Zero-sum games with charges

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Abstract

We consider two-player zero-sum games with countably infinite action spaces and bounded payoff functions. The players’ strategies are finitely additive probability measures, called charges. Since a strategy profile does not always induce a unique expected payoff, we distinguish two extreme attitudes of players. A player is viewed as pessimistic if he always evaluates the range of possible expected payoffs by the worst one, and a player is viewed as optimistic if he always evaluates it by the best one. This approach results in a definition of a pessimistic and an optimistic value for each player. We provide an extensive analysis of the relation between these values, and connect them to the classical values. In addition, we also examine existence of optimal strategies with respect to these values.

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1 Introduction

Countable additivity is a usual assumption of probability, but from a conceptual point of view, the weaker assumption of finite additivity was also argued for (e.g. de Finetti (1975), Savage (1972), Bingham (2010), Dubins and Savage (2014)). Finitely additive measures are briefly called charges, and a notable advantage of them is that they avoid the problem of measure (cf. Aliprantis and Border (2005)). Moreover, translation and rotation invariant charges can be defined on all subsets of the real line and of the plane. The existence of a charge that is not countably additive however requires the axiom of choice, or at least a non-constructive method (cf. Lauwers (2010)).

In decision theory, charges have been used in various models (e.g. de Finetti (1975), Savage (1972), Dubins and Savage (2014)). They also gained recognition in game theory (e.g. Marinacci (1997), Capraro and Scarsini (2013), Maitra and Sudderth (1998)), but to a lesser extent. One possible reason is that in a game setting it can occur that multiple players choose actions simultaneously. If each of them uses a probability charge for the choice of his action, then on the space of action profiles, these probability charges induce a unique charge only on the algebra generated by the cylinders. This algebra is not very rich, and consequently, the set of payoff functions for which the expected payoff is unique becomes rather restricted. For instance, Marinacci (1997) and Harris et al (2005) consider only payoff functions that can be uniformly approximated by simple functions on this algebra. We refer to Capraro and Scarsini (2013) for a more detailed summary of the relevant literature.

Our setup: We examine two-player zero-sum games with countably infinite action spaces. Our starting point is however different, as we allow for any bounded payoff function. Each player’s strategy, which is now a probability charge, is defined on all subsets of his action space. Then, as pointed out above, a strategy profile induces a whole collection of probability charges that are defined on all subsets of the action profiles. Any probability charge in this collection assigns an expected payoff to each player. Since the expected payoff is not necessarily unique, we develop new optimality notions. We distinguish two extreme attitudes of players. A player is viewed as pessimistic if he always evaluates the range of possible expected payoffs by the worst one, or equivalently, he always considers the worst extension in the induced collection of probability charges. In contrast, a player is viewed as optimistic if he always evaluates the
range of possible expected payoffs by the best one, or equivalently, he always considers the best extension in the induced collection of probability charges. This approach results in a definition of a pessimistic and an optimistic value for each player.

Our contribution: We provide an extensive analysis of the relation between these values, and connect them to the classical values. In addition, we also examine existence of optimal strategies with respect to these values. Our main finding for the values is that:

[1] The pessimistic value for player 1, the optimistic value for player 2 and the classical value for player 1 all coincide. Moreover, they can be calculated through taking a specific expected payoff for each strategy profile, which arises by first integrating the payoff function with respect to player 2’s strategy and then integrating with respect to player 1’s strategy. This specific expected payoff, which is not always the worst or the best one for the players, plays an important role in our analysis. It was also used in Kindler (1983), Schervish and Seidenfeld (1996) and Capraro and Scarsini (2013).

[2] Similarly, the optimistic value for player 1, the pessimistic value for player 2 and the classical value for player 2 all coincide. They are connected to the expected payoff which arises by first integrating the payoff function with respect to player 1’s strategy and then integrating with respect to player 2’s strategy.

We prove that the players always have optimal strategies with respect to the optimistic value.

Section 2 provides the mathematical tools that we use. In section 3 we introduce the games that we study, and show the different kinds of expected payoffs that players might consider. In section 4 we define values connected to the focal expected payoffs, and show our main findings. In section 5 we discuss optimal strategies. Section 6 is dedicated to truly zero-sum games. In section 7 we conclude. Most of the proofs are suspended to the appendices.

2 Preliminaries

In this section we provide a brief summary on probability charges. For further reading, we refer to Rao and Rao Rao and Rao (1983).

Let \( \Omega \) be a countably infinite set and let \( \mathcal{P}(\Omega) \) denote the set of all subsets of \( \Omega \). A probability charge on \( \Omega \) is a function \( \mu: \mathcal{P}(\Omega) \to [0, 1] \) such that \( \mu(\Omega) = 1 \) and for all disjoint sets \( E, F \in \mathcal{P}(\Omega) \) it holds that \( \mu(E \cup F) = \mu(E) + \mu(F) \). \(^1\) We denote the set of all probability charges on \( \Omega \) by \( \mathcal{C}(\Omega) \). We denote the set of countably additive probability measures on \( \Omega \) by \( \Delta(\Omega) \).

\(^1\)A probability charge is thus finitely additive, but not necessarily countably additive.
Thus, $\Delta(\Omega) \subseteq C(\Omega)$. A probability charge $\mu \in C(\Omega)$ with $\mu(\omega) = 0$ for all $\omega \in \Omega$ is called a pure charge. We denote the set of all pure probability charges on $\Omega$ by $Q(\Omega)$.  

A function $s: \Omega \rightarrow \mathbb{R}$ is called a simple function if there are $c_1, \ldots, c_m \in \mathbb{R}$ and a partition $\{F_1, \ldots, F_m\}$ of $\Omega$ such that $s = \sum_{i=1}^{m} c_i \mathbb{1}_{F_i}$, where $\mathbb{1}_{F_i}$ is the characteristic function of the set $F_i$. With respect to a probability charge $\mu$ on $\Omega$, the integral of $s$ is defined by $\int_{\Omega} s d\mu = \sum_{i=1}^{k} c_i \cdot \mu(F_i)$.

Let $\mu$ be a probability charge on $\Omega$, and let $f: \Omega \rightarrow \mathbb{R}$ be a bounded function. The integral $\int_{\Omega} f d\mu$ is defined as the supremum of all real numbers $\int_{\Omega} s d\mu$, where $s$ is a simple function and $s \leq f$. Note that, since $f$ is bounded, the integral is finite.

Let $\mu_1$ and $\mu_2$ be probability charges on $\Omega$. A probability charge $\mu$ on $\mathcal{P}(\Omega \times \Omega)$ is called an extension of $(\mu_1, \mu_2)$ if for all $E_1, E_2 \in \mathcal{P}(\Omega)$ it holds that $\mu(E_1 \times E_2) = \mu_1(E_1) \cdot \mu_2(E_2)$. The set of extensions of $(\mu_1, \mu_2)$ is denoted by $<\mu_1, \mu_2>$. All extensions of a given pair $(\mu_1, \mu_2)$ coincide on the algebra generated by sets $E_1 \times E_2$, where $E_1, E_2 \in \mathcal{P}(\Omega)$. They may differ on other subsets of $\mathcal{P}(\Omega \times \Omega)$. 

## 3 Games with charges

In this section, we define the model of games with charges.

Let $u: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}$ be a bounded payoff function. The infinite zero-sum game $g(u)$ is a game that is played as follows. The game is played by two players. Player 1 chooses an action $m \in \mathbb{N}$ and independently player 2 chooses an action $n \in \mathbb{N}$. Subsequently, player 1 receives a payoff $u(m, n)$ from player 2. Then the game ends.

A special class of (infinite) zero-sum games are binary games. Consider a set $W \subseteq \mathbb{N} \times \mathbb{N}$. The set $W$ is called the winning set of player 1. In the binary game defined by $W$ the payoff function $u: \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ is the characteristic function of $W$. That is, $u(m, n) = 1$ when $(m, n) \in W$, and $u(m, n) = 0$ otherwise. We use binary games to illustrate our concepts and results.

### 3.1 Basic definitions

Consider an infinite zero-sum game $g(u)$ with payoff function $u$. A strategy for a player is a probability charge on $\mathbb{N}$. A pair of strategies, one for each player, is called a strategy profile.
Consider a strategy profile \((\mu_1, \mu_2)\). In order to define the resulting payoffs, we need to calculate the expected payoff for the players when they use this profile. As said before, the extension of \((\mu_1, \mu_2)\) to the space \(\mathbb{N} \times \mathbb{N}\) of action profiles need not be unique. Consequently, the expected payoff for the players will generally depend on the extension that we consider. In this paper we closely study several specific ways to extend such a strategy profile \((\mu_1, \mu_2)\).

