Noisy Beliefs Equilibrium*

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April 14, 2017

Abstract

Quantal response equilibrium (QRE) (McKelvey and Palfrey, 1995) relaxes the rationality requirement of Nash equilibrium by “adding noise to actions”. We introduce noisy beliefs equilibrium (NBE), which instead relaxes the belief consistency requirement of Nash by “adding noise to beliefs”. In other words, in an NBE of a game, players best respond to their beliefs, and their beliefs are a noisy version of the true distribution of actions.

We establish existence and basic properties of general NBE, and show that within the $2 \times 2$ games commonly played in the lab, NBE is able to explain the same deviations from Nash as QRE as well as the fact of high dispersion in elicited beliefs. We also show that unlike QRE, NBE predictions are invariant to changes in the payoff magnitude of games, which is consistent with experimental evidence. Hence, NBE performs just as well as QRE in-sample and much better out-of-sample across these games. We develop a one-parameter specification of NBE based on the logit transform and apply it to experimental data from existing studies. Unlike the rationality parameter $\lambda$ of logit QRE, estimates of the noise parameter $\sigma$ of NBE are invariant to the arbitrary “exchange rate” between utility and money. We adjust these exchange rates across 5 studies to build a dataset of 21 comparable $2 \times 2$ games. We find that the value $\sigma = 1$ fits the pooled data from these games much better than the best-fit QRE, resulting in 54% of the QRE prediction error.

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*Very preliminary. I thank Alessandra Casella, Yeon-Koo Che, Mark Dean, Navin Kartik, Judd Kessler, RC Lim, Pietro Ortoleva, Andrea Prat, Mike Woodford and participants at the Columbia Theory Colloquium, Columbia Cognition and Decisions lab, and the Columbia Experimental Lunch for helpful comments. I also thank the Columbia Experimental Laboratory in the Social Sciences (CELSS) for its generous support, and Thorsten Chmura for providing data. All errors are my own.

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1 Introduction

Game theory has been extremely influential in many areas of academia as well as in policy debates. Despite its influence in shaping our understanding of the “real world,” the actual predictive power of classical game theory is quite limited. Much of our understanding of these limitations come from laboratory experiments, where it is clear that even under idealized conditions, they persist. In fact, it is well-documented that there are systematic deviations from the predictions of classical Game Theory in even the simplest of games—where the equilibrium is unique and fully mixed, the action space and number of players is small, and there are no obvious confounds\(^1\). This is a puzzle of fundamental importance: if we cannot predict behavior in simple games, what hope is there to predict behavior in realistic games of broader interest?

Game theory rests on Nash equilibrium (NE) as its central concept. In an NE

(NE 1) **Rationality**: Players *best respond* to their beliefs\(^2\), and

(NE 2) **Belief consistency**: beliefs equal the equilibrium distribution of actions.

There are several theories, both equilibrium and non-equilibrium theories, that relax these conditions that have had some success in explaining deviations from Nash. Most notably, quantal response equilibrium (QRE) (McKelvey and Palfrey [1995]) relaxes (NE 1) by assuming that despite having correct beliefs (i.e. maintaining (NE 2)), players make mistakes, failing to play an optimal action with probability one. In a QRE

(QR 1) **Better response**: Players *make mistakes* in responding to their beliefs\(^3\), and

(NE 2) **Belief consistency**: beliefs equal the equilibrium distribution of actions.

Since we refer to QRE many times, we review it in Section 11.1.

Relaxing (NE 2) on the other hand has been explored most prominently in the context of non-equilibrium models. Indeed, the consistency of beliefs notion of (NE 2) is almost synonymous with equilibrium. The leading example is Level-\(K\) (Nagel [1995] and Stahl and Wilson [1995]) which, along with its successors (Camerer et al. [2004b], Alaoui and Penta [2015], and others), assumes that players mistakenly believe other players are drawn from a particular distribution of types with different “depths of reasoning”: a Level 0 player plays some default action\(^4\), Level 1 best responds to Level 0, and Level \(K\) best responds to Level \(K - 1\). The types of beliefs available to players are

\(^1\)Here, we mean anything at all that induces subjects to rely on non-strategic heuristics or otherwise alters the utility payoffs to outcomes in the game. For instance, confounds could be related to social preferences, salient strategies, etc.

\(^2\)That is, players take an action to maximize their expected utility, where the expected utility to each action is pinned down by beliefs over the distribution of other players’ actions.

\(^3\)Though they tend to take better actions (by expected utility) with higher probability according to a “quantal response function”.

\(^4\)A Level 0 player is usually assumed to be *uniformly mixing* over his pure actions.
“anchored” to the Level 0 strategy, and hence predictive power rests heavily on a non-equilibrium notion with little epistemic foundation. It is well-known that certain types of games, such as the Beauty Contest game\(^5\) or the “ring games” of Kneeland [2015], seem to invite in experimental subjects the type of thinking prescribed by these models. Aside from the troubling need to anchor beliefs, these models do not seem to capture behavior in simple games that were not specifically constructed. Non-equilibrium models also cannot typically be supported as the stationary outcome of games played repeatedly with learning\(^6\).

We have established that equilibrium models have desirable properties, and also that equilibrium models tend to maintain that beliefs are correct in the form of (NE 2). It is well-documented in the laboratory, however, that elicited beliefs have high dispersion and differ from the empirical distribution of actions, so the (NE 2) assumption is broadly counterfactual. For instance, Palfrey and Wang [2009] and Nyarko and Schotter [2002] document this in \(2 \times 2\) games with unique mixed strategy equilibria. This seems to motivate an equilibrium model in which beliefs are noisy and mistaken and yet also responsive to the true distribution of actions. This is exactly the role noisy beliefs equilibrium (NBE) is intended to fill. In an NBE

(NE 1) **Rationality**: Players best respond to their beliefs, and

(NB 2) **Noisy Belief consistency**: beliefs are a noisy version of the equilibrium distribution of actions.

It is readily seen that QRE and NBE are both minimal deviations from NE, the former adding noise to actions and the latter adding noise to beliefs. As an appeal to intuition, we ask: which is more difficult—forming correct beliefs or taking the best action given beliefs? Whatever the answer is, it should be observable in aggregate data and have implications for the interpretation of experimental findings.

In the leading logit specification of QRE, estimates of the rationality parameter \(\lambda\) are sensitive to the arbitrary “exchange rate” between utility and money. That is, holding the data fixed but scaling utilities in the payoff matrix for one or more players leads to different estimates, and if not all players’ utilities are scaled by the same factor, this leads to different predicted actions as well. It is unclear what to make of this observation. At worst, it raises fundamental questions about the information content of QRE and at best, it makes it difficult to compare \(\lambda\) estimates across studies. To compare \(\lambda\) estimates from experiments run in different currencies and years would first require adjustments for currency-to-currency exchange rates and inflation. That is, the distinction between

\(^5\)Nagel [1995] ran the original Beauty Contest experiment, and Camerer et al. [2004b] gives a survey of such games played in the lab.

\(^6\)Since learning gives information on the correct distribution of actions, beliefs will tend in that direction and actions will typically converge to some equilibrium. It is well-known, for instance, that rational learning generically leads to NE. See, for example, Kalai and Lehrer [1993]. Other equilibrium concepts, such as QRE and NBE, can be similarly microfounded under the assumption that some “noise” remains even after many rounds of learning. Some other equilibrium concepts are specifically based on sampling past actions or payoffs.
Analogous to logit QRE, we develop a parametric version of NBE based on the logit transform, where the parameter $\sigma$ represents the noisiness of beliefs. Importantly, unlike QRE-$\lambda$, estimates of NBE-$\sigma$ are invariant to multiplying one or more player’s arbitrary utility numbers by positive constants. Hence, $\sigma$ estimates can be compared across all games ever played in the lab without any adjustments. So that we can compare NBE to QRE, we make the utility adjustments for 5 prominent studies on $2 \times 2$ games, giving a dataset of 21 comparable games. We find that NBE with a value of $\sigma$ close to 1 fits the pooled data with only 54% of the prediction error of the best-fit QRE. We are encouraged by this result because the stability of parameter estimates across different contexts is often interpreted as evidence that the parameters capture behavioral constants, indicating a well-specified theory.

The evidence to support QRE’s ability to explain deviations from Nash is overwhelming, and it is hard to overstate the impact this has had on the interpretation of experimental results. The following quotation from Camerer et al. [2004a] supports the assertion:

*Quantal Response Equilibrium (QRE), a statistical generalization of Nash, almost always explains the direction of deviations from Nash and should replace Nash as the static benchmark to which other models are routinely compared.*

This quotation refers to in-sample performance in which parameters are chosen to best explain data using the data itself. However, out-of-sample predictability, the ability to explain behavior in one game from parameters estimated in another, is a different matter. Despite the remarkable ability of QRE to explain data in-sample, its ability to explain data out-of-sample is well-known to be poor. We view this as a puzzle of considerable importance for two reasons. First, since out-of-sample predictability requires that exogenous parameters are stable across contexts, its failure is evidence of misspecification. Second, out-of-sample predictability is necessary for applying the insights from laboratory experiments to the “real world”.

Recent work by Friedman [2016] attempts to salvage the out-of-sample predictability of QRE by endogenizing the $\lambda$ parameter, though finds mixed results. McKelvey et al. [2000] tests the out-of-sample predictability of QRE using experimental $2 \times 2$ games. It finds that despite QRE’s sensitivity to changes in payoff magnitude, subject behavior is statistically invariant to these changes, and hence QRE makes poor out-of-sample predictions. We show that NBE and QRE make similar predictions for these games, but consistent with the evidence, NBE predictions are invariant to changes in payoff magnitude. Revisiting the data from McKelvey et al. [2000], we show that this makes NBE perform much better than QRE out-of-sample.

The rest of the paper is organized as follows. Section 2 gives the NBE model, Section 3 explains the $2 \times 2$ games that we use throughout the paper, Section 4 gives the parametric logit transform
NBE, Section 5 explores a version of NBE that imposes only minimal structure, Section 6 shows that NBE is invariant to payoff magnitude and explores its empirical implications, Section 7 interprets NBE as a model of learning, Section 8 fits NBE and QRE to games from several studies to compare the stability of parameter estimates and overall fit, Section 9 considers an extension in which beliefs are systematically biased, and Section 10 concludes.

2 Noisy Beliefs Equilibrium

A finite, normal form game $\Gamma = \{N, A, u\}$, is defined by a set of agents $N = \{1, ..., n\}$, action space $A = A_1 \times ... \times A_n$ with $A_i = \{a_{i1}, ..., a_{iJ(i)}\}$ such that each player $i$ has $J(i)$ possible pure actions, and a vector of payoff functions $u = (u_1, ..., u_n)$ with $u_i : A \rightarrow \mathbb{R}$.

Let $\Delta A_i$ be the set of probability measures on $A_i$. Elements of $\Delta A_i$ are of the form $p_i : A_i \rightarrow \mathbb{R}$ where $\sum_{j=1}^{J(i)} p_i(a_{ij}) = 1$ and $p_i(a_{ij}) \geq 0$. Henceforth define $p_{ij} \equiv p_i(a_{ij})$. We abuse notation by using $a_{ij}$ to refer to the action $p_i \in \Delta A_i$ with $p_{ij} = 1$. Define $\Delta A = \Delta A_1 \times ... \times \Delta A_n$ with $p \in \Delta A$ a typical element. Extend payoffs to be defined over $\Delta A$ by $\bar{u}_i(p) = \sum_{a \in A} p(a)u_i(a)$. For every $a_{-i} \in \times_{k \neq i} A_k \equiv A_{-i}$ and $p_{-i} \in \times_{k \neq i} \Delta A_k \equiv \Delta A_{-i}$, define $u_{ij}(a_{-i}) \equiv u_i(a_{ij}, a_{-i})$ and $\bar{u}_{ij}(p_{-i}) \equiv u_i(a_{ij}, p_{-i})$ to be the utilities to player $i$ of taking action $j$ given pure and mixed actions respectively of the other players.

