Computing the Strong Nash Equilibrium For Conforming Coalitions

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***Abstract*—Computing the equilibrium point of games plays an important in computer science. A large number of methods are known for finding a Nash equilibrium. Nevertheless, Nash equilibrium can be adopted only for non-cooperative games. In the last years, there has been a substantial effort in the development methods for finding the Strong Nash Equilibrium useful when coalitions are a fundamental issue.**

In this paper we present a new method for computing strong Nash equilibria in multiplayer games for a class of ergodic controllable Markov chains. For solving the problem we propose a two steps approach: a) we employ a regularized Lagrange principle to construct the Pareto front and b) we regularized the resulting Pareto front using the Tikhonov's regularization method for ensuring the existence of a unique equilibrium and make use of the Newton method for converging to the Strong Nash equilibrium. A numerical example illustrates the efficiency of the approach.

Keywords: Strong Nash equilibrium, Pareto front, game theory, Markov processes, optimization.

I. Introduction

The most common solution concept in game theory is the Nash equilibrium (NE) [13]. The NE concept is ineffective when players can form coalitions and deviate multilaterally in a coordinated way. We consider that each coalition has a set of feasible costs allocations, and use an explicit set of rules for the interaction between the players The strong Nash equilibrium (SNE) is a more refined concept that strengthens the NE concept taking into account joint deviations of coalitions. The SNE was proposed by Aumann [1] as an alternative to the Nash equilibrium, which considers the fact that some of the players, although having no unilateral incentive to deviate, may benefit from forming coalitions with other players. In a SNE there is no coalition of players that can improve their payoffs (by collective deviation). Then, the SNE presents the benefits of a cooperative behavior in a non-cooperative environment. The importance of SNE is studied for classes of games that allow the characterization of SNE, such as congestion games [8], [9], connection games [4], maxcut games [7], voting models [3], [12], coalition formation [2], [10],other fields [11]. Pareto efficiency plays a fundamental role in game theory for finding the SNE [5]. Each player is related with one objective function. The Pareto front identifies the trade-offs among the different players' goals. The Pareto front is defined as the image of the collection of Pareto efficient solutions. The bargaining procedure is based on the SNE on the Pareto front due to Selten [15]. A number of solution concepts in cooperative and non-cooperative game theory are based on Pareto efficiency, prescribing the selection of specific points on the Pareto front (see [6]).

The aim of this article is to compute SNE. We divide the problem in two different sub-problems. First, we construct the Pareto front finding Pareto optimal strategies. For ensuring Pareto efficiency we formulate the problem based on the Lagrange principle adding a Tikhonov's regularizator for making certain the existence of Pareto optimal policies. In the second sub-problem, we introduce an additional Tikhonov's regularizator over the cost-functions making sure that the Pareto front is strict convex. The introduction of a Tikhonov's regularizator is a necessary condition for ensuring the existence of a unique strong Nash equilibrium. Then, we propose a method that tracks the surface of the Pareto front by picking points from the continuous space and converges to an equilibrium point based on the Newton optimization method. The result is illustrated with an application example.

The rest of the article is organized as follows. In the next Section we present the formulation of the Pareto optimality. In Section III we suggest an algorithm for finding the SNE. The methods to obtain the optimal Pareto policies and the optimal scalar values in order to compute the SNE are described in Section IV. A numerical example is presented in Section V. We close the paper with some remarks in Section VI.

II. Pareto optimality

Thus, we have that the regularized average cost function is given by

$$\mathbf{V}_{\delta}^{l}(\mathbf{c}) = \sum_{i_1,k_1} \cdots \sum_{i_l,k_l} \lambda^l W^l_{(i_1,k_1,\ldots i_l,k_l)} \prod_l c^l_{(i_l,k_l)} + \frac{\delta}{2}\left[\sum_l \left\| c^l_{(i_l,k_l)} \right\|^2\right]$$

s.t.