Formally, for a given extension \(\mu\) of profile \((\mu_1, \mu_2)\), the expected payoff \(U(\mu)\) is defined by

\[
U(\mu) = \int_{\mathbb{N} \times \mathbb{N}} u(m, n) d\mu.
\]

Thus, given a zero-sum game \(g(u)\), this construction gives rise to many ways to define a game with charges. Let each player \(i\) be endowed with a function \(\tau_i : \mathcal{C}(\mathbb{N}) \times \mathcal{C}(\mathbb{N}) \rightarrow \mathcal{C}(\mathbb{N} \times \mathbb{N})\). If for every profile \((\mu_1, \mu_2)\) we have that \(\tau_i(\mu_1, \mu_2)\) is an extension of \((\mu_1, \mu_2)\), such a function is called an extension function.

For given extension functions \(\tau_1, \tau_2\), the expected payoffs \(U^{\tau_1}(\mu_1, \mu_2)\) and \(-U^{\tau_2}(\mu_1, \mu_2)\) of profile \((\mu_1, \mu_2)\) are defined by

\[
U^{\tau_1}(\mu_1, \mu_2) = U(\tau_1(\mu_1, \mu_2)) \quad \text{and} \quad -U^{\tau_2}(\mu_1, \mu_2) = -U(\tau_2(\mu_1, \mu_2)).
\]

Thus, in this paper we record payments for player 2 instead of rewards. This way we constructed a new two-player game \(G(\tau_1, \tau_2) = (U^{\tau_1}, -U^{\tau_2})\). When \(\tau_1 = \tau_2\), the resulting game is again a zero-sum game. But in general this need not be the case.

**EXAMPLE** Let \(W\) be the winning set of a binary game. Consider a strategy profile \((\mu_1, \mu_2)\) in this game. Then, for any extension \(\tau_1\) of \((\mu_1, \mu_2)\), the payoff for player 1 is exactly

\[
U^{\tau_1}(\mu_1, \mu_2) = \int_{\mathbb{N} \times \mathbb{N}} u(m, n) d\tau_1(\mu_1, \mu_2) = \tau_1(\mu_1, \mu_2)(W).
\]

The payment that player 2 expects to make is \(U^{\tau_2}(\mu_1, \mu_2) = \tau_2(\mu_1, \mu_2)(W)\). Hence, his utility is \(-U^{\tau_2}(\mu_1, \mu_2)\).

We will hardly ever explicitly specify the extension function in question, but simply explain the specific way extensions are constructed, and then leave it implicit how the game with charges is precisely defined.

### 3.2 Optimistic and pessimistic players

In this subsection we introduce two extreme cases of behavior, that depend on the attitude of the players. Our approach is based on the following Lemma.
Lemma 3.1 Let $\mu_1, \mu_2 \in \mathcal{C}(N)$ be given. Then there exist extensions $\mu^*, \mu_* \in \langle \mu_1, \mu_2 \rangle$ with
\[ U(\mu^*) \geq U(\kappa) \quad \text{and} \quad U(\mu_*) \leq U(\kappa) \quad \text{for all } \kappa \in \langle \mu_1, \mu_2 \rangle. \]

Proof. We construct $\mu^*$. The construction of $\mu_*$ is similar. For every $k \in \mathbb{N}$, let $\nu_k$ denote an extension of $(\mu_1, \mu_2)$ such that $U(\nu_k) \geq \sup_{\kappa \in \langle \mu_1, \mu_2 \rangle} U(\kappa) - \frac{1}{k}$. Take any pure probability charge $\tau$ on $\mathbb{N}$. Define $\mu^*$ by
\[ \mu^*(E) = \int_{k \in \mathbb{N}} \nu_k(E)d\tau \quad \text{for every } E \in \mathcal{P}(\mathbb{N}). \]
Take a $K \in \mathbb{N}$ and $\kappa \in \langle \mu_1, \mu_2 \rangle$. We have
\[ U(\mu^*) = \int_{N \times N} u(m, n)d\mu^* = \int_{k \in \mathbb{N}} \int_{N \times N} u(m, n)d\nu_k d\tau = \int_{k \in \mathbb{N}} U(\nu_k) d\tau \geq U(\kappa) - \frac{1}{K}, \]
where the second equality comes from Lemma 8.2, and the inequality is due to the fact that $\tau(\{K, K + 1, \ldots\}) = 1$. Since $K$ was arbitrary, $U(\mu^*) \geq U(\kappa)$.

Player 1 is optimistic if, given profile $(\mu_1, \mu_2)$, he selects an extension $\mu^*$ with
\[ U(\mu^*) \geq U(\kappa) \quad \text{for all } \kappa \in \langle \mu_1, \mu_2 \rangle. \]
Due to the previous Lemma, such a selection is possible. The optimistic utility for player 1 is then defined by $U^o(\mu_1, \mu_2) = U(\mu^*)$.

In the same way, player 1 is pessimistic, if he selects an extension $\mu_*$ with
\[ u(\mu_*) \leq u(\kappa) \quad \text{for all } \kappa \in \langle \mu_1, \mu_2 \rangle. \]
The pessimistic utility of player 1 is then defined by $U^p(\mu_1, \mu_2) = U(\mu_*)$.

Thus, an optimistic player 1 selects an extension $\mu^*$ of $(\mu_1, \mu_2)$ that, amongst all possible extensions, offers him the best expected payoff. A pessimistic player 1 selects an extension $\mu_*$ that offers him the worst expected payoff. However, since player 2 has to pay the amounts specified by the payoff function, an optimistic player 2 selects an extension $\mu_*$ that offers player 1 the worst expected payoff, and a pessimistic player 2 selects an extension $\mu^*$ of $(\mu_1, \mu_2)$ that, amongst all possible extensions, offers player 1 the best expected payoff.

3.3 Double integrals

We study a second method to extend $(\mu_1, \mu_2)$, which is defined as follows. For every set $E \in \mathcal{P}(N \times N)$, define
\[ \mu(E) = \int_{m \in \mathbb{N}} \int_{n \in \mathbb{N}} \mathbb{I}_E(m, n)d\mu_2 d\mu_1, \]
where \( \mathbb{I}_E \) denotes the characteristic function of the set \( E \). We define \( U^{mn}(\mu_1,\mu_2) = U(\mu) \). We have the following basic observation.

**Lemma 3.2** It holds that
\[
U^{mn}(\mu_1,\mu_2) = \int_{m \in \mathbb{N}} \int_{n \in \mathbb{N}} u(m,n) d\mu_2 d\mu_1.
\]

**Proof.** Let \((\mu_1,\mu_2)\) be a strategy profile. Let \( \mu \) be the extension of \((\mu_1,\mu_2)\) as defined above. Take \( \Omega = \mathbb{N} \times \mathbb{N} \), and let \( S \) be the set of all simple functions on \( \Omega \). Define for any bounded function \( u \) on \( \Omega \)
\[
\phi(u) = \int u(m,n) d\mu.
\]
Define \( \psi(u) \) by
\[
\psi(u) = \int_{m \in \mathbb{N}} \int_{n \in \mathbb{N}} u(m,n) d\mu_2 d\mu_1.
\]
It is straightforward to check that Lemma 8.1 applies. This completes the proof. \( \blacksquare \)

We can also reverse the order of integration, which may lead to a different extension. \(^4\) The resulting expected utility function is denoted by \( U^{nm}(\mu_1,\mu_2) \).

### 3.4 All games are different

Thus, a single zero-sum game may generate different expected utility functions. In this section we show that different choices for extensions indeed result in different games with charges.

In general, of course games induced by different extension functions might coincide, even though the extension functions are different. An extreme case are constant-payoff games, where all possible extension functions generate the same game. However, we have the following observation.

**Proposition 3.3** Let \((\mu_1,\mu_2)\) be a profile. Let \( \kappa \) and \( \mu \) be two different extensions of this profile. Then there is a zero-sum game \( u \) with \( U(\kappa) \neq U(\mu) \).

