such that \( e_y(R^N, R') \leq e_y(R^N) + 1 < e_x(R^N) + 1 = e_x(R^N, R') \). Hence, \( x \not\in Y(R^N) \).\[\]  

\[\]A verbal version of this proof can be found in [4]. A similar version can be found in [11].
In what follows, we elaborate on the question, whether there is a selection from \( H \), which is not from \( M \) that satisfies TP.

**Theorem 4** There is no selection \( f \) from \( H \) such that \( f \) satisfies IIA with respect to \( H \) and TP.

**Proof.** Proof by counterexample. Let \( N = \{1, \ldots, 10\} \) and \( A = \{a, b, c\} \). Consider the following profile \( R^N \):

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<td>c</td>
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Let \( f \) select from \( H \) according to the following ordering \( aQbQc \). Following from the above profile we have \( m_c(R^N) = n_{bc}(R^N) = 4, m_b(R^N) = n_{ab}(R^N) = 4 \). As \( m_c(R^N) < m_a(R^N) = m_b(R^N) \), we have that \( c \in H(R^N) \). Furthermore, \( a \not\in H(R^N) \) as \( m_a(R^N) = m_c(R^N) > m_c(R^N) \). For \( b \) we have that \( n_{ab}(R^N) < m_c(R^N) = n_{bc}(R^N) \). Therefore, \( H(R^N) = \{b, c\} \) and according to \( Q \), \( f(R^N) = b \). Now add a preference \( R' \) such that \( bR'aR'c \). We then have \( m_a(R^N, R') = m_b(R^N, R') = m_c(R^N, R') = 3 \). Now \( H(R^N, R') = \{a, b, c\} \) and according to \( Q \), \( f(R^N, R') = a \). Thus we have a contradiction to TP. \( \square \)

This result is due to the fact, that by adding a voter whose favorite alternative has been chosen, \( H(R^N, R') \supset H(R^N) \) for some \( N \subseteq N \), \( R^N \in L^N \) and \( R' \in L \). If we choose according to a fixed order \( Q \), there may be \( x \in H(R^N, R') \setminus H(R^N) \) such that \( xQy \) for all \( y \in H(R^N) \), which then leads to a violation of TP.

Notwithstanding, we show that there is a selection \( \tilde{f} \) from \( H \) that satisfies CC and TP and that is not a selection from \( M \). Fix \( A = \{a, b, c\} \) and take \( Q \in L \) as follows: \( bQcQa \), let \( R^* \) be the profile initially given in Theorem 4 and let \( \tilde{f} \) select from \( H \) as follows:

- For all \( N \subseteq N \) and \( R^N \neq R^* \), \( \tilde{f} \) selects from \( M \) according to \( Q \)
- For \( R^* \), \( \tilde{f}(R^*) = b \in H(R^*) \setminus M(R^*) \).
Subsequently, we show that \( \tilde{f} \) satisfies CC and TP. As \( H \) satisfies SCC, \( \tilde{f} \) satisfies CC by definition. In what follows, we show that \( \tilde{f} \) satisfies TP. Due to Theorem 1 we only have to consider the situations where, by adding a voter we yield the profile \( R^* \) and the situation in which we add to \( R^* \) a new voter whose favorite alternative is \( b \).

First, let \( N = \{1, \ldots, 9\} \) and suppose the following profile \( R^N \): three voters prefer \( a \) to \( b \) to \( c \), three voters prefer \( b \) to \( c \) to \( a \) and three voter prefer \( c \) to \( a \) to \( b \). Then \( m_a(R^N) = m_b(R^N) = m_c(R^N) = 3 \). Hence, \( M(R^N) = \{a, b, c\} \) and \( \tilde{f}(R^N) = b \). By adding a preference \( R' \in L \) such that \( t(R') = b \), we have that \( (R^N, R') \neq R^* \). By Theorem 1 we are done.

Now, consider the profile \( R^N \): two voters prefer \( a \) to \( b \) to \( c \), three voters prefer \( b \) to \( c \) to \( a \) and four voters prefer \( c \) to \( a \) to \( b \). Then \( m_a(R^N) = 5 \), \( m_b(R^N) = 3 \) and \( m_c(R^N) = 1 \). Thus, \( M(R^N) = c = \tilde{f}(R^N) \). Add a voter whose favorite alternative is \( c \). Again, in any case \( (R^N, R') \neq R^* \) and by Theorem 1 we are done.

Suppose the following profile \( R^N \): three voters prefer \( a \) to \( b \) to \( c \), two voters prefer \( b \) to \( c \) to \( a \) and four voter prefer \( c \) to \( a \) to \( b \). Then \( m_a(R^N) = 3 \), \( m_b(R^N) = 5 \), \( m_c(R^N) = 1 \), hence \( M(R^N) = \{c\} \). Then \( \tilde{f}(R^N) = c \). Again, by adding a voter whose favourite alternative is \( c \) we have that \( (R^N, R') \neq R^* \). By Theorem 1 we are done.

