Global Games With Strategic Substitutes*

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Abstract

We study global games with strategic substitutes. Specifically, for a class of binary action, N-player games with strategic substitutes, we prove that under commonly known payoff asymmetry, as incomplete information vanishes, the global games approach selects a unique equilibrium. We provide simple examples that illustrate our result and the connection with dominance solvability. Our work extends the global game literature, which has been developed so far for games with strategic complementarities, to new applications in industrial organization, collective action problems, finance, etc.

JEL codes: C72, D82, H41

1 Introduction

In general, game-theoretic models are developed under the assumption that the rational behavior of the players and the structure of the game are common knowledge. Since these assumptions might be too stringent, it is important to know whether the prediction of a game substantially changes in comparison to the predictions of a slightly altered version of the same game.1 If indeed it turns out that only certain of the game’s equilibria survive this “robustness check” then we may reasonably refine our prediction of what happens in such games.

This paper examines the dual issues of equilibrium selection and robustness in a class of games with strategic substitutes. These are games in which each player’s marginal payoff from increasing her own action is decreasing in the other players’ actions. An illustrative example is the game of voluntary contribution to the provision of a public good. The equilibria exhibit a classical free rider problem: an individual is less willing to contribute the larger is the total contribution of others. If one’s contribution is an indivisible choice such as a unit of time or effort, then voluntary contribution games typically exhibit multiple Nash equilibria, each corresponding to a distinct configuration of contributors and non-contributors.2

To examine equilibrium selection in games such as these, we follow the global games approach pioneered by Carlsson and van Damme (1993) (CvD hereafter).3 The idea of this approach is to examine Nash equilibria as a limit of equilibria of payoff-perturbed games. More formally, suppose g is a standard game of complete information where the payoffs depend on a parameter x ∈ R, and also suppose that for some subset of the parameter x, g has a strict Nash equilibrium. Rather than observing the parameter x, suppose instead that each player i observes a private noisy signal xi = x + σεi where σ > 0 is a scale factor and εi is a random variable. Denote this “perturbed game” by G(σ), and let NE(g) and BNE(G(σ)) denote the sets of Nash and Bayesian Nash equilibria of the unperturbed and perturbed

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1Examples in this direction are the seminal contributions of Harsanyi’s games with randomly disturbed payoffs (Harsanyi, 1973), and Selten’s concept of trembling hand perfection (Selten, 1975).

2Another good examples are entry games followed by Cournot competition, which we will further use to illustrate our main result; collective action problems; investment decisions in portfolio choice models; etc.

3For an excellent description and survey of the early literature on global games and their applications see Morris and Shin (2003).
games, respectively. Equilibrium selection is obtained when point-wise on \( x \), \( \lim_{\sigma \to 0} \text{BNE}(G(\sigma)) \) is a strict subset of \( \text{NE}(g) \). CvD show that in binary actions two-player games there is a unique equilibrium selected through the iterative elimination of strictly dominated strategies. This result was generalized by Frankel, Morris, and Pauzner (2003) (FMP hereafter) to games with many players and many actions in the context of strategic complementarities. The global games selection approach has strong experimental support (see for instance Heinemann et al. (2009) and Elbittar et al. (2014)).

This approach has proved to be very useful for games such as bank run models (Goldstein and Pauzner, 2004, 2005), currency crises games (Morris and Shin, 1998), herding behavior (Chamley, 1999), platform pricing (Jullien and Pavan, 2016), among others. Nevertheless, the literature on global games has been mainly developed assuming strategic complementarities. Exceptions include Karp et al. (2007) where they study a game with a continuum of players that presents both strategic substitutes and strategic complements; Hoffmann (2014) where he uses a \( p \)-dominant condition as a selection criteria, which does not necessarily deliver a unique solution over the entire set of uncertainty; and Morris and Shin (2009) and Harrison and Jara-Moroni (2015) where they explore the concept of strong rationality (Guesnerie, 1992) in the context of global games with strategic substitutes. Yet, there is a wide class of games with strategic substitutes that are not covered by these settings. The voluntary contribution to the provision of a public good mentioned above and entry games in oligopolistic markets, are two examples. Of course the two player case described in CvD can be represented either as a game of strategic complements or substitutes by just reordering the set of actions. However, in games with more than two players, the analysis has not been extended to games of strategic substitutes.

It is important to mention that global games as a refinement device have been under some criticism. In particular, Weinstein and Yildiz (2007) present a very general way of introducing perturbations in complete information games (more general than traditional global games) replicating the global game uniqueness but reversing the selection result: players’ beliefs can be slightly perturbed in a specific way so that any given rationalizable strategy can become the unique rationalizable strategy. In a first view, the Weinstein and Yildiz (2007) result weakens the predictive power of the global games methodology. However, very recently, Morris, Shin, and Yildiz (2016) studied in detail the connection of this literature for general type spaces. They characterize rationalizable actions in terms of the properties of the belief hierarchies and show that there is a unique rationalizable action played whenever there is approximate common certainty of rank beliefs, arguing that this is the driving force behind selection results for the specific type spaces in the global games literature.

The key insight of the present paper is to show that, when the players’ payoffs display a certain commonly known asymmetry, the global games ideas can be applied to games of strategic substitutes. Specifically, we show that as the noise goes to zero, a process of iterated elimination of conditionally dominated strategies converges to a single profile of switching strategies. In such a profile, each player has a threshold, cutoff signal, above which she takes the “higher” action, and below which the “lower” action is taken. A very important characteristic of this profile is that each player has a different cutoff point. Interestingly, the order of these cutoff points is directly determined by payoff asymmetry. Therefore, depending on the specific payoff structure of the game, the equilibrium profile structure might play an interesting role from an efficiency point of view. As we will see below, our result suggests that common knowledge of payoff asymmetry among players and global games structure are sufficient conditions to select a unique and, depending on the application, efficient equilibrium.

As an introductory example, in Section 2, we present a game of public good provision with commonly known asymmetry in cost of contributing and uncertainty on the value of the good. The main result for this game is that for general distributional properties of the signal noise, there exists a unique strategy profile played in equilibrium. This profile induces a provision of the public good where the contribution comes from the lowest cost contributor. Moreover, in this example, when there are multiple equilibria, the selected equilibrium is socially optimal. This result suggests that inefficient contribution equilibria survive under stringent assumptions: common knowledge of the fundamentals or perfect symmetry in the players’ characteristics. In section 3 we present the general framework and in section 4 we establish our main result. In section 5 we develop the main steps of the proof. In section 6 we discuss about the relation between our result and the question of dominance solvability, concept that is directly related to...
uniqueness of equilibrium under strategic complements (Guesnerie and Jara-Moroni, 2011; Frankel et al., 2003; Milgrom and Roberts, 1990). Finally, in section 7 we conclude. Proofs of propositions and lemmas are relegated to the appendix.

2 Example: Public Good Provision

In many collective action problems multiple Nash equilibria may exist, each corresponding to a different configuration of contributors. Many of these equilibria are inefficient since individuals with a higher marginal cost of contributing end up contributing disproportionately. Here, we prove a result that suggests that these inefficient Nash equilibria are not robust.

We develop a binary action game of incomplete information in which the mechanism for public good provision utilizes both government and voluntary contributions. In particular, to fund a public good, a government pledges “seed money” which must be augmented by funds from private contributors. Each contributor, upon receiving a private signal of the amount of this pledge, then chooses whether to contribute. Players have costs of contributing.

2.1 The Game

Consider the following complete information game: a social planner decides to provide a public good, requiring society’s contribution. The value of the public good depends on the social planner’s amount of contribution and the aggregate contribution of society. The society is composed by 2 different individuals (players) indexed by $i \in \{1, 2\}$. Each player $i$ has to decide whether to contribute (action 1) or not (action 0). Contribution is costly. Players payoffs are equal to the value of the public good minus the contribution cost.

Specifically, players choose an indivisible action $a_i$ from the binary set $A_i = \{1, 0\}$. Let $Y(x, n) = x^{1-\alpha} n^\alpha$, with $0 < \alpha < 1$, denote the public good technology, where $x \in [0, X]$ is the (commonly known) government contribution and $n$ is the number of people who decide to contribute. Without loss of generality we can characterize the payoffs as follows: if player $i$ chooses to contribute, she has to provide an effort (contribution) $c_i > 0$, and receives a utility $Y(x, n+1) - c_i$, where $n = 1$ if the other player contributes and $n = 0$ if not. On the other hand, if the same player chooses not to contribute (free ride), she will receive a utility $Y(x, n)$. Let

$$\Delta \pi_i(x, n) := Y(x, n+1) - c_i - Y(x, n) = x^{1-\alpha} ((n+1)^\alpha - n^{\alpha}) - c_i$$

be player $i$’s net payoff from contributing, given a value $n \in \{0, 1\}$.

Notice that although this is a $2 \times 2$ game, its generalization to more players is a game of strategic substitutes. In general, the greater the other players’ strategy, the smaller is player $i$’s incentive to increase her strategy. Also, we have that the higher the social planner’s contribution, the greater the player’s incentive to contribute. Finally, note that for sufficiently high (low) values of the social planner’s contribution, player $i$ will always (never) contribute, i.e. (not) contributing is a strictly dominant strategy.

We call these ranges of values dominance regions.

For a given $x$ we can represent this game in the following normal form:

First suppose $c_1 = c_2 = c$, the symmetric case. Then both players have the same payoff function.

Consider the following equations and their unique solutions $\tilde{k}$ and $\hat{k}$:

$$\Delta \pi_i(\tilde{k}, 0) = 0 \implies \tilde{k} = c\frac{1}{1-\alpha}$$

and

$$\Delta \pi_i(\hat{k}, 1) = 0 \implies \hat{k} = \left(\frac{c}{2^\alpha} - 1\right)^{\frac{1}{1-\alpha}}.$$

Note that if $x < \tilde{k}$, then action 0 is strictly dominant for both players and if $x > \hat{k}$ then action 1 is strictly dominant for both players. The values $\tilde{k}$ and $\hat{k}$ define the lower and upper dominance regions (respectively) shown in Figure 1. Thus, if $x > \hat{k}$ (resp. $x < \tilde{k}$) then the parameter $x$ is in the upper (resp. lower) dominance region for both players.

The set of Nash equilibria as a function of $x$ has the following structure:
Figure 1: Dominance regions and equilibria in the symmetric case.

• For values of $x$ in the dominance regions, both players choose the dominant action. In Figure 1 the solid lines represent the dominant action of player 1 as a function of $x$, and dashed lines represent the dominant action of player 2. Therefore in each dominance region there exists a unique equilibrium: $(1, 1)$ in the upper dominance region and $(0, 0)$ in the lower dominance region.

• If $x$ takes values in the interval $[k, \bar{k}]$, there are two pure strategy Nash equilibria. In these two equilibria one player chooses to contribute and the other one chooses not to contribute. Then, the equilibrium profiles are $(1, 0)$ and $(0, 1)$.

2.2 Incomplete Information

Suppose now that the game is characterized by incomplete information in the payoff structure. Instead of observing the actual value of the social planner’s contribution $x$, each player just observes a private signal $x_i$, which contains diffuse information about $x$. The signal has the following structure: $x_i = x + \sigma \varepsilon_i$, where $\sigma > 0$ is a scale factor, $x$ is drawn from $[X, \bar{X}]$, where $X > 0$ and $\bar{X}$ is sufficiently large,\(^6\) with uniform density and each $\varepsilon_i$ is randomly selected independently of $x$ on the interval $[-1, 1]$. We assume that the random vector $\varepsilon = (\varepsilon_1, \varepsilon_2)$ admits a continuous density.

In this context of incomplete information, a Bayesian pure strategy for player $i$, is a function $s_i : [X - \sigma, X + \sigma] \to A_i$. A pure strategy profile is a vector $s = (s_1, s_2)$.