**Proof.** Since \( \kappa \neq \mu \), there is a set \( W \) with \( \kappa(W) \neq \mu(W) \). Let \( u \) be the binary game with winning set \( W \). Then \( U(\kappa) = \kappa(W) \neq \mu(W) = U(\mu) \). \( \blacksquare \)

For the specific cases of \( U^o \), \( U^p \), \( U^{mn} \) and \( U^{nm} \), we present a zero-sum game \( g(u) \) for which all four extensions are different. More precisely, we present a binary game \( u \) and strategies \( \mu_1 \in C(\mathbb{N}), \mu_2 \in C(\mathbb{N}), \) and \( \kappa_1 \in C(\mathbb{N}) \) with \( U^p(\mu_1,\mu_2) = 0, U^o(\mu_1,\mu_2) = 1, U^{mn}(\mu_1,\mu_2) = U^{nm}(\mu_1,\mu_2) = \frac{1}{2}, U^{mn}(\kappa_1,\mu_2) = 1 \) and \( U^{nm}(\kappa_1,\mu_2) = 0 \). Thus, \( U^o, U^p, U^{mn} \) and \( U^{nm} \) define four different zero-sum games, even though they are derived from the same zero-sum game.

\(^4\)Thus, also Fubini’s theorem does not hold in general for charges.
EXAMPLE  Consider the following game. When player 1 plays \( m \) and player 2 plays \( n \), the payoff \( u(m, n) \) is defined by

\[
u(m, n) = \begin{cases} 
1 & \text{if } m + 1 \geq 2n \text{ and } m \text{ is odd} \\
1 & \text{if } m < 2n \text{ and } m \text{ is even} \\
0 & \text{otherwise.}
\end{cases}
\]

The game can be represented as follows.

\[
\begin{array}{cccccc}
u & 1 & 2 & 3 & 5 & 6 & \ldots \\
1 & 1 & 0 & 0 & 0 & 0 & \ldots \\
2 & 0 & 1 & 1 & 1 & 1 & \ldots \\
3 & 1 & 1 & 0 & 0 & 0 & \ldots \\
4 & 0 & 0 & 1 & 1 & 1 & \ldots \\
5 & 1 & 1 & 1 & 0 & 0 & \ldots \\
6 & 0 & 0 & 0 & 1 & 1 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

Write \( E = \{ n \in \mathbb{N} : n \text{ is even} \} \) and \( O = \{ n \in \mathbb{N} : n \text{ is odd} \} \). Take a pure probability charge \( \mu_1 \) for player 1 such that

\[
\mu_1(E) = \mu_1(O) = \frac{1}{2}.
\]

Let \( \mu_2 \) be an arbitrary pure probability charge for player 2. Then for the double integrals we have

\[
U^{mn}(\mu_1, \mu_2) = \int_{n \in \mathbb{N}} \int_{m \in \mathbb{O}} u(m, n) d\mu_1 d\mu_2 + \int_{n \in \mathbb{N}} \int_{m \in \mathbb{E}} u(m, n) d\mu_1 d\mu_2
\]

\[
= \int_{n \in \mathbb{N}} \frac{1}{2} d\mu_2 + \int_{n \in \mathbb{N}} 0 d\mu_2 = \frac{1}{2} + 0 = \frac{1}{2}.
\]

Similarly

\[
U^{nm}(\mu_1, \mu_2) = \int_{m \in \mathbb{O}} \int_{n \in \mathbb{N}} u(m, n) d\mu_2 d\mu_1 + \int_{m \in \mathbb{E}} \int_{n \in \mathbb{N}} u(m, n) d\mu_2 d\mu_1
\]

\[
= \int_{m \in \mathbb{O}} 0 d\mu_1 + \int_{m \in \mathbb{E}} 1 d\mu_1 = 0 + \frac{1}{2} = \frac{1}{2}.
\]

We verify that \( U^p(\mu_1, \mu_2) = 0 \). We say that a set \( R \subseteq \mathbb{N} \times \mathbb{N} \) is a rectangle if there are \( A, B \subseteq \mathbb{N} \) such that \( S = A \times B \). Let \( \mathcal{F} \) be the smallest field on \( \mathbb{N} \times \mathbb{N} \) that contains all rectangles. It is straightforward to check that \( \mathcal{F} \) equals the collection of all finite unions of rectangles. Write

\[
D = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m \text{ is even and } n \geq m \}.
\]

We show that \( \alpha(\mu_1, \mu_2)(D) = 0 \), where \( \alpha(\mu_1, \mu_2)(D) \) is defined in Appendix C. Let \( A \times B \) be a non-empty rectangle with \( A \times B \subseteq D \). We show that \( \mu_1(A) \cdot \mu_2(B) = 0 \). Suppose that \( A \) is

\[\text{By Theorem 10.3, we can in fact construct such a pure charge.}\]
infinite. Take any \( n \in B \). Since \( A \) is infinite, there is \( m \in A \) with \( m > n \). Then \( m > n \) and \((m, n) \in D\), which contradicts the definition of \( D \). So, \( A \) is finite. Hence, since \( \mu_1 \) is a pure charge, \( \mu_1(A) \cdot \mu_2(B) = 0 \). It follows that \( \alpha(\mu_1, \mu_2)(D) = 0 \).

Now define
\[
E = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m \text{ is odd and } n \leq m\}.
\]
Since also \( \mu_2 \) is a pure charge, we can show that \( \alpha(\mu_1, \mu_2)(E) = 0 \). However, \( W = D \cup E \), so that also \( \alpha(\mu_1, \mu_2)(W) = 0 \). The claim that \( U^\sigma(\mu_1, \mu_2) = 0 \) then follows from Proposition 10.4.

Now we turn to the definition of a value in probability charges. We first briefly discuss the definition for general extension functions. Let \( g(u) \) be a zero-sum game, and let \( \tau_1 : \mathcal{C}(\mathbb{N}) \times \mathcal{C}(\mathbb{N}) \to \mathcal{C}(\mathbb{N} \times \mathbb{N}) \) and \( \tau_2 : \mathcal{C}(\mathbb{N}) \times \mathcal{C}(\mathbb{N}) \to \mathcal{C}(\mathbb{N} \times \mathbb{N}) \) be two extension functions. The resulting game is \((U^{\tau_1}, -U^{\tau_2})\).

**Definition** The \( \tau_1 \)-value \( v_1^{\tau_1} \) for player 1, and the \( \tau_2 \)-value \( v_2^{\tau_2} \) for player 2 are defined by
\[
\left. \begin{array}{ll}
  v_1^{\tau_1} = & \sup_{\mu_1 \in \mathcal{C}(\mathbb{N})} \inf_{\mu_2 \in \mathcal{C}(\mathbb{N})} U^{\tau_1}(\mu_1, \mu_2) \quad \text{and} \\
  v_2^{\tau_2} = & \inf_{\mu_2 \in \mathcal{C}(\mathbb{N})} \sup_{\mu_1 \in \mathcal{C}(\mathbb{N})} U^{\tau_2}(\mu_1, \mu_2).
\end{array} \right. 
\]

Thus, given a strategy profile \((\mu_1, \mu_2)\), even though the players do not need to have an agreement on the outcome of the game, we still assume that player 1 expects to receive an amount \( U^{\tau_1}(\mu_1, \mu_2) \), while player 2 expects to pay the amount \( U^{\tau_2}(\mu_1, \mu_2) \).
In the specific cases that we mainly study, optimistic and pessimistic extensions, and double integrals, the definitions look as follows.

Definition The optimistic value $v_o^1$ for player 1, and the optimistic value $v_o^2$ for player 2 are defined by

$$v_o^1 = \sup_{\mu_1 \in C(N)} \inf_{\mu_2 \in C(N)} U_o(\mu_1, \mu_2) \quad \text{and} \quad v_o^2 = \inf_{\mu_2 \in C(N)} \sup_{\mu_1 \in C(N)} U_o(\mu_1, \mu_2).$$

The pessimistic value $v_p^1$ for player 1, and the pessimistic value $v_p^2$ for player 2 are defined by

$$v_p^1 = \sup_{\mu_1 \in C(N)} \inf_{\mu_2 \in C(N)} U_p(\mu_1, \mu_2) \quad \text{and} \quad v_p^2 = \inf_{\mu_2 \in C(N)} \sup_{\mu_1 \in C(N)} U_p(\mu_1, \mu_2).$$

The $mn$-value $v_{1mn}$ for player 1, and the $mn$-value $v_{2mn}$ for player 2 are defined by

$$v_{1mn}^1 = \sup_{\mu_1 \in C(N)} \inf_{\mu_2 \in C(N)} U_{1mn}(\mu_1, \mu_2) \quad \text{and} \quad v_{2mn}^2 = \inf_{\mu_2 \in C(N)} \sup_{\mu_1 \in C(N)} U_{1mn}(\mu_1, \mu_2).$$

In the same way we can define $v_{1mn}^1$ and $v_{2mn}^2$. The fact that the pessimistic value for player 2 involves $U^o$ and the optimistic value involves $U^p$ is due to the fact that player 2 still views the amounts specified by the utility function as payments, not as rewards.