Now, let \( N = \{1, \ldots, 10\} \) and let \( R^N = R^* \). Then \( m_a(R^*) = 4 = m_b(R^*) \) and \( m_c(R^*) = 2 \). Further, \( \tilde{f}(R^N) = b \). Add a preference \( R' \) with \( t(R') = b \), i.e. either \( bR'aR'c \) or \( bR'cR'a \). First suppose \( R' \) such that \( bR'aR'c \). Then \( m_a(R^*, R') = 3 \), \( m_b(R^*, R') = 3 \) and \( m_c(R^*, R') = 3 \). Hence, \( M(R^*, R') = \{a, b, c\} \) and \( \tilde{f}(R^*, R') = b \). Now suppose \( R' \) such that \( bR'cR'a \). Then \( m_a(R^*, R') = 5 \), \( m_b(R^N, R') = 3 \) and \( m_c(R^N) = 3 \), hence \( M(R^*, R') = \{b, c\} \) and \( \tilde{f}(R^*, R') = b \). Thus, we conclude that \( \tilde{f} \) satisfies TP.

Now let us examine whether there is a selection from \( H \), which is not from \( M \) that satisfies BP.

**Theorem 5** There is no selection \( f \) from \( H \) that satisfies IIA with respect to \( H \) and BP.

**Proof.** Proof by counterexample. Consider the profile given in Theorem 4. Suppose \( f \) selects from \( H \) according to the following order \( bQcQa \). As it is shown in the proof of Theorem 4, \( H(R^N) = \{b, c\} \). Suppose \( f(R^N) = b \), hence \( f(R^N) \neq c \). Add a preference \( R' \in L \) such that \( aR'bR'c \) and note that \( b(R') = c \). Then \( m_b(R^N, R') = 5 = n_{ab}(R^N, R'), m_a(R^N, R') = 3 = n_{ac}(R^N, R'), m_c(R^N, R') = 3 = n_{cb}(R^N, R') \).
We yield the profile \( R^* \). Thus, we only have to consider the situations where, by adding a voter we yield the profile \( R^* \) and the situation in which we add to \( R^* \) a new voter whose least favorite alternative is \( a \) or \( c \).}

First, let \( N = \{1, \ldots, 9\} \) and suppose the following profile \( R^N \): three voters prefer \( a \) to \( b \) to \( c \), three voters prefer \( b \) to \( c \) to \( a \) and three voter prefer \( c \) to \( a \) to \( b \). Then \( m_a(R^N) = m_b(R^N) = m_c(R^N) = 3 \). Hence, \( M(R^N) = \{a, b, c\} \) and \( f'(R^N) = b \). Now add a preference \( R' \in L \) such that \( b(R') = a \). Then either \( bR'aR'c \) or \( cR'bR'a \). In any case \( (R^N, R') \neq R^* \). By adding a preference \( R' \) such that \( b(R') = c \) we have that \( (R^N, R') \neq R^* \). By Theorem 1 we are done.

Now, consider the profile \( R^N \): two voters prefer \( a \) to \( b \) to \( c \), three voters prefer \( b \) to \( c \) to \( a \) and four voter prefer \( c \) to \( a \) to \( b \). Then \( m_a(R^N) = 5, m_b(R^N) = 3 \) and \( m_c(R^N) = 1 \). Thus, \( M(R^N) = c = f'(R^N) \). Add a voter whose least favorite alternative is \( b \), i.e. we add a preference \( R' \) such that \( cR'aR'b \) or \( aR'cR'b \). In any case \( (R^N, R') \neq R^* \). Now suppose we add a preference \( R' \) such that \( b(R') = a \), i.e. either \( bR'cR'a \) or \( cR'bR'a \). Again, \( (R^N, R') \neq R^* \) and by Theorem 1 we are done.

Suppose the following profile \( R^N \): three voters prefer \( a \) to \( b \) to \( c \), two voters prefer \( b \) to \( c \) to \( a \) and four voter prefer \( c \) to \( a \) to \( b \). Then \( m_a(R^N) = 3, m_b(R^N) = 5, m_c(R^N) = 1 \), hence \( M(R^N) = \{c\} \). Then \( f'(R^N) = c \).

Again, by adding a voter whose least favourite alternative is \( b \) we have that \( (R^N, R') \neq R^* \). Now suppose we add a preference \( R' \) such that \( b(R') = a \). If \( cR'bR'a \) then \( (R^N, R') \neq R^* \). By Theorem 1 we are done. If \( bR'cR'a \) then \( (R^N, R') = R^* \) and \( f'(R^*) = b \).