Calling $G_{(1, 2)}(\sigma)$ this game of incomplete information, let $\text{BNE}(G_{(1, 2)}(\sigma))$ be the set of Bayesian Nash equilibria of $G_{(1, 2)}(\sigma)$.

\(^6\)Such that $\bar{X}$ is contained in every player’s upper dominance region. We care about games in which $\sigma < \frac{x}{X}$ so that $x_i > 0$ in all the support.
At a first glance the incomplete information game \( G_{(1,2)}(\sigma) \) has a similar structure to the class of games studied in CvD. One may wish to apply their Theorem to obtain an equilibrium selection result in \( G_{(1,2)}(\sigma) \). However, the Nash equilibria structure described above suggests two important observations. First, the CvD global game equilibrium selection result cannot be applied to this game because it requires that a selected equilibrium be a unique Nash equilibrium for some subset of values of the exogenous parameter \( x \) in this case). Moreover, the selected equilibrium should be the risk dominant equilibrium. In this game, when we have multiplicity, none of the equilibria profiles (neither \((0,1)\) nor \((1,0)\)) is a unique Nash equilibrium for some value of \( x \) nor we have a risk dominant equilibrium (since \( c_1 = c_2 \)). Second, for \( x \in [k_1, k_2] \) this symmetric case not only entails multiplicity of equilibria, but also that each of the equilibria has an asymmetric structure where just one of the players contributes. This suggests that asymmetry may play an important role in any equilibrium selection attempt.

Let us now introduce a commonly known asymmetry in the payoff structure of the game, by assuming that \( c_2 > c_1 > 0 \). This generates different dominance regions for each player, determined by the values:

\[
k_i = c_i^{1/\alpha} \quad \text{and} \quad \bar{k}_i = \left( \frac{c_i}{2^\alpha - 1} \right)^{1/\alpha}, \quad i \in \{1, 2\}.
\]

Note that \( k_1 < k_2 \) and \( k_1 < \bar{k}_2 \).

A very important consequence of the introduction of cost asymmetry is the generation of a subset of values of \( x \) where the profile \((1, 0)\) is the unique equilibrium. Indeed, in Figure 2 we can observe that if \( x \in [\bar{k}_1, k_2] \cup [k_1, \bar{k}_2] \), the unique equilibrium of the game parametrised on \( x \) is \((1, 0)\). Moreover, \((1, 0)\) is the risk dominant equilibrium. This enables us to apply the CvD global game equilibrium selection result.

Define \( s^* \) as a particular profile of switching strategies, such that player 1 and player 2 switch from action 0 to action 1 at the cutoff points \( k_1 \) and \( \bar{k}_2 \) respectively:

\[
s_1^*(x_1) = \begin{cases} 0 & \text{if } x_1 < k_1 \\ 1 & \text{if } x_1 > k_1 \end{cases} \quad s_2^*(x_2) = \begin{cases} 0 & \text{if } x_2 < \bar{k}_2 \\ 1 & \text{if } x_2 > \bar{k}_2 \end{cases}.
\]

For this game we can restate the CvD result, in terms of Theorem 1 in FMP, in the following proposition:

**Proposition 1.** The profile \( s^* \) is the essentially unique strategy profile of \( G_{(1,2)}(\sigma) \) that survives iterated deletion of strictly dominated strategies as \( \sigma \to 0 \).

There are two results in Proposition 1: uniqueness of equilibrium and dominance solvability, when \( \sigma \to 0 \). Our interest is on uniqueness of equilibrium in games of strategic substitutes with more than
2 players. As we will see below, starting from the setting of FMP, but with two actions and strategic substitutes, it is not possible to obtain dominance solvability. However, introducing certain natural player asymmetry it is possible to prove unique selection of equilibrium when $\sigma \to 0$. Thus, uniqueness of equilibrium in Proposition 1 is a direct consequence of our Theorem 1 stated in Section 4.

Figure 3 shows the structure of the equilibrium profile $s^*$: player 1 switches from not contributing to contributing at $k_1$; and player 2 switches at $\bar{k}_2$. It is important to notice that this strategy profile selects an efficient provision of the public good, and that the contributions come from the lowest cost contributors. However, where there is no multiplicity, the unique equilibrium may not provide the optimal amount of public good.

![Figure 3: Global game equilibrium and equilibrium selection: two players case.](image)

This illustrative result in the context of collective action problems suggests that generalizing this payoff structure, under the global games approach, it is possible to prove unique selection of equilibrium in a class of games with strategic substitutes.

3 General Framework

Consider the following general setup for an $N$-player complete information game. The set of players is $\mathcal{N}$ and each player $i \in \mathcal{N}$ has a binary set of actions $A_i = \{0, 1\}$. The game is parametrized by the exogenous variable $x$, which takes values in the interval $[X, \bar{X}] \subset \mathbb{R}$. Player $i$’s payoff function is $u_i(a_i, a_{-i}, x) = \pi_i(a_i, \sum_{j \neq i} a_j, x)$, where $\pi_i : \{0, 1\} \times \{0, \ldots, N-1\} \times [X, \bar{X}] \to \mathbb{R}$ is an auxiliary function that depends on other players’ actions through the number of players (other than $i$) that are choosing action 1. We call this game $g_N(x)$.

Let us define $\Delta \pi_i(n, x) = \pi_i(1, n, x) - \pi_i(0, n, x)$ as player $i$’s payoff difference when she is choosing action 1 rather than action 0. We consider that the payoff structure of the class of games $g_N(x)$, $x \in [X, \bar{X}]$, satisfies the following assumptions:

A1. **Strategic Substitutes (SS).** Conditional on the value of $x$ the greater the other players’ action profile, the smaller is player $i$’s incentive to choose the higher action:

\[
\text{if } n > n', \text{ then } \Delta \pi_i(n, x) < \Delta \pi_i(n', x) \quad \forall x.
\]

A2. **Continuity (C).** $\pi_i(a_i, n, x)$ is a continuous function of $x$.

A3. **Monotonicity (M).** The greater the value of the exogenous variable $x$, the greater the player $i$’s incentive to choose the higher action:

\[
\exists c > 0 \text{ such that, if } x, x' \in [X, \bar{X}] \text{ and } x \geq x', \text{ then } \Delta \pi_i(n, x) - \Delta \pi_i(n, x') \geq c(x - x'), \forall n.
\]

We will also refer to $a_i = 0$ as the “lower” action and $a_i = 1$ as the “higher” action.
A4. Indifference Points (IP). If other players are choosing identical actions, there exists a value of $x$ such that player $i$ is indifferent between the two actions: for each player $i$,

$$\exists k_i \geq X \text{ such that } \Delta \pi_i(0, k_i) = 0 \text{ and } \exists k_i < X \text{ such that } \Delta \pi_i(N-1, k_i) = 0.$$  

A5. Payoff asymmetry (PA). Payoffs are asymmetric, in the sense that for every pair of players $i, j \in N$ either

$$\Delta \pi_i(n, x) - \Delta \pi_j(n, x) > 0, \forall n, \forall x \quad \text{or} \quad \Delta \pi_j(n, x) - \Delta \pi_i(n, x) > 0, \forall n, \forall x.$$  

Assumption A5 (PA) induces a complete order in the set $N$ of players. We will say that player $j$ is “greater” than player $i$ if, when both players observe the same value of $x$ and face the same $n$, player $j$ has less incentive than $i$ to pick the higher action. That is,

$$j > i \iff \Delta \pi_i(n, x) - \Delta \pi_j(n, x) > 0, \forall n, \forall x.$$  

We may, thus, conveniently define the set of players as $N = \{1, \ldots, N\}$. A simple framework where A5 (PA) holds is when $\Delta \pi_i(n, x) = \Delta \pi(n, x) - c_i$ where $\Delta \pi(n, x)$ is the same for all $i \in N$ and the values $c_i$ are all different.

An important remark is that assumptions A1 (SS), A3 (M) and A4 (IP) provide sufficient conditions for the existence of dominance regions, along which each action is strictly dominant, i.e. $\forall x < k_i$, $\Delta \pi_i(n, x) < 0$ and $\forall x > k_i$, $\Delta \pi_i(n, x) > 0 \forall n$.

Additionally, these assumptions allow us to state a more general single crossing property, which will help to characterize the equilibrium profile:

**Lemma 1.** For all $i \in N$ and for all $n \in \{0, \ldots, N-1\}$ there exists a unique $k_i(n) \in [X, X]$ solving $\Delta \pi_i(n, k_i(n)) = 0$. Moreover, $\Delta \pi_i(n, x) < 0, \forall x < k_i(n)$; and $\Delta \pi_i(n, x) > 0, \forall x > k_i(n)$.

![Figure 4: Player i’s payoffs dependence on x.](image)

In Figure 4, we can observe how player $i$’s payoffs depend on $x$. From Lemma 1 we know that for each $n$ there exists a unique $k_i(n)$ such that player $i$ is indifferent between the two actions; and that given $n$, player $i$’s best response is to switch from the lower action to the higher action at a unique value of the signal. Note then that $k_i = k_i(0)$ and $k_i = k_i(N-1)$. Given assumption A3 (M) we know that the net payoff function is monotonic in $x$ and by assumption A1 (SS) we know that for different $n$ the net payoff functions do not intersect each other.

Additionally A5 (PA) implies that if $j > i$ then $k_j(n) > k_i(n)$ for every $n \in \{0, \ldots, N-1\}$. In particular $k_j > k_i$ and $k_j > k_i$. In Figure 5, for a three player case, we can observe a direct consequence

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9Note that because of A3 (M) these indifference values are unique, and A1 (SS) along with A3 (M) imply that $k_i > k_i$.

10Note that from compactness of $[X, X]$, this implies that $\exists \alpha > 0$ such that $\Delta \pi_i(n, x) - \Delta \pi_j(n, x) > \alpha$, for all $j > i$.

11Without loss of generality in the analysis we will assume the case where $k_N < k_1$, excluding the trivial situations where $k_N > k_1$, i.e. player $N$’s lower dominance region does not overlap player 1’s upper dominance region.
of this assumption: sequentially overlapped dominance regions. Therefore assumption A5 (PA) provides the necessary asymmetry in the game.

\[
\begin{array}{c}
\text{Player } i\text{'s action} \\
(0, 0, 0) \\
\end{array}
\begin{array}{c}
\text{Player 1} \\
\text{Player 2} \\
\text{Player 3} \\
\end{array}
\begin{array}{c}
k_1 \\
k_2 \\
k_3 \\
\end{array}
\begin{array}{c}
0 \\
1 \\
\end{array}
\]

Figure 5: Overlapped Dominance Regions: Three Players Case.

The last important remark about the assumptions is contained in the following lemma:

**Lemma 2.** There exists a value \( \sigma_0 > 0 \) such that \( \forall \sigma \in (0, \sigma_0), \) if \( j > i \) and \( x_j - x_i \leq \sigma \), then \( \Delta \pi_i(n, x_i) - \Delta \pi_j(n, x_j) > 0 \) \( \forall n \).

From assumption A5 (PA), we know that if two players face the same strategy profile and the same value of \( x \), the “greater” player will get a lower net payoff. This lemma states that this is still true even when they face different values of \( x \), such that their difference is less than \( \sigma_0 \).

### 3.1 Incomplete Information

We now endow the class of games \( g_N(x), x \in [\underline{X}, \overline{X}] \), with incomplete information in the payoff structure. Instead of observing the actual value of \( x \), each player just observes a private signal \( x_i \), which contains diffuse information about \( x \).

The signal has the following structure: \( x_i = x + \sigma \varepsilon_i \), where \( \sigma > 0 \) is a scale factor, \( x \) is drawn from the interval \( [\underline{X}, \overline{X}] \) with uniform density and each \( \varepsilon_i \) is randomly selected independently of \( x \) on the interval \( \left[ -\frac{1}{2}, \frac{1}{2} \right] \). In this context, signals \( x_i \) belong to the set \( X(\sigma) = [\underline{X} - \frac{1}{2} \sigma, \overline{X} + \frac{1}{2} \sigma] \). We call this game of incomplete information \( G_N(\sigma) \).