$$g^l_{j_l}(c^l) = \sum_{k_l} c^l_{(j_l,k_l)} - \sum_{i_l,k_l} \pi^l_{(j_l|i_l,k_l)} c^l_{(i_l,k_l)} = 0 \quad (1)$$

$$\text{and} \sum_{i_l,k_l} c^l_{(i_l,k_l)}=1,\ c^l_{(i_l,k_l)}\geq 0$$

and

$$\sum_l \left\| c^l_{(i_l,k_l)} \right\|^2 = \left(\sum_{i_1,k_1} c^1_{(i_1,k_1)} \right)^2 + ... + \left(\sum_{i_l,k_l} c^l_{(i_l,k_l)} \right)^2$$

Now, the function $\mathbf{V}^l_\delta(\mathbf{c})$ is strictly convex if $\delta > 0$.

Let

$$w^l := col\ \left(c^{(l)}\right),\ W^l := C^{(l)}_{adm}\ \left(l = \overline{1,\mathcal{N}}\right)$$

where col is the column operator and define

$$w = \begin{pmatrix} w^1 \\ ... \\ w^{\mathcal{N}} \end{pmatrix}$$

In a multi-objective problem, instead of minimizing the multi-objective function, we can minimizing the correspond Lagrange function

Definition 1: Given a multi-objective function (1) $\mathbf{V}^l_\delta(w)$ then

$$L_\delta(w,\lambda) =$$

$$\sum_l^{\mathcal{N}} \lambda^l \left[\mathbf{V}^l_\delta(w^l) + \sum_{j_l}^{N_l} \mu^l_{j_l} g^l_{j_l}(w^l) + \frac{\delta}{2}(\mu^l)^2 \right] + \frac{\delta}{2}\left\|\lambda\right\|^2$$

is the Lagrange function.

Let us also consider $\varrho > 0$ a distance measure that restrict the cost-functions allowing points in the Pareto front to have a small distance from one another

$$L^l_\delta(w^l_{prev},\lambda) - \varrho \leq L^l_\delta(w,\lambda) \leq L^l_\delta(w^l_{prev},\lambda) + \varrho \text{ for } \left(l = \overline{1,\mathcal{N}}\right)$$

III. SNE Algorithm

We consider a regularized Lagrange function consisting of two variables $\mathcal{L}_\delta(w,\lambda)$. We describe the dependence of the saddle points $(w^*(\delta),\lambda^*(\delta))$ of the regularized Lagrange function on the regularizing parameter δ [14].

Theorem 2: **[Uniqueness of the SNE]** Let $L_\delta(w,\lambda)$ regularized Lagrange function consisting of two variables w and λ with nonnegative components, then when $0 < \delta_n \to 0$ we have

$$(w^*_{\delta_n}, \lambda^*_{\delta_n}) \to (w^{**},\lambda^{**}) \text{ as } n \to \infty$$

and$(w^{**},\lambda^{**}) = \underset{(w^*,\lambda^*)}{\arg\min} (\|w^*\|^2 + \|\lambda^*\|^2)$.

In addition, the equilibrium point $w^{**}_\delta(\lambda^{**}_\delta)$ is a strong Pareto policy.

Now we construct a method of converge to a SNE. We employ the Newton optimization method for finding the SNE. The approximation of the first and second derivative is computed using the Euler method [14].

Let us define the regularized penalty function Φ as follows

$$\Phi_\delta(\lambda) := \Phi(\lambda) + \frac{\delta}{2}\left\|\lambda\right\|^2,\ 0 < \delta \ll 1$$

$$\Phi(\lambda) := \textstyle\sum_l^{\mathcal{N}} \Phi_l(\lambda)$$

$$\Phi_l(\lambda) :=$$

$$\frac{1}{2+\varepsilon}\left[J^l(w^*(\lambda)) - J^{l+}\right]_+^{2+\varepsilon} + \frac{1}{2+\varepsilon}\left[J^{l-} - J^l(w^*(\lambda))\right]_+^{2+\varepsilon}$$

where

$$[z]_+ := \begin{cases} z & \text{if} \quad z \geq 0 \\ 0 & \text{if} \quad z < 0 \end{cases}$$