4.2 The main result

We provide the precise relationship between all values defined above in the following Theorem. The proof can be found in Appendix B, Theorem 9.2.

Theorem 4.1 Let $u$ be any infinite zero-sum game. The values of the players satisfy

$$v_1 = v_o^1 = v_p^1 = v_o^2 = v_{1mn} = v_{2mn}^1 \quad \text{and} \quad v_2 = v_o^2 = v_p^2 = v_{1mn} = v_{2mn}^2.$$

We always have $v_1 \leq v_2$. Theorem 4.1 has the following corollary for the case when $v_1 = v_2$ holds, that is, when the classical value exists.

Corollary 4.2 If the classical value exists in a game with charges, then all values in Theorem 4.1 coincide.

We can also identify a specific class of games for which the above conclusion holds. Following Harris et al (2005) in section 2.1.3, we say that a function is simple when it is a finite linear combination of indicator functions of sets of the form $A \times B$, with $A, B \subseteq N$. A payoff function $u: N \times N \to \mathbb{R}$ is called integrable when it is the uniform limit of simple functions.

We have the following result, which extends the classic result of von Neumann ?.
Corollary 4.3 Let $g(u)$ be a zero-sum game, where $u$ is an integrable function. Then all values are equal. In particular, for any finite zero-sum game all values are equal.

Proof. For an integrable function, Fubini applies. This implies that $v_1^{mn} = v_1^{nm}$. The result is now a direct consequence of Theorem 4.1.

5 Optimality

When using countably additive strategies, even though the value of a game may exist, players may have only approximate optimal strategies. We investigate to which extent the use of charges guarantees existence of optimal strategies.

Let $g(u)$ be a zero-sum game, and let $\tau_1 : C(\mathbb{N}) \times C(\mathbb{N}) \rightarrow C(\mathbb{N} \times \mathbb{N})$ and $\tau_2 : C(\mathbb{N}) \times C(\mathbb{N}) \rightarrow C(\mathbb{N} \times \mathbb{N})$ the extension functions employed by the respective players. Thus the resulting game is $(U^{\tau_1}, -U^{\tau_2})$.

Definition A probability charge $\mu_1 \in C(\mathbb{N})$ is a $\tau_1$-optimal strategy for player 1 if for all strategies $\mu_2 \in C(\mathbb{N})$ it holds that

$$U^{\tau_1}(\mu_1, \mu_2) \geq v_{\tau_1}^{1}.$$ 

A probability charge $\mu_2 \in C(\mathbb{N})$ is a $\tau_2$-optimal strategy for player 2 if for all strategies $\mu_1 \in C(\mathbb{N})$ it holds that

$$U^{\tau_2}(\mu_1, \mu_2) \leq v_{\tau_2}^{2}.$$ 

When player 1 uses $U^o$ to evaluate outcomes, we say that an optimal strategy for player 1 is optimistic optimal. When player 2 uses $-U^p$ to evaluate outcomes, we say that an optimal strategy is optimistic optimal. When player 1 (player 2) uses $U^{mn}$ to evaluate outcomes, we say that an optimal strategy for player 1 (player 2) is $mn$-optimal.

We have the following result, which follows from Lemma 9.1 and Theorem 4.1.

Theorem 5.1 Both players have optimistic optimal strategies.

Example Let $g(u)$ be an infinite zero-sum game. Suppose that $g(u)$ has a classical value $v$. Despite the existence of the classical value, players may only have approximate optimal strategies in countable additive strategies. We show how to construct optimistic optimal strategies.

For each $k \in \mathbb{N}$, let $p_{1k} \in \Delta(\mathbb{N})$ be such that, for every $p_2 \in \Delta(\mathbb{N})$,

$$U((p_{1k}, p_2)) \geq v - \frac{1}{k}.$$
Let $\mu_1$ be a pure charge. Define the probability charge $\kappa_1$ by

$$
\kappa_1(E) = \int_{k \in \mathbb{N}} p_{1k}(E) d\mu_1 \quad \text{for all } E \subseteq \mathbb{N}.
$$

Let $\mu_2$ be any probability charge of player 2. Take a $K \in \mathbb{N}$. Then for every $n \in \mathbb{N},$

$$
\int_{m \in \mathbb{N}} u(m, n) d\kappa_1 = \int_{k \in \mathbb{N}} \int_{m \in \mathbb{N}} u(m, n) d\mu_1 (E) \mu_2(E) \geq \int_{k \geq K} (v - \frac{1}{k}) d\mu_1 \geq v - \frac{1}{K}.
$$

The first equality comes from Lemma 8.2, and the second equality is due to the fact that $\mu_1(\{K, K + 1, \ldots\}) = 1$. Since $K$ was chosen arbitrarily, we find that

$$
\int_{m \in \mathbb{N}} u(m, n) d\kappa_1 \geq v.
$$

Then for every probability charge $\mu_2$ of player 2,

$$
U^o(\kappa_1, \mu_2) \geq \int_{n \in \mathbb{N}} \int_{m \in \mathbb{N}} u(m, n) d\kappa_1 d\mu_2 \geq \int_{n \in \mathbb{N}} v d\mu_2 = v.
$$

Hence, $\kappa_1$ is optimistic optimal for player 1.

**Example** We show that an optimistic optimal strategy may be neither countably additive nor a pure probability charge, and may have to be a real mixture of the two.

Consider the game

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First we show that $v_2 = \frac{1}{2}$. Suppose player 1 plays the first row with probability $\frac{1}{2}$ and the second row with probability $\frac{1}{2}$. Then the expected payoff is exactly $\frac{1}{2}$ regardless of what player 2 does. So, $v_1 \geq \frac{1}{2}$.

Conversely, let $p \in \Delta(\mathbb{N})$ be any countably additive probability distribution for player 1. Let $\varepsilon > 0$. Take a number $m$ such that $\sum_{k=m}^{\infty} p_k < \varepsilon$. Take $n$ with $2n > m$, and let player 2 play the first column with probability $\frac{1}{2}$ and column $n$ with probability $\frac{1}{2}$. Then the expected payoff
Zero-sum games with charges

for player 1 is at most $\frac{1}{2} + \varepsilon$. These observations imply that $v_1 = \frac{1}{2}$. Hence, by Theorem 9.2, also $v_2 = \frac{1}{2}$.

Now notice that for any pure probability charge $\mu_1$ with $\mu_1(O) = 1$, and for any countably additive strategy $p_2$, we have $u(< \mu_1, p_2>) = 1$. Hence, countably additive strategies are not optimistic optimal strategy for player 2.

Next, if player 1 uses the Dirac measure $\delta_2$ on action 2 (so, for $A \subseteq \mathbb{N}$ we have $\delta_2(A) = 1$ if $2 \in A$, and $\delta_2(A) = 0$ otherwise) and player 2 uses a pure probability charge $\mu_2$, then $U(< \delta_2, \mu_2>) = 1$. Hence, an optimistic optimal strategy for player 2 cannot be a pure probability charge either.

Consequently, any optimistic optimal strategy for player 2 must be a real mixture of a countably additive strategy and a pure probability charge. For completeness, we provide an optimistic optimal strategy for player 2. Let $\mu_2 = \frac{1}{2} \cdot \delta_2 + \frac{1}{2} \cdot \kappa_2$ where $\delta_2$ is the Dirac measure on action 1 and $\kappa$ is any pure probability charge on $\mathbb{N}$. Take a strategy $\mu_1$ of player 1. We write $d_1 = \sum_{m \in \mathbb{N}} \mu_1(m)$, and $\mu_1 = d_1 p_1 + (1 - d_1) \kappa_1$. Then

$$U^p(\mu_1, \mu_2) = \frac{1}{2} \cdot d_1 \cdot U(< p_1, \delta_2 >) + \frac{1}{2} \cdot (1 - d_1) \cdot U(< \kappa_1, \delta_2 >) + \frac{1}{2} \cdot d_1 \cdot U(< p_1, \kappa_2 >) + \frac{1}{2} \cdot (1 - d_1) \cdot U^p(\k_1, \kappa_2)$$

$$= \frac{1}{2} \cdot d_1 \cdot p_1(O) + \frac{1}{2} \cdot (1 - d_1) \cdot \kappa_1(O) + \frac{1}{2} \cdot d_1 \cdot p_1(E) + \frac{1}{2} \cdot (1 - d_1) \cdot 0$$

$$= \frac{1}{2} \cdot d_1 + \frac{1}{2} \cdot (1 - d_1) \cdot \kappa_1(O)$$

$$\leq \frac{1}{2} \cdot d_1 + \frac{1}{2} \cdot (1 - d_1)$$

$$= \frac{1}{2}.$$

Here, the inequality follows from $\kappa_1(O) \leq 1$. Since $v_2^o = \frac{1}{2}$, the strategy $\mu_2$ is optimistic optimal for player 2.