This general noise structure has been used in the global game literature, allowing the conditional distribution of the opponents signal to be modelled in a simple way, i.e. given a player’s own signal, the conditional distribution of an opponent’s signal \( x_j \) admits a continuous density \( f_\sigma \) and a cdf \( F_\sigma \) with support in the interval \( [x_i - \sigma, x_i + \sigma] \). Moreover, this literature establishes a significant result: when the prior is uniform, players’ posterior beliefs about the difference between their own observation and other players’ observations are the same, i.e. \( F_\sigma(x_i | x_j) = 1 - F_\sigma(x_j | x_i) \).\(^{12}\) To obtain continuity of the cdf \( F_\sigma \), in this paper we assume, following CvD, that the random vector \( (\varepsilon_1, \ldots, \varepsilon_N) \) admits a continuous density.

### 3.2 Strategies and Equilibrium

A Bayesian pure strategy for a player \( i \), is a function \( s_i : X(\sigma) \rightarrow A_i \), i.e. conditional on receiving a signal \( x_i \), player \( i \) takes an action \( s_i(x_i) \in \{0, 1\} \). A pure strategy profile is denoted as \( s = (s_1, s_2, \ldots, s_N) \) and \( s_i \in S_i \); the set of all functions from \( X(\sigma) \) to \( A_i \); equivalently we use the usual notation \( s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N) \in S_{-i} \).

\(^{12}\)This property holds approximately when \( x \) is not distributed with uniform density but \( \sigma \) is small, i.e. \( F(x_i | x_j) \approx 1 - F(x_j | x_i) \) as \( \sigma \) goes to zero. See details in Lemma 4.1 in CvD.
In this context of incomplete information, player \(i\)'s payoff is characterized by her beliefs about her opponents strategies and about the true value of \(x\) (common values). In general, if player \(i\) is observing a signal \(x_i\) and is facing a strategy \(s_{-i}\) her expected payoff of choosing action \(a_i\) is
\[
\Pi_i(a_i, s_{-i}, x_i) := \int_{x_i - \frac{\sigma}{2}}^{x_i + \frac{\sigma}{2}} \int_{x_i} \pi_i \left( a_i, \sum_{j \neq i} s_j(x_j), x \right) dF(\sigma_{-i})(x - x_i) dP_{\sigma_i}(x | x_i)
\]
and her expected net gain of choosing action 1 instead of action 0 can be written as
\[
\Delta \Pi_i(s_{-i}, x_i) := \int_{x_i - \frac{\sigma}{2}}^{x_i + \frac{\sigma}{2}} \int_{x_i} \Delta \pi_i \left( \sum_{j \neq i} s_j(x_j), x \right) dF(\sigma_{-i})(x - x_i) dP_{\sigma_i}(x | x_i)
\]
where \(dP_{\sigma_i}(x | x_i)\) is the posterior belief over \(x\) after receiving the signal \(x_i\).

The relevant solution concept in this environment is Bayesian Nash equilibrium, whose definition we provide next.

**Definition 1.** A strategy profile \(s^*\) is a Bayesian Nash Equilibrium of \(G_N(\sigma)\) if for every player \(i \in N\) we have
\[
\Pi_i(s^*_i(x_i), s^*_{-i}, x_i) \geq \Pi_i(a_i, s^*_{-i}, x_i) \quad a_i \in \{0, 1\} \quad \text{and} \quad \forall x_i \in X(\sigma).
\]

We will denote by \(\text{BNE}(G_N(\sigma))\) the set of Bayesian Nash equilibria of \(G_N(\sigma)\).

It is important to highlight that the incomplete information game \(G_N(\sigma)\) inherits the dominance regions structure from the complete information setting. For each player \(i \in N\), the dominance regions in the game \(G_N(\sigma)\) are obtained using the function
\[
\Delta \Pi_i(n, x_i) := \int_{x_i - \frac{\sigma}{2}}^{x_i + \frac{\sigma}{2}} \Delta \pi_i(n, x) dP(x | x_i).
\]

This function represents the expected net gain of choosing action 1 instead of action 0 when there is certainty that there are exactly \(n\) players taking action 1. Using \(\Delta \Pi_i(n, x_i)\) we define \(k^\sigma_i\) as the unique\(^{13}\) value of the signal of player \(i\) for which the function \(\Delta \Pi_i(0, x_i)\) is equal to zero:
\[
\Delta \Pi_i(0, k^\sigma_i) = 0;
\]
and \(k^\sigma_i\) as the unique\(^{13}\) value of the signal of player \(i\) for which the function \(\Delta \Pi_i(N - 1, x_i)\) is equal to zero:
\[
\Delta \Pi_i(N - 1, k^\sigma_i) = 0.
\]

Then, if \(x_i < k^\sigma_i\) player \(i\) is certain that \(\forall s_{-i}, \Delta \Pi_i(s_{-i}, x_i) < \Delta \Pi_i(0, x_i) < 0\) and so the optimal action is 0 regardless of the strategies of her rivals. Analogously, if \(x_i > k^\sigma_i\) player \(i\) is certain that \(\forall s_{-i}, \Delta \Pi_i(s_{-i}, x_i) > \Delta \Pi_i(N - 1, x_i) > 0\) and so the optimal action is 1 regardless of the strategies of her rivals.\(^{14}\)

More generally, we define for every \(n \in \{1, \ldots, N - 1\}\) and every player \(i \in N\) the value \(k^\sigma_i(n)\) as the unique solution in \(x_i\) of:
\[
\Delta \Pi_i(n, x_i) = 0.
\]

Existence of \(k^\sigma_i(n)\) is guaranteed by Lemma 1 and uniqueness, when \(\sigma > 0\) is small, by Lemma 2. Clearly, \(k^\sigma_i(0) = k^\sigma_i\) and \(k^\sigma_i(N - 1) = k^\sigma_i\).

\(^{13}\)From Lemma 2 it is unique for \(\sigma\) sufficiently small.

\(^{14}\)Note that these "new" dominance regions preserve the order of the complete information dominance regions. This is, \(k^\sigma_{N-1} > k^\sigma_{N-2} > \ldots > k^\sigma_1\) and \(k^\sigma_{N-1} > k^\sigma_{N-2} > \ldots > k^\sigma_1\).
4 Main Result

A switching strategy of a player $i$ is a Bayesian pure strategy $s_i$ satisfying: $\exists \ y$ s.t.

$$s_i(x_i) = \begin{cases} 
0 & \text{if } x_i < y \\
1 & \text{if } x_i > y 
\end{cases}$$

(8)

Abusing notation, we write $\hat{s}_i(\cdot; y)$ to denote the switching strategy of player $i$ with switching threshold $y$.

Let $s^*$ be the profile such that each player is using a switching strategy $\hat{s}_i(\cdot; x^*_i)$ where the threshold $x^*_i$ solves the following equation:

$$\Delta \pi_i(i-1, x^*_i) = 0.$$  \hspace{1cm} (9)

In this profile, player $i$ will switch from 0 to 1 at $x^*_i$, where $x^*_i$ is the indifference point when she faces a strategy profile such that all players “lower” than her play action 1 and all “higher” players play action 0. From Lemma 1 we know that for all $i$, $x^*_i = k_i(i-1)$ not only exists, but it is also unique. We will prove that $s^*$ is the unique limit of equilibria profiles of $G_N(\sigma)$ as $\sigma$ goes to 0.

Note that the strategy $s^*_i$ does not depend on the number of players. This is, the player that is in the $i$-th position (player $i$) determines her switching point depending only on her position in the set of players and not on how many players interact with her.\(^{15}\)

We now formally state our main result.

**Theorem 1.** The profile $s^*$ is the essentially unique equilibrium strategy profile of $G_N(\sigma)$ as $\sigma \to 0$. More precisely, if for each $\sigma > 0$, $s^0 \in \text{BNE}(G_N(\sigma))$, then for each player $i \in N$ and for almost all $x_i \in X(\sigma)$ we have $\lim_{\sigma \to 0} s^0_i(x) = s^*_i(x)$.

This result allows us to analyze a wide class of games of strategic substitutes where multiplicity is a problem, extending the global game literature. For instance, Theorem 1 generalizes the analysis and conclusion developed in the public good example of Section 2; now, lower cost players are represented by a “higher” position in the induced players order (according to A5 (PA)), and they will switch between the actions at a higher threshold according to equation (9).

Strategic substitutes are present in entry games with quantity competition and multiplicity of equilibria might arise naturally in this type of interaction. Consequently, our result might be useful for selection of equilibrium in this class of models. Consider the following example.

**Example 1.** Consider an entry game where a number $N$ of firms choose whether to enter a market or not. Firms that decide to enter the market face Cournot competition under demand uncertainty.\(^{16}\)

Specifically, suppose that each firm $i \in N = \{1, \ldots, N\}$ faces an entry cost equal to $f_i > 0$. Call $\pi^C_i(n + 1, x)$ the Cournot equilibrium profits of a firm $i$ that enters the market when other $n$ firms have entered as well and the demand parameter is $x$. Thus, firm $i$’s payoffs are

$$\pi_i(a_i, n, x) = \begin{cases} 
\pi^C_i(n + 1, x) - f_i & \text{if } a_i = 1 \\
0 & \text{if } a_i = 0
\end{cases}$$

where $a_i = 1$ if firm $i$ enters the market and $a_i = 0$ if she does not. Consequently,

$$\Delta \pi_i(n, x) = \pi^C_i(n + 1, x) - f_i.$$
then all $N$ firms would enter. The problem of multiplicity arises for intermediate values of $x$ at which different sets of firms may enter the market in equilibrium.

Now, if $\Delta \pi_i(n, x) = \pi_i^c(n + 1, x) - f_i$ satisfies assumption A5 (PA), this entry game fits our setting and Theorem 1 can be applied. Therefore, when uncertainty about demand vanishes, the global game approach selects a unique equilibrium that determines, for each value of $x$, which firms enter the market.

For instance, suppose that firms have equal constant marginal costs, $c > 0 \forall i \in N$, that $f_i \neq f_j \forall i, j \in N$ and that inverse demand is linear and equal to $p = x - Q$ when positive and equal to 0 otherwise. In this setting, the $m$-firm Cournot equilibrium quantity and profits for firm $i \in N$ are respectively:

$$q_i(m) = \frac{x - c}{m + 1} \quad \text{and} \quad \pi_i^C(m, x) = \left(\frac{x - c}{m + 1}\right)^2.$$

We see that $\Delta \pi_i(n, x)$ satisfies assumptions A1 to A3 and A5.

To have assumption A4 fulfilled, we may explicitly solve for $k_i(n)$ (see Lemma 1):

$$\Delta \pi_i(n, k_i(n)) = 0 \quad \pi_i^C(n + 1, k_i(n)) - f_i = 0 \quad \left(\frac{k_i(n) - c}{n + 2}\right)^2 - f_i = 0 \quad k_i(n) = (n + 2) \sqrt{f_i} + c.$$

Thus, to assure the presence of dominance regions, we need that for every $i \in N$, $k_i(0) > X$ and $k_i(N - 1) > X$, which translates into

$$\min_{i \in N} \left\{2\sqrt{f_i}\right\} + c > X \quad \max_{i \in N} \left\{2\sqrt{f_i}\right\} + c < X. \quad (10)$$

Theorem 1 states that when uncertainty about demand vanishes, the uniquely selected strategy profile prescribes that firm $i$ will enter if

$$x < (i + 1) \sqrt{f_i} + c$$

and she will not enter if

$$x > (i + 1) \sqrt{f_i} + c.$$

For a given $x$, in the selected equilibrium all firms that have switching points lower than $x$ enter the market. In this example firms are ranked according to $f_i$ and in the selected equilibrium firms that enter the market are the ones with lower entry costs.