In addition to the standard objects previously defined, we introduce the notion of “noisy beliefs”. Player $i$’s noisy beliefs over $i$’s actions are modeled as a random variable that is parametrized in the actions of player $i$. That is, for all $p_i \in \Delta A_i$, noisy beliefs $p_i^s(p_i) = (p_{i1}^s(p_i), ..., p_{iJ(i)}^s(p_i))$ are a random variable on $(\Delta A_i, \mathcal{B}(\Delta A_i))$, where $\mathcal{B}(\Delta A_i)$ is the Borel $\sigma$-algebra on $\Delta A_i$. Use $p^s = (p_i^s)_{i \neq l}$ to refer to all players’ noisy beliefs. We assume that all of $i$’s opponents draw their beliefs about $i$’s actions independently. We further impose, though it is not necessary for any results, that these beliefs are drawn from the same distribution. In particular, we assume that for all $l \neq i$, $p_i^s(p_i) \sim iid b_l(\cdot|p_i)$ for some probability measure $b_l(\cdot|p_i)$. For convenience, we also define $p_{-i}^s(p_{-i}) = (p_k^s(p_k))_{k \neq i}$ to be player $i$’s noisy beliefs about his opponents with corresponding measure $b_{-i}(\cdot|p_{-i}) = (b_k(\cdot|p_k))_{k \neq i}$. Further define $b(\cdot|p) = (b_1(\cdot|p_1), ..., b_n(\cdot|p_n))$ and the whole family of probability measures by $b = (b(\cdot|p))_{p \in \Delta A}$.

Note that in modeling noisy beliefs, we have assumed that a player’s noisy beliefs about another player only depend on that player’s actions. This rules out the possibility that player $i$’s noisy beliefs about two of his opponents are correlated. We view this as a sensible restriction, though it can easily be relaxed.

Define the $ij$-response set $R_{ij} \subseteq \Delta A_{-i}$ by

$$R_{ij} = \{p_{-i}^' | \bar{u}_{ij}(p_{-i}^') \geq \bar{u}_{ik}(p_{-i}) \ \forall k = 1, ..., J(i)\}. \tag{1}$$

This defines the set of beliefs about $i$’s opponents for which action $a_{ij}$ is a best response. Note that
$R_{ij} \in \mathcal{B}(\Delta A_{-i})$, i.e. that the response sets are measurable. With the next definition, we begin to add structure.

**Definition 1.** We say that the pair $\{\Gamma, p^*\}$ is *admissible* if the following properties hold:

1. **No indifference.** For all $i, j, k$, and $p_{-i}$
   \[ b_{-i}(\{p'_{-i}|\bar{u}_{ij}(p'_{-i}) = \bar{u}_{ik}(p'_{-i})\}|p_{-i}) = 0. \tag{2} \]

2. **Continuity.** For all $i$ and $R_{ij}$,
   \[ b_{-i}(R_{ij}|p_{-i}) \text{ is continuous in } p_{-i}. \tag{3} \]

Note that admissibility is a condition on the game and noisy beliefs *jointly*. Even the most well-behaved noisy beliefs will lead to violations of the *no indifference* condition for some games. For instance, *any* noisy beliefs will lead to violations for any game with “duplicate actions” such that $u_{ij}(a_{-i}) = u_{ik}(a_{-i})$ for all $a_{-i}$. The next example shows that even the most well-behaved games will lead to violations of *both conditions* for some noisy beliefs. Consider a $2 \times 2$ game with a unique mixed strategy NE, the simplest possible non-trivial game. Noisy beliefs that are *correct*, such that $b(p|p) = 1$ for all $p$, will lead to violations of *both* conditions.

If (2) holds and we assume rationality (NE 2), we can define the probability player $i$ chooses action $j$ given actions $p_{-i} \in \Delta A_{-i}$ of his opponents as

\[ Q_{ij}(p_{-i}) = \int_{R_{ij}} db_{-i}(x|p_{-i}). \tag{4} \]

We call $Q_i = (Q_{i1}, ..., Q_{iJ(i)}) : \Delta A_{-i} \to \Delta A_i$ player $i$’s reaction function and $Q = (Q_1, ..., Q_n) : \Delta A \to \Delta A$ the reaction function of all players. If (3) holds, several facts about the reaction function are immediate.

1. $Q \in \Delta A$ is non-empty.

2. $Q$ is continuous on $\Delta A$.

Hence, admissibility makes the reaction functions *well-defined* and *well-behaved*. While admissibility is easy to prove or disprove given specific games and noisy beliefs, it is difficult to come up with a very sensible and general sufficient condition. For instance, consider the following sufficient condition:

1. **Genericness.** For all $i, j, k$ and $a_{-i}$
   \[ u_{ij}(a_{-i}) \neq u_{ik}(a_{-i}). \tag{5} \]
2. Uniformly continuous densities. \( b \) emits a family of densities \((f(\cdot | p))_{p \in \Delta A}\) that are full support and uniformly continuous with respect to \( p \).\(^8\)

That this leads to admissibility is obvious since genericness implies the set of beliefs that make a player indifferent over actions is measure zero and uniformly continuous densities implies that \( b \) can be derived by integrating over densities in a way that preserves continuity. However, the logit transform specification of NBE of Section 4 involves \( b_i(p_i|p_i) = 1 \) if and only if \( p_i \) is degenerate, which we argue is very natural but rules out the representation using densities. We will show that the logit transform model satisfies admissibility for an interesting class of games in Section 4, but it is difficult to generalize the sufficient condition that applies in that case.

A noisy beliefs equilibrium (NBE) is defined for any admissible \( \{\Gamma, p^*\} \) as a fixed point of \( Q \). Formally,

**Definition 2.** Let \( \{\Gamma, p^*\} \) be admissible. A noisy beliefs equilibrium (NBE) is any \( p \in \Delta A \) such that for all \( i \in 1,...,n \) and all \( j \in 1,...,J(i) \),

\[ p_{ij} = Q_{ij}(p_{-i}). \]

Existence follows immediately from properties of the reaction function

**Theorem 1.** For any \( \{\Gamma, p^*\} \), there exists an NBE.

**Proof.** An NBE is a fixed point of \( Q \). By admissibility of \( \{\Gamma, p^*\} \), \( Q : \Delta A \to \Delta A \) is a continuous function mapping from a compact and convex set to itself. By Brouwer’s fixed point theorem, there exists a fixed point of \( Q \). \(\square\)

A few points merit discussion. By only defining NBE for admissible \( \{\Gamma, p^*\} \), we have effectively embedded into the definition that the equilibria are pure. That is, in any equilibrium, the probability a player has beliefs such that he is willing to mix is zero. Of course, we can develop notions of mixed NBE, but since their analysis would generally be intractable, we stick to our definition. This approach mirrors that for quantal response equilibrium (QRE), which in its general form is only defined for a notion of admissible “errors”. See Appendix 11.1 for a review of QRE. At this level of generality, QRE is much simpler than NBE since QRE “adds noise to actions” by adding noise to expected utilities which live in an unbounded space. Since NBE adds noise to beliefs which live in a bounded space, it is possible that no noise is capable of effecting actions in certain ways. Without a careful treatment, discontinuities may arise.

The following remark immediately distinguishes NBE from QRE.

**Remark 1.** No NBE involves strictly dominated actions.

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\(^8\)That is, for all \( p \in \Delta A \) and \( \delta > 0 \), there exists \( \epsilon > 0 \) such that \( \sup_{\|p' - p'\| < \epsilon} \left( \max_{p \in \Delta A} |f(p) - f(p')| \right) < \delta \).
A strictly dominated action is one that is never a best response to any beliefs. Since NBE requires players best respond to their beliefs, such actions are ruled out. QRE models on the other hand imply that all actions, even strictly dominated ones, are played with positive probability. Whether this is a virtue of NBE or not relative to QRE is up for debate. Certainly, subjects sometimes play strictly dominated actions in the laboratory\(^9\), so to allow for that possibility may be better for “data fitting”. Nevertheless, QRE assumes this possibility without much theoretical justification. Note that even though NBE rules out strictly dominated actions, players still may believe that others will play strictly dominated actions. In a QRE, since beliefs are correct, players must believe that others will play strictly dominated actions. Also assumed in QRE is a sensitivity of actions to expected utility differences, such that the probability of playing a strictly dominated action is increasing in its expected utility. This we view as an unrealistic implication.

At this level of generality, NBE is no better than Rationalizibility\(^{10}\) for predicting behavior. Only by adding additional structure to noisy beliefs and hence the reactions they induce does NBE make interesting predictions. This is the subject of Sections 4 and 5, but first we take a detour to introduce the $2 \times 2$ games that form the basis of our analysis later on.

### 3 $2 \times 2$ Games

$2 \times 2$ games, in which there are two players with two pure actions each are the simplest non-trivial games. Of particular interest are the “asymmetric Matching Pennies” games, which have had prominence in experimental economics since at least Ochs [1995]. In these games, the NE are unique and fully mixed, and no outcome is Pareto dominant. These features ensure that any observed deviations from NE are not easily explained away by subject misconceptions, social preferences, or other confounds. These games thus provide a natural testing bed for any number of theories, and have been used as such for decades.

In this paper, all $2 \times 2$ games are assumed to have the structure given in Figure 1, which guarantees unique and fully mixed NE. The parameters $a_L, a_R, b_U, b_D$ give the “base payoffs”. For behavioral considerations, these are typically greater than or equal to zero in experiments, which is the case for all games considered in later sections. The parameters $c_L, c_R, d_U, d_D$ are the “payoff differences”, which are assumed strictly positive without loss of generality.

For convenience, we specialize notation for $2 \times 2$ games.

**Notation 1.** Denote the probabilities that player 1 plays $U$ and player 2 plays $L$ as $p$ and $q$ respectively. Use $\bar{u}_U(q), \bar{u}_D(q), \bar{u}_L(p), \bar{u}_R(p)$ for the expected utilities to the pure actions of either player.

\(^9\)It is well known that subjects often cooperate in finitely repeated Prisoner’s Dilemma, which is strictly dominated. However, these games come with important considerations such as social preferences which may explain some of the effect. See, for example, Andreoni and Miller [1993].

\(^{10}\)The set of rationalizable actions are those which are a best response to some beliefs. See for example Bernheim [1984] and Pearce [1984].
as a function of the other player’s mixed action. Use \( Q_U(q) \) and \( Q_L(p) \) to refer to the NBE reaction functions of either player, which give the probabilities that noisy beliefs induce actions \( U \) and \( L \) respectively.

\[
\begin{array}{c|c|c}
  & L & R \\
\hline
U & a_L + c_L & a_R \\
    & b_U & b_U + d_U \\
D & a_L & a_R + c_R \\
    & b_D + d_D & b_D \\
\end{array}
\]

\( U: \) up \hspace{1cm} \( D: \) down
\( L: \) left \hspace{1cm} \( R: \) right

player 1’s payoff in upper-left corner
player 2’s payoff in lower-right corner

\( a_L, a_R, b_U, b_D \geq 0 \)
\( c_L, c_R, d_U, d_D > 0 \)

Figure 1: Structure of 2 \( \times \) 2 Games. The restrictions imposed ensure that the NE is unique and fully mixed.

4 Logit Transform NBE

There is a standard and elegant way of adding noise to actions, first developed in the context of random utility models by McFadden [1976]. The idea is to add a random error term to each element of a vector of expected utilities, each element corresponding to an action. Assuming player \( i \) chooses an action to maximize noisy expected utility induces a distribution over his actions. It is well-known that if each error term is independently drawn from an extreme value “logit” distribution with precision \( \lambda \in [0, \infty) \), the resulting probability of taking action \( a_{ij} \in A_i \) is given by the logit quantal response function

\[
K_{ij}(u_i; \lambda) = \frac{e^{\lambda u_{ij}}}{\sum_{k=1}^{J(i)} e^{\lambda u_{ik}}},
\]

where \( u_i = (u_{i1}, \ldots, u_{iJ(i)}) \) is the vector of expected utilities. When the game \( \Gamma = \{N, A, u\} \) is clear from the context, we also define \( G_i = (G_{i1}, \ldots, G_{iJ(i)}) = ((K_{i1}(\bar{u}_i), \ldots, K_{iJ(i)}(\bar{u}_i))) : \Delta A_{-i} \to \Delta A_i \) to give the mapping from \( i \)’s opponents’ actions to \( i \)’s actions. We call this the QRE reaction function, which we will compare to the NBE reaction function. Since \( \lambda \) controls the probability of making mistakes, it is also called the rationality parameter in the context of QRE.

Logit QRE assumes noise in actions given by (6) and is defined by the fixed point \( p_{ij} = G_{ij}(p_{-i}; \lambda) \) for all \( i \) and \( j \). We give the definition in the special case of 2 \( \times \) 2 games.
Definition 3. Logit QRE for $2 \times 2$ games is given as the solution to the simultaneous equation system

$$
\begin{align*}
p &= \frac{e^{\lambda q_U(q)}}{e^{\lambda q_U(q)} + e^{\lambda q_D(q)}} \equiv G_U(q; \lambda) \\
q &= \frac{e^{\lambda q_L(p)}}{e^{\lambda q_L(p)} + e^{\lambda q_R(p)}} \equiv G_L(p; \lambda).
\end{align*}
$$

(7)

For more details on logit QRE, see Section 11.1.