Now, let \( N = \{1, \ldots, 10\} \) and let \( R^N = R^* \). Then \( m_a(R^*) = 4 = m_b(R^*) \) and \( m_c(R^*) = 2 \). Further, \( f'(R^N) = b \). Add a preference \( R' \) with \( b(R') = c \),
i.e. either $bR'c$, or $aR'bR'c$. First suppose $R'$ such that $bR'c$. Then $m_a(R^*, R') = 3$, $m_b(R^*, R') = 3$ and $m_c(R^*, R') = 3$. Hence, $M(R^*, R') = \{a, b, c\}$ and $f'(R^*, R') = b \neq c$. Now suppose $R'$ such that $aR'bR'c$. Then $M(R^*, R') = \{a, c\}$ and $f'(R^*, R') = a \neq c$. As $H$ satisfies BP, we do not have to consider profiles $R'$ such that $b(R') = a$. We conclude that $f'$ satisfies BP.

In what follows we show that if considering selections from $H$, a selection satisfying BP need not satisfy TP and the other way around. First, we show that there is a selection from $H$ but not from $M$ that satisfies TP but not BP, namely, $\tilde{f}$. Consider the voting function $\tilde{f}$ defined above. Remember, that $\tilde{f}$ selects from $M(R^N)$ according to $Q : bQcQa$ under any profile $R^N \neq R^*$ and $\tilde{f}(R^*) = b$. Now add a preference $R'$ such that $aR'bR'c$ and note that $b(R') = c$. If $\tilde{f}$ satisfies BP it must hold that $\tilde{f}(R^*, R') \neq c$. As it is shown above, $M(R^*, R') = \{a, c\}$. Hence, $\tilde{f}(R^*, R') = c$. Thus, we have a contradiction against BP.

Now we show that there is a selection from $H$ that is not from $M$ that satisfies BP but not TP. Fix $A = \{a, b, c\}$ and take $Q \in L$ as follows: $cQaQb$, let $R^*$ be the profile initially given in Theorem 4. Let

$$
\bar{M}(R^N) = \{x \in M(R^N) : b_m(R^N)(x) \leq b_m(R^N)(y) \forall y \in A\}
$$

for all $N \subseteq \mathbb{N}$, $R^N \neq R^* \in L^N$. Let $\tilde{f}$ select from $H$ as follows:

- For all $N \subseteq \mathbb{N}$ and $R^N \neq R^*$, $\tilde{f}$ selects from $M$ as follows:

$$
\tilde{f}(R^N) = \begin{cases} 
 x \in \bar{M}(R^N) : xQy & \text{for all } y \in \bar{M}(R^N) \text{ if } |N| \text{ is even} \\
 x \in \bar{M}(R^N) : yQx & \text{for all } y \in \bar{M}(R^N) \text{ if } |N| \text{ is odd}
\end{cases}
$$

- For $R^*$, $\tilde{f}(R^*) = b \in H(R^*) \setminus M(R^*)$.

As we have already shown in section 3.1, $\tilde{f}$ from $M$ satisfies BP. Hence, we only have to consider profiles from which by adding a voter we yield $R^*$ and situations in which we add to $R^*$ a preference with $a$ or $c$ ranked last.

Consider $R^*$ and add a preference $R'$ such that $b(R') = a$. First suppose $bR'cR'a$. Then $m_a(R^*, R') = 5$, $m_b(R^*, R') = m_c(R^*, R') = 3$. Then $\bar{M}(R^*, R') = \{b, c\}$, $\tilde{f}(R^*, R') = b \neq a$. Now suppose $cR'bR'a$. Then $m_a(R^*, R') = 5$, $m_b(R^*, R') = 3$, $m_c(R^*, R') = 1$. Hence, $\bar{M}(R^*, R') = \{c\}$, $\tilde{f}(R^*, R') = c \neq a$. 

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Add a preference $R'$ such that $b(R') = c$. First suppose $bR' a R'c$. Then $m_a(R^*, R') = m_b(R^*, R') = m_c(R^*, R') = 3$. Thus $\bar{M}(R^*, R') = \{a\}$, $\bar{f}(R^*, R') = a \neq c$. Notice that this shows that $\bar{f}$ violates TP. Secondly, suppose $aR'bR'c$. Then $m_a(R^*, R') = 3$, $m_b(R^*, R') = 5$, $m_c(R^*, R') = 3$. Hence, $\bar{M}(R^*, R') = \{b, c\}$, $\bar{f}(R^*, R') = a \neq c$.