As an illustration of Theorem 1, in Figures 6 and 7 we show the equilibria and the global game equilibrium selection respectively in a three players case. Figure 6 depicts the type of equilibria depending on the value of $x$ and the dominance regions for each player, while in Figure 7 we show the selected equilibrium profile. The strategy profile in equilibrium shows the higher player switching at the beginning of her upper dominance region, $x_i^* = k_3$. The lower player switches at the end of her lower dominance region $x_1^* = k_1$, and player 2 switches at $x_2^* = k_2(1)$ where $k_1 < x_2^* < k_3$.

5 Proof of the Theorem

In this section we develop the proof of the Theorem stated in Section 4. We will argue that the profile $s^*$ is the limit when $\sigma \to 0$ of an equilibrium profile $s^\sigma$. We prove that the equilibrium profile $s^\sigma$ is in fact the unique equilibrium of $G_N(\sigma)$ for $\sigma$ sufficiently small.

To do this we introduce a particular process of iterated deletion of non-equilibrium strategies. The sets of surviving strategy profiles are not the standard undominated sets used to define iteratively undominated strategies. Instead these sets are defined by an alternative process that eliminates profiles

\footnote{This would be the case for instance if firms had identical productions costs and asymmetric entry costs.}
that are not part of any equilibrium. These strategies are strictly dominated when we restrict ourselves to considering some subset of others' strategies that are “potentially” part of some equilibrium profile. We call these sets the conditionally iteratively undominated sets.\(^{18}\) We will prove that \(s^\sigma\) is in fact an equilibrium of \(G_N(\sigma)\) and that this process does not rule out any Bayesian Nash equilibria. We then proceed to show that under our assumptions, \(s^\sigma\) is the unique strategy profile surviving the iterated deletion when \(\sigma\) is small.

We begin by defining \(s^\sigma\). Consider for each \(i \in N\) the function \(\Delta \tilde{\Pi}_i(n, x_i)\) defined in (4) and let \(x_i^\sigma\) be the unique\(^{19}\) value of the signal of player \(i\) for which the function \(\Delta \tilde{\Pi}_i(i-1, x_i)\) is equal to zero; this is:

\[
\Delta \tilde{\Pi}_i(i-1, x_i^\sigma) = 0. \tag{11}
\]

Define now the strategy profile \(s^\sigma\), where each player \(i \in N\) uses a switching strategy on \(x_i^\sigma\), this is

\[
s_i^\sigma(x_i) = \tilde{s}_i(x_i; x_i^\sigma). \tag{12}
\]

Note that \(x_i^\sigma \in [x_i^* - \frac{\sigma}{2}, x_i^* + \frac{\sigma}{2}]\) and so as \(\sigma \to 0\) the profile \(s^\sigma\) converges pointwise to \(s^*\). This is, for each player \(i \in N\) and for almost all \(x_i \in X(\sigma)\) we have \(\lim_{\sigma \to 0} s_i^\sigma(x_i) = s_i^*(x_i)\).

The first Lemma states that the strategy profile \(s^\sigma\) is in fact a Bayesian Nash Equilibrium of \(G_N(\sigma)\).

---

\(^{18}\)Since the elimination proceeds upon players receiving the signal, then formally these sets contain strategies that are interim strictly undominated.

\(^{19}\)From Lemma 2 it is unique for \(\sigma\) sufficiently small.

\(^{20}\)Note that \(x_i^* = k_i^*\) and \(x_N^* = k_N^*\).
Lemma 3. For $\sigma > 0$ sufficiently small, $s^\sigma \in \text{BNE}(G_N(\sigma))$.

This is clear since if the signal of player $i$ is below the threshold $x_i^\sigma - \sigma$ and her rivals play $s^\sigma_{-i}$, then she sees at most $i - 1$ players playing action 1. Therefore, by definition of $x_i^\sigma$, the optimal action is 0. Analogously, if the signal of player $i$ is greater than $x_i^\sigma + \sigma$, the optimal action is 1. The proof of Lemma 3 is relegated to the appendix.

In order to define the process of elimination, we first need to introduce some concepts and notation.

5.1 Previous definitions

In this section we define the concept of parametrized strategy set, the concept of parametrized mutually best response strategy profiles set and the notion of extremal profiles.

**Definition 2** (Parametrized strategy set). For each player $i \in N$ and a number $z \in [k_i^\sigma, \bar{k}_i^\sigma]$ define the parametrized (on $z$) strategy set of player $i$, $S_i(z)$, as follows:

$$S_i(z) := \{ s_i \in S_i : s_i(x_i) = 0 \text{ if } x_i < \min\{z, x_i^\sigma\} \text{ and } s_i(x_i) = 1 \text{ if } x_i \in (x_i^\sigma, z) \cup ( \bar{k}_i^\sigma, x_i) \}$$

All strategies in $S_i(z)$ prescribe action 0 for signals less than the minimum between $z$ and $x_i^\sigma$; and prescribe action 1 for signals in player $i$’s upper dominance region and for signals in the interval $(x_i^\sigma, z)$, if not empty. In the interval $(z, \bar{k}_i^\sigma)$ the strategies in $S_i(z)$ may take any value (see Figure 8). The sets $S_i(z)$ are well defined when $\sigma$ is small.

**Definition 3** (Parametrized mutually best response strategy profiles set). For each player $i \in N$ and a tuple of signal values $x \in \prod_j [k_j^\sigma, \bar{k}_j^\sigma]$ define the parametrized (on $x$) mutually best response strategy

To define the parametrized mutually best response strategy profiles set we need to consider for each $i \in N$ player $i$’s best response correspondence $\text{BR}_i : S_{-i} \Rightarrow S_i$:

$$\text{BR}_i(s_{-i}) := \{ s_i \in S_i : \Pi_i(s_i(x_i), s_{-i}, x_i) \geq \Pi_i(a_i, s_{-i}, x_i) \ \forall \ x_i \in X(\sigma) \ \forall \ a_i \in A_i \}$$

**Definition 3** (Parametrized mutually best response strategy profiles set). For each player $i \in N$ and a tuple of signal values $x \in \prod_j [k_j^\sigma, \bar{k}_j^\sigma]$ define the parametrized (on $x$) mutually best response strategy
profiles set of the rivals of player $i$, $S_{-i}(x)$, as follows:

$$S_{-i}(x) := \left\{ s_{-i} \in \prod_{j \neq i} S_j(x_j) : \exists s_i \in S_i(x_i) \text{ such that } s_j \in BR_j(s_{-j}) \forall j \neq i \right\}$$

For a given player $i \in \mathcal{N}$ the set $S_{-i}(x)$ is the set of strategy profiles of all other players (i.e. excluding player $i$) that consist of strategies that are mutually best responses, for some strategy $s_i \in S_i(x_i)$ of player $i$.

Now we turn to the concept of extremal profiles. In general, for a player $i \in \mathcal{N}$, given a set $M_{-i} \subseteq S_{-i}$ of strategy profiles of player $i$’s rivals, we may define two types of extremal profiles. The set $M_{-i} \subseteq M_{-i}$ contains the strategy profiles of the rivals of player $i$, that give player $i$ the most incentive to play action 1 at every $x_i$, while the set $M_{-i} \subseteq M_{-i}$ contains the strategy profiles of the rivals of player $i$, that give player $i$ the less incentive to play action 1 at every $x_i$. It is important to note that extremal profiles are determined by the set $M_{-i}$. More formally:

**Definition 4 (Extremal Profiles).** For each player $i \in \mathcal{N}$ and a set $M_{-i} \subseteq S_{-i}$, we define the sets of Upper Extremal Profiles, $\mathcal{M}_{-i}$, and Lower Extremal Profiles, $\mathcal{M}_{-i}$, faced by player $i$ associated to $M_{-i}$ as follows:

$$\mathcal{M}_{-i} := \bigcap_{x_i \in [\underline{x}, \overline{x}]} \arg\max \Delta \Pi_i(s_{-i}, x_i)$$

$$\mathcal{M}_{-i} := \bigcap_{x_i \in [\underline{x}, \overline{x}]} \arg\min \Delta \Pi_i(s_{-i}, x_i)$$

Note that $\forall s_{-i} \in M_{-i}$ and $\forall x_i \in [\underline{x}, \overline{x}]$ we have

$$\Delta \Pi_i(\bar{s}_{-i}, x_i) \geq \Delta \Pi_i(s_{-i}, x_i) \geq \Delta \Pi_i(\bar{s}_{-i}, x_i)$$

with $\bar{s}_{-i} \in \mathcal{M}_{-i}$ and $\bar{s}_{-i} \in \mathcal{M}_{-i}$. In words, by strategic substitutes (A1) if player $i$, upon receiving a signal $x_i$ and assuming that her opponents are using the strategy profile $\bar{s}_{-i}$ ($\bar{s}_{-i}$), chooses action 0 (resp. 1), then she will choose action 0 (resp. 1) for all $s_{-i} \in M_{-i}$.

In the next section, making use of the previously defined concepts we describe the process of iterated elimination of strategies.

5.2 Iterated Elimination of Strictly Conditionally Dominated Strategies

We now define the process of Iterative Elimination of (interim) Strictly Conditionally Dominated Strategies (IESCDS). Since we have the dominance regions, given $\sigma > 0$ we know that in any reasonable strategy player $i$ plays 0 when the signal is below $k^\sigma_i$ and plays 1 if the signal is above $\bar{k}^\sigma_i$, where $k^\sigma_i$ and $\bar{k}^\sigma_i$ are defined in equations (5) and (6) respectively. Thus, without loss of generality, for each player $i$ the initial set of conditionally undominated strategies is:

$$S^0_i := S_i(k^\sigma_i) = \left\{ s_i \in S_i : s_i(x_i) = 0 \text{ for all } x_i < k^\sigma_i \text{ and } s_i(x_i) = 1 \text{ for all } x_i > \bar{k}^\sigma_i \right\}$$

and $S^0 := \times_{j \in \mathcal{N}} S^0_j$.

At each step $t > 0$ the set of conditionally undominated strategies is defined as:

$$S^t_i := S_i(\underline{x}^t)$$

and $S^t := \times_{j \in \mathcal{N}} S^t_j$, where for each player $i \in \mathcal{N}$, $\{\underline{x}^t\}_{t=0}^{\infty}$ is a sequence of signals such that $\underline{x}^0 = k^\sigma_i$, and for $t > 0$ each element of the sequence is calculated as follows:

21Note that for a given $x_i \in [\underline{x}, \overline{x}]$, the extremal strategy of every rival of player $i$ for which she is uncertain about their play in $M_{-i}$ at $x_i$, will take the same value in a $\sigma$ neighbourhood of $x_i$ (this is, every rival plays 0 - in $\mathcal{M}_{-i}$ - or every rival plays 1 - in $\mathcal{M}_{-i}$ - in the extremal profile). Moreover, as long as there exists an $x_i \in [\underline{x}, \overline{x}]$ such that player $i$ is uncertain about all her rivals play in $M_{-i}$ at a neighbourhood of $x_i$, the intersection is unique and thus the extremal profiles are well defined.
• If $\bar{x}_{i}^{t-1} < x_{i}^{t}$ then
  
  $$x_{i}^{t} := \min\{x_{i} : \Delta \Pi_{i}(\bar{s}_{-i}, x_{i}) = 0\},$$

  with $\bar{s}_{-i} \in \mathcal{S}_{-i}(\bar{x}_{i}^{t-1})$.