While adding noise to actions is standard, we are unaware of applications in economics that add noise exogenously to beliefs. Hence, we propose a novel use of the logit transform to parametrize noisy beliefs. In $2 \times 2$ games, player 2’s action is given by $q$. Given $q$, we derive the noisy beliefs of player 1 through the following procedure:

1. Map $q \in [0, 1]$ to the real line via the logit transform

$$
L(q) = \ln \left( \frac{q}{1-q} \right),
$$

using the convention that $L(0) = -\infty$ and $L(1) = \infty$.

2. Add $\sigma \varepsilon_1$ to $L(q)$ where $\varepsilon_1 \sim \mathcal{N}(0, 1)$ and $\sigma \in [0, \infty)$.

3. Map $L(q) + \sigma \varepsilon_1$ back to $[0, 1]$ via the inverse logit transform

$$
L^{-1}(L(q) + \sigma \varepsilon_1) = \frac{\exp \left( \ln \left( \frac{q}{1-q} \right) + \sigma \varepsilon_1 \right)}{1 + \exp \left( \ln \left( \frac{q}{1-q} \right) + \sigma \varepsilon_1 \right)}.
$$

And following a similar procedure for player 2, noisy beliefs for both players are given as

$$
\begin{align*}
q^*(q; \sigma) &= \frac{\exp \left( \ln \left( \frac{q}{1-q} \right) + \sigma \varepsilon_1 \right)}{1 + \exp \left( \ln \left( \frac{q}{1-q} \right) + \sigma \varepsilon_1 \right)}, \\
p^*(p; \sigma) &= \frac{\exp \left( \ln \left( \frac{p}{1-p} \right) + \sigma \varepsilon_2 \right)}{1 + \exp \left( \ln \left( \frac{p}{1-p} \right) + \sigma \varepsilon_2 \right)},
\end{align*}
$$

(8)

where $\varepsilon_1, \varepsilon_2 \sim_{iid} \mathcal{N}(0, 1)$. The parameter $\sigma \in [0, \infty)$ in (8) is the standard deviation of the noise added to the logit transformed beliefs, and thus can be interpreted as the “noisiness” of beliefs. The expressions in (8) are very tractable, allowing for explicit expressions for the CDFs of noisy beliefs and the induced reaction functions.

Theorem 2. Noisy beliefs given by (8) with $\sigma > 0$ have CDFs

$$
\begin{align*}
F_1(\bar{q}|q; \sigma) &= \Phi \left( \frac{1}{\sigma} \left[ \ln \left( \frac{q}{1-q} \right) - \ln \left( \frac{\bar{q}}{1-\bar{q}} \right) \right] \right) \\
F_2(\bar{p}|p; \sigma) &= \Phi \left( \frac{1}{\sigma} \left[ \ln \left( \frac{p}{1-p} \right) - \ln \left( \frac{\bar{p}}{1-\bar{p}} \right) \right] \right)
\end{align*}
$$

(9)

\(^{11}\)To make these CDFs well-defined, we have to resolve indeterminacies as follows: $-\infty - (-\infty) = \infty$ and $\infty - \infty = \infty$. As is standard, we also need $\Phi(-\infty) = 0$ and $\Phi(\infty) = 1$. 

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and induce reaction functions

\begin{align*}
Q_U(q; \sigma) &= 1 - \Phi \left( \frac{1}{\sigma} \left[ \ln \left( \frac{c_{ik}}{c_{ij}} \right) - \ln \left( \frac{q}{1-q} \right) \right] \right) \\
Q_L(p; \sigma) &= \Phi \left( \frac{1}{\sigma} \left[ \ln \left( \frac{d_{ik}}{d_{ij}} \right) - \ln \left( \frac{p}{1-p} \right) \right] \right).
\end{align*}

(10)

**Proof.** See Appendix 11.2. \qed

In $2 \times 2$ games, the logit transform model is admissible and hence we have existence of NBE. Alternatively, existence can be shown directly using the reaction functions (10) and Brouwer's fixed point theorem, but we find the admissibility proof instructive.

**Theorem 3.** $2 \times 2$ games and noisy beliefs given by (8) with $\sigma > 0$ are admissible.

**Proof.** For all $2 \times 2$ games and any player $i$, $u_{ij}(p_{-i}) = \bar{u}_{ik}(p_{-i})$ for exactly one $p_{-i} \in (0, 1)$, which we denote $\bar{p}_{-i}$. $F_i(\bar{p}_{-i}|p_{-i}; \sigma) = \Phi \left( \frac{1}{\sigma} \left[ \ln \left( \frac{\bar{p}_{-i}}{1-\bar{p}_{-i}} \right) - \ln \left( \frac{p_{-i}}{1-p_{-i}} \right) \right] \right)$ is continuous in $p_{-i}$ for any interior $\bar{p}_{-i}$ and hence $b_{-i}(\bar{p}_{-i}|p_{-i}) = 0$, which shows the no indiherence condition (2). Notice that we can also write $R_{ij} = [0, \bar{p}_{-i}]$ and $R_{ik} = [\bar{p}_{-i}, 1]$. Thus, for all $p_{-i}$, $b_{-i}(R_{ij}|p_{-i}) = F_i(\bar{p}_{-i}|p_{-i}; \sigma) = \Phi \left( \frac{1}{\sigma} \left[ \ln \left( \frac{\bar{p}_{-i}}{1-\bar{p}_{-i}} \right) - \ln \left( \frac{p_{-i}}{1-p_{-i}} \right) \right] \right)$, which is continuous in $p_{-i}$. Since player $i$ and action $j$ are arbitrary, this shows continuity (3), and we are done. \qed

**Corollary 1.** An NBE exists for $2 \times 2$ games and noisy beliefs given by (8) with $\sigma > 0$.

**Definition 4.** logit transform NBE for $2 \times 2$ games is given as the solution to the simultaneous equation system

\begin{align*}
p &= 1 - \Phi \left( \frac{1}{\sigma} \left[ \ln \left( \frac{c_{ik}}{c_{ij}} \right) - \ln \left( \frac{q}{1-q} \right) \right] \right) \equiv Q_U(q; \sigma) \\
q &= \Phi \left( \frac{1}{\sigma} \left[ \ln \left( \frac{d_{ik}}{d_{ij}} \right) - \ln \left( \frac{p}{1-p} \right) \right] \right) \equiv Q_L(p; \sigma).
\end{align*}

(11)

Appendix 11.3 generalizes the logit transform model to arbitrary normal form games.

In $2 \times 2$ games, noisy beliefs given by (8) have several nice properties, given by Theorem 4.

**Theorem 4.** For $r \in \{p, q\}$

1. $r^*(r'; \sigma) \succ_{FOSD} r^*(r; \sigma)$\footnote{That is, the distribution of $r^*(r'; \sigma)$ is strictly greater than the distribution of $r^*(r; \sigma)$ in the sense of first-order stochastic dominance. CDF $F$ first-order stochastically dominates CDF $G$ if and only if $F(t) \leq G(t)$ for all $t$ and strictly so if the inequality is strict.} for all $r' > r$.

2. The median of $r^*(r; \sigma)$ is $r$.

3. $r^*(r; \sigma)$ has full support for $r \in (0, 1)$.

4. $r^*(1; \sigma) = 1$, $r^*(0; \sigma) = 0$, and $r^*(r; 0) = r$. 

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Proof. See Appendix 11.4.

Theorem 4.1 establishes that logit transform noisy beliefs are responsive to the other player’s actions in a very strong sense. Comparing two equilibria, one in which player $j$ plays $B \in \{U, L\}$ with probability $r \in \{p, q\}$ and another in which it is $r' > r$, the theory predicts the distribution of player $i$’s beliefs over $B$ in the latter will first-order stochastically dominate that in the former. Theorem 4.2 establishes that beliefs are correct on median. Compared with QRE which assumes estimations of expected utility are correct on average (see Section 11.1), this is an obvious analogue. The implication of both models is that if player $j$’s mixed action equalizes the expected utility to both of player $i$’s actions, the probability of playing either is exactly one half. In this sense, logit transform beliefs are unbiased. To gain intuition for how the NBE and QRE reactions compare, consider the game in Figure 2. This matching pennies game is symmetric for $X = 1$, and asymmetric otherwise. For player 1, $\bar{u}_U(q) = qX$ and $\bar{u}_D(q) = 1 - q$, and hence $\bar{u}_U(q) = \bar{u}_D(q)$ if and only if $q = \frac{1}{1+X}$. Figure 3 gives the NBE and QRE reactions of player 1 for $X \in \{1, 4, 9, 19\}$ with parameters $(\sigma, \lambda) = (0.8, 5)$, which are chosen so that the reactions are as similar as possible in the symmetric $X = 1$ case. From the figure, it is clear that the reactions coincide for all $X$ when $q = \frac{1}{1+X}$.

Theorem 4.4 establishes the extremes of noisy beliefs. When there is no noise in beliefs ($\sigma = 0$), beliefs are correct and NBE collapses to NE. For any noise ($\sigma \in [0, \infty)$), beliefs are correct for extreme actions $r = 0$ and $r = 1$. We justify this feature in two ways. First, we note that this is a corollary of beliefs being correct on median, which seems sensible. Second, we argue that such a feature is easily justified as a reduced form of a learning model. Often equilibrium theories are interpreted as the stationary distribution of actions arising from a process of learning. That is, players observe a history of their opponents’ actions or their own payoffs which effects their behavior. In the long run, the process converges to some distribution of actions called an equilibrium. If player $i$ observes player $j$’s actions drawn from some stationary distribution $r$ and calculates the sample proportion of $B \in \{U, L\}$, the standard error of the statistic is given by $\sqrt{r(1-r)/m}$ where $m$ is the observed number of actions. Hence, for $r = 0$ and $r = 1$, beliefs will be correct with probability one. We discuss learning in greater detail in Section 7.

![Figure 2: Asymmetric Matching Pennies.](image-url)
Figure 3: Reaction Functions. This figure plots player 1’s reaction functions for NBE ($\sigma = 0.8$) and QRE ($\lambda = 5$) for the game whose payoff matrix is in Figure 2 for $X \in \{1, 4, 9, 19\}$. 
With the logit transform model defined and its properties explored, it is convenient to index equilibria by the parameter $\sigma$. In the $2 \times 2$ case, it is very easy to plot the set of equilibria.

**Definition 5.** The logit transform NBE correspondence is given by

$$\pi(\sigma) = \{ p \in \Delta \mid p_{ij} = Q_{ij}(p_{-i}; \sigma) \ \forall i, j \}$$

The logit QRE correspondence is analogously defined in terms of $\lambda$.

## 5 Regular NBE in $2 \times 2$ Games

We now consider the empirical content of a more general model that does not rely on any specific parametrization. Rather than a “structural” approach, we take a “reduced form” approach by taking as primitive the noisy beliefs and imposing minimal, economically sensible restrictions. We show that these restrictions greatly limit the set of actions that are consistent with the model. Our approach closely follows that for “regular QRE” (Goeree et al. [2004]), which imposes restrictions on the quantal response function. A full treatment for general normal form games is beyond the scope of this paper. For now, we focus on $2 \times 2$ games and the example of the game in Figure 2.

**Definition 6.** Noisy beliefs in $2 \times 2$ games are regular if they are admissible and satisfy properties 1 and 2 of Theorem 4. Any NBE derived from regular noisy beliefs is called regular.

**Theorem 5.** Any regular NBE in the general $2 \times 2$ games of Figure 1 is unique.

**Proof.** For player 1, there exists some unique, interior $\bar{q} \in (0, 1)$ such that $\bar{u}(\bar{q}) = \bar{d}(\bar{q})$. Hence, property 1 implies that $\tilde{Q}_U(q) = b_2(\{q \mid q > \bar{q}\} | q)$ is strictly increasing in $q$. Similarly, $\tilde{Q}_L(p)$ is strictly decreasing in $p$, and thus any fixed point of $\tilde{Q} = (\tilde{Q}_U, \tilde{Q}_L) : \Delta \to \Delta$ is unique.

**Remark 2.** By construction, the logit transform model is regular and hence predicts unique NBE in the general $2 \times 2$ games of Figure 1.