The solution λ^* can be expressed mathematically as follows:

$$\lambda^* \in Arg \min_{\lambda \in \mathcal{S}^N} \Phi(\lambda) = Arg \min_{\lambda \in \mathcal{S}^N} \Phi_{\delta=0}(\lambda) \quad (2)$$

As it has been mentioned above λ^*, satisfying (2), may be not unique that provokes several problems for the numerical procedure implementation. Taking $\delta > 0$ we can guarantee the uniqueness of the solution so that we will try to find

$$\lambda^{**}_\delta = \arg \min_{\lambda \in \mathcal{S}^N} \Phi_\delta(\lambda),\ \delta > 0 \quad (3)$$

This solution corresponds to one of the solutions λ^* (2) which has the minimal norm of the vector $\|\lambda\|^2$.

To find λ^{**}_δ let us apply the Newton's optimization method related to the following numerical procedure

$$\lambda_{n+1} = \Pr_{\mathcal{S}^N}\left[\lambda_n - \Gamma_n \left[\Phi''_\delta(\lambda_n) + \varepsilon\right]^{-1} \Phi'_\delta(\lambda_n)\right],$$

$$\lambda_0 = (1/\mathcal{N}, ...1/\mathcal{N}),\ n = 0,1,...$$

$$\gamma_n > 0,\ \sum_{n=0}^{\infty} \gamma_n = \infty,$$

where $\Pr_{\mathcal{S}^N}$ is the projection operator into the simplex.

IV. Computing the Pareto optimality and the SNE

The aim of this section is to present the methods to obtain the optimal Pareto policies w^* and the optimal scalar values of λ^* in order to compute the SNE. We consider:

1) Construct the Pareto front finding the Pareto optimal strategies.
2) Find the SNE.

1) The method proposed to compute the Pareto front is as follows:

$$w_{n+1} = \Pr\left[w_n - \gamma_n \frac{\partial}{\partial w}\mathcal{L}_{\delta,\varrho}(w_n,\lambda_n,\mu_n,\theta_n)\right]$$

$$\lambda_{n+1} = \Pr\left[\lambda_n - \gamma_n \frac{\partial}{\partial \lambda}\mathcal{L}_{\delta,\varrho}(w_n,\lambda_n,\mu_n,\theta_n)\right], \quad \lambda_0 = 0.5,\ n = 0,1,...$$

$$\mu_{n+1} = \left[\mu_n + \gamma_n \frac{\partial}{\partial \mu}\mathcal{L}_{\delta,\varrho}(w_n,\lambda_n,\mu_n,\theta_n)\right]_+$$

$$\theta_{n+1} = \left[\theta_n + \gamma_n \frac{\partial}{\partial \theta}\mathcal{L}_{\delta,\varrho}(w_n,\lambda_n,\mu_n,\theta_n)\right]_+$$

$$\gamma_n > 0, \quad \sum_{n=0}^{\infty}\gamma_n = \infty$$

where Pr is the projection operator into the simplex.