• If $\bar{x}_{i}^{t-1} \geq x_{i}^{t}$ then
  
  $$x_{i}^{t} := \min\{x_{i} : x_{i} > x_{i}^{\sigma} \text{ and } \Delta \Pi_{i}(\bar{s}_{-i}, x_{i}) = 0\},$$

  with $\bar{s}_{-i} \in \mathcal{S}_{-i}(\bar{x}_{i}^{t-1})$.

Each element of the sequence represents the minimum signal among which player $i$ is indifferent between the two actions, but just considering strategy profiles of her opponents that belong to the set $\mathcal{S}_{-i}(\bar{x}_{i}^{t-1})$. This is, considering only the strategy profiles that, for some strategy $s_{i} \in S_{i}(\bar{x}_{i}^{t-1})$, consist of strategies that are mutually best responses (excluding player $i$). As is the convention in the literature on optimization, if there is no solution for problem (15) we set $x_{i}^{t} = x_{i}^{\sigma}$ and if there is no solution for problem (16) we set $x_{i}^{t} = \bar{k}_{i}$.

It is direct to verify that $\{x_{i}^{t}\}_{t=0}^{\infty}$ is a non-decreasing sequence and thus $S_{i}^{t} \subseteq S_{i}^{t-1}$, and that $\mathcal{S}_{-i}(\bar{x}_{i}^{t}) \subseteq \mathcal{S}_{-i}(\bar{x}_{i}^{t-1})$.

In Figures 9 and 10 we illustrate the structure of the surviving strategies for the three player case at $t > 0$. Given $x_{i}^{t}$, every strategy in $S_{i}^{t}$ prescribes action 0 for signals less than the minimum between $x_{i}^{t}$ and $x_{i}^{\sigma}$; and prescribes action 1 for signals in player $i$’s upper dominance region and for signals in the interval $(x_{i}^{\sigma}, x_{i}^{t})$ (if not empty). Figure 9 shows the case where $x_{3}^{t} < x_{2}^{\sigma}$ and Figure 10 shows the case where $x_{3}^{t} \geq x_{2}^{\sigma}$.

![Figure 9: Case $x_{3}^{t} < x_{2}^{\sigma}$](image)

**Definition 5.** We will say that a strategy $s_{i}$ survives IESCDS if $s_{i} \in \bigcap_{t=0}^{\infty} S_{i}^{t}$.

5.3 Proof

We now proceed with the proof of the Theorem, stating Lemma 4 and Proposition 2, the proofs of which are relegated to the appendix.

First, Lemma 4 establishes a very natural and intuitive property: the process of IESCDS described above does not rule out any Bayesian Nash equilibrium of $G_{N}(\sigma)$.

**Lemma 4.** $\forall t$, $\text{BNE}(G_{N}(\sigma)) \subseteq S^{t}$.

This is clear since for every player $i$ all equilibrium strategy profiles of her rivals belong, by definition, to the set $\mathcal{S}_{-i}(\bar{x}_{i}^{0})$ and so equilibrium profiles are not eliminated in the first round. For the same reason, they are never eliminated.

Lemmata 3 and 4 imply that $s^{\sigma} \in S^{t} \forall t \geq 0$ and therefore survives the process of IESCDS.
Proposition 2. For $\sigma$ sufficiently small, the profile $s^\sigma$ is the only strategy profile that survives the process of IESCDs.

Since Lemma 3 states that $s^\sigma$ is an equilibrium and Lemma 4 that all equilibria survive the process of IESCDs, Proposition 2 closes the proof of the Theorem.

6 Dominance Solvability

The proof of Theorem 1 was not developed using iterative elimination of strictly dominated strategies, as is usual in the literature under strategic complements. In this section we address by examples why this is not possible and shed some light on how dominance solvability can be obtained.

Our setting is based on Frankel, Morris, and Pauzner (2003). Our assumption A1 replaces the strategic complements assumption of FMP by strategic substitutes, and we further consider two variations:

(i) we work with binary action games while in FMP the set of actions is any countable union of closed intervals and points in the unit interval of $R$, and contains 0 and 1; and

(ii) payoff dependence on the other player’s actions is through an aggregate value, while in FMP there is no particular specification of payoff dependence on the action profile.

Assumptions A2, A3 and A4 are also present in FMP, namely: continuity of $\pi_i$, existence of upper and lower dominance regions and strict monotonicity of $\Delta \pi_i(n, x)$ on $x$.

In the FMP framework, they prove that when $\sigma \to 0$ there is a unique strategy profile that is the limit of the unique profile that survives Iterative Elimination of Strictly Dominated Strategies. In our framework this is not possible as the following example shows.

Example 2 (from Morris (2009)). In this example $N = \{1, 2, 3\}$ and the net payoff gain of the players are:

$$\Delta \pi_1(n, x) = 1 - n + x + \xi$$
$$\Delta \pi_2(n, x) = 1 - n + x$$
$$\Delta \pi_3(n, x) = 1 - n + x - \xi$$

for some $\xi \in [0, \frac{1}{4}]$, and considering private values. It is direct to check that this example satisfies our assumptions A1 to A5 and therefore we may apply Theorem 1. The unique selected equilibrium’s switching points are

$$x_1^* = -1 - \xi \quad x_2^* = 0 \quad x_3^* = 1 + \xi.$$
Nevertheless, when \( \sigma \to 0 \) both actions are rationalizable when the signals are in the interval \( ] -1 + 2\xi, 1 - 2\xi [ \), and thus any strategy profile of the form:

\[
\begin{align*}
s_1(x_1) &= \begin{cases} 0 & \text{if } x_1 < -1 - \xi \\ 1 & \text{if } -1 - \xi < x_1 < -1 + 2\xi \text{ or } 1 - 2\xi < x_1 \end{cases} \\
s_2(x_2) &= \begin{cases} 0 & \text{if } x_2 < -1 + 2\xi \\ 1 & \text{if } 1 - 2\xi < x_2 \end{cases} \\
s_3(x_1) &= \begin{cases} 0 & \text{if } x_2 < -1 + 2\xi \text{ or } 1 - 2\xi < x_2 < 1 + \xi \\ 1 & \text{if } 1 + \xi < x_1 \end{cases}
\end{align*}
\]

survives the iterative elimination of strictly dominated strategies. Note that the value of the strategies is not specified in the interval \( ] -1 + 2\xi, 1 - 2\xi [ \) and thus they may take any value (0 or 1).

However, it is possible to provide a condition that assures that the uniquely selected equilibrium is also the only rationalizable outcome. In this example, if we allow \( \xi \) to be greater than \( \frac{1}{2} \) the interval \( ] -1 + 2\xi, 1 - 2\xi [ \) would be empty, thus the only remaining strategy profile would be \( s^* \).

This type of result has been obtained in Harrison and Jara-Moroni (2015). We reproduce the main result in Example 3.

**Example 3.** In this example \( \mathcal{N} = \{1, 2, 3\} \) and the net gain of player \( i \) is:

\[
\Delta \pi_i(n, x) = \frac{d}{2} (2 - n) + mx - c_i
\]

where \( m > 0, d > 0 \) represents the degree of strategic substitution and \( c_i \) may be interpreted as player \( i \)'s specific cost, and considering private values. Clearly this setting satisfies assumptions A1 to A4 and if \( 0 < c_1 < c_2 < c_3 \) then it also satisfies A5. Thus we may apply Theorem 1 which provides the following switching points:

\[
x_1^* = \frac{c_1 - d}{m} \\
x_2^* = \frac{c_2 - d}{m} \\
x_3^* = \frac{c_3}{m}
\]

in this context Harrison and Jara-Moroni (2015) prove that if \( c_3 - c_1 > \frac{d}{2} \) then for \( \sigma \) sufficiently small the profile \( s^* \) is the unique profile that survives the iterative elimination of strictly dominated strategies.

This is, for a given degree of substitution \( (d) \) if players' costs are sufficiently different, then we may get dominance solvability for sufficiently small \( \sigma \).

Examples 2 and 3 lead us to think that sufficient asymmetry among players’ payoffs may play an important role in obtaining dominance solvability in global games with strategic substitutes.

### 7 Conclusions

The global game approach has proven to be a simple method that is rich enough to capture the important role of higher-order beliefs in economic settings. Under general payoff structures, the approach examines Nash equilibria as a limit of equilibria of payoff-perturbed games. Originally, CvD show that in binary action two-player games, there exists a unique equilibrium profile surviving iterated deletion of strictly dominated strategies. Later, this result was generalized by FMP to many players and actions, but limiting the analysis to games with strategic complementarities.

Continuing with this line of research, we extend the literature proving an equilibrium selection result for a class of global games with strategic substitutes. Assuming a certain commonly known asymmetry in players’ payoffs, we prove that for a general class of binary action, \( N \)-player games, each such game has a unique equilibrium strategy profile selected as the noise goes to 0. In this equilibrium profile, each player has a different cutoff signal, whose order is directly determined by payoff asymmetry. In the selected equilibrium, each player will take action 1 if the signal is above the cutoff point and she will take action 0 if the signal is below the cutoff point.
This result might allow us to analyze a wide class of games with strategic substitutes such as collective action problems, entry-exit models in industrial organization, investment decisions in portfolio choice models and hiring labor decisions in standard productions economies (Veldkamp, 2011). In particular, we might apply the result to a model of public good provision as the one described in the example of Section 2. The interesting conclusion to this application is that the equilibrium profile induces an efficient provision of the public good, and the contributions come from the lowest cost contributors. Additionally, we provided an application in the context of entry games under Cournot competition where the firms that enter the market in the selected equilibrium are the ones with the lower entry costs. In general, the result provides a useful tool for applications.

Regarding the question of dominance solvability and in the light of the examples provided in Section 6, our result is in line with the literature on expectational coordination. Following Guesnerie (1992), an equilibrium is strongly rational if it is the unique rationalizable strategy profile. Under complete information, it is well known that expectational coordination (presence of a strongly rational equilibrium) is linked to uniqueness of equilibrium under strategic complements, but that uniqueness is not sufficient for expectational coordination under strategic substitutes (Milgrom and Roberts, 1990; Guesnerie, 2005; Guesnerie and Jara-Moroni, 2011). This is analogous to what we obtain in our global games framework. When passing from strategic complements to substitutes under incomplete information, uniqueness of Bayesian equilibrium is uncoupled from strongly rational equilibrium. However, as is shown in Example 3, Harrison and Jara-Moroni (2015) identify further requirements on the primitives of the model that indeed enhances predictive power to the unique equilibrium in the world of strategic substitutes. Further research should be devoted to the study of dominance solvability in global games with strategic substitutes.

Appendix

A Proofs of Lemmata 1 to 4

To ease the exposition, we now repeat the formal statements of Lemmata 1 to 4 before each of their proofs.

Lemma (Lemma 1). For all \( i \in \mathcal{N} \) and for all \( n \in \{0, \ldots, N-1\} \) there exists a unique \( k_i(n) \in [\bar{X}, X] \) solving \( \Delta_\pi(n, k_i(n)) = 0. \) Moreover, \( \Delta_\pi(n, x) < 0, \forall x < k_i(n); \) and \( \Delta_\pi(n, x) > 0, \forall x > k_i(n). \)

Proof. For any \( n, \) from assumption A3 (monotonicity of \( \Delta_\pi(n, \cdot) \)), if there exists a solution to \( \Delta_\pi(n, \cdot) = 0, \) it is unique. Furthermore, by assumptions A1 (SS) and A4 (IP) we have that for any \( 0 < n < N - 1: \)

\[
\Delta_\pi(n, \bar{k}_i) < \Delta_\pi(0, \bar{k}_i) = 0
\]

\[
\Delta_\pi(n, \bar{k}_i) > \Delta_\pi(n - 1, \bar{k}_i) = 0.
\]

Thus, assumption A2 (C) provides the existence of a point \( \bar{x} \in [\bar{k}_i, k_i[ \) such that \( \Delta_\pi(n, \bar{x}) = 0. \) Clearly \( \bar{x} = k_i(n). \)

Again, assumption A3 (M) implies that if \( x < k_i(n) \), then \( \Delta_\pi(n, x) < \Delta_\pi(n, k_i(n)) = 0; \) and if \( x > k_i(n) \), then \( \Delta_\pi(n, x) > \Delta_\pi(n, k_i(n)) = 0. \)

Lemma (Lemma 2). There exists a value \( \sigma_0 > 0 \) such that \( \forall \sigma \in (0, \sigma_0), \) if \( j > i \) and \( x_j - x_i \leq \sigma, \) then \( \Delta_\pi(n, x_i, x_j) > \Delta_\pi(n, x_i) > 0 \) \( \forall n. \)

Proof. From assumption A5 (PO) we know that there exists a players order \( \{1, \ldots, N\} \) and a number \( \alpha > 0 \) such that if \( j > i \) then \( \Delta_\pi(n, x_j) - \Delta_\pi(n, x_i) > \alpha \) for any \( n \) and any \( x \).\(^{22}\)

Hence from A3 (M), if \( x_j - x_i < 0 \) monotonicity implies that \( \Delta_\pi(n, x_i) - \Delta_\pi(n, x_j) > \alpha \).