**Theorem 6.** In the game of Figure 2 for $X > 1$, in any regular NBE

$$\begin{cases} p \leq \frac{1}{2} & \text{if } q \leq \frac{1}{1+X} \\ p \geq \frac{1}{2} & \text{if } q \geq \frac{1}{1+X} \end{cases}$$

and

$$\begin{cases} q \geq \frac{1}{2} & \text{if } p \leq \frac{1}{2} \\ q \leq \frac{1}{2} & \text{if } p \geq \frac{1}{2} \end{cases}$$

**Proof.** In any NBE, rationality requires that player 1 play $U$ if his beliefs $q'$ are such that $q' > \frac{1}{1+X}$ and that he play $D$ if his beliefs are such that $q' < \frac{1}{1+X}$. By property 2 of noisy beliefs, the
probability that he play \( U \) when player 2 is playing \( q = \frac{1}{1+X} \) is exactly \( p = \frac{1}{2} \). By property 1 of noisy beliefs, if \( q < \frac{1}{1+X} \), the probability player 1 plays \( U \) is strictly less than \( \frac{1}{2} \). The rest of the proof is similar.

For any \( X > 1 \), the set of possible regular NBE is given by the inequalities in Theorem 6 and have measure \( \frac{1}{4} - \frac{1}{2(1+X)} \). Figure 4 shows the set of regular NBE when \( X = 4 \), in which case the set has measure 0.15, i.e. only 15% of all possible outcomes are consistent with the model.

![Figure 4: Regular NBE in Asymmetric Matching Pennies.](image)

The next result is a simple comparative static in \( X \). It shows that only minimal structure is required to make predictions across games.

**Theorem 7.** In a regular NBE of the game in Figure 2, \( p \) is strictly increasing in \( X \) and \( q \) is strictly decreasing in \( X \).

**Proof.** Any regular NBE of this game is given as the fixed point

\[
p = \hat{Q}_U(q, X) \tag{12}
\]

\[
q = \hat{Q}_L(p, X) \tag{13}
\]

for some reactions \( \hat{Q}_U(q, X), \hat{Q}_L(p, X) \) derived from regular noisy beliefs. It is trivial to verify that regularity of noisy beliefs imply from (12) that \( p \) is strictly increasing in \( q \) and \( X \), and from (13) that \( q \) is strictly decreasing in \( p \). This implies that the solution to the fixed point system is unique. From (13), as \( X \) increases, it must be that either \( p \) increases and \( q \) decreases, \( p \) decreases and \( q \) increases, or that both \( p \) and \( q \) remain constant. The latter two cases are impossible since (12) implies that as \( X \) increases, \( p \) increases if \( q \) is constant or increases. Thus, as \( X \) increases, \( p \) must strictly increase and \( q \) must strictly decrease.
The predictions of Theorems 6 and 7 find strong support in data (see for example Ochs [1995], McKelvey and Palfrey [1995], Goeree and Holt [2001], and Goeree et al. [2003]), and hold for regular QRE also. Thus, the most basic and robust empirical patterns can be explained equally well by adding noise to actions (QRE) or adding noise to beliefs (NBE). The results of the next section, however, distinguish between the two models.

6 Effects of Payoff Magnitude: Theory and Evidence

An understanding of the effects of payoff magnitude on behavior in games is potentially of great value. Consider a policy maker who is designing a mechanism, such as an auction, with a specific goal in mind, such as maximizing revenue or welfare. Before implementing the mechanism, she may first wish to run an experiment in the lab. Since her budget is limited, the experiment will be similar to its real world counterpart except for very low stakes, say denominated in pennies as opposed to hundreds of dollars. The value of the experiment lies in being able to extrapolate behavior from the lab to the real world.

In this section, we show that NBE in its most general form makes predictions that are invariant to changes in the payoff magnitude of games in a way that will be made precise. This is important for deriving tests of NBE, as even if NBE is the “true” model to describe behavior, we can never recover noisy beliefs (the random variable) from data. Nevertheless, for any noisy beliefs, the invariance properties hold, and these can be used to test the theory. Unlike NBE, QRE is sensitive to changes in payoff magnitude, so the results of this section allow one to discriminate between the two models.

We present data from McKelvey et al. [2000] that strongly reject the QRE predictions in favor of NBE.

6.1 Theory

We introduce two families of games. The first describes games that are the same up to some positive “scaling” of one or more players’ payoffs.

**Definition 7.** The *scale family* of game $\Gamma = \{N, A, u\}$, denoted $S(\Gamma)$, consists of all games $\Gamma' = \{N', A', u'\}$ such that $N' = N$, $A' = A$, and for all $i$, there exists $\beta_i > 0$ such that $u'_i = \beta_i u_i$. We say that all games in the same scale family are related by changes in *payoff magnitude*.

The next family describes games where utility differences across each player’s actions are the same, holding fixed the actions of others. Payoffs are thus “shifted” across games.

**Definition 8.** The *shift family* of game $\Gamma = \{N, A, u\}$, denoted $H(\Gamma)$, consists of all games $\Gamma' = \{N', A', u'\}$ such that $N' = N$, $A' = A$, and $u'$ is constructed from $u$ as follows: and for all $i$ and $a_{-i} \in A_{-i}$, there exists $\gamma_i(a_{-i}) \in \mathbb{R}$ such that $u'_{ij}(a_{-i}) = u_{ij}(a_{-i}) + \gamma_i(a_{-i})$ for all $j$. 

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Remark. Definitions 7 and 8 are given in terms of pure actions, which makes them easier to verify. However, it is obvious by linearity that $\beta_i$ in Definition 7 satisfies $\bar{u}_i = \beta_i \bar{u}_i$, and $\bar{\gamma}_i(p_{-i}) = \sum_{a_{-i}} p_{-i}(a_{-i}) \gamma_i(a_{-i})$ satisfies $\bar{u}_{ij}(p_{-i}) = \bar{u}_{ij}(p_{-i}) + \bar{\gamma}_i(p_{-i})$ where $\gamma_i(a_{-i})$ is as in Definition 8.

Theorem 8 describes the invariance properties of NBE and QRE.

**Theorem 8.** Fix game $\Gamma \in \{N, A, u\}$. (i) Fix noisy beliefs $p^*$. The set of NBE is the same for all $\Gamma' \in \mathcal{S}(\Gamma) \cup \mathcal{H}(\Gamma)$. (ii) Fix payoff disturbances $(\epsilon_i)_{n=1}^N$ (see Appendix 11.1). The set of QRE is the same for all $\Gamma' \in \mathcal{H}(\Gamma)$. This is not true for all $\Gamma' \in \mathcal{S}(\Gamma)$.

**Proof.** We show that holding fixed $p^*$, the NBE response sets (1) under $\Gamma'$, given by $R'_{ij}$, are equal to those under $\Gamma$, given by $R_{ij}$. This implies that $p^*$ induces the same reactions in both games and hence also the same equilibria. Suppose $\Gamma' \in \mathcal{S}(\Gamma)$. Then for all $i$ and $j$, there exists $\beta_i$ such that

$$R_{ij} = \{\bar{p}_{-i} | \bar{u}_{ij}(\bar{p}_{-i}) \geq \bar{u}_{ik}(\bar{p}_{-i}) \forall k = 1, ..., J(i)\}$$

$$= \{\bar{p}_{-i} | \beta_i \bar{u}_{ij}(\bar{p}_{-i}) \geq \beta_i \bar{u}_{ik}(\bar{p}_{-i}) \forall k = 1, ..., J(i)\}$$

$$= \{\bar{p}_{-i} | \bar{u}_{ij}(\bar{p}_{-i}) \geq \bar{u}_{ik}(\bar{p}_{-i}) \forall k = 1, ..., J(i)\}$$

$$= R_{ij}.$$ 

Suppose $\Gamma' \in \mathcal{H}(\Gamma)$. Then for all $i$, $j$, and $\bar{p}_{-i}$, there exists $\bar{\gamma}_i(\bar{p}_{-i})$ such that

$$R'_{ij} = \{\bar{p}_{-i} | \bar{u}_{ij}(\bar{p}_{-i}) \geq \bar{u}_{ik}(\bar{p}_{-i}) \forall k = 1, ..., J(i)\}$$

$$= \{\bar{p}_{-i} | \bar{u}_{ij}(\bar{p}_{-i}) + \bar{\gamma}_i(\bar{p}_{-i}) \geq \bar{u}_{ik}(\bar{p}_{-i}) + \bar{\gamma}_i(\bar{p}_{-i}) \forall k = 1, ..., J(i)\}$$

$$= \{\bar{p}_{-i} | \bar{u}_{ij}(\bar{p}_{-i}) \geq \bar{u}_{ik}(\bar{p}_{-i}) \forall k = 1, ..., J(i)\}$$

$$= R_{ij}.$$ 

This establishes shows (i). To show (ii), we use a similar approach using the QRE response sets (21) of Appendix 11.1. Suppose $\Gamma' \in \mathcal{H}(\Gamma)$. Then for all $i$, $j$, and $\bar{p}_{-i}$, there exists $\bar{\gamma}_i(\bar{p}_{-i})$ such that

$$W'_{ij}(\bar{p}_{-i}) = \{\epsilon_i \in \mathbb{R}^{J(i)} | \bar{u}_{ij}(\bar{p}_{-i}) + \epsilon_{ij} \geq \bar{u}_{ik}(\bar{p}_{-i}) + \epsilon_{ik} \forall k = 1, ..., J(i)\}$$

$$= \{\epsilon_i \in \mathbb{R}^{J(i)} | \bar{u}_{ij}(\bar{p}_{-i}) + \bar{\gamma}_i(\bar{p}_{-i}) + \epsilon_{ij} \geq \bar{u}_{ik}(\bar{p}_{-i}) + \bar{\gamma}_i(\bar{p}_{-i}) + \epsilon_{ik} \forall k = 1, ..., J(i)\}$$

$$= \{\epsilon_i \in \mathbb{R}^{J(i)} | \bar{u}_{ij}(\bar{p}_{-i}) + \epsilon_{ij} \geq \bar{u}_{ik}(\bar{p}_{-i}) + \epsilon_{ik} \forall k = 1, ..., J(i)\}$$

$$= W_{ij}(\bar{p}_{-i}),$$

and hence the QRE response set is the same for both $\Gamma$ and $\Gamma'$. That QRE can be different for $\Gamma', \Gamma' \in \mathcal{S}(\Gamma)$, we note that an axiom of regular QRE is *responsiveness* to payoff differences. We will also show examples of logit QRE in the next subsection in which payoff magnitude has very large effects. 

\[\square\]
6.2 Evidence

The effects of payoff magnitude have been explored experimentally by McKelvey et al. [2000] using the $2 \times 2$ asymmetric Matching Pennies games in Figure 5. The first three of these games, $A$, $B$, and $C$ are part of the same scale family. Relative to $A$, player 2’s payoffs are scaled by 4 in $B$ and both players’ payoffs are scaled by 4 in $C$. Game $D$, though similar in form, is not part of this family. The predictions of play in these games from logit transform NBE and logit QRE are given by the correspondences in Figure 6, where the correspondences are as defined in (5). Notice that the NBE predictions are the same for games $A$, $B$, and $C$, which is to be expected given Theorem 8. The QRE predictions, however, are sensitive to payoff magnitude, and hence are different for each of these games. The empirical frequencies of actions (i.e. the data) from these games can be visualized in the unit square of $(p, q)$-space. The data from all 4 games is shown in Figure 7. Visually, it appears as though the data from games $A$, $B$, and $C$ are tightly clustered, with only the data from game $D$ significantly different. This seems to favor NBE and is confirmed by statistical tests.

![Figure 5: Asymmetric Matching Pennies from McKelvey et al. [2000].](image)

Table 1 reports the results of standard $z$-tests of predictions of NBE and QRE, and is taken verbatim from Table 6 of McKelvey et al. [2000]. The predictions come in several forms. Some predictions are about the relative action frequencies across games and some are predictions about action frequencies within a game relative to some benchmark. We label these two kinds of predictions as “IS” for in-sample and “OOS” for out-of-sample. Following McKelvey et al. [2000], we mark the OOS predictions across games $A$, $B$, and $C$ with a “P” since they are related to changes in payoff magnitude. We also label predictions relative to the NE prediction with an “NE”. The predictions can be confirmed algebraically, but are also clear from Figure 6. Importantly, the NBE predictions are from the logit transform model and hold for any value of $\sigma$. Similarly, the QRE predictions are from the logit model and hold for any value of $\lambda$. In the $z$-tests, the QRE prediction, which is always an inequality, is the alternative hypothesis. The NBE prediction is either an equality, in which case it is the null hypothesis and opposed to the QRE prediction, or it is an inequality, in which case it is the alternative and agrees with the QRE prediction.