1) Fix the parameters and initial conditions: $\epsilon, \lambda_0, \delta_0, \theta_0, \rho, \mu_0, \gamma_0, n_0$
2) Let
$$\gamma_n = \begin{cases} \gamma_0 & n \le \frac{1}{10}n_0 \\ \frac{\gamma_0}{(n-n_0)^{2/3}} & n > \frac{1}{10}n_0 \end{cases}$$
$$\delta_n = \begin{cases} \delta_0 & n \le \frac{1}{10}n_0 \\ \frac{\delta_0}{(n-n_0)^{1/3}} & n > \frac{1}{10}n_0 \end{cases}$$
3) Compute
$$\tilde{w}_{n+1} = \left[v_n - \gamma_n \frac{\partial}{\partial v}\mathcal{L}_{\delta,\varrho}(w_n,\lambda_n,\mu_n,\theta_n)\right]$$
$$\tilde{\lambda}_{n+1} = \left[\lambda_n - \gamma_n \frac{\partial}{\partial \lambda}\mathcal{L}_{\delta,\varrho}(w_n,\lambda_n,\mu_n,\theta_n)\right],$$
$$\lambda_0 = 1/\mathcal{N},\ n = 0,1,...$$
4) Normalize w_{n+1} projecting the values to the simplex $\mathcal{S}^{N_l M_l}$, i.e. $w_{n+1} = \Pr_{\mathcal{S}^{N_l M_l}}\!:\!\tilde{w}_{n+1} \to \mathcal{S}^{N_l M_l}$, where $\Pr_{\mathcal{S}^{N_l M_l}}\{\cdot\}$ is the projection operator of the vector from $\mathbb{R}^{N_l M_l}$ into the simplex $\mathcal{S}^{N_l M_l}$. As well, normalize λ_{n+1} projecting the values to the simplex $\mathcal{S}^N$, i.e. $\lambda_{n+1} = \Pr_{\mathcal{S}^N}\!:\!\tilde{\lambda}_{n+1} \to \mathcal{S}^N$ where $\Pr_{\mathcal{S}^N}\{\cdot\}$ is the projection operator of the vector from $\mathbb{R}^N$ into the simplex $\mathcal{S}^N$
5) Verify descent direction
$$\left\|w_n - \Pr_{\mathcal{S}^{N_l M_l}}\{\tilde{w}_{n+1}\}\right\| \le \left\|w_n - \tilde{w}_{n+1}\right\|$$
$$\left\|\lambda_n - \Pr_{\mathcal{S}^N}\left\{\tilde{\lambda}_{n+1}\right\}\right\| \le \left\|\lambda_n - \tilde{\lambda}_{n+1}\right\|$$
for any $w^l_{n+1} \in \mathbb{R}^{N_l M_l}$ and any $\lambda_{n+1} \in \mathcal{S}^N$.
6) Check the convergence criteria $\left\|w^l_{n+1} - w^l_n\right\| < \varepsilon_c$ and $\|\lambda_{n+1} - \lambda_n\| < \varepsilon_\lambda$ then stop. Otherwise, set $n = n+1$ and return to Step 4. (v^l_{n+1} and λ_{n+1} are computed independently)
7) Compute the values of μ_{n+1} and θ_{n+1}
$$\mu_{n+1} = \left[\mu_n + \gamma_n \frac{\partial}{\partial \mu}\mathcal{L}_{\delta,\varrho}(w_n,\lambda_n,\mu_n,\theta_n)\right]_+$$
$$\theta_{n+1} = \left[\theta_n + \gamma_n \frac{\partial}{\partial \theta}\mathcal{L}_{\delta,\varrho}(w_n,\lambda_n,\mu_n,\theta_n)\right]_+$$
8) Let $\lambda \leftarrow \lambda + \epsilon$ and go to step 2

Algorithm 3: **Pareto front.**

2) The method proposed to compute the strong Nash equilibrium is as follows:

$$\lambda_{n+1} = \Pr_{\mathcal{S}^N}\left[\lambda_n - \Gamma_n\left[\Phi''_\delta(\lambda_n) + \varepsilon\right]^{-1}\Phi'_\delta(\lambda_n)\right]$$