Moreover, from continuity of \( \Delta_\pi(n, \cdot) \) and compactness of \( [\bar{X}, X] \) we know that there exist for each \( j > i \) and \( n \in \{0, \ldots, N-1\}, \) numbers \( \sigma_{j,i, n} > 0 \) such that if \( x_j - x_i \leq \sigma_{j,i, n} \) then \( \Delta_\pi(n, x_i) - \Delta_\pi(n, x_j) > \alpha. \)

Let

\[
\sigma_0 \equiv \min \{\sigma_{j,i, n} : j > i, n \in \{0, \ldots, N - 1\}\}.
\]

Clearly, since the set \( \{j, i, n : j > i, n \in \{0, \ldots, N - 1\}\} \) is finite, \( \sigma_0 > 0. \) Then, if \( x_j - x_i \leq \sigma_0 \) we have \( \Delta_\pi(n, x_i) - \Delta_\pi(n, x_j) > \alpha > 0 \) for all \( n \) and any \( j > i. \)

\(^{22}\)See Footnote 10
Lemma (Lemma 3). For $\sigma > 0$ sufficiently small, $s^\sigma \in \text{BNE}(G_N(\sigma))$.

Proof. Define $\sigma_i$ as the distance between player $i$’s switching point in $s^\sigma_i$, $x^*_i = k_i(i - 1)$, and the value $k_{i+1}(i - 1)$. This is:

$$\sigma_i := k_i(i - 1) - k_{i+1}(i - 1).$$

And now define $\hat{\sigma}$ as the minimum among all the $\sigma_i$:

$$\hat{\sigma} := \min_{i \in N} \{\sigma_i\}.$$

Consider $\sigma < \hat{\sigma}$ and take a player $i \in N$. We have to prove that if $x_i < x^\sigma_i$ then $\Delta \Pi_i(s^\sigma_{-i}, x_i) \leq 0$ and if $x_i > x^\sigma_i$ then $\Delta \Pi_i(s^\sigma_{-i}, x_i) \geq 0$.

Take a player $j < i$ and consider the interval $[x^\sigma_j - \sigma, x^\sigma_j + 1 - \sigma]$.

If $x^\sigma_j + \sigma \leq x_i \leq x^\sigma_{j+1} - \sigma$ then player $i$ knows that the number of players that are playing action 1 is exactly $j$ (all players less or equal to $j$ are playing 1 and every player greater or equal to $j + 1$, except $i$, is playing 0). Thus,

$$\Delta \pi_i \left( \sum_{\ell \neq i} s^\sigma_{\ell}(x_j), x \right) = \Delta \pi_i(j, x)$$

if $x_{-i}$ is in the support of $dF(\sigma_{-j})(\cdot | x_i)$ and $x$ is in the support of $dP(\cdot | x_i)$. Moreover, since $x_i \leq x^\sigma_{j+1} - \sigma \leq k_{j+1}(j) - \frac{\sigma}{2}$ then, from A5 (PO), $x_i \leq k_i(j) - \frac{\sigma}{2}$ and so

$$\Delta \pi_i(j, x) < 0 \quad \forall \ x \in \left[ x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2} \right].$$

Thus,

$$\Delta \Pi_i(s^\sigma_{-i}, x_i) = \int_{x_i - \frac{\sigma}{2}}^{x_i + \frac{\sigma}{2}} \int_{x_{-i}} \Delta \pi_i \left( \sum_{\ell \neq i} s^\sigma_{\ell}(x_j), x \right) dF(\sigma_{-j})(x_{-i} | x_i) dP(x | x_i) < 0.$$

If $x^\sigma_j - \sigma < x_i \leq x^\sigma_j + \sigma$ then there is a positive probability of player $j$ playing action 0 instead of 1 (all the other players actions remain constant). Therefore $\Delta \Pi_i(s^\sigma_{-i}, x_i)$ is a weighted average between $\Delta \pi_i(j - 1, x)$ and $\Delta \pi_i(j, x)$. However, since

$$\sigma < k_{j+1}(j - 1) - k_j(j - 1) \leq k_i(j - 1) - k_j(j - 1)$$

we know that

$$\Delta \pi_i(j - 1, x) < 0 \quad \forall \ x \in \left[ x_i - \frac{\sigma}{2}, x_i + \frac{\sigma}{2} \right].$$

We conclude that in this case we also have

$$\Delta \Pi_i(s^\sigma_{-i}, x_i) < 0.$$

We illustrate this analysis in Figure 11.

Summarising, for any $j < i$ we have that for all $x_i \in [x^\sigma_j - \sigma, x^\sigma_{j+1} - \sigma]$, $\Delta \Pi_i(s^\sigma_{-i}, x_i) < 0$ and so, for all $x_i < x^\sigma_j - \sigma$, $\Delta \Pi_i(s^\sigma_{-i}, x_i) < 0$.

With analogous arguments, we may prove that for all $x_i > x^\sigma_j + \sigma$, $\Delta \Pi_i(s^\sigma_{-i}, x_i) > 0$.

To conclude the proof we just need to note that if $x_i \in [x^\sigma_j - \sigma, x^\sigma_j + \sigma]$ then we have

$$\Delta \Pi_i(s^\sigma_{-i}, x_i) = \Delta \Pi_i(i - 1, x_i)$$

and so the strategy $s^\sigma_i$ is a best response to $s^\sigma_{-i}$.

Lemma (Lemma 4). \( \forall i, \text{BNE}(G_N(\sigma)) \subseteq S^i. \)
Before commencing the proof, let us state some useful remarks.

B.1 The two player case

Consider a player \( i \in \mathcal{N} \). The strategies in \( S_i(\bar{x}_i^t) \) prescribe the same action than the strategies in \( S_i(\bar{x}_j^{t-1}) \) for all \( x_i < \bar{x}_j^{t-1} \) and all \( x_i > k_i + \frac{\sigma}{2} \). The only discrepancies may appear in the interval \((\bar{x}_i^{t-1}, \bar{x}_i^t)\). Now, the actions prescribed by strategies in \( S_i(\bar{x}_i^t) \) in this interval, are best reply actions even to the worst case scenarios in \( S_j(\bar{x}_j^{t-1}) \). Since \( s_i^t \in S_{-i}(\bar{x}_j^{t-1}) \) and the action prescribed by \( s_i^t \) in this interval is a best reply to \( s_{-i}^t \), it must be the one prescribed by the set \( S_i(\bar{x}_i^t) \). Thus \( s_i^t \in S_i(\bar{x}_i^t) \).

Hence it must be the case that \( s^t \in S^t \).

Proof. Let \( s^t \in \text{BNE}(G_N(\sigma)) \).

It is direct to see that \( s^t \in S^0 \).

For \( t > 0 \) let us suppose that \( s^t \in S^{t-1} \). Then, since it is an equilibrium, we have \( \forall i \in \mathcal{N} \), \( s_{-i}^t \in S_{-i}(\bar{x}_j^{t-1}) \).

To prove the proposition we will prove that for all \( i \in \mathcal{N} \), \( \lim_{t \to \infty} x_i^t = \bar{x}_i^\sigma \). Indeed, if this is the case, then for \( i \in \mathcal{N} \setminus \{N\} \), any strategy \( s_i \in S_i^t \) for sufficiently large \( t \) has the form:

\[
s_i(x_i) = \begin{cases} 0 & \text{if } x_i < x_i^\sigma \\ 1 & \text{if } x_i^\sigma < x_i < x_i^t \text{ or } \bar{x}_i^t < x_i \end{cases}
\]

and in the limit when \( t \to \infty \), the only strategy that satisfies this is \( s_i^\sigma \). Moreover, we will show that the limit is attained in a finite \( t \), thus for player \( i = N \) we have that, once every other player has reached the upper dominance region in a certain \( t' \), then for \( x_N > x_N^{t-1} + \frac{\sigma}{2} \) the only best reply is 0 if \( x_N < x_N^t \) and 1 if \( x_N > x_N^t \) and thus in \( t' + 1 \), the only remaining strategy for player \( N \) in the process of IESCDs is \( s_N^\sigma \) (see footnote 20 in page 12). Thus, for every player \( i \in \mathcal{N} \) the only strategy in \( \bigcap_{t=0}^{\infty} S_i^t \) is \( s_i^\sigma \).

We prove Proposition 2 by induction on the number of players. We begin then by proving that it is true for the game \( G_{(1,2)}(\sigma) \).

B.1 The two player case

Before commencing the proof, let us state some useful remarks.

In the two player case we have two non-decreasing sequences \( \{x_i^t\}_{t=0}^{\infty} \) and \( \{x_j^t\}_{t=0}^{\infty} \) that by construction are bounded from above by \( k_i^\sigma \) and \( k_j^\sigma \) (respectively). The extremal profiles faced by a player are just strategies of her rival, so we call them extremal strategies.

Regarding the sets \( S_i(x_i^t) \), we know that for all \( t \geq 0 \), \( x_i^t \geq k_i^\sigma = x_i^\sigma \) and so the update of the sequence for player 1 is governed by equation (16) and the set \( S_1(x_i^t) \) has the following structure:

\[
S_1(x_i^t) = \{ s_1 \in S_1 : s_1(x_1) = 0 \text{ if } x_1 < k_1^\sigma \text{ and } s_1(x_1) = 1 \text{ if } x_1 \in (k_1^\sigma, x_i^t) \cup (k_1^\sigma, \infty + \sigma) \}.
\]  (18)

Figure 11: Optimal action against \( s_{-i}^t \) as a function of \( x_i \), for \( x_i < x_i^\sigma \).
For player 2 we know that $x^t_2 < k^\sigma_2 = x^*_2$ and so the update of the sequence is governed by equation (15) and the set $S_2(x^t_2)$ has the structure:

$$S_2(x^t_2) = \{s_2 \in S_2 : s_2(x_2) = 0 \text{ if } x_2 < x^t_2, \text{ and } s_2(x_2) = 1 \text{ if } x_2 > k^\sigma_2 \}.$$ (19)

Since the update for player 1 is governed by equation (16), the relevant extremal strategies are the lower extremal strategies, which are the ones that give player 1 the less incentive to play action 1. These, from A1, are the strategies of player 2 that belong to $S_2(x^{t-1}_2)$ and prescribe action 1 in the interval $(x^{t-1}_2, k^\sigma_2)$. Therefore at each $t \geq 1$ there is a unique extremal strategy which is the switching strategy on $x^{t-1}_2$:

$$s^{t-1}_2(x_2) := s_2(x_2; x^{t-1}_2),$$ (20)

where $s_2(\cdot; x^{t-1}_2)$ is defined in (8).