\begin{proposition}
Let $\kappa_1$ be a pessimistic optimal strategy for player 1. Then it is an $\infty$-optimal strategy for player 1.
\end{proposition}

Proof. This is an immediate consequence of Theorem 4.1.

\begin{example}
We do not know whether pessimistic optimal strategies always exist. We do know that they may exist, even sometimes in pure charges. Consider the following binary game. Let the payoff function be

$$u(m, n) = \begin{cases} 
0 & \text{if } m = n \\
1 & \text{otherwise}.
\end{cases}$$
\end{example}
The payoffs are displayed in the following figure.

\[
\begin{array}{cccccc}
  & 1 & 2 & 3 & \ldots \\
1 & 0 & 1 & 1 & \ldots \\
2 & 1 & 0 & 1 & \ldots \\
3 & 1 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

First we argue that \( v_1^p = 1 \). Take a \( k \in \mathbb{N} \) and consider the countably additive strategy \( p_k \) for player 1 such that

\[
p_k(m) = \begin{cases} 
\frac{1}{k} & \text{if } m \leq k \\
0 & \text{otherwise.} 
\end{cases}
\]

It is clear that \( u(<p_k, \mu_2>) \geq \frac{k-1}{k} \) for every \( \mu_2 \in \mathcal{C}(\mathbb{N}) \). Therefore \( v_1^p = 1 \) indeed.

For \( k \in \mathbb{N}, k \geq 1, \) and \( \ell \in \mathbb{N} \) with \( 0 \leq \ell \leq 2^k - 1 \), define

\[
E_{k}^{\ell} = \{ 2^k \cdot n + \ell \mid n \in \mathbb{N} \}.
\]

There exists a pure charge \( \kappa_1 \) with the property that, for all \( k \) and \( \ell \), \( \kappa(E_{k}^{\ell}) = (\frac{1}{2})^{k} \).

Suppose player 1 plays charge \( \kappa_1 \). Player 2, being the minimizer of probability of the winning set, tries to maximize the probability of the diagonal. Thus, let us look at the maximum probability of the diagonal. This is in its turn the same as calculating the minimum probability of the winning set for player 1. That is, we compute the pessimistic payoff for player 1. Let \( \mu_2 \) be any charge for player 2.

One possible cover of the diagonal is

\[
(E \times E) \cup (O \times O).
\]

This cover has a probability of \( \frac{1}{2} \cdot \mu_2(E) + \frac{1}{2} \cdot \mu_2(O) = \frac{1}{2} \). In general, for fixed \( k \),

\[
\bigcup_{\ell=1}^{2^k-1} (E_{k}^{\ell} \times E_{k}^{\ell})
\]

covers the diagonal. The probability of this cover is

\[
\sum_{\ell=0}^{2^k-1} \left( \frac{1}{2} \right)^k \cdot \mu_2(E_{k}^{\ell}) = \left( \frac{1}{2} \right)^k.
\]

\text{\footnote{Since for each } k \text{ the collection of sets } E_{k}^{\ell} \text{ is a partition, it is straightforward to define } \kappa_1 \text{ on the smallest field containing all singletons and all sets } E_{k}^{\ell}. \text{ The existence of such a charge } \kappa_1 \text{ defined on all subsets of } \mathbb{N} \text{ then follows from Theorem 10.3.}}
Thus, the infimum of the probabilities of these covers is 0. To sum up, even in the pessimistic case for player 1, the diagonal has a probability of 0 for any extension of the strategy profile, so the expected payoff must be 1. Thus, \( \kappa_1 \) is pessimistic optimal for player 1.

As a final remark, since \( v_1^p = 1 \), we obviously have \( v_1^o = 1 \).

\[ \triangleleft \]

6 Zero-sum games with charges

When both players evaluate extensions of profiles in the same way, the resulting game with charges is also zero-sum.

**Corollary 6.1** Suppose that player 1 is optimistic, and player 2 is pessimistic. Then every infinite zero-sum game has a value.

**Corollary 6.2** Suppose that player 1 evaluates via \( U^{mn} \) and player 2 via \( -U^{mn} \). Then every infinite zero-sum game has a value.

Both these results are a direct consequence of Theorem 4.1.

It is tempting to think that, whenever the players agree on the extension, all zero-sum games have a value. However, the next example shows this is not true.

**Example** Even when the players agree on the extension function, and the resulting game with charges is zero-sum, the value of the game need not exist.

**Example** Also in infinite zero-sum games that do not have a classical value, players have optimal strategies. Consider the following binary version of Wald’s game (Wald (1945)). The payoff is \( u(m,n) = 1 \) if \( m \geq n \) and \( u(m,n) = 0 \) if \( m < n \). The payoffs are given in the following matrix, where player 1 is the row player and player 2 is the column player.

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We define \( \tau \) as follows. By AC, there exists a total ordering \( \succeq \) of the set \( \mathcal{C}(\mathbb{N}) \) of probability charges on \( \mathbb{N} \) such that, for every \( \mu \in \mathcal{C}(\mathbb{N}) \), there is \( \nu \in Q(\mathbb{N}) \) with \( \nu \succ \mu \). Next, for any pair of probability charges \( (\mu_1, \mu_2) \) we define an extension \( \tau(\mu_1, \mu_2) \) by letting for all \( K \subseteq \mathbb{N} \)

\[
\tau(\mu_1, \mu_2)(K) = \begin{cases} 
\int_{n \in \mathbb{N}} \int_{m \in \mathbb{N}} I_K(m,n) d\mu_1 d\mu_2 & \text{if } \mu_1 \succeq \mu_2 \\
\int_{m \in \mathbb{N}} \int_{n \in \mathbb{N}} I_K(m,n) d\mu_2 d\mu_1 & \text{if } \mu_2 \succ \mu_1.
\end{cases}
\]
Note that $\tau(\mu_1, \mu_2)$ is indeed an extension of $(\mu_1, \mu_2)$. When we apply this extension function to Wald’s game, by Lemma 3.2 the resulting zero-sum on charges is

$$U(\mu_1, \mu_2) = \begin{cases} U_{\text{mm}}(\mu_1, \mu_2) & \text{if } \mu_1 \succeq \mu_2 \\ U_{\text{mn}}(\mu_1, \mu_2) & \text{if } \mu_2 \succ \mu_1. \end{cases}$$

Intuitively, if $\mu_1$ is equal to $\mu_2$ or $\mu_1$ is enumerated later than $\mu_2$ by $\succeq$, then the inner integral is taken with respect the strategy of player 1. As we know, this favors player 1 in Wald’s game. Otherwise, the inner integral is taken with respect to the strategy of player 2.

Take any $\mu_1 \in \mathcal{C}(\mathbb{N})$. Then, by our assumption above, there is a pure charge $\mu_2$ such that $\mu_2 \succ \mu_1$. Then $U^\tau(\mu_1, \mu_2) = 0$. So, $\sup_{\mu_1 \in \mathcal{C}(\mathbb{N})} \inf_{\mu_2 \in \mathcal{C}(\mathbb{N})} U^\tau(\mu_1, \mu_2) = 0$. Similarly, one can show that $\inf_{\mu_2 \in \mathcal{C}(\mathbb{N})} \sup_{\mu_1 \in \mathcal{C}(\mathbb{N})} U^\tau(\mu_1, \mu_2) = 0$.

Thus there exists an extension $\tau(\mu_1, \mu_2)$ for each pair of probability charges $(\mu_1, \mu_2)$ with

$$\sup_{\mu_1 \in \mathcal{C}(\mathbb{N})} \inf_{\mu_2 \in \mathcal{C}(\mathbb{N})} U^\tau(\mu_1, \mu_2) < \inf_{\mu_2 \in \mathcal{C}(\mathbb{N})} \sup_{\mu_1 \in \mathcal{C}(\mathbb{N})} U^\tau(\mu_1, \mu_2).$$

So, the supremum and the infimum are not interchangeable for $\tau$, and the value $v^\tau$ does not exist.