1) Fix the parameters and initial conditions: $\epsilon, \lambda_0, \delta_0, \gamma_0, n_0$
2) Let
$$\gamma_n = \begin{cases} \gamma_0 & n \le \frac{1}{10}n_0 \\ \frac{\gamma_0}{(n-n_0)^{2/3}} & n > \frac{1}{10}n_0 \end{cases}$$
$$\delta_n = \begin{cases} \delta_0 & n \le \frac{1}{10}n_0 \\ \frac{\delta_0}{(n-n_0)^{1/3}} & n > \frac{1}{10}n_0 \end{cases}$$
3) Compute
$$\tilde{\lambda}_{n+1} = \left[\lambda_n - \Gamma_n\left[\Phi''_\delta(\lambda_n) + \varepsilon\right]^{-1}\Phi'_\delta(\lambda_n)\right]$$
4) Normalize λ_{n+1} projecting the values to the simplex $\lambda_{n+1} = \Pr_{\mathcal{S}^N}\!:\!\tilde{\lambda}_{n+1} \to \mathcal{S}^N$ where $\Pr_{\mathcal{S}^N}\{\cdot\}$ is the projection operator of the vector from $\mathbb{R}^N$ into the simplex $\mathcal{S}^N$
5) Verify descent direction
$$\left\|\lambda_n - \Pr_{\mathcal{S}^N}\left\{\tilde{\lambda}_{n+1}\right\}\right\| \le \left\|\lambda_n - \tilde{\lambda}_{n+1}\right\|$$
6) Check the convergence criteria $\|\lambda_{n+1} - \lambda_n\| < \varepsilon_\lambda$ then stop. Otherwise, set $n = n+1$ and return to Step 2. (v^l_{n+1} and λ_{n+1} are computed independently)

Algorithm 4: **Strong Nash equilibrium.**

V. NUMERICAL EXAMPLE

Let us consider a game with two players ($l = 1, 2$) trying to conform a coalition. Let $N_1 = N_2 = 5$ and $M_1 = M_2 = 2$. The transition matrices of the example are as follows

$$\pi^{(1)}_{ij|1} = \begin{bmatrix} 0.9988 & 0.0002 & 0.0000 & 0.0010 & 0.0001 \\ 0.1498 & 0.5371 & 0.1758 & 0.0332 & 0.1042 \\ 0.1095 & 0.3825 & 0.0484 & 0.4495 & 0.0101 \\ 0.1147 & 0.2723 & 0.2470 & 0.2554 & 0.1105 \\ 0.0002 & 0.0006 & 0.0004 & 0.9988 & 0.0000 \end{bmatrix}$$

$$\pi^{(1)}_{ij|2} = \begin{bmatrix} 0.3475 & 0.0973 & 0.0144 & 0.0637 & 0.4771 \\ 0.1836 & 0.0553 & 0.2155 & 0.0407 & 0.5048 \\ 0.2224 & 0.4401 & 0.1062 & 0.1559 & 0.0756 \\ 0.0000 & 0.9983 & 0.0008 & 0.0008 & 0.0001 \\ 0.1447 & 0.4308 & 0.2800 & 0.0250 & 0.1194 \end{bmatrix}$$

$$\pi^{(2)}_{ij|1} = \begin{bmatrix} 0.0103 & 0.1959 & 0.6735 & 0.0649 & 0.0554 \\ 0.0371 & 0.1915 & 0.1700 & 0.3360 & 0.2654 \\ 0.0387 & 0.2916 & 0.2437 & 0.1997 & 0.2263 \\ 0.9998 & 0.0001 & 0.0000 & 0.0000 & 0.0000 \\ 0.3300 & 0.0822 & 0.2201 & 0.0256 & 0.3420 \end{bmatrix}$$

$$\pi^{(2)}_{ij|2} = \begin{bmatrix} 0.0198 & 0.3242 & 0.3319 & 0.0496 & 0.2745 \\ 0.9983 & 0.0007 & 0.0001 & 0.0006 & 0.0002 \\ 0.4956 & 0.0018 & 0.0024 & 0.4946 & 0.0056 \\ 0.0444 & 0.1901 & 0.3024 & 0.1888 & 0.2744 \\ 0.1541 & 0.1359 & 0.2243 & 0.1936 & 0.2920 \end{bmatrix}$$