Analogously, since the update for player 2 is governed by equation (15), the relevant extremal strategies are the upper extremal strategies, which are the ones that give player 2 the less incentive to play action 0 (equivalently, the most incentive to play action 1). These, in turn, are the strategies of player 1 that belong to $S_1(x^{t-1}_1)$ and prescribe action 0 in the interval $(x^{t-1}_1, k^\sigma_1)$. Therefore at each $t \geq 1$ there is a unique extremal strategy which is the strategy:

$$s^{t-1}_1(x_1) := \begin{cases} 
0 & \text{if } x_1 < k^\sigma_1 \\
1 & \text{if } k^\sigma_1 < x_1 < x^{t-1}_1 \\
0 & \text{if } x^{t-1}_1 < x_1 < k^\sigma_1 \\
1 & \text{if } k^\sigma_1 < x_1
\end{cases}$$ (21)

We now turn to the analysis of the process of IESCD. Let us define $\sigma \equiv \frac{1}{2} \min \{ (k_2 - k_1), (k_2 - k_1), \sigma_0 \}$, where $\sigma_0$ is defined in equation (17) in the proof of Lemma 2, and take $0 < \sigma < \sigma$.

As long as $x^t_2 < k^\sigma_2$, for $t > 0$ the sequences $\{x^t_1\}_{t=0}^{\infty}$ and $\{x^t_2\}_{t=0}^{\infty}$ satisfy:

$$|x^t_1 - x^t_2| < \sigma$$ (22)

Since they are bounded from above, there exist limit points $x^\infty_1 \leq \bar{k}^\sigma_1$ and $x^\infty_2 \leq \bar{k}^\sigma_2$.

By construction $\Delta \Pi_1(x^{t-1}_1, x^t_1) = 0$ and $\Delta \Pi_2(x^{t-1}_2, x^t_2) = 0$ where the strategies $s^{t-1}_1$ and $s^{t-1}_2$ are defined in equations (20) and (21), respectively. From (22) we know that $x^t_1$ would reach $\bar{k}^\sigma_1$ before $x^t_2$ reaches $\bar{k}^\sigma_2$.

If there exists $t$ such that $x^t_1 = \bar{k}^\sigma_1$ then $s^{t}_1 = s^\sigma_1$. Thus, for $x_2 > \bar{k}^\sigma_2 + \sigma$ we get

$$\Delta \Pi_2(s^{t}_1, x_2) = \Delta \Pi_2(s^\sigma_1, x_2)$$

and so $x^\infty_2 = \bar{k}^\sigma_2$ which is what we wanted to prove.

If not, then either $x^\infty_1 < k^\sigma_1$ or $x^\infty_1 = k^\sigma_1$ and $x^\infty_2 < k^\sigma_2$, $\forall \ t > 0$.

If $x^\infty_1 < k^\sigma_1$ then, considering the equivalence $F_\sigma(x_1 \mid x_2) = 1 - F_\sigma(x_1 \mid x_2)$, we have

$$\Delta \Pi_1(s^{t-1}_1, x^t_1) = \Delta \tilde{\Pi}_1(1, x^t_1) F_\sigma(x^t_1 \mid x^{t-1}_2) + \Delta \tilde{\Pi}_1(0, x^t_1) (1 - F_\sigma(x^t_1 \mid x^{t-1}_2))$$

$$\Delta \Pi_2(s^{t}_1, x^t_2) = \Delta \tilde{\Pi}_2(0, x^t_2) (1 - F_\sigma(x^t_1 \mid x^{t-1}_2)) F_\sigma(k^\sigma_2 \mid x^t_2) + \Delta \tilde{\Pi}_2(1, x^t_2) [1 - (1 - F_\sigma(x^t_1 \mid x^{t-1}_2)) F_\sigma(k^\sigma_2 \mid x^t_2)]$$

thus, the conditions are

$$0 = \Delta \tilde{\Pi}_1(1, x^t_1) F_\sigma(x^t_1 \mid x^{t-1}_2) + \Delta \tilde{\Pi}_1(0, x^t_1) (1 - F_\sigma(x^t_1 \mid x^{t-1}_2))$$

$$0 = \Delta \tilde{\Pi}_2(0, x^t_2) (1 - F_\sigma(x^t_1 \mid x^{t-1}_2)) F_\sigma(k^\sigma_2 \mid x^t_2) + \Delta \tilde{\Pi}_2(1, x^t_2) [1 - (1 - F_\sigma(x^t_1 \mid x^{t-1}_2)) F_\sigma(k^\sigma_2 \mid x^t_2)].$$ (23)
Then, taking limit when \( t \to \infty \) we get
\[
0 = \Delta \Pi_1(1, x_1^\infty) F_\sigma(x_1^\infty | x_2^\infty) + \Delta \Pi_1(0, x_1^\infty) (1 - F_\sigma(x_1^\infty | x_2^\infty))
\]
\[
0 = \Delta \Pi_2(0, x_2^\infty) (1 - F_\sigma(x_1^\infty | x_2^\infty)) F_\sigma(k_1^\sigma | x_2^\infty) + \Delta \Pi_2(1, x_2^\infty) \left[ 1 - (1 - F_\sigma(x_1^\infty | x_2^\infty)) F_\sigma(k_1^\sigma | x_2^\infty) \right].
\]

Multiplying the first equation by \( F_\sigma(k_1^\sigma | x_2^\infty) > 0 \) and subtracting the second one we get
\[
\left[ \Delta \Pi_1(1, x_1^\infty) - \Delta \Pi_2(1, x_2^\infty) \right] F_\sigma(x_1^\infty | x_2^\infty) F_\sigma(k_1^\sigma | x_2^\infty) + \left[ \Delta \Pi_1(0, x_1^\infty) - \Delta \Pi_2(0, x_2^\infty) \right] (1 - F_\sigma(x_1^\infty | x_2^\infty)) F_\sigma(k_1^\sigma | x_2^\infty) + \Delta \Pi_2(1, x_2^\infty) \left[ 1 - (1 - F_\sigma(x_1^\infty | x_2^\infty)) F_\sigma(k_1^\sigma | x_2^\infty) \right].
\]

From Lemma 2 we get that each term in the left hand side of equation (24) is positive and since \( x_2^\infty < k_2^\sigma \) the last term is strictly positive, which gives us a contradiction.

Thus, the only remaining option is that \( x_1^\infty = k_1^\sigma \) and \( x_2^\infty < k_2^\sigma \), \( \forall t > 0 \).

If this is the case, the update equations (23) are still valid and taking limit when \( t \to \infty \) we get
\[
0 = \Delta \Pi_1(1, k_1^\sigma) F_\sigma(k_1^\sigma | x_2^\infty) + \Delta \Pi_1(0, k_1^\sigma) (1 - F_\sigma(k_1^\sigma | x_2^\infty))
\]
\[
0 = \Delta \Pi_2(0, x_2^\infty) (1 - F_\sigma(k_1^\sigma | x_2^\infty)) F_\sigma(k_1^\sigma | x_2^\infty) + \Delta \Pi_2(1, x_2^\infty) \left[ 1 - (1 - F_\sigma(k_1^\sigma | x_2^\infty)) F_\sigma(k_1^\sigma | x_2^\infty) \right].
\]

By definition, \( \Delta \Pi_1(1, k_1^\sigma) = 0 \) and so from the first equation we have that \( F_\sigma(k_1^\sigma | x_2^\infty) = 1 \). Replacing in the second equation, we get that \( \Delta \Pi_2(1, x_2^\infty) = 0 \) which in turn implies that \( x_2^\infty = k_2^\sigma \) which can not be since \( \sigma < k_2^\sigma - k_1^\sigma \).

We conclude then that it must be the case that there exists \( t \) such that \( x_1^t = k_1^\sigma \) and \( x_2^t = k_2^\sigma \).

### B.2 The N-player case

In the previous section what we actually proved is that in the game \( G_{\{1,2\}}(\sigma) \) there exists a finite \( t \) such that \( x_1^t = k_1^\sigma = k_1^\sigma(1) \) for both players \( i \in \{1,2\} \).

For the \( N \)-player case, since we will induction on the number of players, we will assume that in the game \( G_{N-1}(\sigma) \) there exists a finite \( t \) such that \( x_1^t = k_1^\sigma(N - 2) \) for all players \( i \in N - 1 = \{1, \ldots, N - 1\} \).

We will have to prove that in a finite \( t \) the sequences reach the limit of the upper dominance region in \( G_N(\sigma), k_1^\sigma(N - 1) \).

In order to develop the proof using an inductive argument, it is first necessary to formally introduce the notion of Reduced Game.

**Definition 6 (Reduced Game).** Consider an incomplete information game \( G_N(\sigma) \) as defined in Section 3.1, and an arbitrary subset of players \( I \subseteq N \). Let \( s_1 = (s_i)_{i \in I} \) and \( s_{-1} = (s_i)_{i \notin I} \). Conditionally on \( s_{-1} \), we define the Reduced Game \( \Gamma_I(\sigma | s_{-1}) \) (reduced from \( N \) players to \#1 players) of the original game \( G_N(\sigma) \) as the game with the same structure as \( G_N(\sigma) \) where every player not in \( I \) sticks to the strategy prescribed by the profile \( s_{-1} \).

It is easy to check that if the underlying structure of \( G_N(\sigma) \) satisfies assumptions A1, A2, A3 and A5, then the same assumptions hold for the underlying structure of the reduced game \( \Gamma_I(\sigma | s_{-1}) \). Moreover, if conditionally on \( s_{-1} \) there exists an interval of signals \( [\underline{x}, \bar{x}] \subseteq [\underline{X}, \bar{X}] \) such that for every player \( i \in I \), there exist upper and lower dominance regions (according to assumption A4), then \( \Gamma_I(\sigma | s_{-1}) \) is a reduced game that holds the same properties of the original game \( G_N(\sigma) \). Moreover, note that the reduced game \( \Gamma_{N-1}(\sigma | s_N^\sigma) \), where player \( N \)'s strategy is the switching strategy \( s_N^\sigma(\cdot; x_N^\sigma) \), is equivalent to the game \( G_{N-1}(\sigma) \). These facts may allow us to use results from games with less players.

Recall that in \( s^\sigma \) each player \( i \) switches from 0 to 1 in \( x_i^\sigma = k_i^\sigma(i - 1) \) and so the equilibrium strategy \( s_i^\sigma \) does not depend on the number of players.\(^{24}\) When we pass from a game with \( N \) players to a game...

\(^{23}\)From (23) and the assumption \( x_1^\infty < k_1^\sigma \) we get that \( x_2^\infty \leq x_1^\infty + \sigma < k_2^\sigma + \sigma \).

\(^{24}\)The player that is in the \( r \)-th position (player \( r \)) determines her switching point depending only on her position in the set of players and not on how many players interact with her.
with $N - 1$ players, what changes is the limits of the upper dominance regions (they are $k^*_N(N - 1)$ in the $N$-player game and $k^*_l(N - 2)$ in the $N - 1$-player game). Thus, in the $N$-player game the switching point of player $N - 1$ is $x^0_{N-1} = k^*_N(N - 2) \neq k^*_{N-1} = k^*_N(N - 1)$. In the $N - 1$ player game, we do have $x^0_{N-1} = k^*_{N-1} = k^*_N(N - 2)$.

We now turn to the analysis of the process of IESCDs.

Let us define $\sigma = \frac{1}{2} \min \{ (\bar{k}_i - k^-_i)_{i=2}^N, (\bar{k}_i - k^-_i)_{i=2}^N, \sigma_0 \}$, where $\sigma_0$ is defined in equation (17) in the proof of Lemma 2, and take $0 < \sigma < \bar{\sigma}$.