We can only have a contradiction agains BP if under the profile $R^N$ where three voters prefer $a$ to $b$, three voter prefer $b$ to $a$ and three voter prefer $c$ to $a$ and $b$, $b$ is not chosen. Given this profile we have that $m_a(R^N) = m_b(R^N) = m_c(R^N) = 3$. Also, $\bar{M}(R^N) = \{a, b, c\}$. As $N$ is odd, $\bar{f}$ selects from $\bar{M}$ according to $\bar{Q}$, i.e. $bQaQc$. Hence, $\bar{f}(R^N) = b$. Thus, we conclude that BP is satisfied.

Notice that there is at least one selection from $H$ and not from $M$ that satisfies both, TP and BP, namely $f'$ defined above.

**Theorem 6** There is a selection $f'$ from $H$ that is not from $M$ and satisfies CC, TP and BP.

*Proof.* It has already been shown that $f'$ satisfies CC and BP. It remains to show that $f'$ satisfies TP. Due to Theorem 1 we only have to consider the situations where, by adding a voter we yield the profile $R^*$ and the situation in which we add to $R^*$ a new voter whose favorite alternative is $b$.

First, let $N = \{1, \ldots, 9\}$ and suppose the following profile $R^N$: three voters prefer $a$ to $b$ to $c$, three voters prefer $b$ to $c$ to $a$ and three voter prefer $c$ to $a$ to $b$. Then $m_a(R^N) = m_b(R^N) = m_c(R^N) = 3$. Hence, $\bar{M}(R^N) = \{a, b, c\}$ and $f'(R^N) = b$. Now add a preference $R' \in L$ such that $t(R') = b$. Then either $bR'cRa$ or $bR'aR'c$. In any case $(R^N, R') \neq R^*$. By Theorem 1 we are done.

Now, consider the profile $R^N$: two voters prefer $a$ to $b$ to $c$, three voters prefer $b$ to $c$ to $a$ and four voter prefer $c$ to $a$ to $b$. Then $m_a(R^N) = 5$, $m_b(R^N) = 3$ and $m_c(R^N) = 1$. Thus, $\bar{M}(R^N) = c = f'(R^N)$. Add a voter whose favorite alternative is $c$. In any case $(R^N, R') \neq R^*$. By Theorem 1 we are done.

Suppose the following profile $R^N$: three voters prefer $a$ to $b$ to $c$, two voters prefer $b$ to $c$ to $a$ and four voter prefer $c$ to $a$ to $b$. Then $m_a(R^N) = 3$, $m_b(R^N) = 5$, $m_c(R^N) = 1$, hence $\bar{M}(R^N) = \{c\}$. Then $f'(R^N) = c$. Again, by adding a voter whose favourite alternative is $c$ in any case we have that $(R^N, R') \neq R^*$. By Theorem 1 we are done.

Now, let $N = \{1, \ldots, 10\}$ and let $R^N = R^*$. Then $m_a(R^*) = 4 = m_b(R^*)$.
and \( m_c(R^*) = 2 \). Further, \( f'(R^N) = b \). Add a preference \( R' \) with \( t(R') = b \), i.e. either \( bR'aR'c \) or \( bR'cR'a \). First suppose \( R' \) such that \( bR'aR'c \). Then \( m_a(R^*, R') = 3, m_b(R^*, R') = 3 \) and \( m_c(R^*, R') = 3 \). Hence, \( M(R^*, R') = \{a, b, c\} \) and \( f'(R^*, R') = b \). Now suppose \( R' \) such that \( bR'cR'a \). Then \( M(R^*, R') = \{b, c\} \) and \( f'(R^*, R') = b \). We conclude that \( f' \) satisfies TP and BP. □

4 Conclusion

There are numerous desiderate one can a require a voting correspondence to satisfy. The Top and Bot properties are perhaps two of the most natural and most important ones. As they can be seen as participation properties, they are crucial for every democracy. As it is already known, there are several voting correspondences that satisfy above properties. Felsenthal and Nurmi [4] for example provide an overview over which of the nine voting rules considered in their paper, are immune to the P-TOP and P-BOT paradoxes. Further, Felsenthal and Tideman [5] investigate the effect of P-BOT paradox on five voting rules. It can be seen that there are several non Condorcet consistent voting rules that are immune to above paradoxes. If however, one considers Condorcet consistent voting rules, all but one suffer from at least one of the above paradoxes.

In this paper we tried to reconcile Condorcet consistency and TP and BP. Following the results of Felsenthal and Nurmi [4], as well as Pérez [11], we provided a necessary condition for a Condorcet consistent voting correspondence to satisfy the above two properties. By doing so, we show that the Minimax rule is not the only Condorcet consistent voting rule one can use, when participation is considered to be important.

References