The results are clear. All predictions that are not based off of payoff magnitude are common to
both NBE and QRE, and are supported. Thus, NBE and QRE are both able to explain deviations from NE within a game as well as qualitative differences across games based on differences unrelated to payoff magnitude. All predictions based off of payoff magnitude, however, are in favor of NBE, statistically rejecting the QRE prediction in favor of NBE at the 0.05 level. That is, what we observe in Figure 7 is confirmed statistically: subjects do not respond to changes in payoff magnitude. Following McKelvey et al. [2000], we note that since each subject plays a game multiple times, there is within-subject correlation between observed actions. This would cause the z-statistic to have variance greater than 1 and as a result overstate significance. If anything, Table 1 understates support for the NBE predictions. We also note that in 4 of 5 predictions based on payoff magnitude, even though the action frequencies are not significantly different, QRE predicts an effect in the wrong
direction.

Figure 7: Data from McKelvey et al. [2000].

We have established that qualitatively, NBE and QRE make similar predictions in-sample, but very different predictions out-of-sample across games that differ in payoff magnitude. We now explore the “economic significance” of these findings by fitting the two models to data. To do this, we choose parameters to minimize the “prediction error”\textsuperscript{13} between the empirical frequency of actions and the corresponding model prediction. For prediction error, we use the mean-squared distance measure, as in Selten and Chmura [2008]. For model $M$ with parameter $\xi$, the prediction error for game $x$ is

$$E_x(M, \xi) = (p_x - p_M(\xi))^2 + (q_x - q_M(\xi))^2,$$

where $p_x, q_x$ are the empirical frequencies of actions and $p_M(\xi), q_M(\xi)$ are the model predictions. Use the notation

$$\hat{\xi}_x = \text{argmin}_\xi E_x(M, \xi)$$

to refer to the resulting best-fit parameter. For each game $x \in \{A, B, C, D\}$, we fit logit transform NBE and logit QRE to the data of game $x$ in-sample, resulting in estimates $\hat{\sigma}_x$ and $\hat{\lambda}_x$. We then use these parameter estimates to make out-of-sample predictions for game $y \in \{A, B, C, D\}$. We define the $xy$-difference in prediction error (QRE minus NBE) as

$$\Delta E(x, y) \equiv E_y(QRE, \hat{\lambda}_x) - E_y(NBE, \hat{\sigma}_x).$$

We use these differences to populate the matrix in Table 2. The $xy$-th entry of the matrix gives

\begin{table}
\centering
\begin{tabular}{lcc}
\hline
$N$ & $p$ & $q$ \\
\hline
$A$ & 1800 & 0.643 & 0.241 \\
$B$ & 1200 & 0.630 & 0.244 \\
$C$ & 1200 & 0.594 & 0.257 \\
$D$ & 600 & 0.550 & 0.328 \\
\hline
\end{tabular}
\end{table}

\textsuperscript{13}Many studies also fit parameters by maximizing the likelihood of the data. We instead focus on prediction error since we are interested in the magnitudes of the differences in prediction error across models. We also note that fitting the parameters by maximum likelihood leads to very similar estimates.
Table 1: Summary of Predictions vs. Actual Behavior. This table reports the results of standard $z$-tests of model predictions. “NE” refers to a prediction relative to the corresponding NE prediction and “P” refers to a prediction across games related to payoff magnitude. “IS” and “OOS” mark predictions within and across games respectively. Positive (negative) $z$-values indicate the direction of the effect predicted by QRE is right (wrong), and “*” indicates significance at the 0.05 level.

The parameter estimates and numerical predictions from fitting the games in-sample are given

<table>
<thead>
<tr>
<th>NBE prediction</th>
<th>QRE prediction</th>
<th>Null</th>
<th>Type</th>
<th>IS/OOS</th>
<th>Actual</th>
<th>$z$-Value</th>
<th>Model supported</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.5 &lt; p_B$</td>
<td>$0.5 &lt; p_B$</td>
<td>$0.5 = p_B$</td>
<td>NE</td>
<td>IS</td>
<td>$0.500 &lt; 0.630$</td>
<td>$9.001^*$</td>
<td>NBE, QRE</td>
</tr>
<tr>
<td>$p_B = p_A$</td>
<td>$p_B &lt; p_A$</td>
<td>$p_B = p_A$</td>
<td>P</td>
<td>OOS</td>
<td>$0.630 &lt; 0.643$</td>
<td>$0.726$</td>
<td>NBE</td>
</tr>
<tr>
<td>$p_B = p_C$</td>
<td>$p_B &lt; p_C$</td>
<td>$p_B = p_C$</td>
<td>P</td>
<td>OOS</td>
<td>$0.630 &gt; 0.594$</td>
<td>$-1.810$</td>
<td>NBE</td>
</tr>
<tr>
<td>$0.5 &gt; q_A$</td>
<td>$0.5 &gt; q_B$</td>
<td>$0.5 = q_A$</td>
<td>NE</td>
<td>IS</td>
<td>$0.500 &gt; 0.241$</td>
<td>$21.977^*$</td>
<td>NBE, QRE</td>
</tr>
<tr>
<td>$q_A = q_B$</td>
<td>$q_A &gt; q_B$</td>
<td>$q_A = q_B$</td>
<td>P</td>
<td>OOS</td>
<td>$0.241 &lt; 0.244$</td>
<td>$-0.188$</td>
<td>NBE</td>
</tr>
<tr>
<td>$q_B = q_C$</td>
<td>$q_B &gt; q_C$</td>
<td>$q_B = q_C$</td>
<td>P</td>
<td>OOS</td>
<td>$0.244 &lt; 0.257$</td>
<td>$-0.735$</td>
<td>NBE</td>
</tr>
<tr>
<td>$q_A = q_C$</td>
<td>$q_A &gt; q_C$</td>
<td>$q_A = q_C$</td>
<td>P</td>
<td>OOS</td>
<td>$0.241 &lt; 0.257$</td>
<td>$-0.995$</td>
<td>NBE</td>
</tr>
</tbody>
</table>

Table 2: Out-of-sample Differences in Prediction Error (QRE minus BBE). The $xy$th entry corresponds to $\Delta E(x,y)$ as in (15) for games $x, y \in \{A, B, C, D\}$. Positive (negative) entries indicate that NBE performs better than (worse than) QRE.
Table 3: In-sample Parameter Estimates and Predictions.

<table>
<thead>
<tr>
<th>Game</th>
<th>Parameters</th>
<th>NBE</th>
<th>QRE</th>
<th>NBE</th>
<th>QRE</th>
<th>Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{\sigma}$</td>
<td>$\hat{\lambda}$</td>
<td>Pred. Error</td>
<td>$p$</td>
<td>$q$</td>
<td>$p$</td>
</tr>
<tr>
<td>A</td>
<td>1.411</td>
<td>6.459</td>
<td>0.0112</td>
<td>0.747</td>
<td>0.221</td>
<td>0.662</td>
</tr>
<tr>
<td>B</td>
<td>1.451</td>
<td>0.800</td>
<td>0.0142</td>
<td>0.748</td>
<td>0.227</td>
<td>0.707</td>
</tr>
<tr>
<td>C</td>
<td>0.436</td>
<td>2.513</td>
<td>0.0217</td>
<td>0.629</td>
<td>0.114</td>
<td>0.104</td>
</tr>
<tr>
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Figure 8: In-Sample Predictions. For games A, B, C, and D, this figure gives the in-sample predictions of NBE as the intersection of the reaction functions, as well as the QRE predictions, the NE predictions, and the empirical frequencies.
in Table 3. Figure 8 shows the NBE predictions as the intersection of the best-fit reaction functions, superimposed with the best-fit QRE predictions, the NE predictions, and the empirical frequencies.

Figure 9: Visualizing Prediction Error. This figure plots the NBE and QRE correspondences of games A and B, superimposed with vertical lines representing the in-sample best-fit parameters and the empirical frequencies of actions.

To gain some intuition for the parameter estimates and sources of prediction error, consider games A and B. The data from games A and B are statistically the same. Since the NBE predictions are the same for both games for any given \( \sigma \), the resulting best-fit parameters, \( \hat{\sigma}_A \approx 1.411 \) and \( \hat{\sigma}_B \approx 1.451 \), are very similar. Therefore, we get immediately that the out-of-sample performance across these games can only be slightly worse than the in-sample performance for each game individually. This is confirmed visually in the top panel of Figure 9, which gives the NBE correspondence that is common to games A and B, superimposed with the empirical frequencies (diamonds) and vertical lines representing the best-fit \( \hat{\sigma} \)s. Where the vertical line representing \( \hat{\sigma}_A \) cuts the corre-
spondence gives the \textit{in-sample} prediction for $A$ as well as the \textit{out-of-sample} prediction for $B$ since the correspondences coincide, and similarly for the vertical line representing $\sigma_B$.

The bottom panel of the Figure 9 gives the analogous plot for QRE. Since 1) QRE is sensitive to payoff magnitude, 2) $A$ and $B$ differ by payoff magnitude, and 3) the data from the two games are statistically the same, the best-fit $\hat{\lambda}$s are very different in the two games: $\hat{\lambda}_A \approx 6.459$ and $\hat{\lambda}_B \approx 0.800$. We also have that the QRE correspondences for games $A$ and $B$ are distinct. Consider the vertical line representing $\hat{\lambda}_B$. Where it cuts the correspondence for $B$ gives the in-sample prediction and where it cuts the correspondence for $A$ gives the out-of-sample prediction (from $B$ to $A$). The vertical distance between where the $\hat{\lambda}_B$-line cuts the $p$-branches of the QRE correspondences for $A$ and $B$ gives the difference between \textit{in-sample} (for $B$) and \textit{out-of-sample} (from $B$ to $A$) predictions of $p$. This difference is close to 0.2 even though the empirical difference is statistically 0. Hence, extrapolating from game $B$ to $A$ dramatically over-estimates $p$. Similar analysis can be done for $q$, as well as for the prediction error in going from game $A$ to $B$.

Of independent interest are the noisy beliefs implied by the best-fit NBE, given by $p^*(p; \hat{\sigma})$ and $q^*(q; \hat{\sigma})$. These are a by-product of the estimation and shown in the histograms of Figure 12 in Appendix 11.5. As a proof of concept, these show that equilibrium models can explain deviations from NE and be consistent with the empirical fact of dispersion in elicited beliefs, as documented by Nyarko and Schotter [2002], Palfrey and Wang [2009], and others.

6.3 Discussion

Here we argue that the invariance of NBE and the sensitivity of QRE to payoff magnitude are economically fundamental—not simply artifacts of the modeling assumptions.

QRE is the natural extension to games of random utility models, which are almost always justified through \textit{rational inattention}. The idea is that a decision maker has uncertainty about the payoffs of his alternatives, modeled as a prior distribution over quality. Information about the quality of those alternatives is gathered optimally subject to costs. Typically, conditions are assumed such that, at optimum, the acquired information is imperfect, inducing a probability distribution over choices. Any reasonable information cost function will be such that the ex-ante better alternatives are chosen with higher probability. For instance, information acquisition costs given by mutual information (entropy-based costs), as in Sims [1998], Sims [2003], and Woodford [2009] have this feature. Also, since the benefits of acquiring information are based on \textit{absolute} payoff differences, sensitivity to payoff magnitude is fundamentally embedded into stochastic choice.

It is well-known from the results of Matejka and McKay [2015] that if the decision maker has a prior that is \textit{a priori homogeneous} over the utilities of his $J(i)$ alternatives, and information costs are given by mutual information, then the optimal stochastic choice is given exactly by the logit quantal response function (6), where $\lambda$ is now interpreted as the inverse of the \textit{per-unit cost of information}. Of course, the issue with extending this interpretation to QRE is that, in equilibrium, the vector of
expected utilities is deterministic. Work by Stahl [1990], however, justifies logit QRE by assuming that players must exert effort and pay a “control cost” to reduce the “trembles” that prevent them from taking the best actions with probability one. Under this interpretation, actions are sensitive to payoff magnitude as well. We thus argue that any reasonable model that adds noise to actions, such as QRE, must be sensitive to payoff magnitude. Indeed, the payoff magnitude hypothesis is taken very seriously and tested in McKelvey et al. [2000], which counts among its authors both of QRE’s founders.

Given correct beliefs, the information to take the best action is available. It is only a matter of processing or gathering the information. If processing is costly, actions will be sensitive to payoff magnitude as a simple consequence of rational inattention. The information necessary to form correct beliefs, however, is not available in the same sense as the information required to take the correct action given beliefs. This is because beliefs depend intimately on higher ordered beliefs about who the opponents are and how they play the game, something for which no information structure could possible be available. Hence, as the stakes are raised to infinity, there is no guarantee of forming correct beliefs. Of course, the higher the stakes, the more effort that will be expended in forming beliefs, but what will be the result? Even supposing a subject’s beliefs limit to some beliefs when the stakes are sufficiently high, there is no reason to suppose that all subjects will have the same limiting beliefs. Hence, in the aggregate, the noisy beliefs may remain unchanged. More fundamentally, without having correct beliefs in the first place, it is impossible to calculate the benefits of forming correct beliefs. In other words, a rational belief-formation problem subject to costs, akin to a rational inattention problem, is not even well-defined.