By definition of $\sigma$, we know that $k^*_N - 1 + \sigma < k^*_N$, then $i \in \{1, \ldots, N - 1\}$ and $\forall i < k^*_N - \sigma$ player $i$’s net payoff is equal to $\Delta \Pi_i(s_{-i}, k^*_N, x_i)$. So at least the $N - 1$ first rounds of elimination of $G_N(\sigma)$ are equal to the $N - 1$ first rounds of elimination of the reduced game $\Gamma_{N-1}(\sigma | s^*_N)$ (see Definition 6), game in turn is equal to $G_{N-1}(\sigma)$. Note that, as long as the sequences $z^i_l, i \in N - 1$ do not reach the threshold $k^*_N - \sigma$, each step of the process in $G_N(\sigma)$ is equivalent to a step in $G_{N-1}(\sigma)$. By the induction hypothesis, we know that in $G_{N-1}(\sigma)$, $z^i_{N-1}$ reaches $k^*_{N-1}(N - 2)$ in a finite number of steps. So, necessarily, it must be the case that in $G_N(\sigma)$, $z^i_{N-1}$ reaches $k^*_N - \sigma < k^*_N(N - 2)$. Let us call $\bar{t}$, the moment when the first sequence $z^i_{\bar{t}}$ reaches $k^*_N - \sigma$ (we know that $z^i_{\bar{t}} = k^*_N$).

Note as well, that as long as the sequences $z^i_l$ that reach $k^*_N - \sigma$, have not surpassed the corresponding player $i$’s switching point, the process for the first $N - 1$ players is equivalent to that in game $G_{N-1}(\sigma)$ and for player $N$ nothing changes, thus we may assume that $\bar{t}$ is the moment when the first sequence that has surpassed the corresponding player’s switching point, reaches $k^*_N - \sigma$.

At any step $t$ of the process, we may partition the set $N$ into two sets: one set containing those players who have reached their equilibrium switching point and those players who have not, i.e.

$$\{i \in N : x^i_t \leq z^i_{t-1}\} \text{ and } \{i \in N : x^i_t < x^i_{t-1}\}$$

Moreover, since the switching points are ordered and $\sigma$ is small, we know that these two sets of players consist of the $l$ first players (those who reached their switching process) and the $N - l$ last players (those who have not), for some $l \in N$. By construction of the process, player $l + 1$’s sequence in step $t$ may at most reach her switching point (and not surpass it). If this is the case, in the next step she will then be part of the set of players that reached their switching point (now the first $l + 1$ players).

Regarding the sets $S_i(z^i_t)$, we know that for all $i \leq l$, $z^i_t \geq x^i_t$ and so the update of the sequence for player $i$ is governed by equation (16) and the set $S_i(z^i_t)$ has the following structure:

$$S_i(z^i_t) = \{s_i \in S_i : s_i(x_i) = 0 \text{ if } x_i < x^i_t \text{ and } s_i(x_i) = 1 \text{ if } x_i \in (x^i_t, z^i_t) \cup (\bar{k}^*_N, \underline{X} + \sigma)\}.$$  \hfill (25)

For players $i > l$ we know that $z^i_t < x^i_t$ and so the update of the sequence is governed by equation (15) and the set $S_i(z^i_t)$ has the structure:

$$S_i(z^i_t) = \{s_i \in S_i : s_i(x_i) = 0 \text{ if } x_i < z^i_t \text{ and } s_i(x_i) = 1 \text{ if } x_i > \bar{k}^*_N\}.$$  \hfill (26)

Let us first consider a player $i > l$. Since the update for these players is governed by equation (15), the relevant extremal profiles are the upper extremal profiles, which are the ones that give these players the less incentive to play action 0. These, in turn, by strategic substitutes, are the profiles of the other players that belong to the set $\prod_{j \neq i} S_j(z^j_{t-1})$ that prescribe action 0 in the intervals $(z^j_{t-1}, \bar{k}^*_N)$, subject to being mutually best responses (profiles of the rivals that prescribe the most zeros, within the mutually best responses). In the worst case, the component of the extremal profile $z^j_{t-1}$, associated to the $N^{th}$ player is the equilibrium strategy $s^*_N$. If this is the case, then the update for player $i$ is equal to the update in the reduced game $\Gamma_{N-1}(\sigma | s^*_N) = G_{N-1}(\sigma)$. If not, then the new term $z^i_t$ would be greater than the one obtained in the update in the reduced game. In any case, in the original game the sequence advances further away from $k^*_N$ than in the reduced game.

Second, let us consider a player $i \leq l$. Since the update for player $i \leq l$ is governed by equation (16), the relevant extremal profiles are the lower extremal profiles, which are the ones that give player $i$ the less incentive to play action 1. These, in turn, by strategic substitutes, are the profiles of the other players that belong to the set $\prod_{j \neq i} S_j(z^j_{t-1})$ that prescribe action 1 in the intervals $(z^j_{t-1}, \bar{k}^*_N)$, subject to being mutually best responses (profiles of the rivals that prescribe the most “ones”, within the mutually best responses)
responses). Nevertheless we will see below that the component of the extremal profile $s_{i}^{t-1}$, associated with player $N$ is $s_N^i$ as well.

Indeed, suppose that the strategy of player $N$ in an extremal profile $s_{-i} \in S_{-i}(s_{i}^{t-1})$ is different from $s_N^i$. This is, there exists an open interval $O \subset |S_N|$, $k_N^i$ such that the strategy $s_{N} \in S_{N}(s_{i}^{t-1})$ in the extremal profile $s_{-i}$ prescribes $s_{N}(x_{N}) = 1$ for $x_{N} \in O$. For signals less than $x_{N}^{t-1}$ we cannot have the $N-1$ dimensional profile $(1, \ldots, 1)$ on $O$, since this profile of actions do not constitute mutually best responses. Therefore, for every $s_{-i} \in S_{-i}(s_{i}^{t-1})$ there must be a player $j \neq i$ such that the $j$-th component of $s_{-i}$ prescribes action 0 in $O$. By anonymity, we may assume that this player is player $N$.

Therefore, the update for player $i \leq l$ in game $G_{N}(\sigma)$ is equivalent to the update in the reduced game $\Gamma_{N-1}(\sigma | s_N^i) = G_{N-1}(\sigma)$.

We conclude that, as long as at $t - 1$ the sequences are less than $x_{N}^{t-1}$, the update of every player $i < N$ in the game $G_{N}(\sigma)$ sends the next term of the sequence at least as far as in the update in the reduced game $\Gamma_{N-1}(\sigma | s_N^i) = G_{N-1}(\sigma)$. For player $N$, the updates of all the other players induce an update of $x_{N}^{t} \geq x_{N}^{t-1}$.

Therefore, by the induction hypothesis, for every player $i < N$ there must be a finite $t$ such that, $x_{N}^{t}$ is equal to the lower bound of the upper dominance region in the game $G_{N-1}(\sigma)$. This is, $\forall i < N$ there exists $t$ such that, $x_{N}^{t} = k_N^i(N-2)$.

Moreover, calling $\tilde{t}$ the step where $x_{N}^{\tilde{t}-1} = k_{N-1}^i(N-2)$, note that for a player for $i < N - 1$, if $x_{N}^{\tilde{t}-1} \geq k_N^i(N-2)$ and $x_{N}^{\tilde{t}-1} \leq k_{N-1}^i(N-2)$ then the update for player $i$ in game $G_{N}(\sigma)$ is equivalent to the update in the reduced game $\Gamma_{N-1}(\sigma | s_N^i) = G_{N-1}(\sigma)$ and so action 1 is strictly dominant for this player (since player $i$ sees at most $N - 2$ players taking action 1). So, we can conclude, that at $\tilde{t}$ we have $x_{N}^{\tilde{t}-1} = k_{N-1}^i(N-2)$ and for every player $i < N - 1$, $x_{N}^{\tilde{t}}$ is in a $\sigma$-neighborhood of $k_{N-1}^i(N-2) = x_{N}^{\tilde{t}-1}$. Again, for player $N$, at each such $t$, the updates of all the other players induce an update of $x_{N}^{t}$, and so we also have $x_{N}^{\tilde{t}}$ in a $\sigma$-neighborhood of $k_{N-1}^i(N-2) = x_{N}^{\tilde{t}-1}$. We illustrate this situation for the 3-player case in Figure 12.

![Figure 12: Situation at $\tilde{t}$ for the 3-player case.](image)

We now describe the process for $t > \tilde{t}$ and conclude the proof. Let us assume that for $t - 1$ we have\footnote{It is true for $t - 1 = \tilde{t}$.}

$$x_{N}^{t-1} \geq x_{N}^{t-1} = k_{N-1}^i(N-2).$$

We will follow the same argument described and used above. In this situation we know that for all players $i \leq N - 1$, $x_{N}^{t-1} \geq x_{N}^{t}$. The update for these players is governed by equation (16) and so the relevant extremal profiles are the lower extremal profiles, which are the ones that give the less incentive to play action 1. These, in turn, by strategic substitutes, are the profiles of the other players that belong...
to the set \( \prod_{j \neq i} S_j(x^t_j) \) that prescribe action 1 in the intervals \( (x^t_j, k^T_j) \), subject to being mutually best responses.

Consider a player \( i \in \{2, \ldots, N-1\} \). In the worst case, the component of the extremal profile \( x^t_{i-1} \), associated to player 1 is the equilibrium strategy \( s_1^t \). If this is the case, then the update for player \( i \) is equal to the update in the reduced game \( \Gamma_{\{2,\ldots,N\}}(\sigma \mid s_1^t) \) which is an \( N-1 \) player game that satisfies assumptions A1 to A5. If \( s_1^t \) is not the component of the extremal profile \( x^t_{i-1} \), associated to player 1, then the new term \( x^t_i \) would be greater than the one obtained in the update in the reduced game. In any case, in the original game the sequence advances further away from \( k^T_i \) than in the reduced game.

The update for player \( N \) is governed by equation (15). The relevant extremal profiles are the upper extremal profiles, which are the ones that give less incentive to play action 0. These, in turn, by strategic substitutes, are the profiles of the other players that belong to the set \( S \) from \( s \). This is, there exists an open interval \( \overline{\sigma} \) such that, \( x^t_{i-1} \) is different from \( s_1^t \). This is, there exists an open interval \( O \subset (x^t_{i-1}, k^T_i) \) such that the strategy \( \overline{x}_1 \in S_1(x^t_{i-1}) \) in the extremal profile \( \overline{s}_N \) prescribes \( \overline{x}_1(x_1) = 0 \) for \( x_1 \in O \). However, for signals greater or equal to \( x^t_{N-1} \) we cannot have the \( N-1 \) dimensional profile \( (0, \ldots, 0) \) on \( O \), since this profile of actions do not constitute mutually best responses. Therefore, for every \( s_1^t \in S_1(x^t_{i-1}) \) there must be a player \( j \neq N \) such that the \( j \)-th component of \( \overline{s}_N \) prescribes action 1 in \( O \). By anonymity, we may assume that this player is player 1.

We conclude that the update of every player \( i > 1 \) in the game \( G_N(\sigma) \) sends the next term of the sequence at least as far as in the update in the reduced game \( \Gamma_{\{2,\ldots,N\}}(\sigma \mid s_i^t) \). For player 1, the updates of all the other players induce an update of \( x^T_1 > x^t_1 \).

Therefore, by the induction hypothesis, for every player \( i > 1 \) there must be a finite \( t \) such that, \( x^T_i \) is equal to the lower bound of the upper dominance region in the game \( \Gamma_{\{2,\ldots,N\}}(\sigma \mid s_i^t) \). These lower bounds, are the exact same lower bounds of the upper dominance regions of the game \( G_N(\sigma) \). This is, \( \forall i > 1 \) there exists \( t \) such that, \( x^T_i = k^T_i(N-1) = k^T_i \). If every player but player 1 has reached the upper dominance region, then in the next step, player 1 reaches her upper dominance region.

This completes the proof.
References