EXAMPLE On the other hand, when both players are optimistic, by Theorem 5.1 both players have optimal strategies. However, the resulting game need not be a zero-sum game. Consider for example Wald’s game defined earlier. Clearly, for every $p_1 \in \Delta(\mathbb{N})$ it holds that $\inf_{n \in \mathbb{N}} u(<p_1, n>) = 0$. Hence, $v_1 = 0$. We obtain similarly that $v_2 = 1$.

7 Concluding remarks

Analysing mixed strategies in zero-sum games has been done since the beginning of game theory. Still, it is not clear how to define them when moving to countable action spaces. When mixed strategies are countably additive, then the expected payoff is uniquely defined. However, it is also reasonable to only require finite additivity from the strategies, that is, assuming charges. In that case the expected payoff is only unique when considering a restricted class of games (e.g. Marinacci (1997) and Harris et al (2005)). To investigate all consequences of using charges, we studied all bounded payoff functions on countable action spaces.

Even though the games that we looked at are zero-sum in pure strategies, depending on which expected payoffs the players take, the game with charges might not be zero-sum anymore. There are many ways to calculate an expected payoff, but we dedicate our attention to the more focal ones. The expected payoffs defined by different orders of integration have been looked at before.
(e.g. Schervish and Seidenfeld (1996) and Capraro and Scarsini (2013)), but to our knowledge the pessimistic and optimistic point of view have not been. We show through an example that these expected payoffs might differ from the ones generated by the double integrals. We define values corresponding to these expected payoffs. Although these expected payoffs might differ, the values connected to them are very closely related, as the main finding of our paper shows. We also find that optimistic optimal strategies always exist, even though this is not the case for countably additive optimal strategies. We are working on extending our analysis to uncountable action spaces, most of our results still seem to hold in the more general case.

8 Appendix A. Tools

Lemma 8.1 Let $\Omega$ be a non-empty set. Let $F$ be the set of all bounded functions $f: \Omega \to \mathbb{R}$. Let $S$ be the set of all simple functions in $F$. Let $\phi: F \to \mathbb{R}$ be a mapping such that for every $f \in F$

$$\phi(f) = \sup \{ \phi(s): s \in S \text{ and } s \leq f \}.$$ 

Let $\psi: F \to \mathbb{R}$ be linear and monotone. Suppose that $\phi(s) = \psi(s)$ for all $s \in S$. Then $\phi = \psi$.

Proof. For every $s \in S$ and $f \in F$ with $s \leq f$ we have $\phi(s) = \psi(s) \leq \psi(f)$. Hence, for every $f \in F$

$$\phi(f) = \sup \{ \phi(s): s \in S \text{ and } s \leq f \} \leq \psi(f).$$

Take $f \in F$. We prove $\phi(f) \geq \psi(f)$. Take $\varepsilon > 0$. It is easy to see that there exists a simple function $s$ such that $s \leq f \leq s + \varepsilon$. Thus

$$\psi(f) \leq \psi(s + \varepsilon) = \psi(s) + \psi(\varepsilon) = \phi(s) + \varepsilon \psi(1) \leq \phi(f) + \varepsilon \psi(1).$$

Since $\varepsilon > 0$ is arbitrary, $\psi(f) \leq \phi(f)$. \hfill \blacksquare

Lemma 8.2 Let $\Omega$ be a non-empty set. Let $I$ be a set of indices. For each $k \in I$, let $\mu_k$ be a probability charge on $\Omega$. Let $\tau$ be a probability charge on $I$. Consider the probability charge $\sigma$ on $\Omega$ defined by

$$\sigma(E) = \int_{k \in I} \mu_k(E) d\tau \quad \text{for all } E \subseteq \Omega.$$ 

Then, for any bounded function $f: \Omega \to \mathbb{R}$,

$$\int_{\omega \in \Omega} f(\omega) d\sigma = \int_{k \in I} \int_{\omega \in \Omega} f(\omega) d\mu_k d\tau.$$ 

---

See p. 272 in the appendix by W. D. Sudderth in Dubins and Savage (2014). Indeed, consider the inverse images $f^{-1}[ze, (z + 1)e)$, where $z$ is an integer. Since $f$ is bounded, only finitely many of them are non-empty. If $f^{-1}[ze, (z + 1)e)$ is non-empty, then let $s$ take value $ze$ on this set. It follows that $s \leq f \leq s + \varepsilon$. 

---
Proof. For any simple function \( s: \Omega \to \mathbb{R} \), where \( s = \sum_{i=1}^{m} c_i I_{F_i} \), we have
\[
\int_{\omega \in \Omega} s(\omega) d\sigma = \sum_{i=1}^{m} c_i \cdot \sigma(F_i) = \sum_{i=1}^{m} c_i \int_{k \in I} \mu_k(F_i) d\tau
\]
\[
= \int_{k \in I} \sum_{i=1}^{m} c_i \mu_k(F_i) d\tau
\]
\[
= \int_{k \in I} \int_{\omega \in \Omega} s(\omega) \mu_k d\tau,
\]
where the first equality is based on the definition of the integral of a simple function and the second equality on the definition of \( \sigma \). In view of Lemma 8.1, taking \( \phi(f) = \int_{\omega \in \Omega} f(\omega) d\sigma \) and \( \psi(f) = \int_{k \in I} \int_{\omega \in \Omega} f(\omega) \mu_k d\tau \), the proof is complete.

9 Appendix B. Proof of Theorem 4.1

Lemma 9.1 Player 2 has a strategy \( \sigma_2 \in \mathcal{C}(\mathbb{N}) \) such that for every \( \mu_1 \in \mathcal{C}(\mathbb{N}) \) we have
\[
U_{mn}(\mu_1, \sigma_2) \leq v_1.
\]
Player 1 has a similar strategy \( \sigma_1 \).

Proof. Take \( m \in \mathbb{N} \). Consider the restricted game \( G_m \), in which player 1 can only choose from the action set \( \{1, 2, \ldots, m\} \), but player 2 can still choose an action from \( \mathbb{N} \). In \( G_m \) player 1 has only finitely many \( m \) actions, player 2 has countably many actions and it is a zero-sum game. Then by Theorem 3.1 in Wald (1945), \( G_m \) has a classical value \( w_m \). It follows that
\[
w_m = \sup_{p_1 \in \Delta(\{1, \ldots, m\})} \inf_{p_2 \in \Delta(\mathbb{N})} U(<p_1, p_2>) \leq \sup_{p_1 \in \Delta(\mathbb{N})} \inf_{p_2 \in \Delta(\mathbb{N})} U(<p_1, p_2>) = v_1.
\]
We construct probability charge \( \sigma_2 \in \mathcal{C}(\mathbb{N}) \) as follows. Take \( m \in \mathbb{N} \). Because \( w_m \) is the value of \( G_m \), player 2 has a strategy \( p_{2m} \in \Delta(\mathbb{N}) \) such that for every \( p_1 \in \Delta(\{1, \cdots, m\}) \)
\[
U(<p_1, p_{2m}>) \leq w_m.
\]
Let \( \tau \in \mathcal{C}(\mathbb{N}) \) be any pure probability charge. Define a probability charge \( \sigma_2 \in \mathbb{C}(\mathbb{N}) \) by
\[
\sigma_2(K) = \int_{k \in \mathbb{N}} p_{2k}(K) d\tau \quad \text{for all } K \subseteq \mathbb{N}.
\]
We show that
\[
\int_{m \in \mathbb{N}} \int_{n \in \mathbb{N}} u(m, n) d\sigma_2 d\mu_1 \leq v_1.
\]
Fix \( m \in \mathbb{N} \). Then
\[
\int_{n \in \mathbb{N}} u(m, n) d\sigma_2 = \int_{k \in \mathbb{N}} \int_{n \in \mathbb{N}} u(m, n) d\mu_2 d\tau = \int_{k \in \mathbb{N}} U(<\delta_m, p_{2k}>) d\tau
\]
Zero-sum games with charges

\[ = \int_{k \geq m} U(<\delta_m, p_{2k}>) d\tau \]
\[ \leq \int_{k \geq m} w_m d\tau \leq v_1. \]

Here the first equality follows from Lemma 8.2 and the definition of \( \sigma_2 \). The third equality follows from \( \tau(\{1, \ldots, m-1\}) = 0 \). To see the first inequality, if \( k \geq m \), then by the definition of \( p_{2k} \), it holds that \( U(<\delta_m, p_{2k}>) \leq w_m \). The last inequality follows from the fact that \( w_m \leq v_1 \).