7 NBE as a Model of Learning: Theory and Evidence

One interpretation of NBE is familiar to all equilibrium models. Abstracting from issues of equilibrium selection, common knowledge of 1) the game, 2) noisy beliefs, and 3) rationality are sufficient for player $i$ to “solve” the game for the distribution of other player’s actions. Either player $i$ solves the game imperfectly, resulting in noisy beliefs, or solves the game perfectly but when it comes time to take an action “trembles in beliefs”.

The issues with this classical interpretation are several fold. First, the noisy beliefs object is mysterious: how can players know the exact process by which opponents’ form noisy beliefs? Second, the premise that players can “solve” the game at all given their limitations in forming beliefs is hard to swallow. Given the drawbacks of the classical interpretation, in this section, we consider NBE as the long-run stationary distribution of a process of “learning”.

24
7.1 Theory

Consider the following model. In games played repeatedly with feedback against opponents randomly drawn from some “population”, players observe the history of their opponents’ actions. Players construct beliefs from the sample frequencies of actions, but imperfectly. Heuristically,

\[
p_{ij}^l(m) = \frac{\# \{ a_{ij} \}}{m} + \varepsilon.
\]  

(16)

That is, player l’s belief of the probability player i plays action j after round m is given as the sample frequency—the proportion of action js observed—plus some random error that does not depend on m. The total error in beliefs can thus be decomposed into sampling error and “intrinsic” error. The sampling error, as measured by the standard error of the sample frequencies, goes to 0 after many rounds, but the intrinsic error term remains. The intrinsic error can be interpreted as the reduced form of information processing or random perturbations due to biases in beliefs (either for a given subject or across subjects in the population). One interpretation of NBE is this long-run situation in which only noise in the perception of the true distribution of actions, estimated from an infinite history of actions, remains.

7.2 Evidence

In light of the learning interpretation of NBE, we are interested to see if estimates of NBE-σ vary systematically with experience. Suppose a dataset of actions is collected from games played over many rounds with feedback, and NBE is fit to the data in “blocks” of rounds. If the true model of noisy beliefs is given by (16), fitting standard logit transform NBE to each block would result in estimates \{ \hat{\sigma}_b \} for blocks \( b = 1, 2, \ldots \) with the following features:

1. The sequence \( \hat{\sigma}_1, \hat{\sigma}_2, \ldots \) is decreasing
2. The sequence \( \hat{\sigma}_1, \hat{\sigma}_2, \ldots \) limits to some \( \sigma \).

The intuition is simple. If the true model is given by (16), standard NBE is misspecified in that the dependence of beliefs on the sample frequency is unmodeled. Hence, in early blocks, the unmodeled sampling error will be picked up in high estimates of \( \hat{\sigma}_b \). With time, as the sample frequencies become more precise, this error decays, but the intrinsic error remains.

We revisit the experiment of Selten and Chmura [2008]15 which played twelve \( 2 \times 2 \) games over 200 rounds with feedback. The data is unique in the sheer number of rounds, and therefore ideally suited for this exercise. For each of the twelve games \( x \in \{1, \ldots, 12\} \), which are given in Figure Appendix 11.6.2, we fit logit transform NBE by estimating \( \{ \sigma_b^x \} \) block-by-block, where

14A “block” is a set of rounds. Some papers look at blocks of ten, so that the first block would be rounds 1-10, the second would be rounds 11-20, etc.

15We are grateful to Thorsten Chmura for generously providing data.
$b \in \{25, 26, 27, \ldots, 199, 200\}$ indexes the block of rounds $(b - 24)$ to $b$. That is, we estimate $\sigma_b^\tau$'s using a moving average of 25 rounds. Figure 13 of Appendix 11.8 shows the evolution of estimates over all blocks for each game.

The results of the exercise are summarized in Figure 10. The left panel plots the average of estimates across games for each block $\hat{\sigma}_b^{avg} = \frac{1}{12} \sum_{x=1}^{12} \hat{\sigma}_b^x$, labelled “Average”, as well as the “Pooled” estimate, which gives for each block the value $\hat{\sigma}_b^{pooled}$ that minimizes the average (across games) of mean-squared distances between empirical frequencies of actions and model predictions:

$$
\hat{\sigma}_b^{pooled} = \arg\min_{\sigma} \frac{1}{12} \sum_{x=1}^{12} E_x(NBE, \sigma),
$$

where $E_x(\cdot, \cdot)$ is as given in (14). The result is consistent with the model of learning introduced in this section: after rounds with feedback, estimates of the noisiness of beliefs systematically decrease and seem to limit to some positive level. Inspection of the individual game plots in Figure 13 of Appendix 11.8 reveals that there is substantial heterogeneity across games in the initial noise as well as the stability of the entire path. Nevertheless, in each game, we find that 1) $\sigma$ estimates from the early samples tend to be significantly higher, and 2) that the limiting as well as whole-sample $\sigma$ estimates (given in Table 4 of Section 8) are close to 1. The right panel of Figure 10 shows the prediction error from fitting the pooled estimate (17), which systematically decreases with learning. This is a simple consequence of the fact that each game seems to have a similar limiting $\sigma$.

8 Survey of Existing Studies: The Magic of $\sigma = 1$.

Logit transform NBE-$\sigma$, and indeed the parameters of any parametrization of NBE, has the property that the estimates are insensitive to the “exchange rate” between utility and money. That is, holding the data fixed but scaling the arbitrary utilities in the payoff matrix for one or more players does
not effect the estimated $\sigma$. The proof is basically the same as that of Theorem 8 and hence omitted. Logit QRE on the other hand, and any reasonable parametrization of QRE, is sensitive to these scalings. If all players’ utility numbers are scaled up by some factor $c > 0$, the estimate $\hat{\lambda}$ is simply replaced with $\hat{\lambda} = (\frac{1}{c})\hat{\lambda}$ and the predicted actions remain unchanged. If, however, not all players’ utilities are scaled by the same factor, the $\lambda$ estimate will change in unpredictable ways and lead to different predicted actions as well.

We do not take the pessimistic view that the arbitrariness of $\lambda$ estimates invalidates the informativeness of QRE, but we acknowledge the difficulties it poses in comparing $\lambda$ estimates across studies. To do so would first require fixing the utility-money exchange rate across studies, which requires adjustments for inflation and currency-to-currency exchange rates. That is, the distinction between real- and nominal-$\lambda$ is material. We are unaware of any survey of studies that makes the necessary adjustments, so we undertake that exercise here. We take McKelvey et al. [2000] as the base-study to derive the rate of 1 utile per 0.10 year-2000 US dollars. Using this rate, we recalculate the utility payoffs in the matrices from 5 studies and a total of 21 $2 \times 2$ games. We explain the exact procedure and give the conversion factors applied to these games in Appendix 11.9. The pre-transformed games are given in Appendix 11.6 along with details of the experimental procedures.

Using the transformed utilities, we fit both logit transform NBE and logit QRE to each game individually. Table 4 gives the individual parameter estimates and resulting prediction error. The table also has the number of rounds played by each subject and an indicator for whether or not feedback was provided, factors likely to influence parameter estimates. Table 5 of Appendix 11.7 gives the predictions themselves as well as the data from these games. Figure 11 gives histograms of the best-fit parameter estimates. From the figure, it is clear that both NBE-$\hat{\sigma}$s and QRE-$\hat{\lambda}$s have significant dispersion, though the claim is somewhat meaningless as a reparametrization via a contraction mapping could make the dispersion in estimates arbitrarily close to zero.$^{16}$

To determine which model best explains the data, we fit single values of $\sigma$ and $\lambda$ to all games pooled together. We reiterate that for this exercise, estimates of pooled NBE-$\sigma$ are invariant to utility scalings across games whereas estimates of pooled QRE-$\lambda$ are sensitive to these scalings. Thus, we must continue using the transformed utilities. We fit the models by minimizing the average (across games) of mean-squared distances between empirical frequencies of actions and model predictions:

$$\hat{\sigma}_{\text{pooled}} = \arg\min_{\sigma} \frac{1}{21} \sum_x E_x(NBE, \sigma)$$

$$\hat{\lambda}_{\text{pooled}} = \arg\min_{\lambda} \frac{1}{21} \sum_x E_x(QRE, \lambda),$$

$^{16}$Consider reparametrizing $\sigma$ and $\lambda$ as $\tilde{\sigma} = \frac{\sigma}{\theta}$ and $\tilde{\lambda} = \frac{\lambda}{\theta}$ respectively for $\theta > 0$. Taking $\theta$ to infinity, the dispersion of estimates of $\tilde{\sigma}, \tilde{\lambda}$ would go to zero.
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Figure 11: Histograms of Parameter Estimates Across Games.
where $E_x(\cdot, \cdot)$ is as given in (14). We find that in explaining all data together, NBE outperforms QRE with pooled prediction errors 0.0233 and 0.0434 respectively. That is, NBE gives only 54% of the prediction error of QRE. This is very strong evidence in favor of NBE over QRE, suggesting that it is much more likely that the data was generated via noisy beliefs as opposed to noisy actions. Said differently, it is much more likely that the estimate $\hat{\sigma}^{pooled} \approx 1.0570$ is capturing a “behavioral constant” than $\hat{\lambda}^{pooled} \approx 6.2330$.

We interpret the performance of NBE relative to QRE across these studies as being strongly related to payoff magnitude in the sense that the variation of payoff magnitude across games is very large and likely driving the result. Most pairs of games in the study are not in the same scale family and thus not related by payoff magnitude in a very pure sense, but this is clearly unimportant. Without any formal results, it is clear that QRE is sensitive to the general scale of games in a way that NBE is not.

It is a valid concern that the pooled estimate of $\lambda$ may be very sensitive to the utility-money exchange rate conversions. If that were the case, even small errors in conversions could effect the QRE prediction error (in either direction). Even holding fixed the year and country across studies might not be sufficient, as different subject pools might value the same amount of real money differently, resulting in different utility valuations. That this is a concern at all highlights the virtue of the utility-money exchange rate invariance of NBE. In any case, as a robustness check, we show in Appendix 11.9 that the result of this section is not sensitive to even very large perturbations of the conversion factors.

9 Extension: Beyond Noise—A Model of Biased Beliefs

Though a full treatment is well beyond the scope of this paper, we discuss an extension of NBE that allows for systematic bias in beliefs, as opposed to pure noise. A bias is any specific tendency to form mistaken beliefs over the actions of others. For instance, tending to believe one action will be played with greater probability than the equilibrium level and tending to believe the distribution of actions is “more uniform” than the equilibrium distribution are two examples of biases.

To model biases, we introduce the notion of a “bias function” or simply a “bias”. A bias for player $i$ about his opponents is given by $b_i : \Delta A_{-i} \rightarrow \Delta A_{-i}$, which maps his opponents’ actions into his beliefs. Specifying a distribution over biases for player $i$ is clearly enough to induce noisy beliefs $p^*_i$ about his opponents, as defined in Section 2. And hence we can define a notion of biased beliefs equilibrium as an NBE in the noisy beliefs induced from a distribution over biases for each player.

The applicability of such a model would depend on whether or not there are biases that are stable across many different games. One difficulty is that unless the distribution over biases is symmetric in a very strong sense, the equilibrium predictions would be sensitive to the arbitrary relabelling of actions. One bias that is symmetric on its own is “conservatism”—the tendency to perceive the
distribution over actions as being more uniform than it actually is. We are also optimistic about such a model’s explanatory power because of evidence from Huck and Weizsacker [2002], which finds a bias toward the uniform prior when “a group of subjects was asked to give estimates of a second group’s choice frequencies in a set of lottery-choice tasks.”

In finite games, one can model linear conservatism bias as a column-stochastic matrix that pre-multiples the probability vector representing an opponent’s actions. In $2 \times 2$ games for example, if player 2’s action is given by $(q, 1 - q)$, player 1’s biased belief is given by

$$
\begin{bmatrix}
1 - c & c \\
c & 1 - c
\end{bmatrix}
\begin{bmatrix}
q \\
1 - q
\end{bmatrix}
= 
\begin{bmatrix}
(1 - c)q + c(1 - q) \\
cq + (1 - c)(1 - q)
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{q}(q, c) \\
1 - \tilde{q}(q, c)
\end{bmatrix},
$$

where $c \in [0, \frac{1}{2}]$ is the conservatism parameter. $c = 0$ corresponds to no bias and $c = \frac{1}{2}$ corresponds to a Level 1 bias (as in Level K models) in which player 1 believes player 2 is uniformly mixing independent of his actual actions. One very tractable possibility is to nest conservatism in the logit transform model.