The individual cost matrices are as follows:

$$J^{(1)}_{ij|1} = \begin{bmatrix} 600 & 54 & 30 & 6 & 24 \\ 6 & 18 & 30 & 0 & 18 \\ 12 & 48 & 30 & 42 & 0 \\ 12 & 6 & 24 & 24 & 36 \\ 42 & 24 & 36 & 54 & 42 \end{bmatrix}$$

$$J^{(1)}_{ij|2} = \begin{bmatrix} 4 & 2 & 6 & 6 & 4 \\ 4 & 2 & 10 & 4 & 6 \\ 6 & 2 & 12 & 6 & 6 \\ 10 & 200 & 6 & 8 & 2 \\ 8 & 10 & 14 & 10 & 2 \end{bmatrix}$$

$$J^{(2)}_{ij|1} = \begin{bmatrix} 510 & 6 & 9 & 3 & 300 \\ 3 & 9 & 24 & 510 & 12 \\ 12 & 15 & 12 & 6 & 15 \\ 3 & 15 & 0 & 9 & 3 \\ 9 & 3 & 0 & 12 & 15 \end{bmatrix}$$

$$J^{(2)}_{ij|2} = \begin{bmatrix} 6 & 4 & 6 & 18 & 4 \\ 6 & 4 & 8 & 6 & 12 \\ 4 & 2 & 4 & 6 & 12 \\ 6 & 4 & 4 & 2 & 12 \\ 2 & 2 & 8 & 16 & 52 \end{bmatrix}$$

Fixing $\delta = 0.001$, $\gamma = 0.8$, $\varrho = 0.1$, $n_0 = 3$ and the bounds $J1^- = 920$, $J1^+ = 1080$, $J2 = 405$, $J2 = 445$. Then,the resulting parameter $\lambda^{(1)*}_{final} = 0.5066$ and $\lambda^{(2)*}_{final} = 0.4934$. The corresponding strong Nash equilibrium (see Figure 1) is as follows:

$$c^{(1)*}_{i|k} = \begin{bmatrix} 0.1222 & 0.1179 \\ 0.1371 & 0.1362 \\ 0.0486 & 0.0475 \\ 0.1286 & 0.1036 \\ 0.1523 & 0.0059 \end{bmatrix} \qquad c^{(2)*}_{i|k} = \begin{bmatrix} 0.1352 & 0.1310 \\ 0.0874 & 0.0672 \\ 0.1220 & 0.1227 \\ 0.0709 & 0.0863 \\ 0.1378 & 0.0395 \end{bmatrix}$$

VI. Conclusion

In this paper we suggested a procedure that can be used to efficiently compute strong Nash equilibrium. We divided the formulation of the problem in two different sub-problems. First, we constructed the Pareto front finding the Pareto optimal strategies. Second, we introduced an additional Tikhonov's regularizator over the cost-functions making sure that the Pareto front is strict convex. The proposed method tracks the surface of the Pareto front by picking points from the continuous space and converges to an equilibrium point using the Newton optimization method. Our numerical example showed that the proposed method is efficient in terms of the conceptualization of the solution of the problem.

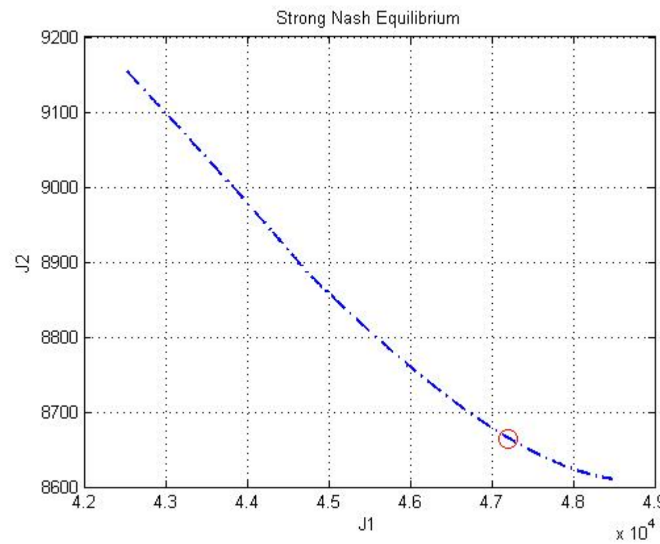

Fig. 1. Strong Nash equilibrium.

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