Hence, since an integral is monotonic, for any \( \mu_1 \in \mathcal{C}(\mathbb{N}) \)
\[ \int_{m \in \mathbb{N}} \int_{n \in \mathbb{N}} u(m, n) d\sigma_2 d\mu_1 \leq \int_{m \in \mathbb{N}} v_1 d\mu_1 = v_1. \]

This concludes the proof.

Theorem 9.2 Let \( u \) be any infinite zero-sum game. The values of the players satisfy
\[ v_1 = v_1^p = v_2^o = v_1^{mn} = v_2^{mn} \quad \text{and} \quad v_2 = v_2^p = v_1^o = v_2^{nm} = v_1^{nm}. \]

Moreover, if \( \sigma_1 \) and \( \sigma_2 \) are strategies as in Lemma 9.1, then \( \sigma_1 \) is optimistic optimal for player 1, and \( \sigma_2 \) is optimistic optimal for player 2.

Proof. We show that \( v_1 \leq v_1^p \leq v_1^{mn} \leq v_2^{mn} \leq v_1, \quad v_1^p \leq v_2^o \leq v_1 \) and that \( \sigma_2 \) is optimistic optimal for player 2.

A. We show that \( v_1 \leq v_1^p \). Take a \( p_1 \in \Delta(\mathbb{N}) \) and a \( \mu_2 \in \mathcal{C}(\mathbb{N}) \). Let \( \delta_n \) denote the Dirac measure on \( n \in \mathbb{N} \). Then
\[ U(<p_1, \mu_2>) = \int_{n \in \mathbb{N}} \int_{m \in \mathbb{N}} u(m, n) dp_1 d\mu_2 = \int_{n \in \mathbb{N}} U(<p_1, \delta_n>) d\mu_2 \geq \inf_{n \in \mathbb{N}} U(<p_1, \delta_n>). \]

We used the fact that the extension for \( (p_1, \mu_2) \) is unique, since one of the strategies in the profile is countably additive. So,
\[ \inf_{\mu_2 \in \mathcal{C}(\mathbb{N})} U(<p_1, \mu_2>) \geq \inf_{n \in \mathbb{N}} U(<p_1, \delta_n>) \geq \inf_{p_2 \in \Delta(\mathbb{N})} U(<p_1, p_2>). \]

It follows that
\[ \sup_{p_1 \in \Delta(\mathbb{N})} \inf_{\mu_2 \in \mathcal{C}(\mathbb{N})} U(<p_1, \mu_2>) \geq \sup_{p_1 \in \Delta(\mathbb{N})} \inf_{p_2 \in \Delta(\mathbb{N})} U(<p_1, p_2>) = v_1. \]

So,
\[ v_1^p = \sup_{\mu_1 \in \mathcal{C}(\mathbb{N})} \inf_{\mu_2 \in \mathcal{C}(\mathbb{N})} U^p(\mu_1, \mu_2) \]
\[ \geq \sup_{p_1 \in \Delta(N)} \inf_{\mu_2 \in C(N)} U(p_1, \mu_2) \]
\[ = \sup_{p_1 \in \Delta(N)} \inf_{\mu_2 \in C(N)} \mu_2 \text{ such that } U(<p_1, \mu_2>) \]
\[ \geq v_1. \]

**B.** We show that \( v_1^p \leq v_1^mn \). Let \( \mu_1, \mu_2 \in C(N) \) be two probability charges. We know that

the probability charge \( \kappa \) defined by

\[ \kappa(E) = \int_{m \in N} \int_{n \in N} 1_E(m, n) d\mu_2 d\mu_1 \quad \text{for all } E \subseteq N \]

is an extension of the strategy profile \((\mu_1, \mu_2)\), and that by definition \( U^{mn}(\mu_1, \mu_2) = U(\kappa) \).

Hence, by definition of \( U(p_1, \mu_2) \), we have \( U(p_1, \mu_2) \leq U(\kappa) = U^{mn}(\mu_1, \mu_2) \). So,

\[ v_1^p = \sup_{\mu_1 \in C(N)} \inf_{\mu_2 \in C(N)} U(p_1, \mu_2) \leq \sup_{\mu_1 \in C(N)} \inf_{\mu_2 \in C(N)} U^{mn}(\mu_1, \mu_2) \leq v_1^mn. \]

**C.** We argue that \( v_1^mn \leq v_2^mn \leq v_1 \). It is clear from the definitions that \( v_1^mn \leq v_2^mn \).

Furthermore, it follows from Lemma 9.1 that \( v_2^mn \leq v_1 \). Hence, \( v_1 = v_1^p = v_1^mn = v_2^mn = v_1 \).

**D.** We show that \( v_2^p \leq v_1^p \leq v_1 \). It is clear from the definitions that \( v_1^p \leq v_2^p \). Let \( \sigma_2 \in C(N) \) be a strategy for player 2 as in Lemma 9.1. Then, for every \( \mu_1 \in C(na) \) we have

\[ U(p_1, \sigma_2) \leq U^{mn}(\mu_1, \sigma_2) \leq v_1. \]

Therefore,

\[ v_2^o = \inf_{\mu_2 \in C(N)} \sup_{\mu_1 \in C(N)} U(p_1, \mu_2) \leq \sup_{\mu_1 \in C(N)} U(p_1, \sigma_2) \leq v_1. \]

**E.** Hence, \( v_1 = v_1^p = v_1^mn = v_2^mn = v_2^o = v_1 \). It now follows from **D** that \( \sigma_2 \) is optimistic optimal for player 2.

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**10 Appendix C. Existence of charges**

A collection \( \mathcal{F} \) of subsets of a non-empty set \( X \) is called a field if:

1. \( \phi \in \mathcal{F} \)
2. if \( A \in \mathcal{F} \) then \( X \setminus A \in \mathcal{F} \) and
3. if \( A_1, \ldots, A_n \) are elements of \( \mathcal{F} \), then \( \bigcup_{i=1}^n A_i \in \mathcal{F} \).
Lemma 10.1 Let $\mathcal{F}$ be a field on $X$, and let $E \subseteq X$ be given. There exists a smallest field $\mathcal{F}(E)$ that contains both $E$ and all elements of $\mathcal{F}$. Moreover,

$$S \in \mathcal{F}(E) \iff \text{there are } A, B \in \mathcal{F} \text{ with } S = (A \cap E) \cup (B \setminus E).$$

Proof. Suppose that $\mathcal{F} \neq \mathcal{P}(X)$ and $E \notin \mathcal{F}$. It is clear that a smallest field $\mathcal{F}(E)$ exists that contains both $E$ and all elements of $\mathcal{F}$. It is also clear that all elements of the form $(A \cap E) \cup (B \setminus E)$ with $A, B \in \mathcal{F}$ are contained in $\mathcal{F}(E)$.

Define the set $\mathcal{G}$ by

$$S \in \mathcal{G} \iff \text{there are } A, B \in \mathcal{F} \text{ with } S = (A \cap E) \cup (B \setminus E).$$

We show that $\mathcal{G}$ is a field that contains both $E$ and all elements of $\mathcal{F}$. Then $\mathcal{F}(E) \subseteq \mathcal{G}$, which concludes the proof.

We show that $\mathcal{F} \subseteq \mathcal{G}$ and $E \in \mathcal{G}$. Take $C \in \mathcal{F}$. Then, by choosing $A = B = C$, we see that $C \in \mathcal{G}$. So, $\mathcal{F} \subseteq \mathcal{G}$. Further, by choosing $A = X$, $B = \phi$, we see that $E \in \mathcal{G}$.

We show that $\mathcal{G}$ is a field. Clearly $\phi \in \mathcal{G}$. Take $S \in \mathcal{G}$. Take $A, B \in \mathcal{F}$ with $S = (A \cap E) \cup (B \setminus E)$. Then $X \setminus S = (A^c \cap E) \cup (B^c \setminus E)$. Since $\mathcal{F}$ is a field, we know that $A^c, B^c \in \mathcal{F}$, and we can conclude that $X \setminus S \in \mathcal{G}$.

Take $S_1, \ldots, S_n \in \mathcal{G}$. Take $A_k, B_k \in \mathcal{F}$ with $S_k = (A_k \cap E) \cup (B_k \setminus E)$. Then

$$\bigcup_k S_k = \left[ \left( \bigcup_k A_k \right) \cap E \right] \cup \left[ \left( \bigcup_k B_k \right) \setminus E \right].$$

Since $\mathcal{F}$ is a field, $\bigcup_k A_k, \bigcup_k B_k \in \mathcal{F}$. Hence, $\bigcup_k S_k \in \mathcal{G}$. 