**Definition 9.** logit transform NBE with conservatism for $2 \times 2$ games is given as the solution to the simultaneous equation system

$$
p = 1 - \Phi \left( \frac{1}{\sigma} \left[ \ln \left( \frac{d_p}{d_L} \right) - \ln \left( \frac{\tilde{q}(q, c)}{1 - \tilde{q}(q, c)} \right) \right] \right) \equiv Q_U(q, c; \sigma)
$$

$$
q = \Phi \left( \frac{1}{\sigma} \left[ \ln \left( \frac{d_p}{d_L} \right) - \ln \left( \frac{\tilde{p}(p, c)}{1 - \tilde{p}(p, c)} \right) \right] \right) \equiv Q_L(p, c; \sigma)
$$

where $\tilde{q}(q, c)$ is as defined in (18) and $\tilde{p}(p, c)$ is defined analogously.

Another potential benefit of using bias functions as opposed to the more general noisy beliefs is that the structure allows one to estimate biases from data on elicited beliefs (albeit with some identification assumptions). Here is a sketch of a possible procedure:

1. Assuming rationality, elicited beliefs for a group of $N$ subjects implies a hypothetical distribution of actions.

2. For each subject $s$, his elicited belief and the hypothetical distribution implies some set of biases that rationalize the belief.

3. An identification assumption can be used to pick out a single bias.

One can then “bootstrap” the empirical distribution of biases to make out-of-sample predictions.

10 Conclusion

It is well-known that Nash equilibrium fails to explain the richness of experimental data. Many models have been proposed as a result. It is clarifying to organize these models into categories.
based on the ways in which they relax the NE assumptions of rationality and belief consistency. We propose a simple equilibrium model that relaxes belief consistency by “adding noise to beliefs” and is tractable enough to estimate using experimental data. We show that unlike the established quantal response equilibrium (QRE), which relaxes rationality by “adding noise to actions”, the novel noisy beliefs equilibrium (NBE) makes predictions that are invariant to changes in payoff magnitude, which has surprisingly large implications. NBE is able to explain the same deviations from NE as QRE in-sample, but the two models make very different predictions out-of-sample. In making out-of-sample predictions across games of differing payoff magnitude, the QRE predictions are rejected and the NBE predictions are supported. As a consequence, NBE performs much better than QRE in explaining the pooled data from many games of varying payoff magnitude. As a consequence of NBE’s invariance properties, unlike QRE, any parametric version has parameters whose estimates are invariant to the arbitrary utility-money exchange rate used in the lab. This, we hope, will encourage fitting NBE to many datasets to explore how parameter estimates are affected by experimental procedures such as feedback and belief elicitation. For a logit transform specification, we find that the parameter $\sigma = 1$ fits the data from many games, and it would be valuable to see just how robust this finding is.

References


Terri Kneeland. Identifying higher-ordered rationality. *Econometrica*, 2015. 1


11 Appendix

11.1 Quantal Response Equilibrium

This section reviews the definition of quantal response equilibrium (QRE) and uses the same notation from Section 2. For an in-depth explanation of QRE, see McKelvey and Palfrey [1995], McKelvey and Palfrey [1998], Goeree et al. [2004], and others.

QRE introduces, for each player $i$ and action $j$, a payoff disturbance or “error”. For player $i$, let

$$
\hat{u}_{ij}(p_{-i}) = \bar{u}_{ij}(p_{-i}) + \varepsilon_{ij},
$$

where the vector $\varepsilon_i = (\varepsilon_{i1}, ..., \varepsilon_{iJ(i)})$ is drawn from a joint density $f_i$. We say that $f = (f_1, ..., f_n)$ is admissible if the marginal distribution of $f_i$ exists for each $\varepsilon_{ij}$ and $\mathbf{E}(\varepsilon_i) = 0$. Each player $i$ follows the basic rationality requirement that he take action $j$ if and only if

$$
\hat{u}_{ij}(p_{-i}) \geq \hat{u}_{ik}(p_{-i}) \quad \forall k = 1, ..., J(i). \quad (20)
$$

Given a vector $u_i' = (u_{i1}', ..., u_{iJ(i)}') \in \mathbb{R}^{J(i)}$, define the $ij$-response set

$$
W_{ij}(u_i') \equiv \{ \varepsilon_i \in \mathbb{R}^{J(i)} : u_{ij}' + \varepsilon_{ij} \geq u_{ik}' + \varepsilon_{ik} \quad \forall k = 1, ..., J(i) \}. \quad (21)
$$

Given a distribution of others’ actions $p_{-i}$, $W_{ij}(\bar{u}_i(p_{-i}))$ is the set of realizations of $\varepsilon_i$ that lead player $i$ to take action $j$. Define

$$
K_{ij}(u_i') \equiv \int_{R_{ij}(u_i')} f_i(\varepsilon_i) d\varepsilon_i
$$

to be the probability that errors are realized in set $W_{ij}(u_i')$, and let $K_i = (K_{i1}, ..., K_{iJ(i)})$. $K_i$ is called player $i$’s quantal response function. A distribution of play of $i$’s opponents and the payoffs over final actions given by $\Gamma$ defines for player $i$ a vector of expected utilities (one for each of $i$’s
actions). $K_i : \mathbb{R}^{J(i)} \rightarrow \Delta A_i$ maps that vector to a distribution over $i$’s pure actions. A quantal response equilibrium is obtained when the distribution over all players’ actions is consistent with their quantal response functions. Letting $K = (K_1, \ldots, K_n)$ and $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$, a QRE is a fixed point of the composite function $K \circ \bar{u} : \Delta A \rightarrow \Delta A$, which maps for each player the beliefs of other players’ actions into a probability distribution over his actions.

**Definition 10.** A QRE is any $p \in \Delta A$ such that for all $i \in 1, \ldots, n$ and all $j \in 1, \ldots, J(i)$, $p_{ij} = K_{ij}(\bar{u}_i(p_{-i}))$.

QRE is applied to data by first specifying joint densities $f_i$ up to a finite set of parameters. The vast majority of applications assume that $\varepsilon_{ij}$ is independently and identically distributed (i.i.d.) across all $i$ and $j$. Typically, it is assumed that every $\varepsilon_{ij}$ is independently drawn from an extreme value “logit” distribution with precision $\lambda \in [0, \infty)$, which gives the familiar logit quantal response function

$$K_{ij}(u_{ij}') = \frac{e^{\lambda u_{ij}'}}{\sum_{k=1}^{J(i)} e^{\lambda u_{ik}'}},$$

a result due to McFadden [1976]. Setting $p_{ij} = K_{ij}(\bar{u}_i(p_{-i}))$, with the empirical frequency of actions observable and all $\bar{u}_{ij}(\cdot)$ known, it is easy to estimate $\lambda$ by maximum likelihood. When the quantal response functions are given by (22), the resulting QRE are called “logit QRE.”

### 11.2 Proof of Theorem 2

Here we derive the CDF of $q^*(q; \sigma)$ for $\sigma > 0$.

$$F_1(\bar{q}|q; \sigma) \equiv P(q^*(q; \sigma) \leq \bar{q}) = P\left(\frac{\exp(\ln \frac{q}{1-q} + \sigma \varepsilon)}{1 + \exp(\ln \frac{q}{1-q} + \sigma \varepsilon)} \leq \bar{q}\right)$$

$$= P\left(\ln \frac{q}{1-q} + \sigma \varepsilon \leq \ln \frac{\bar{q}}{1-\bar{q}}\right)$$

$$= P\left(\varepsilon \leq \frac{1}{\sigma} \left[\ln \frac{\bar{q}}{1-\bar{q}} - \ln \frac{q}{1-q}\right]\right)$$

$$= \Phi\left(\frac{1}{\sigma} \left[\ln \frac{\bar{q}}{1-\bar{q}} - \ln \frac{q}{1-q}\right]\right)$$

Analogously, for player 2, we have

$$F_2(\bar{p}|p; \sigma) = \Phi\left(\frac{1}{\sigma} \left[\ln \frac{\bar{p}}{1-\bar{p}} - \ln \frac{p}{1-p}\right]\right).$$
Here we derive the reaction of player 1.

\[ Q_U(q; \sigma) = P(q^*(q; \sigma)(a_L + c_L) + (1 - q^*(q; \sigma))a_R \geq q^*(q; \sigma)(a_L) + (1 - q^*(q; \sigma))(a_R + c_R)) \]
\[ = P(q^*(q; \sigma) \geq \frac{c_R}{c_L + c_R}) \]
\[ = P \left( \frac{\exp(\ln \frac{q}{1-q} + \sigma \varepsilon)}{1 + \exp(\ln \frac{q}{1-q} + \sigma \varepsilon)} \geq \frac{c_R}{c_L + c_R} \right) \]
\[ = P \left( \ln \frac{q}{1-q} + \sigma \varepsilon \geq \ln \left( \frac{c_R}{c_L + c_R} \right) \right) \]
\[ = P \left( \sigma \geq \frac{1}{\varepsilon} \left[ \ln \frac{c_R}{c_L} - \ln \frac{q}{1-q} \right] \right) \]
\[ = 1 - \Phi \left( \frac{1}{\varepsilon} \left[ \ln \frac{c_R}{c_L} - \ln \frac{q}{1-q} \right] \right) \]

Analogously, for player 2, we have

\[ Q_L(p; \sigma) = \Phi \left( \frac{1}{\varepsilon} \left[ \ln \frac{d_D}{d_U} - \ln \frac{p}{1-p} \right] \right). \]

### 11.3 Logit Transform NBE for Normal Form Games

For arbitrary normal form games, we generalize (8) by parametrizing noisy beliefs for all \( l \in 1, \ldots, n, i \neq l \) and \( j \in 1, \ldots, J(i) \) as

\[ p_{ij}^l(p_i; \sigma) = \frac{\exp \left( \ln \frac{p_{ij}}{1-p_{ij}} + \sigma \varepsilon_{ij}^l \right)}{1 + \exp \left( \ln \frac{p_{ij}}{1-p_{ij}} + \sigma \varepsilon_{ij}^l \right)} \cdot \left( \sum_{k=1}^{J(i)} \frac{\exp \left( \ln \frac{p_{ik}}{1-p_{ik}} + \sigma \varepsilon_{ik}^l \right)}{1 + \exp \left( \ln \frac{p_{ik}}{1-p_{ik}} + \sigma \varepsilon_{ik}^l \right)} \right)^{-1}, \] (23)

where \( \varepsilon_{ij}^l \sim \text{iid} \ \mathcal{N}(0, 1) \), and \( \sigma \in [0, \infty) \) determines the noisiness of beliefs. These noisy beliefs are derived through the following procedure:

1. Take each \( p_{ij} \in [0, 1] \) and map it to the real line via the logit transform

\[ L(p_{ij}) = \ln \left( \frac{p_{ij}}{1-p_{ij}} \right), \]

using the convention that \( L(0) = -\infty \) and \( L(1) = \infty \).

2. Add \( \sigma \varepsilon_{ij}^l \sim \text{iid} \ \mathcal{N}(0, \sigma) \) to each \( L(p_{ij}) \) and \( \sigma \in [0, \infty) \).
3. Map each $L(p_{ij}) + \varepsilon_{ij}^l$ back to $[0, 1]$ via the inverse logit transform

$$L^{-1}(L(p_{ij}) + \varepsilon_{ij}^l) = \frac{\exp\left(ln \frac{P_{ij}}{1-P_{ij}} + \sigma \varepsilon_{ij}^l\right)}{1 + \exp\left(ln \frac{P_{ij}}{1-P_{ij}} + \sigma \varepsilon_{ij}^l\right)}.$$  

4. Normalize the set of $\{L^{-1}(L(p_{ij}) + \varepsilon_{ij}^l)\}_{j=1}^{J(i)}$ so that they sum to 1 by dividing each $L^{-1}(L(p_{ij}) + \varepsilon_{ij}^l)$ by the sum

$$\frac{\sum_{k=1}^{J(i)} \exp\left(ln \frac{P_{ik}}{1-P_{ik}} + \sigma \varepsilon_{ik}^l\right)}{1 + \exp\left(ln \frac{P_{ik}}{1-P_{ik}} + \sigma \varepsilon_{ik}^l\right)}.$$  

It is easily verified that noisy beliefs given by (23) satisfy the following properties.

1. For all $p_i \in \Delta A_i$, the distribution of $p_{ij}^{l*}(p_i; \sigma)$ strictly first order stochastically dominates the distribution of $p_{ij}^{l*}(p_i - \epsilon e_j + \epsilon e_k; \sigma)$ where $\epsilon > 0$ and $e_j$ is the standard $j$th basis vector $(0, ..., 1, ..., 0)$, so long as $p_i - \epsilon e_j + \epsilon e_k \in \Delta A_i$.

2. $p_{ij}^{l*}(p_i; \sigma) = 0$ if and only if $p_{ij} = 0$.

3. $p_{ij}^{l*}(p_i; \sigma)$ emits a density for all $p_i$, unless $p_{ik} = 1$ for some $k$, in which case $p_{ik}^{l*}(p_i; \sigma) = 1$.

11.4 Proof of Theorem 4

1. $r^*(r'; \sigma) \succ_{FOSD} r^*(r; \sigma)$ for all $r' > r$.

We show that $F_i(\bar{r}|r'; \sigma) < F_i(\bar{r}|r; \sigma)$ for all $\bar{r}$ if and only if $r' > r$.

$$F_i(\bar{r}|r'; \sigma) < F_i(\bar{r}|r; \sigma) \iff \Phi \left(\frac{1}{\sigma} \left[ln \frac{\bar{r}}{1-\bar{r}} - ln \frac{r'}{1-r'}\right]\right) < \Phi \left(\frac{1}{\sigma} \left[ln \frac{\bar{r}}{1-\bar{r}} - ln \frac{r}{1-r}\right]\right) \iff \frac{1}{\sigma} \left[ln \frac{\bar{r}}{1-\bar{r}} - ln \frac{r'}{1-r'}\right] < \frac{1}{\sigma} \left[ln \frac{\bar{r}}{1-\bar{r}} - ln \frac{r}{1-r}\right] \iff r' > r$$

2. The median of $r^*(r; \sigma)$ is $r$.

$$F_i(r|r; \sigma) = \Phi \left(\frac{1}{\sigma} \left[ln \frac{r}{1-r} - ln \frac{r}{1-r}\right]\right) = \Phi(0) = \frac{1}{2}$$

3. $r^*(r; \sigma)$ has full support for $r \in (0, 1)$.

This is obvious from the construction of $r^*(r; \sigma)$.  

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4. \( r^*(1; \sigma) = 1, \) \( r^*(0; \sigma) = 0, \) and \( r^*(r; 0) = r. \)

To show these, simply plug into

\[
\frac{\exp\left(\ln\left(\frac{r}{1-r}\right) + \sigma \varepsilon_i\right)}{1 + \exp\left(\ln\left(\frac{r}{1-r}\right) + \sigma \varepsilon_i\right)},
\]

and use the convention that \( L(0) = -\infty \) and \( L(1) = \infty \) where \( L(r) = \ln\left(\frac{r}{1-r}\right). \)
11.5 Histograms of Beliefs Implied by Best-fit NBE in McKelvey et al. [2000]

Figure 12: Histograms of beliefs implied by best-fit NBE.
11.6 Description of Papers and Games

11.6.1 **MPW 2000 (McKelvey et al. [2000])**

For any one game, each subject takes an action 50 times against randomly matched opponents with feedback. Subjects play multiple games, and maintain their role as either player 1 or player 2 for the duration of the experiment. The exchange rate is $0.10 (year 2000 Dollars).

\[
\begin{array}{c|c|c|c}
| & A & B & C & D \\
|---|---|---|---|---|
| U | 9 & 0 & 36 & 4 \\
| D | 0 & 1 & 0 & 1 \\
| D | 1 & 0 & 4 & 0 \\
\end{array}
\]

11.6.2 **SC 2008 (Selten and Chmura [2008])**

Each subject plays just one game, and takes an action 200 times against randomly matched opponents with feedback. Subjects maintain their role as either player 1 or player 2 for the duration of the experiment. The exchange rate is €0.016 (year 2008 Euros), and the experiment took place in Germany.

\[
\begin{array}{c|c|c|c|c}
| & 1 & 2 & 3 & 4 \\
|---|---|---|---|---|
| U | 10 & 9 & 8 & 7 \\
| D | 8 & 6 & 5 & 4 \\
| U | 18 & 13 & 14 & 11 \\
| D | 19 & 8 & 7 & 2 \\
\end{array}
\]

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Each subject plays only one of the two games. Those who play game 2 (game 3) take an action 56 (64) times against randomly matched opponents with feedback. Subjects maintain their role as either player 1 or player 2 for the duration of the experiment. Following McKelvey and Palfrey [1995], who note that “The subjects in the Ochs experiments were paid using a lottery procedure,” we convert the payoffs described in Ochs [1995] to those in the matrices below before estimation. The exchange rate is $0.01 (expected 1982 Dollars).
11.6.4  **GH 2001** (Goeree and Holt [2001])

Each subject takes just one action against an anonymous opponent, and the exchange rate is $0.01 (year 2001 dollars). *AMP* refers to “asymmetric matching pennies” and *RA* refers to “reversed asymmetry.”

\[
\begin{array}{c|cc|c|cc}
 & \text{AMP} & & \text{RA} & \\
\hline
\text{L} & 320 & 40 & \text{U} & 44 & 40 \\
40 & 80 & 40 & 80 & \\
\text{D} & 40 & 80 & \text{D} & 80 & 40 \\
\end{array}
\]

11.6.5  **NS 2002** (Nyarko and Schotter [2002])

Each subject plays one of four treatments, which differ in terms of details regarding a belief-elicitation procedure. Since there are no significant differences in empirical frequencies of actions across treatments, we pool all data together. Each subject takes an action 60 times against randomly matched opponents with feedback. Subjects maintain their role as either player 1 or player 2 for the duration of the experiment. The exchange rate is $0.05 (year 2000 Dollars). *AMP* refers to “Asymmetric Matching Pennies”.

\[
\begin{array}{c|cc|c|cc}
 & \text{AMP} & & \\
\hline
\text{L} & 6 & 3 & \\
2 & 5 & \\
\text{D} & 3 & 5 & \\
5 & 3 & \\
\end{array}
\]
### 11.7 Data and Model Predictions for All Games

<table>
<thead>
<tr>
<th>Paper</th>
<th>Game</th>
<th>Data</th>
<th>NBE</th>
<th>QRE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$p$</td>
<td>$q$</td>
<td>$p$</td>
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<td><strong>MPW 2000</strong></td>
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<td>0.241</td>
<td>0.747</td>
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<td></td>
<td>$B$</td>
<td>0.630</td>
<td>0.244</td>
<td>0.748</td>
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<tr>
<td></td>
<td>$C$</td>
<td>0.594</td>
<td>0.257</td>
<td>0.629</td>
</tr>
<tr>
<td></td>
<td>$D$</td>
<td>0.550</td>
<td>0.328</td>
<td>0.661</td>
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<td><strong>SC 2008</strong></td>
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</tr>
<tr>
<td></td>
<td>1</td>
<td>0.079</td>
<td>0.690</td>
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<td>0.217</td>
<td>0.527</td>
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<tr>
<td></td>
<td>3</td>
<td>0.163</td>
<td>0.793</td>
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<td></td>
<td>4</td>
<td>0.286</td>
<td>0.736</td>
<td>0.266</td>
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<td>5</td>
<td>0.327</td>
<td>0.664</td>
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<td></td>
<td>6</td>
<td>0.445</td>
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<td>12</td>
<td>0.439</td>
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<td>0.425</td>
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<tr>
<td><strong>Ochs 1995</strong></td>
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</tr>
<tr>
<td></td>
<td>2</td>
<td>0.595</td>
<td>0.258</td>
<td>0.632</td>
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<td></td>
<td>3</td>
<td>0.542</td>
<td>0.336</td>
<td>0.667</td>
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<td><strong>GH 2001</strong></td>
<td>AMP</td>
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<td>0.719</td>
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<td></td>
<td>RA</td>
<td>0.080</td>
<td>0.800</td>
<td>0.242</td>
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<td><strong>NS 2002</strong></td>
<td>AMP</td>
<td>0.458</td>
<td>0.390</td>
<td>0.459</td>
</tr>
</tbody>
</table>

11.8 Learning in Selten and Chmura [2008]

Figure 13: Learning in Selten and Chmura [2008].
11.9 Utility-Money Exchange Rate Conversions: Procedure and Robustness

11.9.1 Procedure

To make estimates of QRE-λ comparable across studies in the exercise in Section 8, we convert all utility-money exchange rates in the studies considered to be consistent with that of McKelvey et al. [2000], the arbitrarily chosen “base study” denominated in U.S. currency. Given an exchange rate of 1 utile per $C_{ft}$, where $C_{ft}$ is a numerical amount denominated in the currency of country $f$ in year $t$, we calculate a “conversion factor” $\gamma_{ft}$ using the formula

$$\gamma_{ft} = \frac{C_{ft}}{E_{ft}} \left( \frac{P_0}{P_t} \right) \frac{1}{C_0},$$

where $E_{ft}$ is the year $t$ PPP adjustment factor to U.S. dollars, $P_0$ and $P_t$ are the U.S. CPI price indices in the base year and year $t$ respectively, and $C_0$ is the dollar value of 1 utile in the base study. Before fitting QRE to a study with an exchange rate of 1 utile per $C_{ft}$, the payoff matrices are multiplied by $\gamma_{ft}$.

PPP adjustment factors are from the World Bank\textsuperscript{17} and CPI is from the St. Louis Federal Reserve Bank\textsuperscript{18}. In all cases, we use annual statistics from the last year in which the studies were conducted (for studies that took place in more than one calendar year). Table 6 gives the conversion factors as well as their components.

<table>
<thead>
<tr>
<th>Paper</th>
<th>Currency ($C_{ft}$)</th>
<th>PPP ($E_{ft}$)</th>
<th>CPI 2000 ($P_0$)</th>
<th>CPI ($P_t$)</th>
<th>$C_0$</th>
<th>$\gamma_{ft}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MPW 2000</td>
<td>$0.10$ (2000 U.S.)</td>
<td>1</td>
<td>172.192</td>
<td>172.192</td>
<td>0.10</td>
<td>1</td>
</tr>
<tr>
<td>SC 2008</td>
<td>€0.016 (2008 Germany)</td>
<td>0.820401</td>
<td>172.192</td>
<td>215.254</td>
<td>0.10</td>
<td>0.1560</td>
</tr>
<tr>
<td>Ochs 1995</td>
<td>$0.01$ (1982 U.S.)</td>
<td>1</td>
<td>172.192</td>
<td>96.533</td>
<td>0.10</td>
<td>0.1784</td>
</tr>
<tr>
<td>GH 2001</td>
<td>$0.01$ (2001 U.S.)</td>
<td>1</td>
<td>172.192</td>
<td>177.042</td>
<td>0.10</td>
<td>0.0973</td>
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<td>NS 2002</td>
<td>$0.05$ (2000 U.S.)</td>
<td>1</td>
<td>172.192</td>
<td>172.192</td>
<td>0.10</td>
<td>0.5</td>
</tr>
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</table>

Table 6: Utility-Money Exchange Rate Conversion Factors.

11.9.2 Robustness

After calculating conversion factors, we consider the effect of perturbing the factors on the overall performance of QRE in the exercise of Section 8. For each study $s$ (other than the base study), we

\textsuperscript{17}http://data.worldbank.org/indicator/PA.NUS.PPP?locations=DE
\textsuperscript{18}https://fred.stlouisfed.org/series/CPIAUCSL
calculate $\gamma^s$ via the procedure in the previous subsection and then we consider all

$$\tilde{\gamma}^s \in \{0.5\gamma^s, 0.6\gamma^s, \ldots, \gamma^s, \ldots, 1.4\gamma^s, 1.5\gamma^s\} \equiv \Pi^s,$$

i.e. perturbed factors that are off by as much as 50% in either direction in 10% increments. We calculate the pooled prediction error from fitting QRE to all 21 games across the 5 studies, just as in Section 8, for every possible combination of perturbed conversion factors:

$$(\tilde{\gamma}^1, \ldots, \tilde{\gamma}^4) \in \Pi^1 \times \ldots \times \Pi^4.$$

Figure 14 plots the histogram of pooled QRE prediction errors for all 14,641 factor combinations, as well as vertical lines representing the pooled NBE prediction error which is invariant to the factors and the minimum QRE prediction error across all factors. Clearly, NBE outperforms QRE by a wide margin for all factors considered. At worst, the prediction error of NBE is still less than 70% of that of QRE.