A function $\mu : \mathcal{F} \to [0, 1]$ is called a charge on $\mathcal{F}$ if $\mu(X) = 1$, and $\mu(A \cup B) = \mu(A) + \mu(B)$ for all disjoint sets $A, B \in \mathcal{F}$. In particular $\mu(\phi) = 0$ for any charge $\mu$.

Take $E \subseteq X$. A charge $\nu$ on $\mathcal{F}(E)$ is called an extension to $\mathcal{F}(E)$ of a charge $\mu$ on $\mathcal{F}$ if $\nu(A) = \mu(A)$ for all $A \in \mathcal{F}$. Take $Y \subset X$.

$$\alpha(\mu)(Y) = \sup \{ \mu(A) \mid A \in \mathcal{F}, A \subseteq Y \} \quad \text{and} \quad \beta(\mu)(Y) = \inf \{ \mu(B) \mid B \in \mathcal{F}, Y \subseteq B \}.$$ 

Clearly $\alpha(\mu)(Y) \leq \beta(\mu)(Y)$. We have the following observation. A similar remark can already be found in Loś and Marczewski (1949).

Lemma 10.2 Let $\mathcal{F}$ be a field, and let $\mu$ be a charge on $\mathcal{F}$. Let $E \in \mathcal{P}(X)$ and $r \in \mathbb{R}$ with $\alpha(\mu)(E) \leq r \leq \beta(\mu)(E)$. Then there exists an extension $\nu$ of $\mu$ to $\mathcal{F}(E)$ such that $\nu(E) = r$. 

Proof. We show there exists a charge \( \nu \) on \( \mathcal{F}(E) \) with \( \nu(A) = \mu(A) \) for all \( A \in \mathcal{F} \) and \( \nu(E) = r \).

**A.** We first define an extension \( \nu \) with \( \nu(E) = \alpha(\mu)(E) \). Define \( \nu : \mathcal{F}(E) \to \mathbb{R} \) as follows. Take \( S \in \mathcal{F}(E) \). Take \( A, B \in \mathcal{F} \) with \( S = (A \cap E) \cup (B \setminus E) \). Define

\[
\nu(S) = \alpha(\mu)(A \cap E) + \mu(B) - \alpha(\mu)(B \cap E).
\]

**A1** We show that this is a valid definition. Take \( A', B' \in \mathcal{F} \) with \( S = (A' \cap E) \cup (B' \setminus E) \). Then \( A \cap E = A' \cap E \). So, clearly

\[
\alpha(\mu)(A \cap E) = \alpha(\mu)(A' \cap E).
\]

Also \( B \setminus E = B' \setminus E \). Write \( H = B \setminus B' \) and \( J = B' \setminus B \). Then \( H, J \in \mathcal{F} \). So, \( \mu(B) - \mu(H) = \mu(B \cap B') = \mu(B') - \mu(J) \). Hence, it remains to show that

\[
\alpha(\mu)(B \cap E) - \mu(H) = \alpha(\mu)(B' \cap E) - \mu(J).
\]

Take \( D \in \mathcal{F} \) with \( D \subseteq B \setminus E \). Write \( G = D \setminus H \). Then \( G \cup J \in \mathcal{F} \), and \( G \cup J \subseteq B' \cap E \). So,

\[
\mu(D) - \mu(H) \leq \mu(\mu) - \mu(H) = \mu(G) + \mu(H) - \mu(H) = \mu(G) + \mu(J) - \mu(J) = \mu(G \cup J) - \mu(J) \leq \alpha(\mu)(B' \cap E) - \mu(J).
\]

This shows that

\[
\alpha(\mu)(B \cap E) - \mu(H) = \alpha(\mu)(B' \cap E) - \mu(J).
\]

The reverse inequality follows similarly. This shows that the definition of \( \nu \) is sound.

**A2.** Take \( F \in \mathcal{F} \). We show that \( \nu(F) = \mu(F) \). Take \( A = B = F \). Then

\[
\nu(F) = \alpha(\mu)(A \cap E) + \mu(B) - \alpha(\mu)(B \cap E) = \mu(B) = \mu(F).
\]

We show that \( \nu(E) = \alpha(\mu)(E) \). Take \( A = X \) and \( B = \phi \). Then

\[
\nu(E) = \alpha(\mu)(X \cap E) + \mu(\phi) - \alpha(\mu)(\phi \cap E) = \alpha(\mu)(E).
\]

**A3.** Take two sets \( A \) and \( B \) in \( \mathcal{F} \) for which \( A \cap E \) and \( B \cap E \) are disjoint. We show that \( \alpha(\mu)(A \cap E) + \alpha(\mu)(B \cap E) = \alpha(\mu)((A \cup B) \cap E) \). It is clear that \( \alpha(\mu)(A \cap E) + \alpha(\mu)(B \cap E) \leq \alpha(\mu)(A \cap E) + \alpha(\mu)(B \cap E) \leq \)
α(µ)((A ∪ B) ∩ E). We show the reverse inequality. Take a C ∈ F with C ⊆ (A ∪ B) ∩ E). Define U = C ∩ A and V = C ∩ B. Then U, V ∈ F, U ∩ V = φ, and C = U ∪ V. So,

\[
\mu(C) = \mu(U) + \mu(V) \leq \alpha(\mu)(A ∩ E) + \alpha(\mu)(B ∩ E).
\]

This completes the proof.

A4. We show that ν is additive on F(E). Take two disjoint sets S and T in F(E). We show that ν(S) + ν(T) = ν(S ∪ T). Take sets A, B, C, D ∈ F such that S = (A ∩ E) ∪ (B \ E) and T = (C ∩ E) ∪ (D \ E). First note that

\[
S ∪ T = ((A ∪ C) ∩ E) ∪ ((B ∪ D) \ E).
\]

Note that A ∩ E and C ∩ E are disjoint. Moreover, B and D can be chosen in such a way that B ∩ D = φ. Then, using A3,

\[
\nu(S) + \nu(T) = \alpha(\mu)(A ∩ E) + \mu(B) - \alpha(\mu)(B ∩ E) + \alpha(\mu)(C ∩ E) + \mu(D) - \alpha(\mu)(D ∩ E) = \nu(S ∪ T).
\]

B. In the same way we can construct κ on G with κ(E) = β(µ)(E). Taking a convex combination shows that we can find a ν on G with ν(E) = r. ■

Let P(X) denote the collection of subsets of X. Note that P(X) is a field. Let µ be a charge on F. An extension of µ is a charge ν on P(X) with ν(A) = µ(A) for all A ∈ F. Let µ be an extension of (µ1, µ2). By monotonicity of µ,

\[
\alpha(µ)(E) \leq µ(E) \leq \beta(µ)(E)
\]

for every set E ∈ P(X × X).

Theorem 10.3 Assume AC. Let F be a field, and let µ be a charge on F. Let E ∈ P(X) and r ∈ R with α(µ)(E) ≤ r ≤ β(µ)(E). Then there exists an extension ν of µ such that ν(E) = r.

Proof. The statement now follows from the Lemma of Zorn. ■

For probability charges µ1 and µ2 on Ω and a set E ∈ P(Ω × Ω), we define

\[
\alpha(µ1, µ2)(E) = \sup \left\{ \sum_{i=1}^{k} µ1(A_i) \cdot µ2(B_i) \mid A_i \times B_i \text{ are mutually disjoint, contained in } E \right\},
\]
and

$$\beta(\mu_1, \mu_2)(E) = \inf \left\{ \sum_{i=1}^{k} \mu_1(A_i) \cdot \mu_2(B_i) \mid A_i \times B_i \text{ are mutually disjoint, and cover } E \right\}.$$  

Clearly $\alpha(\mu_1, \mu_2)(E) \leq \beta(\mu_1, \mu_2)(E)$. Let $\mu$ be an extension of $(\mu_1, \mu_2)$. By monotonicity of $\mu$,

$$\alpha(\mu_1, \mu_2)(E) \leq \mu(E) \leq \beta(\mu_1, \mu_2)(E)$$

for every set $E \in \mathcal{P}(\Omega \times \Omega)$

Conversely, as shown in Loš and Marczewski (1949) (see also Rao and Rao (1983)), we have

**Proposition 10.4** Let $\mu_1, \mu_2 \in \mathcal{C}(\Omega)$. Let $E \in \mathcal{P}(\Omega \times \Omega)$ and $r \in \mathbb{R}$ with $\alpha(\mu_1, \mu_2)(E) \leq r \leq \beta(\mu_1, \mu_2)(E)$. Then there exists an extension $\mu$ of $(\mu_1, \mu_2)$ such that $\mu(E) = r$.

**References**


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